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The young structure based on $\mathcal{N}$-structure and its application in BCI/BCK-algebras

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Abstract. A new structure called a young structure based on the $\mathcal{N}$-structure is introduced and applied to BCK/BCI-algebra. The concepts of young subalgebra, $\in$-subset, q-subset, and O-subset are introduced, and their related properties are studied. The relationship between $\mathcal{N}$-subalgebra and young subalgebra is discussed. It is observed that every $\mathcal{N}$-subalgebra is a young subalgebra, and an example is give to show that the converse does not hold. The characterization of young subalgebra is considered, and conditions are given to show that a young structure based on $\mathcal{N}$-structure is a (strong) young subalgebra. Conditions under which $\in$-subset, q-subset, and O-subset can be subalgebra are explored.

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1. Introduction

It is a well-known fact that fuzzy set, which is generalizations of (crisp) set, are used in almost every field. The fuzzy set was introduced to extend the information related to the clear point $\{1\}$ to the closed interval $[0, 1]$. So a fuzzy set is very useful tool for dealing with positive information, but it is not suitable for dealing with negative information. We feel the need for a mathematical vessel that can contain the negative aspects of information. So Jun et al. [1] developed the so-called $\mathcal{N}$-structure and applied it to BCK/BCI-algebra. It was also applied to CI-algebra, subtraction algebra, BCH-algebra, and IS-algebra (See [2, 3, 4, 5]).

The purpose of this paper is to introduce a novel structure called a *young structure* based on the $\mathcal{N}$-structure and apply it to BCK/BCI-algebra. We introduce the concepts of young subalgebra, $\in$-subset, q-subset, and O-subset and study their related properties. We discuss the relationship between N-subalgebra and young subalgebra. We show that every $\mathcal{N}$-subalgebra is a young subalgebra, and we provide an example that shows that the converse does not hold. We consider the characterization of young subalgebra. We provide conditions for a young structure (W, Y_f^{ε}) based on (W, f) to be a (strong) young subalgebra. We explore the conditions under which $\in$-subset, q-subset, and O-subset can be subalgebra.

2. Preliminaries

A BCI/BCK-algebra is introduced by Iséki (See [6] and [7]), and it is an important class of logical algebras. We list the necessary definitions and basic results in this paper. For more information about BCK-algebra and BCI-algebra, refer to the books [8, 9].

If a set W has a binary operation " $*$ " and a special element "0" satisfying the conditions:

(I_1) $(\forall a, b, c \in W)\ (((a * b) * (a * c)) * (c * b) = 0)$,
(I_2) $(\forall a, b \in W)\ ((a * (a * b)) * b = 0)$,
(I_3) $(\forall a \in W)\ (a * a = 0)$,
(I_4) $(\forall a, b \in W)\ (a * b = 0,\ b * a = 0\ \Rightarrow\ a = b)$,

then we say that W is a *BCI-algebra*. If a BCI-algebra W has the following identity:

(K) $(\forall a \in W)\ (0 * a = 0)$,

then it is called a *BCK-algebra*. The BCI/BCK-algebra is written as $(W, 0)_*$.

A BCI-algebra W is said to be *p-semisimple* (See [8]) if $0 * (0 * a) = a$ for all $a \in W$.

The order relation "$\leq$" in a BCI/BCK-algebra $(W, 0)_*$ is defined as follows:

$$(\forall a, b \in W)(a \leq b\ \Leftrightarrow\ a * b = 0). \tag{2.1}$$

Every BCI/BCK-algebra $(W, 0)_*$ satisfies the following assertions (See [8, 9]):

$$(\forall a \in W)\,(a * 0 = a)\,, \tag{2.2}$$

$$(\forall a, b, c \in W)\,(a \leq b\ \Rightarrow\ a * c \leq b * c,\ c * b \leq c * a)\,, \tag{2.3}$$

$$(\forall a, b, c \in W)\,((a * b) * c = (a * c) * b)\,. \tag{2.4}$$

Every BCI-algebra $(W, 0)_*$ satisfies (See [8]):

$$(\forall a, b \in W)\,(a * (a * (a * b)) = a * b)\,, \tag{2.5}$$

$$(\forall a, b \in W)\,(0 * (a * b) = (0 * a) * (0 * b))\,. \tag{2.6}$$

A subset D of a BCI/BCK-algebra $(W, 0)_*$ is called a *subalgebra* of $(W, 0)_*$ (See [8, 9]) if $a * b \in D$ for all $a, b \in D$.

The collection of functions from a set W to $[-1, 0]$ is denoted by $\mathcal{F}(W, [-1, 0])$. We say that an element of $\mathcal{F}(W, [-1, 0])$ is a *negative-valued function* from W to $[-1, 0]$ (briefly, *$\mathcal{N}$-function* on W). By an *$\mathcal{N}$-structure* we mean an ordered pair (W, f) of W and an $\mathcal{N}$-function f on W.

By a *subalgebra* of a BCI/BCK-algebra $(W, 0)_*$ based on $\mathcal{N}$-function f (briefly, *$\mathcal{N}$-subalgebra* of $(W, 0)_*$), we mean an $\mathcal{N}$-structure (W, f) in which f satisfies the following assertion:

$$(\forall a, b \in W)(f(a * b) \leq \max\{f(a), f(b)\}). \tag{2.7}$$

Recall that every $\mathcal{N}$-subalgebra (W, f) of a BCI/BCK-algebra $(W, 0)_*$ satisfies $f(0) \leq f(x)$ for all $x \in W$.

