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Abstract. Neighborhood systems present a general model for the theory of granular computing. Consistent functions can be used to study the communication between neighborhood systems. In this paper, we propose the extended notions of system consistent (sys-consistent) functions and granule consistent (gra-consistent) functions for fuzzifying neighborhood systems. The definition of the transformation of a fuzzifying neighborhood system by a mapping is presented. Then, the structure preservation properties for interior and closure operators of fuzzifying neighborhood systems are investigated.

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Corresponding Author: Keyun Qin (Keyunqin@263.net)

1. Introduction

Rough set theory (RST) proposed by Pawlak [1] is an important mathematical tool for dealing with uncertain knowledge in information systems. The RST is established based on an approximate space which is constituted by a universe and an equivalence relation on it. The uncertain concepts are approximated by using a pair of approximation operators, namely the lower and upper approximation operators. RST has been widely used in decision analysis [2], machine learning [3] and data mining [4] and so on.

Generalization of RST based on neighborhood systems(NSs) [5] was introduced by Lin as a tool to describe granular computing(GrC)[6, 7]. It was shown in [8] that neighborhood systems can provide a uniform framework for studying classical rough set, binary relation based rough set, covering based rough set and Ziarko's variable precision rough set. In addition, the combinations of rough set with some other mathematical structures for modelling vagueness and uncertainty have received much attention from researchers worldwide. Soft set theory was proposed by Molodtsov [9] which provide us a new way of coping with uncertainty from the viewpoint of parameterization. The topological structures of soft sets as well as their applications in decision making problems have been extensively studied [10, 11, 12, 13, 14, 15].

With the development of fuzzy mathematics, many concepts have been endowed with fuzzy set theory [16, 17, 18, 19]. Also, the concept of rough sets has been generalized to fuzzy environment. In the framework of fuzzy rough set theory, various fuzzy generalization approximation operators have been proposed and investigated in both of the constructive approach and axiomatic approach. These works are concerned with investigating internally the structures and properties of generalized approximation operators. In recent years, a few studies aimed at exploring the relationship between two generalized approximation spaces via homomorphisms or mappings [20, 21, 22, 23].

To study the communication between approximation spaces, wang et al. [24] proposed the concepts of consistent functions and investigated their properties from different perspective. Zhu et al. [25] unified and extended the consistent functions into the framework of neighborhood systems. Then Liau et al. [26] investigated the properties of structure-preservation under consistent functions for interior and closure operators of NSs. We noticed that there are fewer studies on the issue of structure-preserving properties of interior and closure operators in two generalized approximate spaces based on fuzzifying neighborhood systems. There is naturally a problem in how the fuzzifying neighborhood systems is structure-preserving. Therefore, the aim of this paper is to explore the structure-preserving properties of interior and closure operators in two generalized approximate spaces based on fuzzifying neighborhood systems by consistent functions or mappings.

In this study, we propose the notions of system consistent functions and granule consistent functions for fuzzifying neighborhood systems. The transformation of a fuzzifying neighborhood system by a mapping between two universes is presented. Furthermore, the structure preservation properties for interior and closure operators of fuzzifying neighborhood systems are investigated.The remainder of the paper is structured as follows. In Section 2, we recall some notions and notations used in this paper. And, we extend the concept of consistent functions to fuzzifying neighborhood systems. In Section 3, by means of the notion of fuzzifying neighborhood systems, we construct a pair of interior and closure operators. Furthermore, we explore the structural preservation of interior and closure operators in fuzzifying neighborhood systems on the universe. In Section 4, we examine some preservation results about interior and closure operators in fuzzifying neighborhood systems on the image universe. Some concluding remarks are presented in Section 5.

2. Preliminaries

In this section, we shall recall some notion and conclusions about fuzzy sets and fuzzifying neighborhood systems for later use.

Let U denote a nonempty set called the universal set. A fuzzy subset μ of U is defined by a function assigning to each element x of U a value $\mu(x) \in [0,1]$. We write $P(U)$ and $F(U)$ for the set of all subsets and fuzzy subsets of U, respectively.

For two fuzzy sets μ, ν on U, by $\mu \subseteq \nu$ we mean that $\mu(x) \leq \nu(x)$ for any $x \in U$, and $\mu = \nu$ we mean $\mu \subseteq \nu$ and $\nu \subseteq \mu$.

Let U and V be two universes and $f : U \to V$ a mapping. We note that $f^{-1}(\{y\}) = \{x \in U | f(x) = y\}$. By Zadeh [27] extension principle for fuzzy sets, the mapping f can be extended to fuzzy power sets $f : F(U) \to F(V)$ and $f^{-1} : F(V) \to F(U)$ given by: for any $y \in V$ and for any $x \in U$,

$$f(\mu)(y) = \begin{cases} \bigvee_{x \in f^{-1}(\{y\})} \mu(x), & f^{-1}(\{y\}) \neq \varnothing \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

$$f^{-1}(\lambda)(x) = \lambda(f(x)),$$

for any $\mu \in F(U)$ and $\lambda \in F(V)$.

Lemma 2.1. *For any $\mu, \nu \in F(U)$ and $\omega, \lambda \in F(V)$, it is obvious that the following properties hold:*

(1) $\mu \subseteq f^{-1}(f(\mu))$,
(2) $f(f^{-1}(\lambda)) \subseteq \lambda$,
(3) $f(\mu) \subseteq f(\nu)$ *if* $\mu \subseteq \nu$,
(4) $f^{-1}(\omega) \subseteq f^{-1}(\lambda)$ *if* $\omega \subseteq \lambda$.

