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Interval-valued ideals in BCK-algebras

S. H. Han, S. M. Mostafa, K. Hur, Jong-Il Baek

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Abstract. In this paper, we introduce the concepts of interval-valued ideals, interval-valued positive implicative ideals, interval-valued positive ideals and commutative ideals, and investigate some of their properties respectively. In particular, we obtain some of their characterizations respectively.

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Keywords: Interval-valued set, Interval-valued (positive implicative, implicative, commutative) ideal, Interval-valued (positive implicative, implicative, commutative) BCK-algebra.

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1. Introduction

In 2009, Yao [1] introduced the notion of interval-valued sets as a generalization of classical sets and a special case of interval-valued sets proposed by Zadeh [2]. After then, Kim et al. [3] studied topological structures via interval-valued sets. Lee et al. [4] defined an interval-valued soft set combined with interval-valued sets and soft sets introduced by Molodtsov [5], and discussed some of its topological structures.

Our research aim is to extend the concepts of (positive implicative, implicative and commutative) ideals to interval-valued sets and to find some of their properties. In order to do our research, our article is composed of six sections. In Section 2, we recall some definitions of interval-valued sets, interval-valued points and BCK-algebras. In Section 3, we define an interval-valued subalgebra and interval-valued ideal of a BCK-algebra, and investigate some of its properties respectively and give some examples. In Section 4, we introduce the notion of interval-valued positive implicative ideals of BCK-algebra, and obtain some of its properties and its characterizations. Moreover, we define an interval-valued BCK-algebra and give its one characterization. In Section 5, we propose the concept of interval-valued implicative ideals of BCK-algebra and study some properties including its characterizations.

Also, we introduce notions of interval-valued implicative BCK-algebras and have its one characterization. In Section 6, We define an interval-valued commutative ideal of a BCK-algebra and discuss some properties including its characterizations.

2. Preliminaries

In this section, we list some basic concepts and one results needed next sections.

Definition 2.1 (See [1]). Let X be an non-empty set. Then the form

$$[A^-, A^+] = \{B \subset X : A^- \subset B \subset A^+\}$$

is called an *interval-valued set* (briefly, IVS) or *interval set* in X, if $A^-,\ A^+ \subset X$ and $A^- \subset A^+$. In this case, A^- [resp. A^+] represents the set of minimum [resp. maximum] memberships of elements of X to A. In fact, A^- [resp. A^+] is a minimum [resp. maximum] subset of X agreeing or approving for a certain opinion, view, suggestion or policy. $[\varnothing, \varnothing]$ [resp. $[X, X]$] is called the *interval-valued empty* [resp. *whole*] set in X and denoted by $\widetilde{\varnothing}$ [resp. $\widetilde{X}$]. We will denote the set of all IVSs in X as $IVS(X)$.

It is obvious that $[A, A] \in IVS(X)$ for classical subset A of X. Then we can consider an IVS in X as the generalization of a classical subset of X. Furthermore, if $A = [A^-, A^+] \in IVS(X)$, then

$$\chi_A = [\chi_{A^-}, \chi_{A^+}]$$

is an interval-valued fuzzy set in X introduced by Zadeh [2]. Thus we can consider an interval-valued fuzzy set as the generalization of an IVS.

Definition 2.2 (See [1]). Let X be a non-empty set and let $A,\ B \in IVS(X)$. Then

(i) we say that A *contained in* B, denoted by $A \subset B$, if $A^- \subset B^-$ and $A^+ \subset B^+$,

(ii) we say that A *equal to* B, denoted by $A = B$, if $A \subset B$ and $B \subset A$,

(iii) the *complement* of A, denoted A^c, is an interval-valued set in X defined by:

$$A^c = [(A^+)^c, (A^-)^c],$$

(iv) the *union* of A and B, denoted by $A \cup B$, is an interval-valued set in X defined by:

$$A \cup B = [A^- \cup A^-, A^+ \cup A^+],$$

(v) the *intersection* of A and B, denoted by $A \cap B$, is an interval-valued set in X defined by:

$$A \cap B = [A^- \cap A^-, A^+ \cap A^+].$$

Definition 2.3 ([3]). Let X be a non-empty set, let $a \in X$ and let $A \in IVS(X)$. Then the form $[\{a\}, \{a\}]$ [resp. $[\varnothing, \{a\}]$] is called an *interval-valued* [resp. *vanishing*] *point* in X and denoted by a_{IVP} [resp. a_{IVVP}]. We will denote the set of all interval-valued points in X as $IVP(X) = IV_P(X) \cup IV_{VP}(X)$, where $IV_P(X)$ [resp. $IV_{VP}(X)$] denotes the set of all interval-valued [resp. vanishing] points in X.

(i) We say that a_{IVP} *belongs to* A, denoted by $a_{IVP} \in A$, if $a \in A^-$.

(ii) We say that a_{IVVP} *belongs to* A, denoted by $a_{IVVP} \in A$, if $a \in A^+$.

Result 2.4 (Proposition 3.11 [3]). *Let X be a non-empty set and let $A \in IVS(X)$. Then*

$$A = A_{IVP} \cup A_{IVVP},$$

where $A_{IVP} = \bigcup_{a_{IVP} \in A} a_{IVP}$ *and* $A_{IVVP} = \bigcup_{a_{IVVP} \in A} a_{IVVP}$.

In fact, $A_{IVP} = [A^-, A^-]$ *and* $A_{IVVP} = [\varnothing, A^+]$

Definition 2.5 ([6]). Let X be a set with a binary operation $*$ and a constant 0. Then $(X; *, 0)$ is called a *BCK-algebra*, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

(BCI_1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCI_2) $(x * (x * y)) * y = 0$,
(BCI_3) $x * x = 0$,
(BCI_4) $x * y = 0$ and $y * x = 0$ imply $x = y$,
(BCK_5) $0 * x = 0$.

We define a binary relation $\leq$ on a BCK-algebra X as follows: for any $x,\ y \in X$,

$$x \leq y \text{ if and only if } x * y = 0.$$

3. Interval-valued ideals

In this section, we define an interval-valued ideal of a BCK-algebra X by modifying the concept of ideals of X introduced by Iséki [6]. And we obtain some of its properties.

Definition 3.1. Let $(X; *, 0)$ be a BCK-algebra. Then the binary operation $\circ_{IV}$ on $\widetilde{X}$ is defined as follows: for any $a_{IVP},\ b_{IVP},\ a_{IVVP},\ b_{IVVP} \in \widetilde{X}$,

$$a_{IVP} \circ_{IV} b_{IVP} = (a * b)_{IVP},$$

$$a_{IVP} \circ_{IV} b_{IVVP} = a_{IVVP} \circ_{IV} b_{IVP} = a_{IVVP} \circ_{IV} b_{IVP} = a_{IVVP} \circ_{IV} b_{IVVP} = (a * b)_{IVP}.$$

The following is an immediate consequence of Definitions 2.5 and 3.1.

Lemma 3.2. *Let $(X; *, 0)$ be a BCK-algebra. Then $(\widetilde{X}; \circ_{IV}, 0_{IVP})$ is a BCK-algebra, i.e., it satisfies the following conditions: for any $x,\ y,\ z \in X$,*

($IVBCI_1$) $((x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} (x_{IVP} \circ_{IV} z_{IVP})) \circ_{IV} (z_{IVP} \circ_{IV} y_{IVP}) = 0_{IVP}$,
$((x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} (x_{IVVP} \circ_{IV} z_{IVVP})) \circ_{IV} (z_{IVVP} \circ_{IV} y_{IVVP}) = 0_{IVVP}$,
($IVBCI_2$) $(x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP})) \circ_{IV} y_{IVP} = 0_{IVP}$,
$(x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP})) \circ_{IV} y_{IVVP} = 0_{IVVP}$,
($IVBCI_3$) $x_{IVP} \circ_{IV} x_{IVP} = 0_{IVP}$, $x_{IVVP} \circ_{IV} x_{IVVP} = 0_{IVVP}$,
($IVBCI_4$) $x_{IVP} \circ_{IV} y_{IVP} = 0_{IVP}$, $y_{IVP} \circ_{IV} x_{IVP} = 0_{IVP}$ *imply* $x_{IVP} = y_{IVP}$,
$x_{IVVP} \circ_{IV} y_{IVVP} = 0_{IVP}$, $y_{IVVP} \circ_{IV} x_{IVVP} = 0_{IVP}$ *imply* $x_{IVVP} = y_{IVVP}$,
($IVBCK_5$) $0_{IVP} \circ_{IV} x_{IVP} = 0_{IVP}$, $0_{IVVP} \circ_{IV} x_{IVVP} = 0_{IVVP}$.

In this case, $(\widetilde{X}; \circ_{IV}, 0_{IVP})$ is called an *interval-valued BCK-algebra*.

We define the binary relation $\leq_{IV}$ on $\widetilde{X}$ as follows: for any $x,\ y \in X$,

$$x_{IVP} \leq_{IV} y_{IVP} \text{ if and only if } x_{IVP} \circ_{IV} y_{IVP} = 0_{IVP}.$$

Also, the following is an immediate consequence of Definitions 2.5 and 3.1.

Lemma 3.3. *Let $(X;*,0)$ be a BCK-algebra. Then $([\varnothing, X]; \circ_{IV}, 0_{IVVP})$ is a BCK-algebra, i.e., it satisfies the following conditions: for any $x,\ y,\ z \in X$,*
(1) $((x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} (x_{IVVP} \circ_{IV} z_{IVVP})) \circ_{IV} (z_{IVVP} \circ_{IV} y_{IVVP}) = 0_{IVVP}$,
(2) $(x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP})) \circ_{IV} y_{IVVP} = 0_{IVVP}$,
(3) $x_{IVVP} \circ_{IV} x_{IVVP} = 0_{IVVP}$,
(4) $x_{IVVP} \circ_{IV} y_{IVVP} = 0_{IVVP}$, $y_{IVVP} \circ_{IV} x_{IVVP} = 0_{IVVP}$ *imply* $x_{IVVP} = y_{IVVP}$,
(5) $0_{IVVP} \circ_{IV} x_{IVVP} = 0_{IVVP}$.

We define the binary relation $\leq_{IV}$ on $[\varnothing, X]$ as follows: for any any $x,\ y \in X$,

$$x_{IVVP} \leq_{IV} y_{IVVP} \text{ if and only if } x_{IVVP} \circ_{IV} y_{IVVP} = 0_{IVVP}.$$

The proofs of the following propositions are easy by Lemmas 3.2 and 3.3.