3. The young structure based on $\mathcal{N}$-structure

In what follows, let $(W, 0)_*$ and (W, f) denote a BCI/BCK-algebra and an $\mathcal{N}$-structure, respectively, unless otherwise specified.

Let W be a set. Given a number ε in $[-1, 0]$, we consider a function so called a *young function* on W as follows:

$$Y_f^\varepsilon : W \to [-1, 0],\ x \mapsto \min\{0, f(x) + \varepsilon + 1\}. \tag{3.1}$$

Definition 3.1. Let W be a set. Given a number ε in $[-1, 0]$, the couple (W, Y_f^ε) of W and Y_f^ε is called a *young structure* based on (W, f).

Example 3.2. Consider the following $\mathcal{N}$-function on $\mathbb{R}$ (the set of real numbers)

$$f : \mathbb{R} \to [-1, 0],\ x \mapsto \tfrac{1}{2}(\sin x - 1).$$

For $\varepsilon = -0.7$, the young structure based on $(\mathbb{R}, f)$ is $(\mathbb{R}, Y_f^\varepsilon)$ with

$$Y_f^\varepsilon : \mathbb{R} \to [-1, 0],\ x \mapsto \min\{0, \tfrac{1}{2}\sin x - 0.2\}.$$

Let (W, Y_f^ε) be a young structure based on (W, f). If $\varepsilon = -1$, then

$$Y_f^\varepsilon(x) = \min\{0, f(x) - 1 + 1\} = \min\{0, f(x)\} = f(x)$$

for all $x \in W$. This shows that if $\varepsilon = -1$, then the young structure (W, Y_f^ε) is the same as the $\mathcal{N}$-structure (W, f), that is, $(W, Y_f^{-1}) = (W, f)$. If $\varepsilon = 0$, then

$$Y_f^\varepsilon(x) = \min\{0, f(x) + 0 + 1\} = \min\{0, f(x) + 1\} = 0$$

for all $x \in W$. Thus, when processing young structure (W, Y_f^ε) based on (W, f), the value of ε can always be considered to be at an open interval $(-1, 0)$.

Given ε in $(-1, 0)$, if $f(x) + \varepsilon \geq -1$ for all $x \in W$, then the young function on W is the 0-constant function, i.e., $Y_f^\varepsilon(x) = 0$ for all $x \in W$. Therefore, in order for the young structure to have a meaningful form, an $\mathcal{N}$-function f in W and $\varepsilon \in (-1, 0)$ must be set to satisfy the condition below:

$$(\exists x \in W)(f(x) + \varepsilon < -1).$$

Proposition 3.3. *For $\varepsilon \in (-1, 0)$, the young structure (W, Y_f^ε) based on (W, f) satisfies:*

$$(\forall x, y \in W)(f(x) \geq f(y)\ \Rightarrow\ Y_f^\varepsilon(x) \geq Y_f^\varepsilon(y)), \tag{3.2}$$

$$(\forall x \in W)(f(x) + \varepsilon < -1\ \Rightarrow\ Y_f^\varepsilon(x) = f(x) + \varepsilon + 1). \tag{3.3}$$

$$(\forall x \in W)(\forall \delta \in (-1, 0))(\varepsilon \geq \delta\ \Rightarrow\ Y_f^\varepsilon(x) \geq Y_f^\delta(x)). \tag{3.4}$$

Proof. Straightforward. □

Proposition 3.4. *If (W, f) and (W, g) are $\mathcal{N}$-structures in a set W, then*

$$(\forall \varepsilon \in (-1, 0))\left(Y_{f\cup g}^\varepsilon = Y_f^\varepsilon \cup Y_g^\varepsilon,\ Y_{f\cap g}^\varepsilon = Y_f^\varepsilon \cap Y_g^\varepsilon\right). \tag{3.5}$$

Proof. For every $\varepsilon \in (-1, 0)$ and $x \in W$, we have

$$\begin{aligned} Y^{\varepsilon}_{f\cup g}(x) &= \min\{0, (f \cup g)(x) + \varepsilon + 1\} \\ &= \min\{0, \max\{f(x), g(x)\} + \varepsilon + 1\} \\ &= \min\{0, \max\{f(x) + \varepsilon + 1, g(x) + \varepsilon + 1\}\} \\ &= \max\{\min\{0, f(x) + \varepsilon + 1\}, \min\{0, g(x) + \varepsilon + 1\}\} \\ &= \max\{Y^{\varepsilon}_f(x), Y^{\varepsilon}_g(x)\} \\ &= (Y^{\varepsilon}_f \cup Y^{\varepsilon}_g)(x) \end{aligned}$$

which proves $Y^{\varepsilon}_{f\cup g} = Y^{\varepsilon}_f \cup Y^{\varepsilon}_g$. Use the same method to derive $Y^{\varepsilon}_{f\cap g} = Y^{\varepsilon}_f \cap Y^{\varepsilon}_g$. □

Let (W, Y^{ε}_f) be a young structure based on (W, f) and let $t \in [-1, 0)$. We consider the sets

(3.6)
$$(Y^{\varepsilon}_f, t)_{\in} := \{x \in W \mid Y^{\varepsilon}_f(x) \leq t\} \text{ and } (Y^{\varepsilon}_f, t)_q := \{x \in W \mid Y^{\varepsilon}_f(x) + t + 1 < 0\}$$

which are called the $\in$-*subset* and the q-*subset*, respectively, of W.