The notion of neighborhood system has been extended to fuzzifying neighborhood system by Li [28] as follows.

Definition 2.2 ([28])**.** Let U be a universal set. A *fuzzifying neighborhood system* on U is a mapping $N : U \to F(P(U))$, where for any $x \in U$, $\bigvee_{H \in P(U)} N(x)(H) = 1$. The value $N(x)(H)$ is interpreted as degree of H being a neighborhood of x.

In [25], the definition of consistent functions has been generalized to neighborhood systems. In this paper, we extend notion of consistent functions with respect to fuzzifying neighborhood systems. Prepare for the later study of the structure-preserving properties of interior and closure operators.

Definition 2.3. Let U and V be two universal sets, let $f : U \to V$ be a mapping and let N be a fuzzifying neighborhood system on U. Then f is called a *granule consistent function* (in short, gra-consistent function) with respect to N, if for any $x, y \in U$, $f(x) = f(y)$ implies that for any $z \in U$, for any $H \in P(U)$ and $N(z)(H) > 0$, $x \in H$ iff $y \in H$.

Definition 2.4. Let U and V be two nonempty universes and let N be a fuzzifying neighborhood system on U. Then a mapping $f : U \to V$ is called a *system consistent function* (in short, sys-consistent function) with respect to N, if for any $x, y \in U$, $f(x) = f(y)$ implies that $N(x) = N(y)$.

3. Preservation of interior and closure in fuzzifying neighborhood systems on the universe

The preservation results to any NS under consistent functions has been explored by Liau [26]. In this section, we present a pair interior and closure operator based on a fuzzifying neighborhood system. Then, we will investigate the structure-preserving properties of interior and closure operators based on fuzzifying neighbor systems on the universe.

Definition 3.1. Let $N : U \to F(P(U))$ be a fuzzifying neighborhood system on U and let $\mu \in F(U)$. The *interior* $\underline{N}(\mu)$ and the *closure* $\overline{N}(\mu)$ of μ with respect to N are fuzzy sets given by:

$$\underline{N}(\mu)(x) = \bigvee_{H \in P(U)} (N(x)(H) \wedge (\bigwedge_{z \in H} \mu(z))) \tag{3.1}$$

$$\overline{N}(\mu)(x) = \bigwedge_{H \in P(U)} ((1 - N(x)(H)) \vee (\bigvee_{z \in H} \mu(z))) \tag{3.2}$$

for any $x \in U$.

Let $N : U \to F(P(U))$ be a fuzzifying neighborhood system on U and let $W \subseteq U$. We define the restriction of N to W as a fuzzifying neighborhood system on W, i.e., $N|_W : W \to F(P(W))$, such that $N|_W(x) = \{N(x)|_W\}$ for each $x \in W$.

We consider the transformation of a fuzzifying neighborhood system by a mapping. Let U and V be two universes, $f : U \to V$ be a mapping and $N : U \to F(P(U))$ be a fuzzifying neighborhood system on U. Then $f(N) : V \to F(P(V))$ is a fuzzifying neighborhood system on V defined by:

$$f(N)(y)(H) = \begin{cases} \bigvee_{f(x)=y} \bigvee_{f(H^{'})=H} N(x)(H^{'}), & f^{-1}(\{y\}) \neq \varnothing \\ 0, & f^{-1}(\{y\}) = \varnothing \end{cases} \tag{3.3}$$

for any $y \in V$, for any $H \in P(V)$. We note that $f(N)(y)(H) = 0$ if $f^{-1}(\{y\}) \neq \varnothing$ and $f^{-1}(\{H\}) = \varnothing$.

Lemma 3.2. *If $f(H^{'}) = H \in P(V)$ for any $H^{'} \in P(U)$, then $\bigwedge_{z \in H} f(\mu)(z) \geq \bigwedge_{z^{'} \in H^{'}} \mu(z^{'})$.*

Proof. By Zadeh's extension principle, we have

$$\bigwedge_{z \in H} f(\mu)(z) = \bigwedge_{z \in f(H^{'})} \bigvee_{f(z^{'})=z} \mu(z^{'}).$$

Then for any $z \in f(H^{'})$,

$$\bigvee_{f(z^{'})=z} \mu(z^{'}) \geq \mu(z^{'}) \geq \bigwedge_{z^{'} \in H^{'}} \mu(z^{'}).$$

Thus $\bigwedge_{z \in H} f(\mu)(z) = \bigwedge_{z \in f(H^{'})} \bigvee_{f(z^{'})=z} \mu(z^{'}) \geq \bigwedge_{z^{'} \in H^{'}} \mu(z^{'})$. □

Lemma 3.3. *Let N be a fuzzifying neighborhood system on U and let f be gra-consistent with respect to N. If for any $H^{'} \in P(U)$, $f(H^{'}) = H \in P(V)$ and for any $z \in U$, $N(z)(H^{'}) > 0$, then $\bigvee_{z \in H} f(\mu)(z) = \bigvee_{z^{'} \in H^{'}} \mu(z^{'})$.*

Proof. By Zadeh's extension principle, we have

$$\bigvee_{z\in H} f(\mu)(z) = \bigvee_{z\in f(H')}\bigvee_{f(z')=z}\mu(z') = \bigvee_{z'\in f^{-1}(f(H'))}\mu(z').$$