Proposition 3.4. *Let $(X;*,0)$ be a BCK-algebra and let $x,\ y,\ z \in X$. Then we have*
(1) $x_{IVP} \leq_{IV} y_{IVP}$ *implies* $z_{IVP} \circ_{IV} y_{IVP} \leq_{IV} z_{IVP} \circ_{IV} x_{IVP}$,
(2) $x_{IVVP} \leq_{IV} y_{IVVP}$ *implies* $z_{IVVP} \circ_{IV} y_{IVVP} \leq_{IV} z_{IVVP} \circ_{IV} x_{IVVP}$,
(3) $x_{IVP} \leq_{IV} y_{IVP}$ *and* $y_{IVP} \leq_{IV} z_{IVP}$ *imply* $x_{IVP} \leq_{IV} z_{IVP}$,
(4) $x_{IVVP} \leq_{IV} y_{IVVP}$ *and* $y_{IVVP} \leq_{IV} z_{IVVP}$ *imply* $x_{IVVP} \leq_{IV} z_{IVVP}$.

Proposition 3.5. *Let $(X;*,0)$ be a BCK-algebra and let $x,\ y,\ z \in X$. Then*
(1) $(x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP} = (x_{IVP} \circ_{IV} z_{IVP}) \circ_{IV} y_{IVP}$,
(2) $(x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} z_{IVVP} = (x_{IVVP} \circ_{IV} z_{IVVP}) \circ_{IV} y_{IVVP}$.

Proposition 3.6. *Let $(X;*,0)$ be a BCK-algebra and let $x,\ y,\ z \in X$. Then $\widetilde{X}$ has the following properties:*
(1) $x_{IVP} \circ_{IV} y_{IVP} \leq_{IV} z_{IVP}$ *implies* $x_{IVP} \circ_{IV} z_{IVP}) \leq_{IV} y_{IVP}$,
(2) $x_{IVVP} \circ_{IV} y_{IVVP} \leq_{IV} z_{IVVP}$ *implies* $x_{IVVP} \circ_{IV} z_{IVVP} \leq_{IV} y_{IVVP}$,
(3) $(x_{IVP} \circ_{IV} z_{IVP}) \circ_{IV} (y_{IVP} \circ_{IV} z_{IVP}) \leq_{IV} x_{IVP} \circ_{IV} y_{IVP}$,
(4) $(x_{IVVP} \circ_{IV} z_{IVVP}) \circ_{IV} (y_{IVVP} \circ_{IV} z_{IVVP}) \leq_{IV} x_{IVVP} \circ_{IV} y_{IVVP}$,
(5) $x_{IVP} \leq_{IV} y_{IVP}$ *implies* $x_{IVP} \circ_{IV} z_{IVP} \leq_{IV} y_{IVP} \circ_{IV} z_{IVP}$,
(6) $x_{IVVP} \leq_{IV} y_{IVVP}$ *implies* $x_{IVVP} \circ_{IV} z_{IVVP} \leq_{IV} y_{IVVP} \circ_{IV} z_{IVVP}$,
(7) $x_{IVP} \circ_{IV} y_{IVP} \leq_{IV} x_{IVP}$, $x_{IVVP} \circ_{IV} y_{IVVP} \leq_{IV} x_{IVVP}$,
(8) $x_{IVP} \circ_{IV} 0_{IVP} = x_{IVP}$, $x_{IVVP} \circ_{IV} 0_{IVVP} = x_{IVVP}$.

Definition 3.7. Let $(X;*,0)$ be a BCK-algebra and let $\widetilde{\varnothing} \neq A \in IVS(X)$. Then A is called an *interval-valued subalgebra* of X, if $x_{IVP} \circ_{IV} y_{IVP},\ x_{IVVP} \circ_{IV} y_{IVVP} \in A$ for any $x_{IVP},\ y_{IVP},\ x_{IVVP},\ y_{IVVP} \in A$.

We will denote the set of all interval-valued subalgebras of X as $IVSA(X)$.

Remark 3.8. (1) From Definitions 3.1 and 3.7, it is obvious that $A \in IVSA(X)$ if and only if A^- and A^+ are subalgebras of X.

(2) If X_0 is a subalgebra of a BCK-algebra X, then clearly, $[X_0, X_0] \in IVSA(X)$. Furthermore, if X_0 and X_1 are subalgebras of X such that $X_0 \subset X_1$, then $[X_0, X_1] \in IVSA(X)$.

(3) It is well-known (Theorem I.2.7 [7]) that for any BCK-algebra with order $n(\geq 1)$,

$$1 \leq N(i) \leq C_{n-1}^{i-1},\ i = 1, 2, \cdots, n,$$

where $N(i)$ denotes the number of subalgebras with order i in X.
Then from (2) and the above fact, we can easily see that

$$n(IVSA(X)) = C_n^2 + \Sigma_{i=1}^n N(i),$$

where $n(IVSA(X))$ denotes the numbers of interval-valued subalgebras of X with order n.

(4) If $A \in IVSA(X)$, then clearly, $\chi_A = [\chi_{A^-}, \chi_{A^+}]$ is an interval-valued fuzzy subalgebra of X in the sense of Jun [8].

From (1) and (3), we can see that an interval-valued subalgebra is a generalization of a classical subalgebra ans a special case of an interval-valued fuzzy subalgebra.

The following is an immediate consequence of Lemma 3.2, Definitions 3.1 and 3.7, and Result 2.4.

Proposition 3.9. *Let $(X; *, 0)$ be a BCK-algebra and let $A \in IVSA(X)$. Then*
(1) $0_{IVP} \in A$,
(2) $(A; \circ_{IV}, 0_{IVP})$ *is an interval-valued BCK-algebra,*
(3) $\widetilde{X} \in IVSA(X)$,
(4) $0_{IVP} \in IVSA(X)$,

Example 3.10 (See Example 1 in 12 page [7]). Consider the BCK-algebra $(X; *, 0)$ with the operation $*$ given by Table 3.1:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Table 3.1

Then from Remark 3.8 (2) and Proposition 3.9, 0_{IVP}, $[\{0\}, \{0, a\}]$ and $\widetilde{X}$ are interval-valued subalgebras of X. Furthermore, we can easily calculate $n(IVSA(X))$:

$$n(IVSA(X)) = C_8^2 + \Sigma_{i=1}^4 N(i) = 36.$$

Definition 3.11. Let $(X; *, 0)$ be a BCK-algebra and let $\widetilde{\varnothing} \neq I \in IVS(X)$. Then I is called an *interval-valued ideal* (briefly, IVI) of X, if it satisfies the following conditions: for any $x,\ y \in X$,
(IVI$_1$) $0_{IVP} \in I$,
(IVI$_2$) $x_{IVP} \circ_{IV} y_{IVP} \in I$ and $y_{IVP} \in I$ imply $x_{IVP} \in I$,
(IVI$_3$) $x_{IVVP} \circ_{IV} y_{IVVP} \in I$ and $y_{IVVP} \in I$ imply $x_{IVVP} \in I$.
We denote the set of all IVIs of X as $IVI(X)$.

It is clear that $0_{IVP},\ \widetilde{X} \in IVI(X)$ and we will call $\widetilde{X}$ an *interval-valued trivial ideal*. An interval-valued ideal I is said to be *proper*, if $I \neq \widetilde{X}$.

Remark 3.12. (1) I is an interval-valued ideal of a BCK-algebra X if and only if I^- and I^+ are ideals of X.
(2) If A is an ideal of a BCK-algebra X, then $[A, A] \in IVI(X)$.

(3) If $A \in IVI(X)$, then we can easily see that $\chi_{_A}$ is an interval-valued fuzzy ideal of X in the sense of Jun [8].

From (2) and (3), we can see that an interval-valued ideal is a characterization of a classical ideal and a special case of an interval-valued fuzzy ideal.

Example 3.13. (1) Consider the implicative BCK-algebra with $*$ (See Example 1 in 64 page [7]) given by Table 3.2:

$*$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Table 3.2

Then we can easily obtain $IVI(X)$:

$$IVI(X) = \{0_{_{IVP}}, \widetilde{X}, [\{0\},\{0,1\}], [\{0\},\{0,2\}], [\{0,1\},\{0,1\}], [\{0,2\},\{0,2\}], [\{0\}, X], \ [\{0,1\}, X], [\{0,2\}, X]\}.$$

(2) Let X be the commutative BCK-algebra with $*$ (See Example 2 in 65 page [7]) given by Table 3.3:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	1	0

Table 3.3

Then clearly, $IVI(X) = \{0_{_{IVP}}, \widetilde{X}, [\{0\}, X]\}$.

(3) Let X be any BCK-algebra and let $\bar{X}$ be the Iséki's extension of X (See Theorem 3.6 in 16 page [7]). Then it is obvious that $\widetilde{X} \in IVI(\bar{X})$.

Proposition 3.14. *Let X be a BCK-algebra, let $I \in IVI(X)$ and let $x,\ y \in X$.*

(1) If $y_{_{IVP}} \leq_{_{IV}} x_{_{IVP}}$ and $x_{_{IVP}} \in I$, then $y_{_{IVP}} \in I$,

(2) If $y_{_{IVVP}} \leq_{_{IV}} x_{_{IVVP}}$ and $x_{_{IVVP}} \in I$, then $y_{_{IVVP}} \in I$.

Proof. (1) Suppose $y_{_{IVP}} \leq_{_{IV}} x_{_{IVP}}$ and $x_{_{IVP}} \in I$. Then we have

$$y_{_{IVP}} \circ_{_{IV}} x_{_{IVP}} = (y * x)_{_{IVP}} = 0_{_{IVP}} \in I.$$

Thus by the condition (IVI_2), $y_{_{IVP}} \in I$.