Proposition 3.5. *Let (W, Y^{ε}_f) be a young structure based on (W, f) and let $t, r \in [-1, 0)$. Then*

$$t \in [-0.5, 0) \;\Rightarrow\; (Y^{\varepsilon}_f, t)_q \subseteq (Y^{\varepsilon}_f, t)_{\in}. \tag{3.7}$$

$$t \in [-1, -0.5) \;\Rightarrow\; (Y^{\varepsilon}_f, t)_{\in} \subseteq (Y^{\varepsilon}_f, t)_q. \tag{3.8}$$

$$t \leq r \;\Rightarrow\; (Y^{\varepsilon}_f, t)_{\in} \subseteq (Y^{\varepsilon}_f, r)_{\in},\; (Y^{\varepsilon}_f, r)_q \subseteq (Y^{\varepsilon}_f, t)_q. \tag{3.9}$$

Proof. For $t \in [-0.5, 0)$, let $x \in (Y^{\varepsilon}_f, t)_q$. Then $Y^{\varepsilon}_f(x) < -1 - t \leq t$, and so $x \in (Y^{\varepsilon}_f, t)_{\in}$. Let $x \in (Y^{\varepsilon}_f, t)_{\in}$ for all $t \in [-1, -0.5)$. Then $Y^{\varepsilon}_f(x) \leq t < -0.5$, and so $Y^{\varepsilon}_f(x) + t \leq 2t < -1$. Thus $x \in (Y^{\varepsilon}_f, t)_q$. The third result (3.9) is straightforward. □

4. The young subalgebra

In what follows, let $(W, 0)_*$ be a BCI-algebra or a BCK-algebra, and ε is an element of $(-1, 0)$ unless otherwise specified.

Definition 4.1. A young structure (W, Y^{ε}_f) based on (W, f) is called a *young subalgebra* of $(W, 0)_*$, if it satisfies:

$$(\forall x, y \in W)(t_x, t_y \in [-1, 0)) \left(\begin{array}{c} x \in (Y^{\varepsilon}_f, t_x)_{\in},\; y \in (Y^{\varepsilon}_f, t_y)_{\in} \\ \Rightarrow x * y \in (Y^{\varepsilon}_f, \max\{t_x, t_y\})_{\in} \end{array} \right). \tag{4.1}$$

Example 4.2. Consider a set $W = \{0, b_1, b_2, b_3\}$ in which a binary operation “$*$” is given by the table below.

$*$	0	b_1	b_2	b_3
0	0	0	0	0
b_1	b_1	0	0	b_1
b_2	b_2	b_1	0	b_2
b_3	b_3	b_3	b_3	0

Then $(W, 0)_*$ is a BCK-algebra (See [9]). Define an $\mathcal{N}$-structure (W, f) as follows:

$$f : W \to [-1, 0],\ x \mapsto \begin{cases} -0.87 & \text{if } x = 0, \\ -0.63 & \text{if } x = b_1, \\ -0.63 & \text{if } x = b_2, \\ -0.44 & \text{if } x = b_3. \end{cases}$$

Given $\varepsilon := -0.59$, the young structure (W, Y_f^ε) based on (W, f) is given as follows:

$$Y_f^\varepsilon : W \to [0, 1],\ x \mapsto \begin{cases} -0.46 & \text{if } x = 0, \\ -0.22 & \text{if } x = b_1, \\ -0.22 & \text{if } x = b_2, \\ -0.03 & \text{if } x = b_3. \end{cases}$$

It is routine to check that (W, Y_f^ε) is a young subalgebra of $(W, 0)_*$.

We discuss the relationship between $\mathcal{N}$-subalgebra and young subalgebra.

Theorem 4.3. *If (W, f) is an $\mathcal{N}$-subalgebra of $(W, 0)_*$, then a young structure (W, Y_f^ε) based on (W, f) is a young subalgebra of $(W, 0)_*$.*

Proof. Assume that (W, f) is an $\mathcal{N}$-subalgebra of $(W, 0)_*$. Let $x, y \in W$ and $t_x, t_y \in [-1, 0)$ be such that $x \in (Y_f^\varepsilon, t_x)_\in$ and $y \in (Y_f^\varepsilon, t_y)_\in$. Then $Y_f^\varepsilon(x) \leq t_x$ and $Y_f^\varepsilon(y) \leq t_y$. Thus we have

$$\begin{aligned} Y_f^\varepsilon(x * y) &= \min\{0, f(x * y) + \varepsilon + 1\} \\ &\leq \min\{0, \max\{f(x), f(y)\} + \varepsilon + 1\} \\ &= \min\{0, \max\{f(x) + \varepsilon + 1, f(y) + \varepsilon + 1\}\} \\ &= \max\{\min\{0, f(x) + \varepsilon + 1\}, \min\{0, f(y) + \varepsilon + 1\}\} \\ &= \max\{Y_f^\varepsilon(x), Y_f^\varepsilon(y)\} \leq \max\{t_x, t_y\}. \end{aligned}$$

So $x * y \in (Y_f^\varepsilon, \max\{t_x, t_y\})_\in$. Hence (W, Y_f^ε) is a young subalgebra of $(W, 0)_*$. □

The following example illustrates the existence of a young subalgebra (W, Y_f^ε) of $(W, 0)_*$ in which its base $\mathcal{N}$-structure (W, f) may not be an $\mathcal{N}$-subalgebra of $(W, 0)_*$.