For any $x \in f^{-1}(f(H'))$, we have $f(x) \in f(H')$. Then there exists $y \in H'$ such that $f(x) = f(y)$. Since f is gra-consistent with respect to N, $N(z)(H') > 0$ for any $z \in U$. Then we have $x \in H'$. Thus by Lemma 2.1 (1), we have $f^{-1}(f(H')) = H'$. So we get

$$\bigvee_{z\in H} f(\mu)(z) = \bigvee_{z\in f(H')}\bigvee_{f(z')=z}\mu(z') = \bigvee_{z'\in f^{-1}(f(H'))}\mu(z') = \bigvee_{z'\in H'}\mu(z').$$

□

Lemma 3.4. *If $f(H') = H \in P(V)$ for any $H' \in P(U)$, then $\bigwedge_{z'\in H'} f^{-1}(\lambda)(z') = \bigwedge_{z\in H}\lambda(z)$.*

Proof. By Zadeh's extension principle, we have

$$\bigwedge_{z'\in H'} f^{-1}(\lambda)(z') = \bigwedge_{z'\in H'}\lambda(f(z')) = \bigwedge_{z=f(z')\in f(H')=H}\lambda(z).$$

□

Theorem 3.5. *Let U and V be two nonempty universes, let $N : U \to F(P(U))$ be a fuzzifying neighborhood system and let $f : U \to V$ ba a mapping. Then for any fuzzy subsets $\lambda \in F(V)$, we have*
(1) $\overline{f(N)}(\lambda) = (\underline{f(N)}(\lambda^c))^c$,
(2) $\underline{f(N)}(\lambda) = \overline{f(N)}(\lambda^c))^c$.

Proof. (1) For any $y \in V$,

$$\begin{aligned}(\underline{f(N)}(\lambda^c))^c(y) &= 1 - (\underline{f(N)}(\lambda^c))(y)\\ &= 1 - \bigvee_{H\in P(V)}(f(N)(y)(H) \wedge (\bigwedge_{z\in H}\lambda^c(z)))\\ &= 1 - \bigvee_{H\in P(V)}(f(N)(y)(H) \wedge (\bigwedge_{z\in H}(1-\lambda(z))))\\ &= 1 - \bigvee_{H\in P(V)}(f(N)(y)(H) \wedge (1-\bigvee_{z\in H}\lambda(z)))\\ &= \bigwedge_{H\in P(V)}((1 - f(N)(y)(H)) \vee (\bigvee_{z\in H}\lambda(z)))\\ &= \overline{f(N)}(\lambda)(y).\end{aligned}$$

(2) The conclusion can be proved similarly. □

Theorem 3.6. *Let U and V be two nonempty universes, let $N : U \to F(P(U))$ be a fuzzifying neighborhood system and let $f : U \to V$ ba a mapping. Then for any fuzzy subsets $\mu \in F(U)$, we have*
(1) $f(\underline{N}(\mu)) \subseteq \underline{f(N)}(f(\mu))$,
(2) $\underline{N}(\mu) \subseteq f^{-1}(\underline{f(N)}(f(\mu)))$,
(3) $\overline{f(N)}|_{f(U)}(f(\mu)) \subseteq f(\overline{N}(\mu))|_{f(U)}$ *if f is gra-consistent with respect to N,*

(4) $f^{-1}(\overline{f(N)}(f(\mu))) \subseteq \overline{N}(\mu)$ *if* f *is gra-consistent with respect to* N.

Proof. (1) For any $y \in V$, if $y \in V \backslash f(U)$, $f(\underline{N}(\mu))(y) = 0$, the conclusion is obvious. We just need to prove the case $y \in f(U)$ in the following steps.

$$f(\underline{N}(\mu))(y) = \bigvee_{f(x)=y} \underline{N}(\mu)(x) = \bigvee_{f(x)=y} \bigvee_{H' \in P(U)} (N(x)(H') \wedge (\bigwedge_{z' \in H'} \mu(z')))$$

According to the Lemma 3.2, for any $f(x) = y$ and for any $H' \in P(U)$, if $f(H') = H \in P(V)$, then we have

$$\begin{aligned} N(x)(H') \wedge (\bigwedge_{z' \in H'} \mu(z')) &\leq N(x)(H') \wedge (\bigwedge_{z \in H} f(\mu)(z)) \\ &\leq \bigvee_{f(x)=y} \bigvee_{f(H')=H} N(x)(H') \wedge (\bigwedge_{z \in H} f(\mu)(z)) \\ &\leq \bigvee_{H \in P(V)} (\bigvee_{f(x)=y} \bigvee_{f(H')=H} N(x)(H') \wedge (\bigwedge_{z \in H} f(\mu)(z))) \\ &= \underline{f(N)}(f(\mu))(y). \end{aligned}$$

Thus $f(\underline{N}(\mu))(y) \leq \underline{f(N)}(f(\mu))(y)$. So the conclusion holds.

(2) It follows immediately from (1), and Lemma 2.1 (1) and (4).