(2) The proof is similar to (1). □

Definition 3.15. Let X be a BCK-algebra. Then for any $a,\ b \in X$, we define the interval-valued set $A(a, b)$ in X as follows:

$$A(a, b) = A(a_{_{IVP}}, b_{_{IVP}}) \bigcup A(a_{_{IVVP}}, b_{_{IVVP}}), \text{ where}$$

$$A(a_{_{IVP}}, b_{_{IVP}}) = \bigcup \{x_{_{IVP}} \in \widetilde{X} : x_{_{IVP}} \circ_{_{IV}} a_{_{IVP}} \leq_{_{IV}} b_{_{IVP}}\},$$

$$A(a_{_{IVVP}}, b_{_{IVVP}}) = \bigcup \{x_{_{IVVP}} \in \widetilde{X} : x_{_{IVVP}} \circ_{_{IV}} a_{_{IVVP}} \leq_{_{IV}} b_{_{IVVP}}\}.$$

It is obvious that $0_{_{IVP}},\ 0_{_{IVVP}},\ a_{_{IVP}},\ a_{_{IVVP}},\ b_{_{IVP}},\ b_{_{IVVP}} \in A(a,b)$.

The following is an characterization of interval-valued ideals.

Theorem 3.16. *Let X be a BCK-algebra and let $\widetilde{\varnothing} \neq I \in IVS(X)$. Then $I \in IVI(X)$ if and only if $A(x,y) \subset I$, i.e., $A(x_{_{IVP}}, y_{_{IVP}}) \subset I$ and $A(x_{_{IVVP}}, y_{_{IVVP}}) \subset I$ for any $x_{_{IVP}},\ y_{_{IVP}},\ x_{_{IVVP}},\ y_{_{IVVP}} \in I$.*

Proof. Suppose $I \in IVI(X)$ and let $x_{_{IVP}},\ y_{_{IVP}} \in I$. Let $z_{_{IVP}} \in A(x_{_{IVP}}, y_{_{IVP}})$. Then clearly, $z_{_{IVP}} \circ_{_{IV}} x_{_{IVP}} \leq_{_{IV}} x_{_{IVP}}$. Thus by Proposition 3.14 (1), $z_{_{IVP}} \circ_{_{IV}} x_{_{IVP}} \in I$. So by ($\mathrm{IVI}_2$), $z_{_{IVP}} \in I$. Hence $A(x_{_{IVP}}, y_{_{IVP}}) \subset I$. Similarly, we have $A(x_{_{IVVP}}, y_{_{IVVP}}) \subset I$.

Conversely, suppose the necessary conditions hold. Since $I \neq \widetilde{\varnothing}$, there is $x \in X$ such that $x_{_{IVP}},\ x_{_{IVVP}} \in I$. Then by ($\mathrm{IVBCK}_5$), we get

$$0_{_{IVP}} \circ_{_{IV}} x_{_{IVP}} \leq_{_{IV}} x_{_{IVP}},\ 0_{_{IVVP}} \circ_{_{IV}} x_{_{IVVP}} \leq_{_{IV}} x_{_{IVVP}}.$$

Thus by the hypothesis, $0_{_{IVP}} \in A(x_{_{IVP}}, x_{_{IVP}}) \subset I$ and $0_{_{IVVP}} \in A(x_{_{IVVP}}, x_{_{IVVP}}) \subset I$. So the condition ($\mathrm{IVI}_1$) holds.

Suppose $x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}} \in I$ and $y_{_{IVP}} \in I$. Then by (IVBCI_2), we have

$$x_{_{IVP}} \circ_{_{IV}} (x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}}) \leq_{_{IV}} y_{_{IVP}}.$$

Thus $x_{_{IVP}} \in A(x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}}, y_{_{IVP}}) \subset I$. So the condition ($\mathrm{IVI}_2$) holds.

Now suppose $x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}} \in I$ and $y_{_{IVVP}} \in I$. Then by (IVBCI_2), we have

$$x_{_{IVVP}} \circ_{_{IV}} (x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}}) \leq_{_{IV}} y_{_{IVVP}}.$$

Thus $x_{_{IVVP}} \in A(x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}}, y_{_{IVVP}}) \subset I$. So the condition ($\mathrm{IVI}_3$) holds. Hence $I \in IVI(X)$. □

The following is an immediate consequence of Theorem 3.16.

Corollary 3.17. *Let X be a BCK-algebra, let $\widetilde{\varnothing} \neq I \in IVS(X)$ and let $z \in X$. Then $I \in IVI(X)$ if and only if for any $x_{_{IVP}},\ y_{_{IVP}},\ x_{_{IVVP}},\ y_{_{IVVP}} \in I$,*

$$(z_{_{IVP}} \circ_{_{IV}} x_{_{IVP}}) \circ_{_{IV}} y_{_{IVP}} = 0_{_{IVP}} \text{ implies } z_{_{IVP}} \in I$$

and

$$(z_{_{IVVP}} \circ_{_{IV}} x_{_{IVVP}}) \circ_{_{IV}} y_{_{IVVP}} = 0_{_{IVVP}} \text{ implies } z_{_{IVVP}} \in I.$$

Proposition 3.18. *Every interval-valued ideal of a BCK-algebra X is an interval-valued subalgebra of X.*

Proof. Let $I \in IVI(X)$ and suppose $x_{_{IVP}},\ y_{_{IVP}},\ x_{_{IVVP}},\ y_{_{IVVP}} \in I$. Then by Proposition 3.6 (7), $x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}} \leq_{_{IV}} x_{_{IVP}}$ and $x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}} \leq_{_{IV}} x_{_{IVVP}}$. Thus by Proposition 3.14, $x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}},\ x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}} \in I$. So $I \in IVSA(X)$. □

4. Interval-valued positive implicative ideals

In this section, we introduce the concept of interval-valued positive implicative ideals of a BCK-algebra X by modifying the notion of positive implicative ideals of X defined by Iséki [6]. We study some of its properties. In particular, we give tow characterizations of interval-valued ideals. Moreover, by modifying the concept of positive implicative BCK-algebras introduced by Iséki [9], we define an interval-valued positive implicative BCK-algebra and obtain a characterization of interval-valued positive implicative BCK-algebras.

Definition 4.1. Let X be a BCK-algebra and let $\widetilde{\varnothing} \neq I \in IVS(X)$. Then I is called an *interval-valued positive implicative ideal* (briefly, IVPII), if it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(IVI$_1$) $0_{_{IVP}} \in I$,
(IVPII$_2$) $(x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}}) \circ_{_{IV}} z_{_{IVP}} \in I$ and $y_{_{IVP}} \circ_{_{IV}} z_{_{IVP}} \in I$ imply $x_{_{IVP}} \circ_{_{IV}} z_{_{IVP}} \in I$,
(IVPII$_3$) $(x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}}) \circ_{_{IV}} z_{_{IVVP}} \in I$ and $y_{_{IVVP}} \circ_{_{IV}} z_{_{IVVP}} \in I$ imply $x_{_{IVVP}} \circ_{_{IV}} z_{_{IVVP}} \in I$.
We will denote the set of all IVPIIs of X as $IVPII(X)$.

Remark 4.2. (1) If A is a positive implicative ideal of a BCK-algebra X, then clearly, $[A, A] \in IVPII(X)$.
(2) If I is an interval-valued positive implicative ideal of a BCK-algebra X, then I^- and I^+ are positive implicative ideals of X.

Proposition 4.3. *Every interval-valued positive implicative ideal a BCK-algebra X is an interval-valued ideal of X but the converse is not true in general* (See Example 4.4).

Proof. Let $I \in IVPII(X)$ and suppose $x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}},\ y_{_{IVP}} \in I$. Then by the condition (IVPII$_1$) and Proposition 3.6 (8), we have

$$(x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}}) \circ_{_{IV}} 0_{_{IVP}},\ y_{_{IVP}} \circ_{_{IV}} 0_{_{IVP}} \in I.$$

Thus by the condition (IVPII$_2$) and Proposition 3.6 (8), $x_{_{IVP}} = x_{_{IVP}} \circ_{_{IV}} 0_{_{IVP}} \in I$. So the condition (IVI$_2$) holds.

Now suppose $x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}},\ y_{_{IVVP}} \in I$. Then by the condition (IVPII$_1$) and Proposition 3.6 (8), we get

$$(x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}}) \circ_{_{IV}} 0_{_{IVVP}},\ y_{_{IVVP}} \circ_{_{IV}} 0_{_{IVVP}} \in I.$$

Thus by the condition (IVPII$_3$) and Proposition 3.6 (8), $x_{_{IVVP}} = x_{_{IVVP}} \circ_{_{IV}} 0_{_{IVVP}} \in I$. So the condition (IVI$_3$) holds. Hence $I \in IVI(X)$. □

Example 4.4. Let us consider the BCK-algebra $(X; *.0)$ having the binary operation $*$ given by Table 4.1 (See Example 1 in 69 page [7]):

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Table 4.1

We can easily check that

$$IVPII(X) = \{[\{0, 1, 3\}, \{0, 1, 3\}], [\{0, 1, 3\}, \{0, 1, 2, 3\}], [\{0, 1, 2, 3\}, \{0, 1, 2, 3\}]\}.$$

Moreover, by Proposition 4.3, we have

$$IVI(X)=\{0_{_{IVP}},[\{0\},\{0,2\}],[\{0,2\},\{0,2\}],[\{0\},\{0,2,4\}],[\{0,2\},\{0,2,4\}],[\{0,2,4\},\{0,2,4\}],[\{0\},\{0,1,3\}],[\{0\},\{0,1,2,3\}],[\{0,2\},\{0,1,2,3\}]\}\cup IVPII(X).$$

However, every member of $IVI(X)\cap IVPII(X)^c$ is not positive implicative. For example, let us the interval-valued ideal $[\{0,2\},\{0,2\}]$ of X. It is obvious that

$$(3_{_{IVP}}\circ_{_{IV}}2_{_{IVP}})\circ_{_{IV}}0_{_{IVP}},\ 2_{_{IVP}}\circ_{_{IV}}0_{_{IVP}}\in[\{0,2\},\{0,2\}].$$

But $3_{_{IVP}}\circ_{_{IV}}0_{_{IVP}}=3_{_{IVP}}\notin[\{0,2\},\{0,2\}]$. So $[\{0,2\},\{0,2\}]\notin IVPII(X)$.

The following is an characterization of interval-valued positive implicative ideals.

Theorem 4.5. *Let X be a BCK-algebra and let $I\in IVI(X)$. For each $a\in X$, let A_a be the interval-valued set in X defined by:*

$$A_a=A_{a_{_{IVP}}}\bigcup A_{a_{_{IVVP}}},\ \textit{where}$$

$$A_{a_{_{IVP}}}=\bigcup\{x_{_{IVP}}\in\widetilde{X}:x_{_{IVP}}\circ_{_{IV}}a_{_{IVP}}\in I\},$$

$$A_{a_{_{IVVP}}}=\bigcup\{x_{_{IVVP}}\in\widetilde{X}:x_{_{IVVP}}\circ_{_{IV}}a_{_{IVVP}}\in I\}.$$

Then $I\in IVPII(X)$ if and only if $A_a\in IVI(X)$ for each $a\in X$.