Example 4.4. Consider a set $W = \{0, b_1, b_2, b_3, b_4\}$ with a binary operation "$*$" by the following Cayley table.

$*$	0	b_1	b_2	b_3	b_4
0	0	0	b_2	b_3	b_4
b_1	b_1	0	b_2	b_3	b_4
b_2	b_2	b_2	0	b_4	b_3
b_3	b_3	b_3	b_4	0	b_2
b_4	b_4	b_4	b_3	b_2	0

Then $(W,0)_*$ is a BCI-algebra (See [8]). Define an $\mathcal{N}$-structure (W,f) as follows:

$$f : W \to [-1,0],\ x \mapsto \begin{cases} -0.83 & \text{if } x = 0, \\ -0.68 & \text{if } x = b_1, \\ -0.62 & \text{if } x = b_2, \\ -0.57 & \text{if } x = b_3, \\ -0.46 & \text{if } x = b_4. \end{cases}$$

If we take $\varepsilon := -0.43$, then the young structure (W, Y_f^ε) based on (W,f) is given as follows:

$$Y_f^\varepsilon : W \to [0,1],\ x \mapsto \begin{cases} -0.26 & \text{if } x = 0, \\ -0.11 & \text{if } x = b_1, \\ -0.05 & \text{if } x = b_2, \\ 0.00 & \text{if } x \in \{b_3, b_4\}. \end{cases}$$

It is easy to check that (W, Y_f^ε) is a young subalgebra of $(W,0)_*$. Since

$$f(b_3 * b_2) = f(b_4) = -0.46 \nleq -0.57 = \max\{f(b_3), f(b_2)\},$$

we know that (W,f) is not an $\mathcal{N}$-subalgebra of $(W,0)_*$.

Proposition 4.5. *Let (W, Y_f^ε) be a young structure based on (W,f). If (W,f) is an $\mathcal{N}$-subalgebra of $(W,0)_*$, then*

(4.2) $\qquad Y_f^\varepsilon(0)$ is a lower bound of $\{Y_f^\varepsilon(x) \mid x \in W\}$,

(4.3) $\qquad (\forall x, y \in W)\left(x * y \in (Y_f^\varepsilon, Y_f^\varepsilon(y))_\in \ \Rightarrow\ Y_f^\varepsilon \text{ is constant on } W\right).$

Proof. Assume that (W,f) is an $\mathcal{N}$-subalgebra of $(W,0)_*$. Then

$$f(0) = f(x * x) \leq \max\{f(x), f(x)\} = f(x)$$

for all $x \in W$. It follows from (3.2) that $Y_f^\varepsilon(0) \leq Y_f^\varepsilon(x)$ for all $x \in W$. Thus $Y_f^\varepsilon(0)$ is a lower bound of $\{Y_f^\varepsilon(x) \mid x \in W\}$. Let $x, y \in W$ be such that $x * y \in (Y_f^\varepsilon, Y_f^\varepsilon(y))_\in$. Then $Y_f^\varepsilon(x*y) \leq Y_f^\varepsilon(y)$, and thus $Y_f^\varepsilon(x) = Y_f^\varepsilon(x*0) \leq Y_f^\varepsilon(0)$. Since $Y_f^\varepsilon(x) \geq Y_f^\varepsilon(0)$ for all $x \in W$ by (4.2), we get $Y_f^\varepsilon(x) = Y_f^\varepsilon(0)$ for all $x \in W$. So Y_f^ε is constant on W. $\square$

Proposition 4.6. *Let (W, Y_f^ε) be a young structure based on (W,f). If (W,f) is an $\mathcal{N}$-subalgebra of a BCI-algebra $(W,0)_*$, then*

(4.4) $\qquad (\forall x \in W)(Y_f^\varepsilon(0 * x) \leq Y_f^\varepsilon(x)),$

(4.5) $\qquad (\forall x, y \in W)(t_x, t_y \in [-1,0))\left(\begin{array}{l} x \in (Y_f^\varepsilon, t_x)_\in,\ y \in (Y_f^\varepsilon, t_y)_\in \\ \Rightarrow x * (0 * y) \in (Y_f^\varepsilon, \max\{t_x, t_y\})_\in \end{array}\right).$

Proof. Assume that (W,f) is an $\mathcal{N}$-subalgebra of a BCI-algebra $(W,0)_*$. Then

$$f(0 * x) \leq \max\{f(0), f(x)\} = f(x),$$

and thus $Y_f^\varepsilon(0*x) \leq Y_f^\varepsilon(x)$ for all $x \in W$ by (3.2). Let $x, y \in W$ and $t_x, t_y \in [-1,0)$ be such that $x \in (Y_f^\varepsilon, t_x)_\in$ and $y \in (Y_f^\varepsilon, t_y)_\in$. Then $Y_f^\varepsilon(x) \leq t_x$ and $Y_f^\varepsilon(y) \leq t_y$.

