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A new fixed point result in double controlled fuzzy metric space with application

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Abstract. In this paper, we introduce the notion of double controlled fuzzy metric spaces which is an extension of the result of Sezen [1]. The paper concerns our sustained efforts for the materialization of double controlled fuzzy metric spaces. Further, we establish a Banach-type fixed point theorem. We provide suitable examples with graphic that validate our result. We also employ an application to substantiate the utility of our established result to show the existence and unique solution of an integral equation.

2020 AMS Classification: 54H25, 47H10

Keywords: Fixed point, Control function, Controlled fuzzy metric spaces, Double controlled fuzzy metric spaces.

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1. Introduction

In 1922, Banach [2] provided an important result to fixed point theory. This topic has been studied, presented and generalized by many researchers in many different spaces. Firstly, the work of Bakhtin [3], Bourbaki [4] and Czerwik [5] expanded the theory of fixed points for b-metric spaces. Also, many authors proved some important fixed point theorems in b-metric spaces (See [6, 7, 8]). Later, Abdeljawad et al. [9] proved some fixed point results in partial b-metric spaces. Mlaiki et al. [10] introduced controlled metric spaces and proved some fixed point theorems. Abdeljawad et al. [11] modified controlled metric spaces called double controlled metric spaces.

On the other hand, after producing the fuzzy subject of Zadeh [12], Kramosil and Michalek [13] introduced the concept of fuzzy metric spaces, which can be regarded as a generalization of the statistical metric spaces. Subsequently, Grabiec [14] defined G-complete fuzzy metric spaces and extended the complete fuzzy metric spaces.

Following Grabiec's work, many authors introduced and generalized the different types of fuzzy contractive mappings and investigated some fixed point theorems in fuzzy metric spaces. George and Veeramani [15] modified the notion of M-complete fuzzy metric spaces with the help of continuous t-norms.

Nădăban [16] introduced the concept of fuzzy b-metric spaces. Kim et al. [17] established some fixed point results in fuzzy b-metric spaces. Recently, Mehmood et al. [18] has defined a new concept called extended fuzzy b-metric spaces, which is the generalization of fuzzy b-metric spaces.

I. Demir [19, 20] introduced the concept of a complex valued fuzzy b-metric space and soft complex valued b-metric space and investigate some of its properties. Also, he established some fixed point theorems in the context of complex valued fuzzy b-metric spaces and soft complex valued b-metric spaces. Most recently Sangurlu [1] introduced controlled fuzzy metric spaces, which is a generalization of extended fuzzy b-metric spaces.

2. Preliminaries

Now, we begin with some basic concepts, notations and definitions. Let $\mathbb{R}$ represent the set of real numbers, $\mathbb{R}^+$ represent the set of all non-negative real numbers and $\mathbb{N}$ represent the set of natural numbers.

We start by the following definition of a fuzzy metric space.

Definition 2.1 ([15]). An ordered triple $(X, M, *)$ is called a *fuzzy metric space* such that X is a nonempty set, $*$ defined a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$, satisfying the following conditions: for all $x, y, z \in X, s, t > 0$,

(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ iff $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $(M(x, y, t) * M(y, z, s)) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is left continuous.

In 2017, Nădăban [16] introduced the idea of a fuzzy b-metric space to generalize the notion of a fuzzy metric spaces introduced by Kramosil and Michalek [13].

Definition 2.2 ([16]). Let X be a non-empty set, let $b \geq 1$ be a given real number and let $*$ be a continuous t-norm. Then a fuzzy set M in $X^2 \times (0, \infty)$ is called a *fuzzy b-metric* on X, if it satisfies the following conditions: for all $x, y, z \in X$ and $s, t > 0$,

(FbM-1) $M(x, y, t) = 0$,
(FbM-2) $M(x, y, t) = 1$ iff $x = y$,
(FbM-3) $M(x, y, t) = M(y, x, t)$,
(FbM-4) $M(x, z, b(t + s)) \geq M(x, y, t) * M(y, z, s)$,
(FbM-5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is left continuous and $\lim_{t\to\infty} M(x, y, t) = 1$.

Mehmood et al. [18] introduced the notion of an extended fuzzy b-metric space following the approach of Grabiec [14].

Definition 2.3 ([18]). Let X be a non-empty set, let $\alpha : X \times X \to [1, \infty)$, let $*$ be a continuous t-norm and let M_α be a fuzzy set on $X^2 \times (0, \infty)$. Then M_α is

called an *extended fuzzy b-metric* on X, if it satisfies the following conditions: for all $x, y, z \in X$ and $s, t > 0$,
(FbM$_\alpha$1) $M_\alpha(x, y, 0) = 0$,
(FbM$_\alpha$2) $M_\alpha(x, y, t) = 1$ iff $x = y$,
(FbM$_\alpha$3) $M_\alpha(x, y, t) = M_\alpha(y, x, t)$,
(FbM$_\alpha$4) $M_\alpha(x, z, \alpha(x, z)(t + s)) \geq M_\alpha(x, y, t) * M_\alpha(y, z, s)$,
(FbM$_\alpha$5) $M_\alpha(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.
Then $(X, M_\alpha, *, \alpha(x, y))$ is an extended fuzzy b-metric space.

In [1], Sezen introduced controlled fuzzy metric space, which is a generalization of extended fuzzy b-metric space.