Proof. Suppose $I\in IVPII(X)$ and $x_{_{IVP}}\circ_{_{IV}}y_{_{IVP}},\ y_{_{IVP}}\in A_a$. Then clearly,

$$(x_{_{IVP}}\circ_{_{IV}}y_{_{IVP}})\circ_{_{IV}}a_{_{IVP}},\ y_{_{IVP}}\circ_{_{IV}}a_{_{IVP}}\in I.$$

Thus by the condition ($IVPII_2$), $x_{_{IVP}}\circ_{_{IV}}a_{_{IVP}}\in I$. So $x_{_{IVP}}\in A_{a_{_{IVP}}}$. Similarly, by the condition ($IVPII_3$), we have $x_{_{IVVP}}\in A_{a_{_{IVVP}}}$. Hence $A_a\in IVI(X)$.

Conversely, suppose the necessary condition holds. Suppose $(x_{_{IVP}}\circ_{_{IV}}y_{_{IVP}})\circ_{_{IV}}z_{_{IVP}},\ y_{_{IVP}}\circ_{_{IV}}z_{_{IVP}}\in I$. Then clearly, $x_{_{IVP}}\circ_{_{IV}}y_{_{IVP}},\ y_{_{IVP}}\in A_{z_{_{IVP}}}$. Since $A_z\in IVI(X)$, $x_{_{IVP}}\in A_{z_{_{IVP}}}$. Thus $x_{_{IVP}}\circ_{_{IV}}z_{_{IVP}}\in I$. Similarly, we get $x_{_{IVVP}}\circ_{_{IV}}z_{_{IVVP}}\in I$. So $I\in IVPII(X)$. □

Corollary 4.6. *Let X be a BCK-algebra. If $I\in IVPII(X)$, then for each $a\in X$, A_a is the least interval-valued ideal of X such that $I\cup\{a_{_{IVP}}\}\subset A_a$.*

Proof. Suppose $I\in IVPII(X)$ and let J be any interval-valued ideal of X such that $I\cup\{a_{_{IVP}}\}\subset J$. Let $x_{_{IVP}}\in A_a$. Then clearly, $x_{_{IVP}}\circ_{_{IV}}a_{_{IVP}}\in I$. Since $I\cup\{a_{_{IVP}}\}\subset J$, $x_{_{IVP}}\circ_{_{IV}}a_{_{IVP}}\in J$ and $a_{_{IVP}}\in J$. Thus $x_{_{IVP}}\in J$. Now let $x_{_{IVVP}}\in A_a$. Then similarly, we have $x_{_{IVVP}}\in J$. So $A_a\subset J$. Hence the result holds. □

We give another characterization of interval-valued positive implicative ideals.

Theorem 4.7. *Let X be a BCK-algebra and let $\widetilde{\varnothing}\neq I\in IVS(X)$. Then the followings are equivalent:*

(1) *$I\in IVPII(X)$,*

(2) *$I\in IVI(X)$ and for any $x,\ y\in X$,*

(2_a) *if $(x_{_{IVP}}\circ_{_{IV}}y_{_{IVP}})\circ_{_{IV}}y_{_{IVP}}$, then $x_{_{IVP}}\circ_{_{IV}}y_{_{IVP}}\in I$,*

(2_b) *if $(x_{_{IVVP}}\circ_{_{IV}}y_{_{IVVP}})\circ_{_{IV}}y_{_{IVVP}}\in I$, then $x_{_{IVVP}}\circ_{_{IV}}y_{_{IVVP}}\in I$,*

(3) *$I\in IVI(X)$ and for any $x,\ y,\ z\in X$,*

(3_a) *if $(x_{_{IVP}}\circ_{_{IV}}y_{_{IVP}})\circ_{_{IV}}z_{_{IVP}}\in I$, then $(x_{_{IVP}}\circ_{_{IV}}z_{_{IVP}})\circ_{_{IV}}(y_{_{IVP}}\circ_{_{IV}}z_{_{IVP}})\in I$,*

(3_b) *if* $(x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} z_{IVVP} \in I$, *then*
$$(x_{IVVP} \circ_{IV} z_{IVVP}) \circ_{IV} (y_{IVVP} \circ_{IV} z_{IVVP}) \in I,$$
(4) $I \in IVI(X)$ *and for any* $x,\ y,\ z \in X$,
(4_a) *if* $((x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP},\ z_{IVP} \in I$, *then* $x_{IVP} \circ_{IV} y_{IVP} \in I$,
(4_b) *if* $((x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} y_{IVVP}) \circ_{IV} z_{IVVP},\ z_{IVVP} \in I$, *then*
$$x_{IVVP} \circ_{IV} y_{IVVP} \in I.$$

Proof. The proofs are almost similar to ones of Theorem II.2.5 [7]. □

Proposition 4.8. *Let* X *be a* BCK*-algebra and let* $I,\ A \in IVI(X)$ *such that* $I \subset A$. *If* $I \in IVPII(X)$, *then* $A \in IVPII(X)$.

Proof. Suppose $I \in IVPII(X)$. Let $(x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP},\ (x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} z_{IVVP} \in A$ and let $u_{IVP} = (x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP}$, $u_{IVVP} = (x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} z_{IVVP}$. Then we have

$$\begin{aligned} &((x_{IVP} \circ_{IV} u_{IVP}) \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP} \\ &= ((x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} u_{IVP}) \circ_{IV} z_{IVP} \text{ [By Proposition 3.5 (1)]} \\ &= ((x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP}) \circ_{IV} u_{IVP} \text{ [By Proposition 3.5 (1)]} \\ &= 0_{IVP}. \text{ [By (IVBCI}_3\text{)]} \end{aligned}$$

Similarly, we get $((x_{IVVP} \circ_{IV} u_{IVVP}) \circ_{IV} y_{IVVP}) \circ_{IV} z_{IVVP} = 0_{IVVP}$. Since $I \in IVPII(X)$, by the condition (IVI$_1$), we get

$$((x_{IVP} \circ_{IV} u_{IVP}) \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP} \in I, \tag{4.1}$$

$$((x_{IVVP} \circ_{IV} u_{IVVP}) \circ_{IV} y_{IVVP}) \circ_{IV} z_{IVVP} \in I. \tag{4.2}$$

Thus by Theorem 4.7 (3_a) and (3_b), we have

$$((x_{IVP} \circ_{IV} u_{IVP}) \circ_{IV} z_{IVP}) \circ_{IV} (y_{IVP} \circ_{IV} z_{IVP}) \in I, \tag{4.3}$$

$$((x_{IVVP} \circ_{IV} u_{IVVP}) \circ_{IV} z_{IVVP}) \circ_{IV} (y_{IVVP} \circ_{IV} z_{IVVP}) \in I. \tag{4.4}$$

On the other hand, from Proposition 3.5, (4.3) and (4.4), we obtain

$$\begin{aligned} &((x_{IVP} \circ_{IV} u_{IVP}) \circ_{IV} z_{IVP}) \circ_{IV} (y_{IVP} \circ_{IV} z_{IVP}) \\ &= ((x_{IVP} \circ_{IV} z_{IVP}) \circ_{IV} (y_{IVP} \circ_{IV} z_{IVP})) \circ_{IV} ((x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP}) \in I, \end{aligned}$$

$$\begin{aligned} &((x_{IVVP} \circ_{IV} u_{IVVP}) \circ_{IV} z_{IVVP}) \circ_{IV} (y_{IVVP} \circ_{IV} z_{IVVP}) \\ &= ((x_{IVVP} \circ_{IV} z_{IVVP}) \circ_{IV} (y_{IVVP} \circ_{IV} z_{IVVP})) \\ &\quad \circ_{IV} ((x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} z_{IVVP}) \in I. \end{aligned}$$

Since $I \subset A$, we have

$$((x_{IVP} \circ_{IV} z_{IVP}) \circ_{IV} (y_{IVP} \circ_{IV} z_{IVP})) \circ_{IV} ((x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP}) \in A, \tag{4.5}$$

$$((x_{IVVP} \circ_{IV} z_{IVVP}) \circ_{IV} (y_{IVVP} \circ_{IV} z_{IVVP})) \circ_{IV} ((x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} z_{IVVP}) \in A. \tag{4.6}$$

Since $(x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP},\ (x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} z_{IVVP} \in A$ and $A \in IVI(X)$, $(x_{IVP} \circ_{IV} z_{IVP}) \circ_{IV} (y_{IVP} \circ_{IV} z_{IVP}),\ (x_{IVVP} \circ_{IV} z_{IVVP}) \circ_{IV} (y_{IVVP} \circ_{IV} z_{IVVP}) \in A$. So the conditions ($3_a$) and ($3_b$) hold. Hence by Theorem 4.7, $A \in IVPII(X)$. □

Definition 4.9 ([9]). Let X be a BCK-algebra. Then X is said to be *positive implicative*, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

$$(x * z) * (y * z) = (x * y) * z.$$

The following is an immediate consequence of Definitions 3.1 and 4.9.

Lemma 4.10. *Let $\widetilde{X}$ be an interval-valued positive implicative BCK-algebra. Then the followings hold: for any $x,\ y,\ z \in X$,*

(1) $(x_{IVP} \circ_{IV} z_{IVP}) \circ_{IV} (y_{IVP} \circ_{IV} z_{IVP}) = (x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} y_{IVP}$,
(2) $(x_{IVVP} \circ_{IV} z_{IVVP}) \circ_{IV} (y_{IVVP} \circ_{IV} z_{IVVP}) = (x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} z_{IVVP}$.

In this case, $\widetilde{X}$ is called an *interval-valued positive implicative BCK-algebra.*

We give a characterization of interval-valued positive implicative BCK-algebras.