Thus we get

$$\begin{aligned} Y_f^\varepsilon(x*(0*y)) &= \min\{0, f(x*(0*y))+\varepsilon+1\} \\ &\le \min\{0, \max\{f(x), f(0*y)\}+\varepsilon+1\} \\ &\le \min\{0, \max\{f(x), \max\{f(0), f(y)\}\}+\varepsilon+1\} \\ &= \min\{0, \max\{f(x), f(y)\}+\varepsilon+1\} \\ &= \min\{0, \max\{f(x)+\varepsilon+1, f(y)+\varepsilon+1\}\} \\ &= \max\{\min\{0, f(x)+\varepsilon+1\}, \min\{0, f(y)+\varepsilon+1\}\} \\ &= \max\{Y_f^\varepsilon(x), Y_f^\varepsilon(y)\} \\ &\le \max\{t_a, t_b\}. \end{aligned}$$

So $x*(0*y) \in (Y_f^\varepsilon, \max\{t_x, t_y\})_\in$. □

Theorem 4.7. *A young structure (W, Y_f^ε) based on (W, f) is a young subalgebra of $(W, 0)_*$ if and only if it satisfies:*

$$(4.6) \qquad (\forall x, y \in W)(Y_f^\varepsilon(x*y) \le \max\{Y_f^\varepsilon(x), Y_f^\varepsilon(y)\}).$$

Proof. Assume that (W, Y_f^ε) is a young subalgebra of $(W, 0)_*$ and let $x, y \in W$. It is obvious that $x \in (Y_f^\varepsilon, Y_f^\varepsilon(x))_\in$ and $y \in (Y_f^\varepsilon, Y_f^\varepsilon(y))_\in$. It follows from (4.1) that $x*y \in (Y_f^\varepsilon, \max\{Y_f^\varepsilon(x), Y_f^\varepsilon(y)\})_\in$. Then $Y_f^\varepsilon(x*y) \le \max\{Y_f^\varepsilon(x), Y_f^\varepsilon(y)\}$.

Conversely, suppose that (W, Y_f^ε) satisfies the condition (4.6). Let $x, y \in W$ and $t_x, t_y \in [-1, 0)$ be such that $x \in (Y_f^\varepsilon, t_x)_\in$ and $y \in (Y_f^\varepsilon, t_y)_\in$. Then $Y_f^\varepsilon(x) \le t_x$ and $Y_f^\varepsilon(y) \le t_y$. It follows from (4.6) that $Y_f^\varepsilon(x*y) \le \max\{Y_f^\varepsilon(x), Y_f^\varepsilon(y)\} \le \max\{t_x, t_y\}$. Thus $x*y \in (Y_f^\varepsilon, \max\{t_x, t_y\})_\in$. So (W, Y_f^ε) is a young subalgebra of $(W, 0)_*$. □

We provide conditions for a young structure (W, Y_f^ε) based on (W, f) to be a young subalgebra.

Theorem 4.8. *Let (W, Y_f^ε) be a young structure based on (W, f). If it satisfies:*

$$(4.7) \qquad (\forall x, y, z \in W)(t_y, t_z \in [-1, 0)) \left(\begin{array}{l} z \le x,\ y \in (Y_f^\varepsilon, t_y)_\in,\ z \in (Y_f^\varepsilon, t_z)_\in \\ \Rightarrow x*y \in (Y_f^\varepsilon, \max\{t_y, t_z\})_\in \end{array} \right),$$

then (W, Y_f^ε) is a young subalgebra of $(W, 0)_*$.

Proof. Let $x, y, z \in W$ and $t_y, t_z \in [-1, 0)$ be such that $z \le x$, $y \in (Y_f^\varepsilon, t_y)_\in$ and $z \in (Y_f^\varepsilon, t_z)_\in$. If we take $z := x$ in (4.7), then $x*y \in (Y_f^\varepsilon, \max\{t_y, t_z\})_\in$. Thus (W, Y_f^ε) is a young subalgebra of $(W, 0)_*$. □

Let (W, Y_f^ε) be a young subalgebra of a BCI-algebra $(W, 0)_*$. Then (W, Y_f^ε) does not satisfy the condition below.

$$(4.8) \qquad (\forall x \in W)(Y_f^\varepsilon(0*x) = Y_f^\varepsilon(x)).$$

In fact, the young subalgebra (W, Y_f^ε) in Example 4.4 does not satisfy the condition (4.8) since $Y_f^\varepsilon(0*b_1) = Y_f^\varepsilon(0) = -0.26 \neq -0.11 = Y_f^\varepsilon(b_1)$. If a young subalgebra (W, Y_f^ε) of a BCI-algebra $(W, 0)_*$ satisfies the condition (4.8), we say it is *strong*.