Definition 2.4 ([1])**.** Let X be a non-empty set, let $\lambda : X \times X \to [1, \infty)$, let $*$ be a continuous t-norm and let M_λ be a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $a, c, d \in X, s, t > 0$,
(FM-1) $M_\lambda(a, c, 0) = 0$,
(FM-2) $M_\lambda(a, c, t) = 1$ iff $a = c$,
(FM-3) $M_\lambda(a, c, t) = M_\lambda(c, a, t)$,
(FM-4) $M_\lambda(a, d, t + s) \geq M_\lambda(a, c, \frac{t}{\lambda(a,c)}) * M_\lambda(c, d, \frac{s}{\lambda(c,d)})$,
(FM-5) $M_\lambda(a, c, \cdot) : [0, \infty) \to [0, 1]$ is continuous.
Then the triple $(X, M_\lambda, *)$ is called a *controlled fuzzy metric space* on X.

Saleema et al. [21] introduced fuzzy double controlled metric space which is a generalization of controlled fuzzy metric space.

Definition 2.5 ([21])**.** Let X be a nonempty set, let $\alpha, \beta : X \times X \to [1, \infty)$ given non-comparable functions and let $*$ a continuous t-norm. Then fuzzy set M_q on $X \times X \times (0, \infty)$ is called a *fuzzy double controlled metric* on X, if it satisfies the following conditions: for all $x, y, z \in X$,
(FM-1) $M_q(x, y, t) > 0$,
(FM-2) $M_q(x, y, t) = 1$ for all $t > 0$, iff $x = y$,
(FM-3) $M_q(x, y, t) = M_q(y, x, t)$,
(FM-4) $M_q(x, z, t + s) \geq M_q(x, y, \frac{t}{\alpha(x,y)}) * M_\lambda(y, z, \frac{s}{\beta(y,z)})$, for all $s, t > 0$,
(FM-5) $M_q(x, y, \cdot) : [0, \infty) \to [0, 1]$ is continuous.
Then the triple $(X, M_q, *)$ is called a *fuzzy double controlled metric space.*

The objective of this work is to prove a Banach type fixed point theorem in double controlled fuzzy metric spaces. Our result generalizes many recent fixed point theorems in the literature (See [16, 17, 18]). We furnish an example to validate our result. An application is also provided in support of our result.

3. Main Result

In this section, we introduce some new definitions and establish a fixed point theorem in double controlled fuzzy metric space.

We employ control functions $\lambda, \mu : X \times X \to [1, \infty)$ on the fuzzy triangle inequality

$$M_{\lambda,\mu}(a, d, t + s) \geq M_{\lambda,\mu}(a, c, \frac{t}{\lambda(a, c)}) * M_{\lambda,\mu}(c, d, \frac{s}{\mu(c, d)})$$

as follows.

Definition 3.1. Let X be a non-empty set, let $\lambda, \mu : X \times X \to [1, \infty), *$ be a continuous t-norm and let $M_{\lambda,\mu}$ be a fuzzy set on $X^2 \times (0, \infty)$. Then $M_{\lambda,\mu}$ is called a *double controlled fuzzy metric* on X, if it satisfies the following conditions: for all $a, c, d \in X, s, t > 0$,

($\text{FM}_{\lambda,\mu}$-1) $M_{\lambda,\mu}(a, c, 0) = 0$,
($\text{FM}_{\lambda,\mu}$-2) $M_{\lambda,\mu}(a, c, t) = 1$ iff $a = c$,
($\text{FM}_{\lambda,\mu}$-3) $M_{\lambda,\mu}(a, c, t) = M_{\lambda,\mu}(c, a, t)$,
($\text{FM}_{\lambda,\mu}$-4) $M_{\lambda,\mu}(a, d, t + s) \geq M_{\lambda,\mu}(a, c, \frac{t}{\lambda(a,c)}) * M_{\lambda,\mu}(c, d, \frac{s}{\mu(c,d)})$,
($\text{FM}_{\lambda,\mu}$-5) $M_{\lambda,\mu}(a, c, \cdot) : [0, \infty) \to [0, 1]$ is continuous.

Then the triple $(X, M_{\lambda,\mu}, *)$ is called a *double controlled fuzzy metric space.*

Now, we display an example to verify our definition.

Example 3.2. Let $X = A \cup C$, where $A = (0, 2)$ and $C = [2, \infty)$. Define a fuzzy set $M_{\lambda,\mu}$ on $X^2 \times (0, \infty)$ as follows:

$$M_{\lambda,\mu}(a, c, t) = \begin{cases} 1 & \text{if } a = c \\ e^{-\frac{2}{ct}} & \text{if } a \in A \text{ and } c \in C \\ e^{-\frac{3}{at}} & \text{if } a \in C \text{ and } c \in A \\ e^{-\frac{2}{t}} & \text{otherwise} \end{cases}$$

with the continuous t-norm $*$ such that $t_1 * t_2 = t_1 t_2$. Define $\lambda, \mu : X \times X \to [1, \infty)$, as

$$\lambda(a, c) = \begin{cases} 1 & \text{if } a, c \in A \\ max\{a, c\} & \text{otherwise,} \end{cases} \quad and \quad \mu(a, c) = \begin{cases} \frac{2}{a} & \text{if } a, c \in A \\ 1 & \text{otherwise} \end{cases}$$

Let us show that $(X, M_{\lambda,\mu}, *)$ is a double controlled fuzzy metric space. It is easy to prove conditions ($\text{FM}_{\lambda,\mu}$-1), ($\text{FM}_{\lambda,\mu}$-2) and ($\text{FM}_{\lambda,\mu}$-3). We have to examine the following cases to show that condition ($\text{FM}_{\lambda,\mu}$-4) holds.