(3) For any $y \in f(U)$,

$$\begin{aligned} f(\overline{N}(\mu))|_{f(U)}(y) &= \bigvee_{f(x)=y} \overline{N}(\mu)(x) \\ &= \bigvee_{f(x)=y} \bigwedge_{\substack{H' \in P(U) \\ N(x)(H') > 0}} ((1 - N(x)(H')) \vee (\bigvee_{z' \in H'} \mu(z'))) \\ &\geq \bigwedge_{\substack{H' \in P(U) \\ N(x)(H') > 0}} ((1 - N(x)(H')) \vee (\bigvee_{z' \in H'} \mu(z'))), \end{aligned}$$

$$\begin{aligned} \overline{f(N)}|_{f(U)}(f(\mu))(y) &= \bigwedge_{H \in P(f(U))} ((1 - f(N)(y)(H)) \vee (\bigvee_{z \in H} f(\mu)(z))) \\ &= \bigwedge_{H \in P(f(U))} ((1 - \bigvee_{f(x)=y} \bigvee_{f(H')=H} N(x)(H')) \vee (\bigvee_{z \in H} f(\mu)(z))) \\ &= \bigwedge_{H \in P(f(U))} (\bigwedge_{f(x)=y} \bigwedge_{f(H')=H} (1 - N(x)(H')) \vee (\bigvee_{z \in H} f(\mu)(z))). \end{aligned}$$

According to the Lemma 3.3, for any $H' \in P(U)$, if $f(H') = H \in P(V)$, then

$$\begin{aligned} &(1 - N(x)(H')) \vee (\bigvee_{z' \in H'} \mu(z')) \\ &= (1 - N(x)(H')) \vee (\bigvee_{z \in H} f(\mu)(z)) \\ &\geq \bigwedge_{f(x)=y} \bigwedge_{f(H')=H} (1 - N(x)(H')) \vee (\bigvee_{z \in H} f(\mu)(z)) \end{aligned}$$

$$\geq \bigwedge_{H\in P(f(U))} (\bigwedge_{f(x)=y} \bigwedge_{f(H^{'})=H} (1-N(x)(H^{'})) \vee (\bigvee_{z\in H} f(\mu)(z))).$$

Thus $f(N)|_{f(U)}(f(\mu))(y) \leq f(\overline{N}(\mu))|_{f(U)}(y)$. So the conclusion holds.

(4) For any $x \in U$,

$$\begin{aligned}
& f^{-1}(\overline{f(N)}(f(\mu)))(x) \\
&= \overline{f(N)}(f(\mu))(f(x)) \\
&= \bigwedge_{H\in P(V)} ((1-f(N)(f(x))(H)) \vee (\bigvee_{z\in H} f(\mu)(z))) \\
&= \bigwedge_{H\in P(V)} ((1-\bigvee_{f(x^{'})=f(x)} \bigvee_{f(H^{'})=H} N(x^{'})(H^{'})) \vee (\bigvee_{z\in H} f(\mu)(z))) \\
&= \bigwedge_{H\in P(V)} (\bigwedge_{f(x^{'})=f(x)} \bigwedge_{f(H^{'})=H} (1-N(x^{'})(H^{'})) \vee (\bigvee_{z\in H} f(\mu)(z))),
\end{aligned}$$

$$\begin{aligned}
\overline{N}(\mu)(x) &= \bigwedge_{H^{'}\in P(U)} ((1-N(x)(H^{'})) \vee (\bigvee_{z^{'}\in H^{'}} \mu(z^{'}))) \\
&= \bigwedge_{\substack{H^{'}\in P(U)\\ N(x)(H^{'})>0}} ((1-N(x)(H^{'})) \vee (\bigvee_{z^{'}\in H^{'}} \mu(z^{'}))).
\end{aligned}$$

According to the Lemma 3.3, for any $H^{'} \in P(U)$, if $f(H^{'}) = H \in P(V)$, then

$$\begin{aligned}
& (1-N(x)(H^{'})) \vee (\bigvee_{z^{'}\in H^{'}} \mu(z^{'})) \\
&= (1-N(x)(H^{'})) \vee (\bigvee_{z\in H} f(\mu)(z)) \\
&\geq \bigwedge_{f(x^{'})=f(x)} \bigwedge_{f(H^{'})=H} (1-N(x^{'})(H^{'})) \vee (\bigvee_{z\in H} f(\mu)(z)) \\
&\geq \bigwedge_{H\in P(V)} (\bigwedge_{f(x^{'})=f(x)} \bigwedge_{f(H^{'})=H} (1-N(x^{'})(H^{'})) \vee (\bigvee_{z\in H} f(\mu)(z))).
\end{aligned}$$

Thus $f^{-1}(\overline{f(N)}(f(\mu)))(x) \leq \overline{N}(\mu)(x)$. So the conclusion holds. □

Corollary 3.7. *Let U and V be two nonempty universes, let $N : U \to F(P(U))$ be a fuzzifying neighborhood system and let $f : U \to V$ ba a mapping. Then for any fuzzy subsets $\mu \in F(U)$, if f is gra-consistent and sys-consistent with respect to N, then we have*

(1) $\overline{f(N)}|_{f(U)}(f(\mu)) = f(\overline{N}(\mu))|_{f(U)}$,

(2) $f^{-1}(\overline{f(N)}(f(\mu))) = \overline{N}(\mu)$.