Theorem 4.9. *If (W, f) is an $\mathcal{N}$-subalgebra of a BCI-algebra $(W, 0)_*$. If $(W, 0)_*$ is p-semisimple, then a young structure (W, Y_f^ε) based on (W, f) is a strong young subalgebra of $(W, 0)_*$.*

Proof. Assume that (W, f) is an $\mathcal{N}$-subalgebra of a p-semisimple BCI-algebra $(W, 0)_*$. Then (W, Y_f^ε) is a young subalgebra of $(W, 0)_*$, and satisfies the condition (4.5) (See Theorem 4.3 and Proposition 4.6). Since $x \in (Y_f^\varepsilon, Y_f^\varepsilon(x))_\in$ and $y \in (Y_f^\varepsilon, Y_f^\varepsilon(y))_\in$ for all $x, y \in W$, we have $x * (0 * y) \in (Y_f^\varepsilon, \max\{Y_f^\varepsilon(x), Y_f^\varepsilon(y)\})_\in$ by (4.5). Thus

$$Y_f^\varepsilon(x * (0 * y)) \leq \max\{Y_f^\varepsilon(x), Y_f^\varepsilon(y)\} \tag{4.9}$$

for all $x, y \in W$. The combination of (2.5), (4.2), (4.4), Theorem 4.7, (4.9) and the p-semisimplicity of $(W, 0)_*$ induces

$$\begin{aligned} Y_f^\varepsilon(x) &= Y_f^\varepsilon(0 * (0 * x)) \leq \max\{Y_f^\varepsilon(0), Y_f^\varepsilon(0 * x)\} \\ &= Y_f^\varepsilon(0 * x) = Y_f^\varepsilon(0 * (0 * (0 * x))) \\ &\leq \max\{Y_f^\varepsilon(0), Y_f^\varepsilon(0 * x)\} = Y_f^\varepsilon(0 * x) \\ &\leq Y_f^\varepsilon(x). \end{aligned}$$

So $Y_f^\varepsilon(x) = Y_f^\varepsilon(0 * x)$ for all $x \in W$. Hence (W, Y_f^ε) is a strong young subalgebra of $(W, 0)_*$. □

We explore the conditions under which $\in$-subset and q-subset can be subalgebra.

Theorem 4.10. *Let (W, Y_f^ε) be a young structure based on (W, f). Then the $\in$-subset $(Y_f^\varepsilon, t)_\in$ of W with value $t \in [-1, -0.5)$ is a subalgebra of $(W, 0)_*$ if and only if (W, Y_f^ε) satisfies:*

$$(\forall x, y \in W)\left(\max\{Y_f^\varepsilon(x), Y_f^\varepsilon(y)\} \geq \min\{Y_f^\varepsilon(x * y), -0.5\}\right). \tag{4.10}$$

Proof. Assume that the $\in$-subset $(Y_f^\varepsilon, t)_\in$ of W with value $t \in [-1, -0.5)$ is a subalgebra of $(W, 0)_*$. If there exist $a, b \in W$ such that $\max\{Y_f^\varepsilon(a), Y_f^\varepsilon(b)\} < \min\{Y_f^\varepsilon(a * b), -0.5\}$, then $t_{a*b} := \max\{Y_f^\varepsilon(a), Y_f^\varepsilon(b)\} \in [-1, -0.5)$, $Y_f^\varepsilon(a) \leq t_{a*b}$, $Y_f^\varepsilon(b) \leq t_{a*b}$ and $Y_f^\varepsilon(a*b) > t_{a*b}$. Thus $a, b \in (Y_f^\varepsilon, t_{a*b})_\in$ and $a*b \notin (Y_f^\varepsilon, t_{a*b})_\in$. This is a contradiction. So $\max\{Y_f^\varepsilon(x), Y_f^\varepsilon(y)\} \geq \min\{Y_f^\varepsilon(x * y), -0.5\}$ for all $x, y \in W$.

Conversely, suppose that (W, Y_f^ε) satisfies (4.10). Let $t \in [-1, -0.5)$ and $x, y \in W$ be such that $x \in (Y_f^\varepsilon, t)_\in$ and $y \in (Y_f^\varepsilon, t)_\in$. Then $Y_f^\varepsilon(x) \leq t$ and $Y_f^\varepsilon(y) \leq t$ which imply from (4.10) that

$$-0.5 > t \geq \max\{Y_f^\varepsilon(x), Y_f^\varepsilon(y)\} \geq \min\{Y_f^\varepsilon(x * y), -0.5\}.$$

Thus $Y_f^\varepsilon(x * y) \leq t$. So $x * y \in (Y_f^\varepsilon, t)_\in$. Hence $(Y_f^\varepsilon, t)_\in$ with value $t \in [-1, -0.5)$ is a subalgebra of $(W, 0)_*$. □

Theorem 4.11. *Let (W, Y_f^ε) be a young structure based on (W, f). If (W, f) is an $\mathcal{N}$-subalgebra of $(W, 0)_*$, then the q-subset $(Y_f^\varepsilon, t)_q$ of W with value $t \in [-1, 0)$ is a subalgebra of $(W, 0)_*$.*

Proof. Assume that (W, f) is an $\mathcal{N}$-subalgebra of $(W, 0)_*$. Then (W, Y_f^ε) is a young subalgebra of $(W, 0)_*$ by Theorem 4.3. Let $x, y \in (Y_f^\varepsilon, t)_q$ for $t \in [-1, 0)$. Then