Case I. If $d = a$ or $d = c$, ($\text{FM}_{\lambda,\mu}$-4) is satisfied.

Case II. If $d \neq a$ and $d \neq c$, ($\text{FM}_{\lambda,\mu}$-4) holds when $a = c$.

Suppose that $a \neq c$. Then we get $a \neq c \neq d$. Now, we can see that ($\text{FM}_{\lambda,\mu}$-4) is satisfied in all the cases below.

(1) Let $a, c, d \in A$ and $a, c, d \in C$, choose $t = 1, s = 1$ and $a = 1, c = \frac{1}{2}, d = \frac{3}{2}$ and $\lambda(a, c) = 1,\ \mu(a, d) = \frac{2}{a}$, then we get $e^{-1} \geq e^{-6}$.
(2) Let $a, d \in A$ and $c \in C$, choose $t = 1, s = 1$ and $a = \frac{1}{2}, c = 3, d = \frac{3}{2}$ and $\lambda(a, c) = max\{a, c\} = 3,\ \mu(a, d) = \frac{2}{a}$, then we get $e^{-0.3} \geq e^{-2.6}$.
(3) Let $a, d \in C$ and $c \in A$, choose $t = 1, s = 1$ and $a = 3, c = \frac{3}{2}, d = 4$ and $\lambda(a, c) = max\{a, c\} = 3,\ \mu(a, d) = 1$, then we get $e^{-0.5} \geq e^{-4}$.
(4) Let $a \in C$ and $c, d \in A$, choose $t = 1, s = 1$, and $a = 4, c = \frac{1}{2}, d = \frac{3}{2}$ and $\lambda(a, c) = max\{a, c\} = 4,\ \mu(a, d) = 1$, then we get $e^{-0.3} \geq e^{-3.7}$.
(5) Let $a, c \in A$ and $d \in C$, choose $t = 1, s = 1$ and $a = \frac{1}{2}, c = \frac{3}{2}, d = 4$ and $\lambda(a, c) = 1,\ \mu(a, d) = \frac{2}{a}$, then we get $e^{-1} \geq e^{-10}$.
(6) Let $a, c \in C$ and $d \in A$, choose $t = 1, s = 1$, and $a = 3, c = 4, d = \frac{1}{2}$ and $\lambda(a, c) = max\{a, c\} = 4,\ \mu(a, d) = 1$, then we get $e^{-1} \geq e^{-10}$.
(7) Let $a, c \in C$ and $d \in A$, choose $t = 1, s = 1$, and $a = \frac{3}{2}, c = 3, d = 4$ and $\lambda(a, c) = max\{a, c\} = 3,\ \mu(a, d) = 1$, then we get $e^{-1} \geq e^{-8}$.

Consequently, $(X, M_{\lambda,\mu}, *)$ is a double controlled fuzzy metric space. Also, for the same functions λ, μ using by ($FM_{\lambda,\mu}$-4), we get

$$M_{\lambda,\mu}(a, d, t+s) \geq M_{\lambda,\mu}(a, d, \frac{t}{\lambda(a,c)}) * M_{\lambda,\mu}(c, d, \frac{t}{\mu(c,d)}).$$

Definition 3.3. Let $(X, M_{\lambda,\mu}, *)$ be a double controlled fuzzy metric space.

(i) A sequence $\{x_n\}$ in X is said to be a *DC convergent to x in X*, if $lim_{n\to\infty} M_{\lambda,\mu}(x_n, x, t) = 1$ for any $n > 0$ and for all $t > 0$.

(ii) A sequence $\{x_n\}$ in X is said to be a *DC-Cauchy sequence*, if $lim_{n\to\infty} M_{\lambda,\mu}(x_n, x_{n+m}, t) = 1$ for any $m > 0$ and for all $t > 0$.

(iii) The double controlled fuzzy metric space is called *DC-complete*, if every DC-Cauchy sequence is convergent.

Now, we present our main result as follows:

Theorem 3.4. *Let $(X, M_{\lambda,\mu}, *)$ be a double controlled fuzzy metric space with $\lambda, \mu : X \times X \to [1, \infty)$ and suppose that*

$$lim_{n\to\infty} M_{\lambda,\mu}(a, c, t) = 1, \tag{3.1}$$

for all $a, c \in X, t > 0$. Let $h : X \to X$ be satisfied the following condition

$$M_{\lambda,\mu}(ha, hc, kt) \geq M_{\lambda,\mu}(a, c, t), \tag{3.2}$$

where $k \in (0, 1)$. Also, for every $a_n \in X$, let

$$lim_{n\to\infty}\lambda(a_n, c) \text{ and } lim_{n\to\infty}\mu(c, a_n) \tag{3.3}$$

exist and be finite. Then h has a unique fixed point in X.