Proof. (1) Since f is sys-consistent with respect to N, for any $y \in f(U)$, we have

$$f(\overline{N}(\mu))|_{f(U)}(y) = \bigwedge_{\substack{H^{'}\in P(U)\\ N(x)(H^{'})>0}} ((1-N(x)(H^{'})) \vee (\bigvee_{z^{'}\in H^{'}} \mu(z^{'}))),$$

$$\overline{f(N)}|_{f(U)}(f(\mu))(y) = \bigwedge_{H\in P(f(U))} (\bigwedge_{f(H^{'})=H} (1-N(x)(H^{'})) \vee (\bigvee_{z\in H} f(\mu)(z))).$$

f is gra-consistent with respect to N, for any $f(x)=y$ and for any $H^{'}\in P(U)$, if $f(H^{'})=H\in P(V)$, then

$$\begin{aligned}\overline{f(N)}|_{f(U)}(f(\mu))(y) &= \bigwedge_{H\in P(f(U))}((1-N(x)(H^{'}))\vee(\bigvee_{z^{'}\in H^{'}}\mu(z^{'})))\\ &= \bigwedge_{H^{'}\in P(U)}((1-N(x)(H^{'}))\vee(\bigvee_{z^{'}\in H^{'}}\mu(z^{'})))\\ &= f(\overline{N}(\mu))|_{f(U)}(y).\end{aligned}$$

(2) The conclusion can be proved similarly. □

Theorem 3.8. *Let U and V be two nonempty universes, let $N:U\to F(P(U))$ be a fuzzifying neighborhood system and let $f:U\to V$ ba a mapping. Then for any fuzzy subset $\lambda\in F(V)$, we have*

(1) $f(\underline{N}(f^{-1}(\lambda)))\subseteq\underline{f(N)}(\lambda)$,
(2) $\underline{N}(f^{-1}(\lambda))\subseteq f^{-1}(\underline{f(N)}(\lambda))$,
(3) $\overline{f(N)}|_{f(U)}(\lambda)\subseteq f(\overline{N}(f^{-1}(\lambda)))|_{f(U)}$,
(4) $f^{-1}(\overline{f(N)}(\lambda))\subseteq\overline{N}(f^{-1}(\lambda))$.

Proof. (1) For any $y\in V$, if $y\in V\backslash f(U)$, then $f(\underline{N}(f^{-1}(\lambda)))(y)=0$. Thus the conclusion is obvious. We just need to prove the case $y\in f(U)$ in the following steps.

$$\begin{aligned}f(\underline{N}(f^{-1}(\lambda)))(y) &= \bigvee_{f(x)=y}\underline{N}(f^{-1}(\lambda))(x)\\ &= \bigvee_{f(x)=y}\bigvee_{H^{'}\in P(U)}(N(x)(H^{'})\wedge(\bigwedge_{z^{'}\in H^{'}}f^{-1}(\lambda)(z^{'}))),\end{aligned}$$

$$\begin{aligned}\underline{f(N)}(\lambda)(y) &= \bigvee_{H\in P(V)}(f(N)(y)(H)\wedge(\bigwedge_{z\in H}\lambda(z)))\\ &= \bigvee_{H\in P(V)}(\bigvee_{f(x)=y}\bigvee_{f(H^{'})=H}N(x)(H^{'})\wedge(\bigwedge_{z\in H}\lambda(z))).\end{aligned}$$

According to the Lemma 3.4, for any $f(x)=y$ and for any $H^{'}\in P(U)$, if $f(H^{'})=H\in P(V)$, then we have

$$\begin{aligned}N(x)(H^{'})\wedge(\bigwedge_{z^{'}\in H^{'}}f^{-1}(\lambda)(z^{'})) &= N(x)(H^{'})\wedge(\bigwedge_{z\in H}\lambda(z))\\ &\leq \bigvee_{f(x)=y}\bigvee_{f(H^{'})=H}N(x)(H^{'})\wedge(\bigwedge_{z\in H}\lambda(z))\\ &\leq \bigvee_{H\in P(V)}(\bigvee_{f(x)=y}\bigvee_{f(H^{'})=H}N(x)(H^{'})\wedge(\bigwedge_{z\in H}\lambda(z)))\\ &= \underline{f(N)}(\lambda)(y).\end{aligned}$$

Thus $f(\underline{N}(f^{-1}(\lambda)))(y)\leq\underline{f(N)}(\lambda)(y)$. So the conclusion holds.

(2) It follows immediately from (1), and Lemma 2.1 (1) and (4).

(3) The conclusion can be proved similarly.

(4) For any $x\in U$,

$$f^{-1}(\overline{f(N)}(\lambda))(x)=\overline{f(N)}(\lambda)(f(x))$$

$$= \bigwedge_{H\in P(V)} ((1 - f(N)(f(x))(H)) \vee (\bigvee_{z\in H} \lambda(z)))$$
$$= \bigwedge_{H\in P(V)} ((1 - \bigvee_{f(x')=f(x)} \bigvee_{f(H')=H} N(x')(H')) \vee (\bigvee_{z\in H} \lambda(z)))$$
$$= \bigwedge_{H\in P(V)} (\bigwedge_{f(x')=f(x)} \bigwedge_{f(H')=H} (1 - N(x')(H')) \vee (\bigvee_{z\in H} \lambda(z))),$$

$$\overline{N}(f^{-1}(\lambda))(x) = \bigwedge_{H'\in P(U)} ((1 - N(x)(H')) \vee (\bigvee_{z'\in H'} f^{-1}(\lambda)(z'))).$$