$Y_f^\varepsilon(x) + t + 1 < 0$ and $Y_f^\varepsilon(y) + t + 1 < 0$. It follows from Theorem 4.7 that

$$\begin{aligned} Y_f^\varepsilon(x * y) + t + 1 &\leq \max\{Y_f^\varepsilon(x), Y_f^\varepsilon(y)\} + t + 1 \\ &= \max\{Y_f^\varepsilon(x) + t + 1, Y_f^\varepsilon(y) + t + 1\} < 0. \end{aligned}$$

Thus $x * y \in (Y_f^\varepsilon, t)_q$. So $(Y_f^\varepsilon, t)_q$ is a subalgebra of $(W, 0)_*$. □

Theorem 4.12. *Let (W, Y_f^ε) be a young structure based on (W, f). If the q-subset $(Y_f^\varepsilon, t)_q$ is a subalgebra of $(W, 0)_*$ for all $t \in [-0.5, 0)$, then (W, Y_f^ε) satisfies:*

$$(4.11) \qquad (\forall x, y \in W)(\forall t_a, t_b \in [-0.5, 0)) \left(\begin{array}{c} x \in (Y_f^\varepsilon, t_a)_q,\ y \in (Y_f^\varepsilon, t_b)_q \\ \Rightarrow x * y \in (Y_f^\varepsilon, \min\{t_a, t_b\})_\in \end{array} \right).$$

Proof. Assume that the q-subset $(Y_f^\varepsilon, t)_q$ is a subalgebra of $(W, 0)_*$ for all $t \in [-0.5, 0)$. Let $x, y \in W$ and $t_a, t_b \in [-0.5, 0)$ be such that $x \in (Y_f^\varepsilon, t_a)_q$ and $y \in (Y_f^\varepsilon, t_b)_q$. Then

$$Y_f^\varepsilon(x) + \min\{t_a, t_b\} + 1 \leq Y_f^\varepsilon(x) + t_a + 1 < 0$$

and

$$Y_f^\varepsilon(y) + \min\{t_a, t_b\} + 1 \leq Y_f^\varepsilon(y) + t_a + 1 < 0.$$

Thus $x, y \in (Y_f^\varepsilon, \min\{t_a, t_b\})_q$. So $x * y \in (Y_f^\varepsilon, \min\{t_a, t_b\})_q$, i.e., $Y_f^\varepsilon(x * y) + \min\{t_a, t_b\} + 1 < 0$. Since $\min\{t_a, t_b\} \geq -0.5$, it follows that

$$Y_f^\varepsilon(x * y) \leq -\min\{t_a, t_b\} - 1 \leq \min\{t_a, t_b\}.$$

Hence $x * y \in (Y_f^\varepsilon, \min\{t_a, t_b\})_\in$. □

Theorem 4.13. *Let (W, Y_f^ε) be a young structure based on (W, f). If the q-subset $(Y_f^\varepsilon, t)_q$ is a subalgebra of $(W, 0)_*$ for all $t \in [-1, -0.5)$, then (W, Y_f^ε) satisfies:*

$$(4.12) \qquad (\forall x, y \in W)(\forall t_a, t_b \in [-1, -0.5)) \left(\begin{array}{c} x \in (Y_f^\varepsilon, t_a)_\in,\ y \in (Y_f^\varepsilon, t_b)_\in \\ \Rightarrow x * y \in (Y_f^\varepsilon, \min\{t_a, t_b\})_q \end{array} \right).$$

Proof. Assume that the q-subset $(Y_f^\varepsilon, t)_q$ is a subalgebra of $(W, 0)_*$ for all $t \in [-1, -0.5)$. Let $x, y \in W$ and $t_a, t_b \in [-1, -0.5)$ be such that $x \in (Y_f^\varepsilon, t_a)_\in$ and $y \in (Y_f^\varepsilon, t_b)_\in$. Then $x \in (Y_f^\varepsilon, t_a)_q \subseteq (Y_f^\varepsilon, \min\{t_a, t_b\})_q$ and $y \in (Y_f^\varepsilon, t_b)_q \subseteq (Y_f^\varepsilon, \min\{t_a, t_b\})_q$ by Proposition 3.5. It follows that $x * y \in (Y_f^\varepsilon, \min\{t_a, t_b\})_q$. □

Theorem 4.14. *Let (W, Y_f^ε) be a young structure based on (W, f). If (W, Y_f^ε) satisfies:*

$$(4.13) \qquad (\forall x, y \in W)(\forall t_a, t_b \in [-1, 0)) \left(\begin{array}{c} x \in (Y_f^\varepsilon, t_a)_q,\ y \in (Y_f^\varepsilon, t_b)_q \\ \Rightarrow x * y \in (Y_f^\varepsilon, \min\{t_a, t_b\})_\in \end{array} \right),$$

then the q-subset $(Y_f^\varepsilon, t)_q$ is a subalgebra of $(W, 0)_*$ for all $t \in [-1, -0.5)$.