Lemma 4.11. *Let X be a BCK-algebra. Then the followings are equivalent: for any $x,\ y \in X$,*

(1) *$\widetilde{X}$ is interval-valued positive implicative,*
(2) $x_{IVP} \circ_{IV} y_{IVP} = (x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} y_{IVP}$,
$x_{IVVP} \circ_{IV} y_{IVVP} = (x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} y_{IVVP}$,
(3) $(x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP})) \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})$
$= x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})))$,
$(x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP})) \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})$
$= x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})))$,
(4) $x_{IVP} \circ_{IV} y_{IVP} = (x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} (x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP}))$,
$x_{IVVP} \circ_{IV} y_{IVVP} = (x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} (x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP}))$,
(5) $x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP}) = (x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP})) \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP})$,
$x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP})$
$= (x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP})) \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP})$,
(5) $(x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP})) \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})$
$= (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})) \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP})$,
$(x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP})) \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})$
$= (y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})) \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP})$.

Proof. The proofs are similar to Theorem I.4.2 in [7]. □

We give a characterization of interval-valued positive implicative BCK-algebras.

Theorem 4.12. *Let X be a BCK-algebra. Then the followings are equivalent:*

(1) *$\widetilde{X}$ is an interval-valued positive implicative BCK-algebra,*
(2) $0_{IVP} \in IVPII(X)$,
(3) *for each* $I \in IVI(C)$, $I \in IVPII(X)$,
(4) *for each* $a \in X$, $A(a) = A(a_{IVP}) \bigcup A(a_{IVVP}) \in IVI(X)$, *where*

$$A(a_{IVP}) = \bigcup \{x_{IVP} \in \widetilde{X} : x_{IVP} \leq_{IV} a_{IVP}\},$$

$$A(a_{IVVP}) = \bigcup \{x_{IVVP} \in \widetilde{X} : x_{IVVP} \leq_{IV} a_{IVVP}\}.$$

Proof. (1)⇒(2): Suppose $\widetilde{X}$ is an interval-valued positive implicative BCK-algebra. It is clear that $0_{IVP} \in IVI(X)$ and $0_{IVVP} \in 0_{IVP}$. For for any $x,\ y,\ z \in X$, let

$$(x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} y_{IVP},\ (x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} y_{IVVP} \in 0_{IVP}.$$

Then by Lemma 4.11 (2), we have

$$(x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} y_{IVP}) = x_{IVP} \circ_{IV} y_{IVP},$$

$$(x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}}) \circ_{_{IV}} y_{_{IVVP}}) = x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}}.$$

Thus $x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}},\ x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}} \in 0_{_{IVP}}$. So by Theorem 4.7 (2), $0_{_{IVP}} \in IVPII(X)$.

(2)⇒(3): The proof is clear from Proposition 4.8.

(3)⇒(4): Suppose the condition (2) holds. For any $a,\ x,\ y \in X$, let $x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}},\ y_{_{IVP}} \in A(a_{_{IVP}})$. Then clearly, $x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}} \leq_{_{IV}} a_{_{IVP}}$ and $y_{_{IVP}} \leq_{_{IV}} a_{_{IVP}}$. Thus $(x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}}) \circ_{_{IV}} a_{_{IVP}} = 0_{_{IVP}} \in 0_{_{IVP}}$ and $y_{_{IVP}} \circ_{_{IV}} a_{_{IVP}} = 0_{_{IVP}} \in 0_{_{IVP}}$. By the hypothesis, $x_{_{IVP}} \circ_{_{IV}} a_{_{IVP}} = 0_{_{IVP}} \in 0_{_{IVP}}$, i.e., $x_{_{IVP}} \leq_{_{IV}} a_{_{IVP}}$. So $x_{_{IVP}} \in A(a_{_{IVP}})$. Now let $x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}},\ y_{_{IVP}} \in A(a_{_{IVVP}})$. Then by the similar way, we can see that $x_{_{IVVP}} \in A(a_{_{IVVP}})$. Thus $x_{_{IVP}},\ x_{_{IVP}} \in A(a)$. Hence $A(a) \in IVI(X)$.

(4)⇒(1): Suppose the condition (4) holds and $(x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}}) \circ_{_{IV}} y_{_{IVP}} = 0_{_{IVP}}$ and $(x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}}) \circ_{_{IV}} y_{_{IVVP}} = 0_{_{IVVP}}$. Then clearly, $x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}},\ x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}} \in A(y)$. By the definition of $A(y)$, $y_{_{IVP}},\ y_{_{IVVP}} \in A(y)$. Since $A(y) \in IVI(X)$, $x_{_{IVP}},\ x_{_{IVVP}} \in A(y)$, i.e., $x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}} = 0_{_{IVP}},\ x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}} = 0_{_{IVVP}}$. Thus by the condition ($IVBCK_5$), we get

$$(x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}}) \circ_{_{IV}} y_{_{IVP}} = x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}},$$

$$(x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}}) \circ_{_{IV}} y_{_{IVVP}} = x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}}.$$

So the condition (1) holds. □

We give another characterization of interval-valued positive implicative BCK-algebras.

Theorem 4.13. *Let X be a BCK-algebra. Then $\widetilde{X}$ is an interval-valued positive implicative BCK-algebra if and only if for each $I \in IVI(X)$ and each $a \in X$, $A_{_a} \in IVI(X)$.*

Proof. Suppose $\widetilde{X}$ is an interval-valued positive implicative BCK-algebra, let $I \in IVI(X)$ and let $a \in X$. Then by Theorem 4.12, $I \in IVPII(X)$. Thus Theorem 4.5, $A \in IVI(X)$.

Suppose the necessary condition holds. In order to show that $\widetilde{X}$ is an interval-valued positive implicative BCK-algebra, from Theorem 4.12, it is sufficient to prove that $I \in IVPII(X)$ for each $I \in IVI(X)$. For any $x,\ y,\ z \in X$, let

$$(x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}}) \circ_{_{IV}} z_{_{IVP}},\ y_{_{IVP}} \circ_{_{IV}} z_{_{IVP}} \in I,$$

$$(x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}}) \circ_{_{IV}} z_{_{IVVP}},\ y_{_{IVVP}} \circ_{_{IV}} z_{_{IVVP}} \in I.$$

Let $A = \{u_{_{IVP}} \in \widetilde{X} : u_{_{IVP}} \circ_{_{IV}} z_{_{IVP}} \in I\} \bigcup \{u_{_{IVVP}} \in \widetilde{X} : u_{_{IVVP}} \circ_{_{IV}} z_{_{IVVP}} \in I\}$. Then clearly, $x_{_{IVP}} \circ_{_{IV}} y_{_{IVP}},\ y_{_{IVP}} \in A$ and $x_{_{IVVP}} \circ_{_{IV}} y_{_{IVVP}},\ y_{_{IVVP}} \in A$. From the hypothesis, it is obvious that $A \in IVI(X)$. Thus $x_{_{IVP}},\ x_{_{IVVP}} \in A$. So we get

$$x_{_{IVP}} \circ_{_{IV}} z_{_{IVP}},\ x_{_{IVVP}} \circ_{_{IV}} z_{_{IVVP}} \in I.$$

Hence $I \in IVPII(X)$. Therefore by Theorem 4.12, $\widetilde{X}$ is an interval-valued positive implicative BCK-algebra. □

5. Interval-valued implicative ideals

In this section, we define an interval-valued implicative ideal of a BCK-algebra X by modifying the notion of implicative ideals of X introduced by Meng [12]. We discuss some of its properties. Moreover, we give tow characterizations of interval-valued implicative ideals. In particular, we propose the notion of interval-valued implicative BCK-algebras and obtain its one characterization.

Definition 5.1. Let X be a BCK-algebra and let $\widetilde{\varnothing} \neq I \in IVS(X)$. Then I is called an *interval-valued implicative ideal* (briefly, IVII), if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

(IVI$_1$) $0_{IVP} \in I$,

(IVII$_2$) $(x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})) \circ_{IV} z_{IVP} \in I$ and $z_{IVP} \in I$ imply $x_{IVP} \in I$,

(IVII$_3$) $(x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})) \circ_{IV} z_{IVVP} \in I$ and $z_{IVVP} \in I$ imply $x_{IVVP} \in I$.

We will denote the set of all IVIIs of X as $IVII(X)$.

Remark 5.2. (1) If A is a positive implicative ideal of a BCK-algebra X, then clearly, $[A, A] \in IVII(X)$.

(2) If I is an interval-valued positive ideal of a BCK-algebra X, then I^- and I^+ are positive implicative ideals of X.

Example 5.3. (1) Let X be a BCK-algebra. Then clearly, $\widetilde{X} \in IVII(X)$. In this case, $\widetilde{X}$ is called the *interval-valued trivial implicative ideal* of X.

(2) Let $(X; *.0)$ be the BCK-algebra with the binary operation $*$ given by Table 5.1 (See Example 3 in 73 page [7]):

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	4	4	4	0

Table 5.1

Then clearly, X is neither positive implicative nor commutative. Furthermore, we can easily check that $[\{0, 1, 2, 3\}, \{0, 1, 2, 3\}],\ [\{0, 1, 2, 3\}, X] \in IVII(X)$.

Proposition 5.4. *Every interval-valued implicative ideal of a BCK-algebra X is an interval-valued ideal of X but the converse is not true in general* (See Example 5.5).

Proof. Let $I \in IVII(X)$ and suppose $x_{IVP} \circ_{IV} z_{IVP},\ z_{IVP} \in I$. Then by (IVBCI$_3$) and Proposition 3.6 (8), we have

$$(x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} x_{IVP}) \circ_{IV} z_{IVP} = x_{IVP} \circ_{IV} z_{IVP}.$$

Since $x_{IVP} \circ_{IV} z_{IVP} \in I$, $(x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} x_{IVP}) \circ_{IV} z_{IVP} \in I$. Thus by the condition (IVII$_2$), $x_{IVP} \in I$. So the condition (IVI$_2$) holds. Similarly, we prove that the condition (IVI$_3$) holds. Hence $I \in IVI(X)$. □

Example 5.5. Let $(X;*.0)$ be the BCK-algebra given in Example 4.4. Then We can easily check that $\{[\{0,2\},\{0,2\}] \in IVI(X)$ but $\{[\{0,2\},\{0,2\}] \notin IVII(X)$. Moreover, $\{[\{0,1,3\},\{0,1,3\}] \in IVPII(X)$ but $\{[\{0,1,3\},\{0,1,3\}] \notin IVII(X)$.

Proposition 5.6. *Every interval-valued implicative ideal of a BCK-algebra X is an interval-valued positive implicative ideal of X but the converse is not true in general* (See Example 5.5).