According to the Lemma 3.4, for any $H' \in P(U)$, if $f(H') = H \in P(V)$, then

$$\begin{aligned} &(1 - N(x)(H')) \vee (\bigvee_{z'\in H'} f^{-1}(\lambda)(z')) \\ &= (1 - N(x)(H')) \vee (\bigvee_{z\in H} \lambda(z)) \\ &\geq \bigwedge_{f(x')=f(x)} \bigwedge_{f(H')=H} (1 - N(x')(H')) \vee (\bigvee_{z\in H} \lambda(z)) \\ &\geq \bigwedge_{H\in P(V)} (\bigwedge_{f(x')=f(x)} \bigwedge_{f(H')=H} (1 - N(x')(H')) \vee (\bigvee_{z\in H} \lambda(z))). \end{aligned}$$

Thus $f^{-1}(\overline{f(N)}(\lambda))(x) \leq \overline{N}(f^{-1}(\lambda))(x)$. So the conclusion holds. □

Corollary 3.9. *Let U and V be two nonempty universes, let $N : U \to F(P(U))$ be a fuzzifying neighborhood system and let $f : U \to V$ ba a mapping. Then, for any fuzzy subsets $\lambda \in F(V)$, if f is gra-consistent and sys-consistent with respect to N, we have*

(1) $f(\underline{N}(f^{-1}(\lambda))) = \underline{f(N)}(\lambda)$,
(2) $\underline{N}(f^{-1}(\lambda)) = f^{-1}(\underline{f(N)}(\lambda))$,
(3) $\overline{f(N)}|_{f(U)}(\lambda) = f(\overline{N}(f^{-1}(\lambda)))|_{f(U)}$,
(4) $f^{-1}(\overline{f(N)}(\lambda)) = \overline{N}(f^{-1}(\lambda))$.

Proof. According to the Theorem 3.8, and f is gra-consistent and sys-consistent with respect to N, the conclusion is obvious. □

4. Preservation of interior and closure in fuzzifying neighborhood systems on the image universe

Similarly, let $f : U \to V$ be a mapping and $N : V \to F(P(V))$ be a fuzzifying neighborhood system on V. Then, $f^{-1}(N) : U \to F(P(U))$ is a fuzzifying neighborhood system on U given by:

$$f^{-1}(N)(x)(H') = f^{-1}(N(f(x)))(H') = N(f(x))(f(H')) \tag{4.1}$$

for any $x \in U$ and for any $H' \in P(U)$.

Some preservation results about closure and interior with respect to N and $f(N)$ are presented before. In this section, we consider the related properties with respect to N and $f^{-1}(N)$.

Lemma 4.1. *If $f(H^{'}) = H \in P(V)$ for any $H^{'} \in P(U)$, then*

$$\bigvee_{z\in H} f(\mu)(z) \geq \bigvee_{z^{'}\in H^{'}} \mu(z^{'}).$$

Furthermore, if f is gra-consistent with respect to $f^{-1}(N)$ and $f^{-1}(N)(z)(H^{'}) > 0$ for any $z \in U$, then $\bigvee_{z\in H} f(\mu)(z) = \bigvee_{z^{'}\in H^{'}} \mu(z^{'})$.

Proof. By Zadeh's extension principle, we have

$$\bigvee_{z\in H} f(\mu)(z) = \bigvee_{z\in f(H^{'})} \bigvee_{f(z^{'})=z} \mu(z^{'}) = \bigvee_{z^{'}\in f^{-1}(f(H^{'}))} \mu(z^{'}).$$

Then by Lemma 2.1 (1), $\bigvee_{z^{'}\in f^{-1}(f(H^{'}))} \mu(z^{'}) \geq \bigvee_{z^{'}\in H^{'}} \mu(z^{'})$.

Thus $\bigvee_{z\in H} f(\mu)(z) \geq \bigvee_{z^{'}\in H^{'}} \mu(z^{'})$.

If f is gra-consistent with respect to $f^{-1}(N)$, then $f^{-1}(N)(z)(H^{'}) > 0$ for any $z \in U$. Thus clearly, $\bigvee_{z\in H} f(\mu)(z) = \bigvee_{z^{'}\in H^{'}} \mu(z^{'})$. □

Theorem 4.2. *Let $N : V \to F(P(V))$ be a fuzzifying neighborhood system on V, let $f : U \to V$ and let $\lambda \in F(V)$. Then we have*

(1) $f(\underline{f^{-1}(N)}(f^{-1}(\lambda))) \subseteq \underline{N}(\lambda)$,

(2) $\underline{f^{-1}(N)}(f^{-1}(\lambda)) \subseteq f^{-1}(\underline{N}(\lambda))$,

(3) $f^{-1}(\overline{N}(\lambda)) \subseteq \overline{f^{-1}(N)}(f^{-1}(\lambda))$,

(4) $\overline{N}(\lambda) \subseteq f(\overline{f^{-1}(N)}|_{f(U)}(f^{-1}(\lambda)))$.