Proof. Let $a_0 \in X$ and define a sequence $\{a_n\}$ by $a_n = ha_{n-1}$ for all $n \in \mathbb{N}$. If $a_n = a_{n-1}$, then a_n is a fixed point of h. Suppose that $a_n \neq a_{n-1}$ for all $n \in \mathbb{N}$. Successively applying inequality (3.2), we get

$$\begin{aligned} M_{\lambda,\mu}(a_n, a_{n+1}, t) &= M_{\lambda,\mu}(ha_{n-1}, ha_n, t) \\ &\geq M_{\lambda,\mu}(a_{n-2}, a_{n-1}, \frac{t}{k}) \\ &\vdots \\ &\geq M_{\lambda,\mu}(a_0, a_1, \frac{t}{k^{n-1}}). \end{aligned} \tag{3.4}$$

Now, using the condition (FM$_{\lambda,\mu}$-4), we have

$$\begin{aligned}
M_{\lambda,\mu}(a_n, a_{n+m}, t) &\geq M_{\lambda,\mu}(a_n, a_{n+1}, \frac{t}{2\lambda(a_n, a_{n+1})}) * M_{\lambda,\mu}(a_{n+1}, a_{n+m}, \frac{t}{2\mu(a_{n+1}, a_{n+m})}) \\
&\geq M_{\lambda,\mu}(a_n, a_{n+1}, \frac{t}{2\lambda(a_n, a_{n+1})}) \\
&\quad * M_{\lambda,\mu}(a_{n+1}, a_{n+2}, \frac{t}{(2)^2\mu(a_{n+1}, a_{n+m})\lambda(a_{n+1}, a_{n+2})}) \\
&\quad * M_{\lambda,\mu}(a_{n+2}, a_{n+m}, \frac{t}{(2)^2\mu(a_{n+1}, a_{n+m})\mu(a_{n+2}, a_{n+m})}) \\
&\geq M_{\lambda,\mu}(a_n, a_{n+1}, \frac{t}{2\lambda(a_n, a_{n+1})}) \\
&\quad * M_{\lambda,\mu}(a_{n+1}, a_{n+2}, \frac{t}{(2)^2\mu(a_{n+1}, a_{n+m})\lambda(a_{n+1}, a_{n+2})}) \\
&\quad * M_{\lambda,\mu}(a_{n+2}, a_{n+3}, \frac{t}{(2)^3\mu(a_{n+1}, a_{n+m})\mu(a_{n+2}, a_{n+m})\lambda(a_{n+2}, a_{n+3})}) \\
&\quad * M_{\lambda,\mu}(a_{n+3}, a_{n+m}, \frac{t}{(2)^3\mu(a_{n+1}, a_{n+m})\mu(a_{n+2}, a_{n+m})\mu(a_{n+3}, a_{n+m})}) \\
&\;\vdots \\
&\geq M_{\lambda,\mu}(a_0, a_1, \frac{t}{2k^{n-1}\lambda(a_n, a_{n+1})}) \\
&\quad * [*_{i=n+1}^{n+m-2} M_{\lambda}(a_0, a_1, \frac{t}{(2)^{m-1}k^{i-1}(\prod_{j=n+1}^{i}\mu(a_j, a_{n+m}))\lambda(a_i, a_{i+1})})] \\
&\quad * [M_{\lambda,\mu}(a_0, a_1, \frac{t}{(2)^{m-1}k^{n+m-1}(\prod_{i=n+1}^{n+m-1}\mu(a_i, a_{n+m}))})]. \qquad (3.5)
\end{aligned}$$

Then, by taking limit as $n \to \infty$ in (**??**), from (**??**) together with (3.1) we have

$$lim_{n\to\infty} M_{\lambda,\mu}(a_n, a_{n+m}, t) \geq 1 * 1 * \cdots * 1 = 1 \ for\ all\ t > 0\ and\ n, m \in \mathbb{N}.$$

Thus $\{a_n\}$ is a DC-Cauchy sequence in X. From the completeness of $(X, M_{\lambda,\mu}, *)$, there exists $a_n, u \in X$ such that

$$lim_{n\to\infty} M_{\lambda,\mu}(a_n, u, t) = 1, \qquad (3.6)$$

for all $t > 0$. Now we show that u is a fixed point of h. For any $t > 0$ and from the condition (FM$_{\lambda,\mu}$-4), we have

$$\begin{aligned}
M_{\lambda,\mu}(u, hu, t) &\geq M_{\lambda,\mu}(u, a_{n+1}, \frac{t}{2\lambda(u, a_{n+1})}) * M_{\lambda,\mu}(a_{n+1}, hu, \frac{t}{2\mu(a_{n+1}, hu)}) \\
&\geq M_{\lambda,\mu}(u, a_{n+1}, \frac{t}{2\lambda(u, a_{n+1})}) * M_{\lambda,\mu}(ha_n, hu, \frac{t}{2\mu(a_{n+1}, hu)}) \\
&\geq M_{\lambda,\mu}(u, a_{n+1}, \frac{t}{2\lambda(u, a_{n+1})}) * M_{\lambda,\mu}(a_n, u, \frac{t}{2\mu(a_{n+1}, hu)k}). \qquad (3.7)
\end{aligned}$$

Letting $n \to \infty$ in (3.7) and using (3.6), we get $M_{\lambda}(u, hu, t) = 1$ for all $t > 0$, that is, $u = hu$.