Proof. Let $I \in IVII(X)$ and suppose $(x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP},\ y_{IVP} \circ_{IV} z_{IVP} \in I$ and $(x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} z_{IVVP},\ y_{IVVP} \circ_{IV} z_{IVVP} \in I$. Then we get

$$\begin{aligned}&((x_{IVP} \circ_{IV} z_{IVP}) \circ_{IV} z_{IVP}) \circ_{IV} (y_{IVP} \circ_{IV} z_{IVP})\\ &\leq_{IV} (x_{IVP} \circ_{IV} z_{IVP}) \circ_{IV} y_{IVP} \text{ [By Lemma 3.6 (3)]}\\ &= (x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP}. \text{ [By Lemma 3.5 (1)]}\end{aligned}$$

Similarly, by Lemmas 3.6 (4) and 3.5 (2), we have

$$((x_{IVP} \circ_{IV} z_{IVP}) \circ_{IV} z_{IVP}) \circ_{IV} (y_{IVP} \circ_{IV} z_{IVP}) \leq_{IV} (x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP}.$$

Since $(x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP} \in I$ and $(x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} z_{IVVP} \in I$, by Proposition 3.14 (1) and (2), we have

$$((x_{IVP} \circ_{IV} z_{IVP}) \circ_{IV} z_{IVP}) \circ_{IV} (y_{IVP} \circ_{IV} z_{IVP}) \in I,$$

$$((x_{IVVP} \circ_{IV} z_{IVVP}) \circ_{IV} z_{IVVP}) \circ_{IV} (y_{IVVP} \circ_{IV} z_{IVVP}) \in I.$$

Since $y_{IVP} \circ_{IV} z_{IVP},\ y_{IVVP} \circ_{IV} z_{IVVP} \in I$ and $I \in IVI(X)$ by Proposition 5.4, by the conditions (IVI_2) and (IVI_3), we get

$$(x_{IVP} \circ_{IV} z_{IVP}) \circ_{IV} z_{IVP} \in I,$$

$$(x_{IVVP} \circ_{IV} z_{IVVP}) \circ_{IV} z_{IVVP} \in I.$$

On the other hand, we have

$$\begin{aligned}&(x_{IVP} \circ_{IV} z_{IVP}) \circ_{IV} (x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} z_{IVP}))\\ &= (x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} z_{IVP}))) \circ_{IV} z_{IVP} \text{ [By Proposition 3.5 (1)]}\\ &= (x_{IVP} \circ_{IV} z_{IVP}) \circ_{IV} z_{IVP} \in I\end{aligned}$$

Similarly, we get

$$(x_{IVVP} \circ_{IV} z_{IVVP}) \circ_{IV} (x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} z_{IVVP})) = (x_{IVVP} \circ_{IV} z_{IVVP}) \circ_{IV} z_{IVVP} \in I.$$

Thus by Proposition 3.6 (8),

$$((x_{IVP} \circ_{IV} z_{IVP}) \circ_{IV} (x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} z_{IVP}))) \circ_{IV} 0_{IVP} \in I,$$

$$((x_{IVVP} \circ_{IV} z_{IVVP}) \circ_{IV} (x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} z_{IVVP}))) \circ_{IV} 0_{IVVP} \in I.$$

Since $0_{IVP} \in I$ and $I \in IVI(X)$, by the conditions (IVI_2) and (IVI_3), we have

$$x_{IVP} \circ_{IV} z_{IVP},\ x_{IVVP} \circ_{IV} z_{IVVP} \in I.$$

So the conditions ($IVPII_2$) and ($IVPII_3$) hold. Hence $I \in IVPII(X)$. □

We give a characterization of interval-valued implicative ideals.

Theorem 5.7. *Let X be a BCK-algebra and let $I \in IVPII(X)$. Then $I \in IVII(X)$ if and only if it satisfies the following conditions: for any $x,\ y \in X$,*

(5.1) $$y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}) \in I \text{ implies } x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP}) \in I,$$

(5.2) $$y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \in I \text{ implies } x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP}) \in I.$$

Proof. Suppose $I \in IVII(X)$ and let $y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}) \in I$ and $y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \in I$. From Proposition 3.5 (1) and (2), Lemma 3.2,

$$x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP}) \leq_{IV} x_{IVP},$$

$$x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP}) \leq_{IV} x_{IVVP}.$$

Then by Proposition 3.4 (1) and (2), we have

$$y_{IVP} \circ_{IV} x_{IVP} \leq_{IV} y_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP})),$$

$$y_{IVVP} \circ_{IV} x_{IVVP} \leq_{IV} y_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP})).$$

Thus we get

$$\begin{aligned} &(x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP})) \circ_{IV} (y_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP}))) \\ \leq_{IV}\ &(x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP})) \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}) \text{ [By Proposition 3.4 (1)]} \\ \leq_{IV}\ &y_{IVP} \circ_{IV} (y_{IVP})) \circ_{IV} x_{IVP}). \text{ [By (IVBCI}_2\text{)]} \end{aligned}$$

Also, by Proposition 3.4 (2) and (IVBCI$_2$), we have

$$\begin{aligned} &(x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP})) \circ_{IV} (y_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP}))) \\ \leq_{IV}\ &y_{IVVP} \circ_{IV} (y_{IVVP})) \circ_{IV} x_{IVVP}). \end{aligned}$$

Since $I \in IVPII(X)$, by Proposition 4.3, $I \in IVI(X)$. So by Proposition 3.6 (8), Proposition 3.14 (1) and (2), we get

$$\begin{aligned} &(x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP})) \circ_{IV} (y_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP}))) \circ_{IV} 0_{IVP} \\ = &(x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP})) \circ_{IV} (y_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP}))) \in I, \end{aligned}$$

$$\begin{aligned} &(x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP})) \circ_{IV} (y_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP}))) \\ &\circ_{IV} 0_{IVVP} \\ = &(x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP})) \circ_{IV} (y_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP}))) \in I. \end{aligned}$$

Since $I \in IVII(X)$, by the conditions (IVII$_2$) and (IVII$_3$),

$$x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP}),\ x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP}) \in I.$$

Hence (5.1) and (5.2) hold.

Conversely, suppose the necessary conditions (5.1) and (5.2) hold. Let

$$(x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})) \circ_{IV} z_{IVP},\ z_{IVP} \in I$$

and

$$(x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})) \circ_{IV} z_{IVVP},\ z_{IVVP} \in I.$$

Since $I \in IVI(X)$, by the conditions (IVI$_2$) and (IVI$_3$), we have

$$x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}),\ x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \in I.$$

On the other hand, by (IVBCI$_2$), we get

$$\begin{aligned} &(y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})) \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}) \\ \leq_{IV}\ &x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}) \in I, \end{aligned}$$

$$\begin{aligned} &(y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})) \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \\ \leq_{IV}\ &x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \in I. \end{aligned}$$

Then $y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}),\ y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \in I$. Thus by Theorem 4.7 ($2_a$) and ($2_b$), we get

$$y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}),\ y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \in I.$$

By the conditions (5.1) and (5.2), we have

(5.3) $$x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP}),\ x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP}) \in I.$$

Now we can easily see that the followings hold:

$$(x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP} \leq_{IV} x_{IVP} \circ_{IV} y_{IVP} \leq_{IV} x_{IVP} \circ_{IV} (y_{IVP}) \circ_{IV} z_{IVP}) \in I,$$

$$(x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} z_{IVVP} \leq_{IV} x_{IVVP} \circ_{IV} y_{IVVP} \leq_{IV} x_{IVVP} \circ_{IV} (y_{IVVP}) \circ_{IV} z_{IVVP}) \in I.$$

So $(x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP},\ (x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} z_{IVVP} \in I$. Since $z_{IVVP} \in I$ and $I \in IVI(X)$, by the conditions (IVI_2) and (IVI_3), we get

$$x_{IVP} \circ_{IV} y_{IVP},\ x_{IVVP} \circ_{IV} y_{IVVP} \in I.$$

Hence from (5.3), $x_{IVP},\ x_{IVVP} \in I$. Therefore $I \in IVII(X)$. □

Also, we give another characterization of interval-valued implicative ideals.

Theorem 5.8. *Let X be a BCK-algebra and let $I \in IVI(X)$. Then $I \in IVII(X)$ if and only if it satisfies the following conditions: for any $x,\ y \in X$,*

(5.4) $$x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}) \in I \text{ implies } x_{IVP} \in I,$$

(5.5) $$x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \in I \text{ implies } x_{IVVP} \in I.$$

Proof. Suppose $I \in IVII(X)$ and let $x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}) \in I$. Then clearly,

$$(x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})) \circ_{IV} 0_{IVP} = x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}) \in I.$$

Since $0_{IVP} \in I$ and $I \in IVII(X)$, by the condition ($IVII_2$), $x_{IVP} \in I$. Thus (5.4) holds. Similarly, we can easily prove that (5.5) holds.

Conversely, suppose the necessary conditions (5.4) and (5.5) hold. Let

$$(x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})) \circ_{IV} z_{IVP},\ z_{IVP} \in I$$

and

$$(x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})) \circ_{IV} z_{IVVP},\ z_{IVVP} \in I.$$

Since $I \in IVI(X)$, $x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}),\ x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \in I$. Then by the hypothesis, $x_{IVP},\ x_{IVVP} \in I$. Thus $I \in IVII(X)$. □

The following is a similar consequence of Proposition 4.8.

Proposition 5.9. *Let X be a BCK-algebra. If $I \in IVII(X)$, then $A \in IVII(X)$ for each $A \in IVI(X)$ such that $I \subset A$.*

Proof. Suppose $I \in IVII(X)$ and let $A \in IVI(X)$ such that $I \subset A$. By Proposition 5.6, $I \in IVPII(X)$. By Proposition 4.8, $A \in IVPII(X)$. In order to show that $A \in IVII(X)$, it is sufficient to prove that A satisfies the conditions (5.1) and (5.2).