Proof. (1) For any $y \in V$, if $y \in V \backslash f(U)$, then $f(\underline{f^{-1}(N)}(f^{-1}(\lambda)))(y) = 0$. Thus the conclusion is obvious. We just need to prove the case $y \in f(U)$ in the following steps.

$$\begin{aligned} & f(\underline{f^{-1}(N)}(f^{-1}(\lambda)))(y) \\ = & \bigvee_{f(x)=y} \underline{f^{-1}(N)}(f^{-1}(\lambda))(x) \\ = & \bigvee_{f(x)=y} \bigvee_{H^{'}\in P(U)} (f^{-1}(N)(x)(H^{'}) \wedge (\bigwedge_{z^{'}\in H^{'}} f^{-1}(\lambda)(z^{'}))) \\ = & \bigvee_{f(x)=y} \bigvee_{H^{'}\in P(U)} (N(f(x))(f(H^{'})) \wedge (\bigwedge_{z^{'}\in H^{'}} f^{-1}(\lambda)(z^{'}))) \\ = & \bigvee_{H^{'}\in P(U)} (N(y)(f(H^{'})) \wedge (\bigwedge_{z^{'}\in H^{'}} f^{-1}(\lambda)(z^{'}))). \end{aligned}$$

According to the Lemma 3.4, if $f(H^{'}) = H \in P(V)$ for any $H^{'} \in P(U)$, then

$$\begin{aligned} f(\underline{f^{-1}(N)}(f^{-1}(\lambda)))(y) &= \bigvee_{H^{'}\in P(U)} (N(y)(f(H^{'})) \wedge (\bigwedge_{z\in f(H^{'})} \lambda(z))) \\ &= \bigvee_{f(H^{'})=H\in P(f(U))} (N(y)(H) \wedge (\bigwedge_{z\in H} \lambda(z))) \\ &\leq \bigvee_{H\in P(V)} (N(y)(H) \wedge (\bigwedge_{z\in H} \lambda(z))) \end{aligned}$$

$$= \underline{N}(\lambda)(y).$$

Thus $f(\underline{f^{-1}(N)}(f^{-1}(\lambda)))(y) \leq \underline{N}(\lambda)(y)$. So the conclusion holds.

(2) It follows immediately from (1) and Lemma 2.1 (1) and (4).

(3) For any $x \in U$,

$$\begin{aligned}\overline{f^{-1}(N)}(f^{-1}(\lambda))(x) &= \bigwedge_{H^{'} \in P(U)} ((1 - f^{-1}(N)(x)(H^{'})) \vee (\bigvee_{z^{'} \in H^{'}} f^{-1}(\lambda)(z^{'}))) \\ &= \bigwedge_{H^{'} \in P(U)} ((1 - N(f(x))(f(H^{'}))) \vee (\bigvee_{z^{'} \in H^{'}} f^{-1}(\lambda)(z^{'}))).\end{aligned}$$

According to the Lemma 3.4, if $f(H^{'}) = H \in P(V)$ for any $H^{'} \in P(U)$, then

$$\begin{aligned}\overline{f^{-1}(N)}(f^{-1}(\lambda))(x) &= \bigwedge_{H^{'} \in P(U)} ((1 - N(f(x))(f(H^{'}))) \vee (\bigvee_{z \in f(H^{'})} \lambda(z))) \\ &= \bigwedge_{f(H^{'})=H \in P(f(U))} ((1 - N(f(x))(H)) \vee (\bigvee_{z \in H} \lambda(z))) \\ &\geq \bigwedge_{H \in P(V)} ((1 - N(f(x))(H)) \vee (\bigvee_{z \in H} \lambda(z))) \\ &= \overline{N}(\lambda)(f(x)) \\ &= f^{-1}(\overline{N}(\lambda))(x).\end{aligned}$$

Thus $f^{-1}(\overline{N}(\lambda))(x) \leq \overline{f^{-1}(N)}(f^{-1}(\lambda))(x)$. So the conclusion holds.

(4) The conclusion can be proved similarly. □

Corollary 4.3. *Let $N : V \to F(P(V))$ be a fuzzifying neighborhood system and let $f : U \to V$ be a mapping. Then for any fuzzy subsets $\lambda \in F(V)$, if f is surjective, we have*

(1) $f(\underline{f^{-1}(N)}(f^{-1}(\lambda))) = \underline{N}(\lambda)$,

(2) $\underline{f^{-1}(N)}(f^{-1}(\lambda)) = f^{-1}(\underline{N}(\lambda))$,

(3) $f^{-1}(\overline{N}(\lambda)) = \overline{f^{-1}(N)}(f^{-1}(\lambda))$,

(4) $\overline{N}(\lambda) = f(\overline{f^{-1}(N)}(f^{-1}(\lambda)))$.

Proof. According to the Theorem 4.2 and the hypothesis that f is surjective, the conclusion is obvious. □

Theorem 4.4. *Let $N : V \to F(P(V))$ be a fuzzifying neighborhood system on V, let $f : U \to V$ and let $\mu \in F(U)$. Then we have*

(1) $f(\underline{f^{-1}(N)}(\mu)) \subseteq \underline{N}(f(\mu))$,

(2) $\underline{f^{-1}(N)}(\mu) \subseteq f^{-1}(\underline{N}(f(\mu)))$,

(3) $f(\overline{f^{-1}(N)}(\mu))|_{f(U)} \subseteq \overline{N}|_{f(U)}(f(\mu))$,

(4) $\overline{f^{-1}(N)}(\mu) \subseteq f^{-1}(\overline{N}|_{f(U)}(f^{-1}(\mu)))$.