For uniqueness, let $w \in X$ is another fixed point of h and there exists $t > 0$ such that $M_\lambda(u, w, t) \neq 1$, then it follows from (3.2) that

$$\begin{aligned} M_{\lambda,\mu}(u, w, t) &= M_{\lambda,\mu}(hu, hw, t) \\ &\geq M_{\lambda,\mu}(u, w, \frac{t}{k}) \\ &\geq M_{\lambda,\mu}(u, w, \frac{t}{k^2}) \\ &\vdots \\ &\geq M_{\lambda,\mu}(u, w, \frac{t}{k^n}), \end{aligned} \tag{3.8}$$

for all $n \in \mathbb{N}$. By taking limit as $n \to \infty$ in (3.8), $M_{\lambda,\mu}(u, w, t) = 1$ for all $t > 0$, that is, $u = w$. This completes the proof. □

Now we furnish an example to validate Theorem 3.4

Example 3.5. Let $X = A \cup C$ where $A = (0, 2)$ and $C = (2, \infty)$. Define $M_{\lambda,\mu} : X \times X \times [0, \infty) \to [0, 1]$ as

$$M_{\lambda,\mu}(a, c, t) = \begin{cases} 1 & \text{if } a = c \\ \frac{t}{(t+\frac{2}{c})} & \text{if } a \in A \text{ and } c \in C \\ \frac{t}{(t+\frac{2}{a})} & \text{if } a \in C \text{ and } c \in A \\ \frac{1}{(t+1)} & \text{otherwise,} \end{cases}$$

with the continuous t-norm $*$ such that $t_1 * t_2 = t_1 t_2$. Define $\lambda, \mu : X \times X \to [1, \infty)$, as

$$\lambda(a, c) = \begin{cases} 1 & \text{if } a, c \in A \\ max\{a, c\} & \text{otherwise.} \end{cases} \quad \textit{and} \quad \mu(a, c) = \begin{cases} 1 + \frac{1}{a} & \text{if } a, c \in A \\ 1 & \text{otherwise.} \end{cases}$$

Clearly $(X, M_{\lambda,\mu}, *)$ is a double controlled fuzzy metric space. Consider $h : X \to X$ by

$$h(x) = \begin{cases} x & \text{if } x \in A \\ x^2 + 1 & \text{if } x \in C, \end{cases}$$

for all $x \in X$ and $k = 0.5$. We have to examine the inequality (3.2) for all the four cases given below.

Case I. If $a = c$ then we have $ha = hc$. In this case:

$$M_{\lambda,\mu}(ha, hc, kt) = 1 = M_{\lambda,\mu}(a, c, t). \tag{3.9}$$

Case II. Let $a \in A$ and $c \in C$, then we have $ha \in A$ and $hc \in C$. In this case:

$$\begin{aligned} M_{\lambda,\mu}(ha, hc, kt) &= \frac{kt}{(kt + \frac{2}{hc})} \\ &= \frac{0.5t}{(0.5t + \frac{2}{c^2+1})} \\ &\geq \frac{t}{(t + \frac{2}{c+1})} \\ &= M_{\lambda,\mu}(a, c, t). \end{aligned} \tag{3.10}$$

Figure 1 (a) shows the illustration of above case on $2D$ view, in which the variation of $M_{\lambda,\mu}(ha, hc, kt)$ as a function of c with fixed values of t, is shown as a red colored curve. A dotted curved line represents the variation of $M_{\lambda,\mu}(a, c, t)$ as a function of c relative to t, the variation of this curve is similar to the red colored line with little smaller values of $M_{\lambda,\mu}(a, c, t)$.
Figure 1 (b) shows the variation of $M_{\lambda,\mu}(ha, hc, kt)$ as a function of t with fixed values of c, is shown as a red colored curve. A dotted curved line represents the variation of $M_{\lambda,\mu}(a, c, t)$ as a function of t fixed to c, the variation of this curve is similar to the red colored line with little smaller values of $M_{\lambda,\mu}(a, c, t)$.

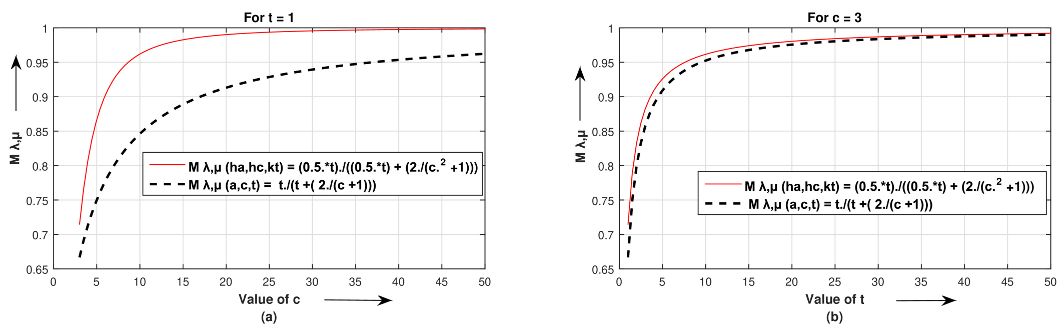

Figure 1. Variation of $M_{\lambda,\mu}(ha, hc, kt)$ with $M_{\lambda,\mu}(a, c, t)$ of Example 3.5, case-II on $2D$ view, for:
(a)$M_{\lambda,\mu}(ha, hc, kt)$ vs $M_{\lambda,\mu}(a, c, t)$ at $t = 1$ and $c \in (3, 50)$.
(b)$M_{\lambda,\mu}(ha, hc, kt)$ vs $M_{\lambda,\mu}(a, c, t)$ at $t \in (1, 50)$ and $c = 3$.