Suppose $y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}),\ y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \in A$ and let $u_{IVP} = y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}),\ u_{IVVP} = y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})$. Then clearly, we have

$$y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}) \circ_{IV} u_{IVP} = 0_{IVP} \in I,$$

$$y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \circ_{IV} u_{IVVP} = 0_{IVVP} \in I.$$

Since $I \in IVPII(X)$, by Theorem 4.7 (3_a) and (3_b), Proposition 3.5 (1) and (2),

$$(y_{IVP} \circ_{IV} u_{IVP}) \circ_{IV} ((y_{IVP} \circ_{IV} x_{IVP}) \circ_{IV} u_{IVP})$$

$$= (y_{IVP} \circ_{IV} u_{IVP}) \circ_{IV} ((y_{IVP} \circ_{IV} u_{IVP}) \circ_{IV} x_{IVP}) \in I,$$

$$\begin{aligned} &(y_{IVP} \circ_{IV} u_{IVP}) \circ_{IV} ((y_{IVP} \circ_{IV} x_{IVP}) \circ_{IV} u_{IVP}) \\ &= (y_{IVVP} \circ_{IV} u_{IVVP}) \circ_{IV} ((y_{IVVP} \circ_{IV} u_{IVVP}) \circ_{IV} x_{IVVP}) \in I. \end{aligned}$$

Since $I \in IVII(X)$, by Theorem 5.7 (5.1) and (5.2),

$$x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} u_{IVP})),\ x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} u_{IVVP})) \in I.$$

Since $I \subset A$, we have

$$x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} u_{IVP})),\ x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} u_{IVVP})) \in A.$$

On the other hand, we get

$$\begin{aligned} &x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} (x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} u_{IVP}))) \\ &\leq_{IV} (x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} u_{IVP})) \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP}) \\ &\leq_{IV} y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} u_{IVP}) \\ &= y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}))) \\ &= y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}) \in A. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} (x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} u_{IVVP}))) \\ &\leq_{IV} y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \in A. \end{aligned}$$

Thus we get

$$x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} (x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} u_{IVP}))) \in A,$$

$$x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} (x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} u_{IVVP}))) \in A.$$

Since $x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} u_{IVP})),\ x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} u_{IVVP})) \in A$ and $A \in IVI(X)$, we have

$$x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP}),\ x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP}) \in A.$$

So by Theorem 5.7, $A \in IVII(X)$. □

Definition 5.10. Let X be a BCK-algebra. Then $\widetilde{X}$ is called an *interval-valued positive BCK-algebra*, if it satisfies the following conditions: for any $x,\ y \in X$,

$$x_{IVP} = x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP}),\ x_{IVVP} = x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP}).$$

Now we give a characterization of interval-valued positive BCK-algebra.

Theorem 5.11. *Let X be a BCK-algebra. Then the followings are equivalent:*
(1) $0_{IVP} \in IVII(X)$,
(2) *for each* $I \in IVI(X)$, $I \in IVPII(X)$,
(3) *for each* $a \in X$, $A(a) \in IVPII(X)$,
(4) $\widetilde{X}$ *is an interval-valued implicative BCK-algebra.*

Proof. (1)⇔(2)⇔(3): The proofs are straightforward from Theorem 5.9.

(4)⇒(1): The proof is clear.

(1)⇒(4): Suppose $0_{IVP} \in IVII(X)$. Then by Proposition 5.6, $0_{IVP} \in IVPII(X)$. By Theorem 4.12, $A(x * (y * x)) \in IVI(X)$ for any $x,\ y \in X$. By the condition (2), $A(x * (y * x)) \in IVII(X)$. From the definition of $A(x * (y * x))$, we have

$$x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}),\ x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \in A(x * (y * x)).$$

Thus $x_{_{IVP}},\ x_{_{IVVP}} \in A(x*(y*x))$. So we get

$$x_{_{IVP}} \leq_{IV} x_{_{IVP}} \circ_{IV} (y_{_{IVP}} \circ_{IV} x_{_{IVP}}),$$

$$x_{_{IVVP}} \leq_{IV} x_{_{IVVP}} \circ_{IV} (y_{_{IVVP}} \circ_{IV} x_{_{IVVP}}).$$

On the other hand, by Proposition 3.5 and (IVBCK_5), we have

$$x_{_{IVP}} \circ_{IV} (y_{_{IVP}} \circ_{IV} x_{_{IVP}}) \leq_{IV} x_{_{IVP}},$$

$$x_{_{IVVP}} \circ_{IV} (y_{_{IVVP}} \circ_{IV} x_{_{IVVP}}) \leq_{IV} x_{_{IVVP}}.$$

Hence $x_{_{IVP}} = x_{_{IVP}} \circ_{IV} (y_{_{IVP}} \circ_{IV} x_{_{IVP}}),\ x_{_{IVVP}} = x_{_{IVVP}} \circ_{IV} (y_{_{IVVP}} \circ_{IV} x_{_{IVVP}})$. Therefore $\widetilde{X}$ is an interval-valued implicative BCK-algebra. □

6. Interval-valued commutative ideals

In this section, by modifying the concept of commutative ideals proposed by Meng [13], we define an interval-valued commutative ideal and investigate some of its properties. Finally, by using the notion of commutative BCK-algebras defined by Tanaka [10, 11], we define an interval-valued commutative BCK-algebra and give its one characterization.

Definition 6.1. Let X be a BCK-algebra and let $\widetilde{\varnothing} \neq I \in IVS(X)$. Then I is called an *interval-valued commutative ideal* (briefly, IVCI), if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

(IVI_1) $0_{_{IVP}} \in I$,

(IVCI_2) $x_{_{IVP}} \circ_{IV} y_{_{IVP}} \in I$ imply $x_{_{IVP}} \circ_{IV} (y_{_{IVP}} \circ_{IV} (y_{_{IVP}} \circ_{IV} x_{_{IVP}})) \in I$,

(IVCI_3) $(x_{_{IVVP}} \circ_{IV} y_{_{IVVP}}) \circ_{IV} z_{_{IVVP}} \in I$ and $z_{_{IVVP}} \in I$ imply
$x_{_{IVVP}} \circ_{IV} (y_{_{IVVP}} \circ_{IV} (y_{_{IVVP}} \circ_{IV} x_{_{IVVP}})) \in I$.

We will denote the set of all IVCIs of X as $IVCI(X)$.

It is obvious that $\widetilde{X} \in IVCI(X)$ and it is called the *interval-valued trivial commutative ideal* of X

Remark 6.2. (1) If A is a commutative ideal of a BCK-algebra X, then clearly, $[A, A] \in IVCI(X)$.

(2) If I is an interval-valued commutative ideal of a BCK-algebra X, then I^- and I^+ are positive commutative ideals of X.

Example 6.3. Consider the BCK-algebra $(X; *.0)$ given in with the binary operation $*$ given in Example 4.4. Then we can easily check that:

$$[\{0,2\},\{0,2\}],\ [\{0,2\},\{0,2,4\}],\ [\{0,2,4\},\{0,2,4\}] \in IVCI(X)$$

but

$$[\{0,2\},\{0,2\}],\ [\{0,2\},\{0,2,4\}],\ [\{0,2,4\},\{0,2,4\}] \notin IVPII(X).$$

Moreover, $[\{0,1,3\},\{0,1,3\}] \in IVPII(X)$ but $[\{0,1,3\},\{0,1,3\}] \notin IVCI(X)$. Also $[\{0,1,2,3\},\{0,1,2,3\}] \in IVII(X)$.

Proposition 6.4. *Every interval-valued commutative ideal of a BCK-algebra X is an interval-valued ideal of X but the converse is not true in general* (See Example 6.3).

Proof. Let $I \in IVCI(X)$ and suppose $x_{IVP} \circ_{IV} y_{IVP},\ y_{IVP} \in I$ and $x_{IVVP} \circ_{IV} y_{IVVP},\ y_{IVVP} \in I$. Then by Proposition 3.6 (8), we have

$$(x_{IVP} \circ_{IV} 0_{IVP}) \circ_{IV} y_{IVP},\ y_{IVP} \in I \text{ and } (x_{IVVP} \circ_{IV} 0_{IVVP}) \circ_{IV} y_{IVVP},\ y_{IVVP} \in I.$$

Thus by Proposition 3.6 (8), and the conditions ($IVCI_2$) and ($IVCI_3$), we get

$$x_{IVP} = x_{IVP} \circ_{IV} (0_{IVP} \circ_{IV} (0_{IVP} \circ_{IV} x_{IVP})) \in I,$$
$$x_{IVVP} = x_{IVVP} \circ_{IV} (0_{IVVP} \circ_{IV} (0_{IVVP} \circ_{IV} x_{IVVP})) \in I.$$

So the conditions (IVI_2) and (IVI_3) hold. Hence $I \in IVI(X)$. □

We give a characterization of interval-valued commutative ideals.

Theorem 6.5. *Let X be a BCK-algebra. Then $I \in IVCI(X)$ if and only if it satisfies the following conditions: for any $x,\ y \in X$,*

(6.1) $$x_{IVP} \circ_{IV} y_{IVP} \in I \text{ implies } x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP}) \circ_{IV} x_{IVP})) \in I,$$

(6.2) $$x_{IVVP} \circ_{IV} y_{IVVP} \in I \text{ implies } x_{IVVP} \circ_{IV} (y_{VIVP} \circ_{IV} (y_{IVVP}) \circ_{IV} x_{IVVP})) \in I.$$

Proof. Suppose $I \in IVCI(X)$ and let $x_{IVP} \circ_{IV} y_{IVP} \in I$ and $x_{IVVP} \circ_{IV} y_{IVVP} \in I$. Then clearly, we get

$$(x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} 0_{IVP},\ 0_{IVP} \in I \text{ and } (x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} 0_{IVVP},\ 0_{IVVP} \in I.$$

Thus by the conditions ($IVCI_2$) and ($IVCI_2$), we have

$$x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}) \in I,$$
$$x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \in I.$$

So (6.1) and (6.2) hold.

Conversely, suppose the necessary conditions (6.1) and (6.2) hold. Let

$$(x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} z_{IVP},\ z_{IVP} \in I \text{ and } (x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} z_{IVVP},\ z_{IVVP} \in I.$$

Since $I \in IVI(X)$, $x_{IVP} \circ_{IV} y_{IVP},\ x_{IVVP} \circ_{IV} y_{IVVP} \in I$. Thus by the conditions (6.1) and (6.2), we have

$$x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP}) \circ_{IV} x_{IVP})),\ x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (y_{IVVP}) \circ_{IV} x_{IVVP})) \in I.$$

So $I \in IVCI(X)$. □

Now we give a relationships of interval-valued commutative ideals, interval-valued implicative ideals and interval-valued positive implicative ideals.