Proof. (1) For any $y \in V$, if $y \in V \backslash f(U)$, then $f(\underline{f^{-1}(N)}(\mu))(y) = 0$. Thus the conclusion is obvious. We just need to prove the case $y \in f(U)$ in the following steps.

$$\begin{aligned}f(\underline{f^{-1}(N)}(\mu))(y) &= \bigvee_{f(x)=y} \underline{f^{-1}(N)}(\mu)(x) \\ &= \bigvee_{f(x)=y} \bigvee_{H^{'} \in P(U)} (f^{-1}(N)(x)(H^{'}) \wedge (\bigwedge_{z^{'} \in H^{'}} \mu(z^{'})))\end{aligned}$$

$$= \bigvee_{f(x)=y} \bigvee_{H^{'}\in P(U)} (N(f(x))(f(H^{'})) \wedge (\bigwedge_{z^{'}\in H^{'}} \mu(z^{'})))$$
$$= \bigvee_{H^{'}\in P(U)} (N(y)(f(H^{'})) \wedge (\bigwedge_{z^{'}\in H^{'}} \mu(z^{'}))).$$

According to the Lemma 3.2, if $f(H^{'}) = H \in P(V)$ for any $H^{'} \in P(U)$, then

$$\begin{aligned} f(\underline{f^{-1}(N)}(\mu))(y) &\leq \bigvee_{H^{'}\in P(U)} (N(y)(f(H^{'})) \wedge (\bigwedge_{z\in f(H^{'})} f(\mu)(z))) \\ &= \bigvee_{f(H^{'})=H\in P(f(U))} (N(y)(H) \wedge (\bigwedge_{z\in H} f(\mu)(z))) \\ &\leq \bigvee_{H\in P(V)} (N(y)(H) \wedge (\bigwedge_{z\in H} f(\mu)(z))) \\ &= \underline{N}(f(\mu))(y). \end{aligned}$$

Thus $f(\underline{f^{-1}(N)}(\mu))(y) \leq \underline{N}(f(\mu))(y)$. So the conclusion holds.

(2) It follows immediately from (1), and Lemma 2.1 (1) and (4).

(3) For any $y \in f(U)$,

$$\begin{aligned} & f(\overline{f^{-1}(N)}(\mu))|_{f(U)}(y) \\ = & \bigvee_{f(x)=y} \overline{f^{-1}(N)}(\mu)(x) \\ = & \bigvee_{f(x)=y} \bigwedge_{H^{'}\in P(U)} ((1 - f^{-1}(N)(x)(H^{'})) \vee (\bigvee_{z^{'}\in H^{'}} \mu(z^{'}))) \\ = & \bigvee_{f(x)=y} \bigwedge_{H^{'}\in P(U)} ((1 - N(f(x))(f(H^{'}))) \vee (\bigvee_{z^{'}\in H^{'}} \mu(z^{'}))) \\ = & \bigwedge_{H^{'}\in P(U)} ((1 - N(y)(f(H^{'}))) \vee (\bigvee_{z^{'}\in H^{'}} \mu(z^{'}))). \end{aligned}$$

According to the Lemma 4.1, if $f(H^{'}) = H \in P(V)$ for any $H^{'} \in P(U)$, then

$$f(\overline{f^{-1}(N)}(\mu))|_{f(U)}(y) \leq \bigwedge_{f(H^{'})=H\in P(f(U))} ((1 - N(y)(H)) \vee (\bigvee_{z\in H} f(\mu)(z))).$$

Thus $f(\overline{f^{-1}(N)}(\mu))|_{f(U)}(y) \leq \overline{N}|_{f(U)}(f(\mu))(y)$. So the conclusion holds.

(4) It follows immediately from (3), and Lemma 2.1 (1) and (4). □

Corollary 4.5. *Let $N : V \to F(P(V))$ be a fuzzifying neighborhood system, and let $f : U \to V$ ba a mapping. Then, for any fuzzy subsets $\mu \in F(U)$, f is gra-consistent with respect to $f^{-1}(N)$, we have*

(1) $f(\overline{f^{-1}(N)}(\mu))|_{f(U)} = \overline{N}|_{f(U)}(f(\mu))$.

(2) $\overline{f^{-1}(N)}(\mu) = f^{-1}(\overline{N}|_{f(U)}(f^{-1}(\mu)))$.

Proof. According to the Theorem 4.4, and f is gra-consistent with respect to $f^{-1}(N)$, the conclusion is obvious. □

5. Conclusions

In the paper, we have provided a natural extension of consistent functions with respect to fuzzifying neighborhood systems. Furthermore, we define a pair of interior and closure operators based on the fuzzifying neighborhood system, and then explore the relationship of interior and closure operators in two generalized approximation spaces based on fuzzifying neighborhood systems by consistent functions or mappings.

In the future, we can explore the topological properties of interior and closure operators based on the fuzzifying neighborhood system. This issue can be considered from two aspects. On the one hand, from the perspective of forming a topology, we can consider what conditions can be satisfied by a definable set of interior and closure operators based on the fuzzifying neighborhood system to constitute a topology. On the other hand, by defining a fuzzifying neighborhood system, given a topological space, discuss what properties the defined fuzzifying neighborhood system has.

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Ting Hou (1394574374@qq.com)
School of Mathematics, Southwest Jiaotong University, Chengdu 611756, China

Longyu He (2267297748@qq.com)
School of Mathematics, Southwest Jiaotong University, Chengdu 611756, China

Keyun Qin (Keyunqin@263.net)
School of Mathematics, Southwest Jiaotong University, Chengdu 611756, China