Figure 2 (a) shows the illustration of case II of example 3.5 on $3D$ view, in which the variation of $M_{\lambda,\mu}(ha, hc, kt)$ as a function of c with different values of t, is shown as a red-yellow surface and a blue-green surface represents the variation of $M_{\lambda,\mu}(a, c, t)$ as a function of c relative to t, the variation of this curve is similar to the red-yellow surface with little smaller values of $M_{\lambda,\mu}(a, c, t)$.
Figure 2 (b) is similar to the variation of $M_{\lambda,\mu}(ha, hc, kt)$ as a function of c with different values of t, is shown as a yellow surface curve and a green surface represents the variation of $M_{\lambda,\mu}(a, c, t)$ as a function of c relative to t.

Table 1 and 2 show the variation between $M_{\lambda,\mu}(ha, hc, kt)$ and $M_{\lambda,\mu}(a, c, t)$ as a function of c with relative to t, this table justifies inequality (3.10), which observed

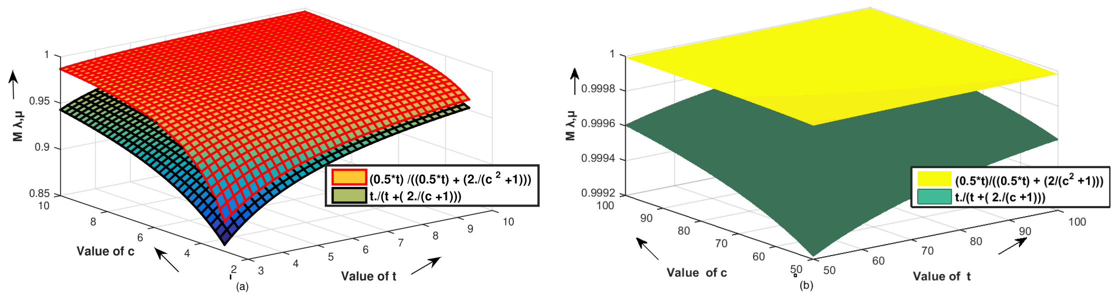

FIGURE 2. Variation of $M_{\lambda,\mu}(ha, hc, kt)$ with $M_{\lambda,\mu}(a, c, t)$ of Example 3.5, case-II on $3D$ view, for:
(a)$M_{\lambda,\mu}(ha, hc, kt)$ vs $M_{\lambda,\mu}(a, c, t)$ at $t \in (1, 10)$ and $c \in (3, 10)$.
(b)$M_{\lambda,\mu}(ha, hc, kt)$ vs $M_{\lambda,\mu}(a, c, t)$ at $t \in (50, 100)$ and $c \in (50, 100)$.

in both the curves for the value of t is a higher than 50 as a function of c. At $t = 70$, $M_{\lambda,\mu}(ha, hc, kt)$ becomes 1 and after higher value of t, it remains constant $(= 1)$. $M_{\lambda,\mu}(a, c, t)$ doesn't become to 1 till $t = 100$, but it approached nearby to 1.

Value of t	Value of c	$M_{\lambda,\mu}(ha, hc, kt)$	$M_{\lambda,\mu}(a, c, t)$
1	3	0.7143	0.6667
	20	0.9901	0.9130
	50	0.9984	0.9623
50	3	0.9921	0.9901
	20	0.9998	0.9981
	50	1.0000	0.9992

TABLE 1. Variation of $M_{\lambda,\mu}(ha, hc, kt)$ with $M_{\lambda,\mu}(a, c, t)$ of inequality (3.10), as a function of c with fixed value of $t = 1$ and $t = 50$.

Value of c	Value of t	$M_{\lambda,\mu}(ha, hc, kt)$	$M_{\lambda,\mu}(a, c, t)$
3	1	0.7143	0.6667
	20	0.9804	0.9756
	50	0.9921	0.9901
50	1	0.9984	0.9623
	20	0.9999	0.9980
	50	1.0000	0.9992

TABLE 2. Variation of $M_{\lambda,\mu}(ha, hc, kt)$ with $M_{\lambda,\mu}(a, c, t)$ of inequality (3.10) as a function of t with fixed value of $c = 3$ and $c = 50$.

Case III.Let $a \in C$ and $c \in A$, then we have $ha \in C$ and $hc \in A$. In this case:

$$
\begin{aligned}
M_{\lambda,\mu}(ha, hc, t) &= \frac{kt}{(kt + \frac{2}{ha})} \\
&= \frac{0.5t}{(0.5t + \frac{2}{a^2+1})} \\
&\geq \frac{t}{(t + \frac{2}{a+1})} \\
&= M_{\lambda,\mu}(a, c, t). \qquad (3.11)
\end{aligned}
$$

Figure 3 (a)Shows the illustration of case II of example 3.5 on $2D$ view, in which the variation of $M_{\lambda,\mu}(ha, hc, kt)$ as a function of a with fixed values of t, is shown as red colored dotted curve and the variation of $M_{\lambda,\mu}(a, c, t)$ as a function of a fixed to t shown as a blue colored curve.

Figure 3 (b) is the variation of $M_{\lambda,\mu}(ha, hc, kt)$ as a function of t with fixed values of a, is shown as red colored dotted curve and the variation of $M_{\lambda,\mu}(a, c, t)$ as a function of t fixed to a shown as a blue colored curve.