Theorem 6.6. *Let X be a BCK-algebra and let $\widetilde{\varnothing} \neq I \in IVS(X)$. Then $I \in IVII(X)$ if and only if $I \in IVCI(X)$ and $I \in IVPII(X)$.*

Proof. Suppose $I \in IVII(X)$. From Proposition 5.6, it is obvious that $I \in IVPII(X)$. It is sufficient to show that $I \in IVCI(X)$ by using Theorem 6.5. Let $x_{IVP} \circ_{IV} y_{IVP},\ x_{IVP} \circ_{IV} y_{IVP} \in I$. From Proposition 3.5 (1) and (2), and ($IVBCK_5$), it is clear that

$$x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} x_{IVP})) \leq_{IV} x_{IVP},$$
$$x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} x_{IVP})) \leq_{IV} x_{IVP}.$$

By Proposition 3.4 (1) and (2), we have

$$y_{IVP} \circ_{IV} x_{IVP} \leq_{IV} y_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}))),$$

$$y_{IVVP} \circ_{IV} x_{IVVP} \leq_{IV} y_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}))).$$

Let $u_{IVP} = x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}))$. Then we get

$$\begin{aligned}
&u_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} u_{IVVP}) \\
&= (x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}))) \\
&\quad \circ_{IV} (y_{IVVP} \circ_{IV} (x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})))) \\
&\leq_{IV} (x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}))) \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}) \\
&= x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})) \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})) \\
&\leq_{IV} x_{IVP} \circ_{IV} y_{IVP} \in I.
\end{aligned}$$

For $u_{IVVP} = x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}))$, similarly, we have

$$u_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} u_{IVVP}) \leq_{IV} x_{IVVP} \circ_{IV} y_{IVVP} \in I.$$

Then $u_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} u_{IVP}),\ u_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} u_{IVVP}) \in I$. Thus by Theorem 5.8, $u_{IVP},\ u_{IVVP} \in I$, i.e., we get

$$x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})) \in I,$$

$$x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})) \in I.$$

So by Theorem 6.5, $I \in IVCI(X)$.

Conversely, suppose $I \in IVCI(X)$ and $I \in IVPII(X)$. In order to prove that $I \in IVII(X)$, we use Theorem 5.8. Let

$$x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}),\ x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \in I.$$

We can easily see that the followings hold:

$$(y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})) \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}) \leq_{IV} x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}),$$

$$(y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})) \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \leq_{IV} x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}).$$

Then we have

$$(y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})) \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}) \in I,$$

$$(y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})) \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \in I.$$

Since $I \in IVPII(X)$, by Theorem 4.7 (2_a) and (2_b), we get

$$y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}),\ y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}) \in I. \tag{6.3}$$

Furthermore, we have

$$x_{IVP} \circ_{IV} y_{IVP} \leq_{IV} x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}),$$

$$x_{IVVP} \circ_{IV} y_{IVVP} \leq_{IV} x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}).$$

Thus $x_{IVP} \circ_{IV} y_{IVP},\ x_{IVVP} \circ_{IV} y_{IVVP} \in I$. Since $I \in IVCI(X)$, by Theorem 6.5,

$$x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})) \in I,$$

$$x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})) \in I.$$

So by (6.3), $x_{IVP},\ x_{IVVP} \in I$. Hence by Theorem 5.8, $I \in IVII(X)$. □

Proposition 6.7. *Let X be a BCK-algebra and let $I,\ A \in IVI(X)$ such that $I \subset A$. If $I \in IVCI(X)$, then $A \in IVCI(X)$.*

Proof. Let $x_{IVP} \circ_{IV} y_{IVP},\ x_{IVVP} \circ_{IV} y_{IVVP} \in A$ and let $u_{IVP} = x_{IVP} \circ_{IV} y_{IVP},\ u_{IVVP} = x_{IVVP} \circ_{IV} y_{IVVP}$. Then we have

$$(x_{IVP} \circ_{IV} u_{IVP}) \circ_{IV} y_{IVP} = (x_{IVP} \circ_{IV} y_{IVP}) \circ_{IV} u_{IVP} = 0_{IVP} \in I,$$

$$(x_{IVVP} \circ_{IV} u_{IVVP}) \circ_{IV} y_{IVVP} = (x_{IVVP} \circ_{IV} y_{IVVP}) \circ_{IV} u_{IVVP} = 0_{IVVP} \in I.$$

Since $I \in IVCI(X)$, by Theorem 6.5, we get

$$(x_{IVP} \circ_{IV} u_{IVP}) \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} u_{IVP}))) \in I,$$

$$(x_{IVVP} \circ_{IV} u_{IVVP}) \circ_{IV} (y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} u_{IVVP}))) \in I.$$

Thus by the hypothesis $I \subset A$ and Proposition 3.5, we have

$$\begin{aligned}&(x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} u_{IVP})))) \circ_{IV} u_{IVP}\\ &= (x_{IVP} \circ_{IV} u_{IVP}) \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} u_{IVP}))) \in A,\end{aligned}$$

$$\begin{aligned}&(x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} u_{IVVP})))) \circ_{IV} u_{IVVP}\\ &= (x_{IVVP} \circ_{IV} u_{IVVP}) \circ_{IV} (y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} u_{IVVP}))) \in A.\end{aligned}$$

Since $u_{IVP} \in A$ and $A \in IVI(X)$, we have

$$(x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} u_{IVP})))) \in A,$$

$$(x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} u_{IVVP})))) \in A.$$

On the other hand, by (IVBCI_1), Proposition 3.5 (1), (IVBCI_3) and (IVBCK_5), we get

$$\begin{aligned}&(x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})))\\ &\quad \circ_{IV} (x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} u_{IVP}))))\\ &\leq_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} u_{IVP}))) \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}))\\ &\leq_{IV} (y_{IVP} \circ_{IV} x_{IVP}) \circ_{IV} (y_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} u_{IVP}))\\ &\leq_{IV} (x_{IVP} \circ_{IV} u_{IVP}) \circ_{IV} x_{IVP}\\ &= (x_{IVP} \circ_{IV} x_{IVP}) \circ_{IV} u_{IVP}\\ &= 0_{IVP} \circ_{IV} u_{IVP}.\\ &= 0_{IVP} \in A.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&(x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})))\\ &\quad \circ_{IV} (x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} u_{IVVP}))))\\ &\leq_{IV} 0_{IVVP} \in A.\end{aligned}$$

Since $A \in IVI(X)$, we get

$$x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})),\ x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})) \in A.$$

So by Theorem 6.5, $A \in IVCI(X)$. □

Definition 6.8. Let X be a BCK-algebra. Then $\widetilde{X}$ is called an *interval-valued commutative BCK-algebra*, if it satisfies the following conditions: for any $x,\ y \in X$,

$$x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP}) = y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP}),$$

$$x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP}) = y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP}).$$

We obtain a characterization of interval-valued commutative BCK-algebras.

Theorem 6.9. *Let X be a BCK-algebra. Then The followings are equivalent: for any $x,\ y \in X$,*
(1) $\widetilde{X}$ *is an interval-valued commutative BCK-algebra,*
(2) $x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP}) \leq_{IV} y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})$,
$x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP}) \leq_{IV} y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})$,
(3) $(x_{IVP} \circ_{IV} (x_{IVP} \circ_{IV} y_{IVP})) \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})) = 0_{IVP}$,
$(x_{IVVP} \circ_{IV} (x_{IVVP} \circ_{IV} y_{IVVP})) \circ_{IV} (y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})) = 0_{IVVP}$.

Proof. The proof is straightforward from Definition 6.8 and (IVBCI_3). □

We give another characterization of interval-valued commutative BCK-algebras.

Theorem 6.10. *Let X be a BCK-algebra. Then The followings are equivalent: for any $x,\ y \in X$,*
(1) $x_{IVP} \leq_{IV} z_{IVP}$ *and* $z_{IVP} \circ_{IV} y_{IVP} \leq_{IV} z_{IVP} \circ_{IV} x_{IVP}$ *imply* $x_{IVP} \leq_{IV} y_{IVP}$,
$x_{IVVP} \leq_{IV} z_{IVVP}$ *and* $z_{IVVP} \circ_{IV} y_{IVVP} \leq_{IV} z_{IVVP} \circ_{IV} x_{IVVP}$ *imply*
$x_{IVVP} \leq_{IV} y_{IVVP}$,
(2) $x_{IVP},\ y_{IVP} \leq_{IV} z_{IVP}$ *and* $z_{IVP} \circ_{IV} y_{IVP} \leq_{IV} z_{IVP} \circ_{IV} x_{IVP}$ *imply*
$x_{IVP} \leq_{IV} y_{IVP}$,
$x_{IVVP},\ y_{IVVP} \leq_{IV} z_{IVVP}$ *and* $z_{IVVP} \circ_{IV} y_{IVVP} \leq_{IV} z_{IVVP} \circ_{IV} x_{IVVP}$ *imply*
$x_{IVVP} \leq_{IV} y_{IVVP}$,
(3) $x_{IVP} \leq_{IV} y_{IVP}$ *implies* $x_{IVP} = y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})$,
$x_{IVVP} \leq_{IV} y_{IVVP}$ *implies* $x_{IVVP} = y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})$,
(4) $\widetilde{X}$ *is an interval-valued commutative BCK-algebra,*
(5) $x_{IVP} \circ_{IV} y_{IVP} = 0_{IVP}$ *implies* $x_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} (y_{IVP} \circ_{IV} x_{IVP})) = 0_{IVP}$,
$x_{IVVP} \circ_{IV} y_{IVVP} = 0_{IVVP}$ *implies*
$x_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} (y_{IVVP} \circ_{IV} x_{IVVP})) = 0_{IVVP}$.

Proof. The proof is similar to Theorem I.5.6 in [7]. □

Now we give a characterization of interval-valued commutative BCK-algebras by interval-valued commutative ideals.

Theorem 6.11. *Let X be a BCK-algebra. Then The followings are equivalent:*
(1) $0_{IVP} \in IVCI(X)$,
(2) $I \in IVCI(X)$ *for each* $I \in IVI(X)$,
(4) $\widetilde{X}$ *is an interval-valued commutative BCK-algebra.*

Proof. (1)⇔(2): The proof is straightforward from Proposition 6.7.
(2)⇔(3): The proof is easy from Theorem 6.10. □

7. Conclusions

Throughout this paper, we could see that classical (positive implicative, implicative and commutative) ideals of a BCK-algebra is naturally extended to interval-valued sets. In the future, we expect that we or one can apply the concept of interval-valued sets to semigroup, group and ring theory, category theory and decision-making problem, etc.

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