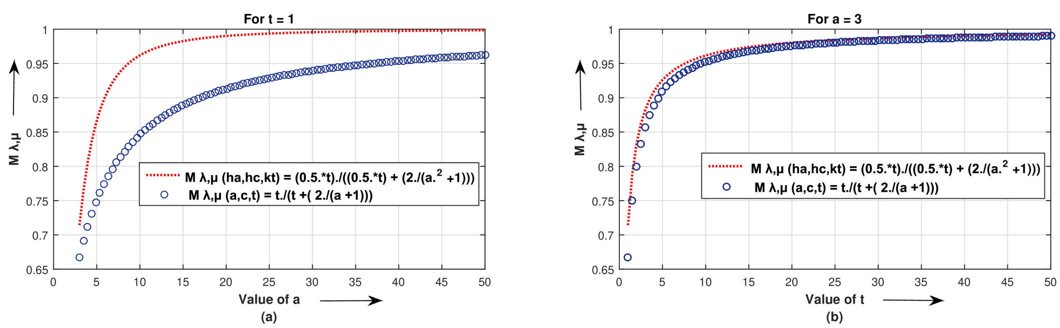

FIGURE 3. Variation of $M_{\lambda,\mu}(ha, hc, kt)$ with $M_{\lambda,\mu}(a, c, t)$ of Example 3.5, Case III on $2D$ view, for:
(a)$M_{\lambda,\mu}(ha, hc, kt)$ vs $M_{\lambda,\mu}(a, c, t)$ at $t = 1$ and $a \in (3, 50)$.
(b)$M_{\lambda,\mu}(ha, hc, kt)$ vs $M_{\lambda,\mu}(a, c, t)$ at $t \in (1, 50)$ and $a = 3$.

Case IV. For other states of a, c and similarly ha, hc. $M_{\lambda,\mu}(ha, hc, kt)$ and $M_{\lambda,\mu}(a, c, t)$ depends on only t. we have,

$$
\begin{aligned}
M_{\lambda,\mu}(ha, hc, kt) &= \frac{1}{(kt + 1)} \\
&= \frac{1}{(0.5t + 1)} \\
&\geq \frac{1}{(t + 1)} \\
&= M_{\lambda,\mu}(a, c, t). \qquad (3.12)
\end{aligned}
$$

Therefore, all the conditions of Theorem 3.4 hold and h has a unique fixed point $x = 1$.

Remark 3.6. By taking $\mu = \lambda$, in Theorem 3.4, we infer the Theorem 2 in [1].

Remark 3.7. We show by the following example, that a controlled fuzzy metric space is also a double controlled fuzzy metric space when $\lambda = \mu$. The converse is not true in general.

Example 3.8. Let $X = A \cup C$ where $A = [0, 1]$ and $C = (2, \infty)$. Define $M_{\lambda,\mu} : X \times X \times [0, \infty) \to [0, 1]$ as

$$M_{\lambda,\mu}(a, c, t) = \begin{cases} 1 & \text{if } a = c \\ \frac{t}{(t+\frac{1}{c})} & \text{if } a \in A \text{ and } c \in C \\ \frac{t}{(t+\frac{1}{a})} & \text{if } a \in C \text{ and } c \in A \\ \frac{t}{(t+1)} & \text{if otherwise.} \end{cases}$$

For $d \in A$ and $a, c \in C$, choose $t = 1, s = 1$ and $c = 50$ and let us define $\lambda, \mu : X \times X \to [1, \infty)$, as $\lambda(a, c) = 2, \mu(c, d) = 50$, then we have

$$\frac{t + s}{(t + s + 1)} \geq \left(\frac{t}{t + \frac{\lambda(a,c)}{c}}\right)\left(\frac{s}{s + \frac{\mu(c,d)}{c}}\right),$$

or

$$\frac{2}{3} \geq \left(\frac{1}{1 + \frac{2}{50}}\right)\left(\frac{1}{1 + \frac{50}{50}}\right).$$

Which satisfies condition of double controlled fuzzy metric space. But, if we take $\lambda = \mu$ in the above, we have

$$\frac{t + s}{(t + s + 1)} \leq \left(\frac{t}{t + \frac{\lambda(a,c)}{c}}\right)\left(\frac{s}{s + \frac{\lambda(c,d)}{c}}\right)$$

or

$$\frac{2}{3} \leq \left(\frac{1}{1 + \frac{2}{50}}\right)\left(\frac{1}{1 + \frac{2}{50}}\right).$$

which is certainly not true.

4. Application

In this section, we give an application of our result. Let $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0, \infty)$ and for each $\epsilon > 0$, we have $\int_0^\epsilon \varphi(t)dt > 0$.

Theorem 4.1. *Let $(X, M_{\lambda,\mu}, *)$ be a double controlled fuzzy metric spaces with $\lambda, \mu : X \times X \to [1, \infty)$, and suppose that*

$$lim_{n\to\infty} M_{\lambda,\mu}(a, c, t) = 1, \tag{4.1}$$

for all $a, c \in X, t > 0$. Let $h : X \to X$ be a mapping, suppose there exists a constant $k \in (0, 1)$ satisfies

$$\int_0^{M_{\lambda,\mu}(ha,hc,t)} \varphi(t)dt \geq k \int_0^{M_{\lambda,\mu}(a,c,t)} \varphi(t)dt, \tag{4.2}$$

for all $a, c \in X, t > 0$. Also, we assume that for every $a_n \in X$,

$$\lim_{n\to\infty} \lambda(a_n, c) \text{ and } \lim_{n\to\infty} \mu(c, a_n) \tag{4.3}$$

exist and are finite. Then h has a unique fixed point in X.

Proof. Let $a_0 \in X$ and define a sequence $\{a_n\}$ by $a_n = ha_{n-1}$ for all $n \in \mathbb{N}$.

If $a_n = a_{n-1}$ then a_n is a fixed point of h. Suppose that $a_n \neq a_{n-1}$ for all $n \in \mathbb{N}$. Successively applying inequality (4.2), we get

$$\int_0^{M_{\lambda,\mu}(a_n,a_{n+1},t)} \varphi(t)dt \geq k^{n-1} \int_0^{M_{\lambda,\mu}(a_0,a_1,t)} \varphi(t)dt.$$

Now, using the condition ($FM_{\lambda,\mu}$-4), we have

$$\begin{aligned} M_{\lambda,\mu}(a_n, a_{n+m}, t) \geq M_{\lambda,\mu}(a_0, a_1, \frac{t}{2\lambda(a_n, a_{n+1})}) \\ * [*_{i=n+1}^{n+m-2} M_\lambda(a_0, a_1, \frac{t}{(2)^{m-1}(\prod_{j=n+1}^{i} \mu(a_j, a_{n+m}))\lambda(a_i, a_{i+1})})] \\ * [M_{\lambda,\mu}(a_0, a_1, \frac{t}{(2)^{m-1}(\prod_{i=n+1}^{n+m-1} \mu(a_i, a_{n+m}))})], \end{aligned}$$

since

$$\begin{aligned} \int_0^{M_{\lambda,\mu}(a_n,a_{n+m},t)} \varphi(t)dt \geq k^{n-1} \int_0^{M_{\lambda,\mu}(a_0,a_1,\frac{t}{2\lambda(a_n,a_{n+1})})} \varphi(t)dt \\ * [*_{i=n+1}^{n+m-2} k^{i-1} \int_0^{M_\lambda(a_0,a_1,\frac{t}{(2)^{m-1}(\prod_{j=n+1}^{i} \mu(a_j,a_{n+m}))\lambda(a_i,a_{i+1})})} \varphi(t)dt] \\ * [k^{n+m-1} \int_0^{M_{\lambda,\mu}(a_0,a_1,\frac{t}{(2)^{m-1}(\prod_{i=n+1}^{n+m-1} \mu(a_i,a_{n+m}))})} \varphi(t)dt]. \end{aligned} \tag{4.4}$$

We assert that $B_n = M_{\lambda,\mu}(a_n, a_{n+1}, t) \to 0$ as $n \to \infty$. Certainly, $\{B_n\}$ is a DC-convergent sequence, because $\{B_n\}$ is a non-negative and decreasing sequence. Assume $B_n \to \epsilon$ as $n \to \infty$, thus we have

$$lim_{n\to\infty} \int_0^{B_n} \varphi(t)dt \geq \int_0^{\epsilon} \varphi(t)dt > 0.$$

Therefore, by taking the limit as $n \to \infty$ in (4.4), we have from (4.2)

$$lim_{n\to\infty} M_{\lambda,\mu}(a_n, a_{n+m}, t) \geq 1 * 1 * \cdots * 1 = 1,$$

for all $t > 0$ and $n, m \in \mathbb{N}$. Thus, $\{a_n\}$ is a DC-Cauchy sequence in X. From Theorem 3.4, h has a unique fixed point in X. This completes the proof. □

Let $\phi : [0, 1] \to [0, 1]$ as $\phi(t) = \int_0^t \varphi(t)dt$ for all $t > 0$, be a decreasing and continuous function. For each $\epsilon > 0, \varphi(\epsilon) > 0$. Which shows that $\varphi(t)$ and $\phi(t) = 0$ iff $t = 0$.

Theorem 4.2. *Let $(X, M_{\lambda,\mu}, *)$ be double controlled fuzzy metric spaces and $h : X \to X$ be a mapping satisfying $\lim_{n\to\infty} M_{\lambda,\mu}(a, c, t) = 1$, for all $a, c \in X, t > 0$. Let $h : X \to X$ be satisfied the following condition*

$$\int_0^{M_{\lambda,\mu}(ha,hc,kt)} \varphi(t)dt \geq \int_0^{M_{\lambda,\mu}(a,c,t)} \varphi(t)dt, \tag{4.5}$$

where $k \in (0, 1)$. Also, we consider that for every $a_n \in X$,

$$lim_{n\to\infty}\lambda(a_n, c) \text{ and } lim_{n\to\infty}\mu(c, a_n)$$

exist and are finite, then h has a unique fixed point in X.

Proof. By taking $\varphi(t) = 1$ in equation (4.5), we obtain Theorem 3.4. □

Conclusions. In this article, we extend the controlled fuzzy metric spaces of Sezen [1] by introducing double controlled fuzzy metric spaces. We employed two control functions λ and μ on fuzzy triangle inequality. We prove a Banach-type fixed point theorem. Our investigations and results obtained were supported by suitable examples with graphic. We also provide an application of our result to show the existence and unique solution of Lebesgue integrable mapping which is summable on each compact subset of $[0, \infty)$. This work provides a new path for researchers in the concerned field.

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