Proof. Assume that (W, Y_f^ε) satisfies the condition (4.13). Let $x, y \in (Y_f^\varepsilon, t)_q$ for every $t \in [-1, -0.5)$. Then $x * y \in (Y_f^\varepsilon, \min\{t, t\})_\in = (Y_f^\varepsilon, t)_\in$ by (4.13). Thus $Y_f^\varepsilon(x * y) \leq t < -1 - t$. So $x * y \in (Y_f^\varepsilon, t)_q$. Hence $(Y_f^\varepsilon, t)_q$ is a subalgebra of $(W, 0)_*$. □

Corollary 4.15. *Let (W, Y_f^ε) be a young structure based on (W, f). If (W, Y_f^ε) satisfies:*

$$(\forall x, y \in W)(\forall t_a, t_b \in [-1, 0)) \left(\begin{array}{l} x \in (Y_f^\varepsilon, t_a)_q,\ y \in (Y_f^\varepsilon, t_b)_q \\ \Rightarrow \left\{ \begin{array}{l} x * y \in (Y_f^\varepsilon, \min\{t_a, t_b\})_\in \text{ or} \\ x * y \in (Y_f^\varepsilon, \min\{t_a, t_b\})_q \end{array} \right. \end{array} \right), \tag{4.14}$$

then the q-subset $(Y_f^\varepsilon, t)_q$ is a subalgebra of $(W, 0)_*$ for all $t \in [-1, -0.5)$.

Let (W, Y_f^ε) be a young structure based on (W, f). We consider the following set

$$O(Y_f^\varepsilon) := \{x \in W \mid Y_f^\varepsilon(x) < 0\} = \{x \in W \mid f(x) + \varepsilon + 1 < 0\}$$

which is called the *O-subset* of W.

We provide conditions for the O-subset to be a subalgebra.

Theorem 4.16. *Let (W, Y_f^ε) be a young structure based on (W, f). If (W, f) is an $\mathcal{N}$-subalgebra of $(W, 0)_*$, then the O-subset $O(Y_f^\varepsilon)$ of W is a subalgebra of $(W, 0)_*$.*

Proof. Assume that (W, f) is an $\mathcal{N}$-subalgebra of $(W, 0)_*$. Then (W, Y_f^ε) is a young subalgebra of $(W, 0)_*$ (See Theorem 4.3). Let $x, y \in O(Y_f^\varepsilon)$. Then $f(x) + \varepsilon + 1 < 0$ and $f(y) + \varepsilon + 1 < 0$, which imply from Theorem 4.7 that

$$Y_f^\varepsilon(x * y) \leq \max\{Y_f^\varepsilon(x), Y_f^\varepsilon(y)\} = \max\{f(x) + \varepsilon + 1, f(y) + \varepsilon + 1\} < 0.$$

Thus $x * y \in O(Y_f^\varepsilon)$. So $O(Y_f^\varepsilon)$ is a subalgebra of $(W, 0)_*$. □

Theorem 4.17. *Let (W, Y_f^ε) be a young structure based on (W, f). If it satisfies:*

$$(\forall x, y \in W)(\forall t_a, t_b \in [-1, 0)) \left(\begin{array}{l} x \in (Y_f^\varepsilon, t_a)_\in,\ y \in (Y_f^\varepsilon, t_b)_\in \\ \Rightarrow x * y \in (Y_f^\varepsilon, \min\{t_a, t_b\})_q \end{array} \right), \tag{4.15}$$

then the O-subset $O(Y_f^\varepsilon)$ of W is a subalgebra of $(W, 0)_*$.

Proof. Suppose that (W, Y_f^ε) satisfies the condition (4.15). Let $x, y \in O(Y_f^\varepsilon)$. Then $f(x) + \varepsilon + 1 < 0$, $f(y) + \varepsilon + 1 < 0$, and note that $x \in (Y_f^\varepsilon, Y_f^\varepsilon(x))_\in$ and $y \in (Y_f^\varepsilon, Y_f^\varepsilon(y))_\in$. Thus

$$x * y \in (Y_f^\varepsilon, \min\{Y_f^\varepsilon(x), Y_f^\varepsilon(y)\})_q \tag{4.16}$$

by (4.15). If $x * y \notin O(Y_f^\varepsilon)$, then $Y_f^\varepsilon(x * y) = 0$. Thus we have

$$\begin{aligned} & Y_f^\varepsilon(x * y) + \min\{Y_f^\varepsilon(x), Y_f^\varepsilon(y)\} = \min\{Y_f^\varepsilon(x), Y_f^\varepsilon(y)\} \\ & = \min\{\min\{0, f(x) + \varepsilon + 1\}, \min\{0, f(y) + \varepsilon + 1\}\} \\ & = \min\{f(x) + \varepsilon + 1, f(y) + \varepsilon + 1\} \\ & = \min\{f(x), f(y)\} + \varepsilon + 1 \\ & \geq -1 + \varepsilon + 1 = \varepsilon > -1, \end{aligned}$$

which shows that (4.16) is not valid. This is a contradiction. So $x * y \in O(Y_f^\varepsilon)$. Hence $O(Y_f^\varepsilon)$ is a subalgebra of $(W, 0)_*$. □

Theorem 4.18. *Let* (W, Y_f^ε) *be a young structure based on* (W, f)*. If it satisfies:*

$$(4.17)\qquad (\forall x, y \in W)(\forall t_a, t_b \in [-1, 0))\left(\begin{array}{c} x \in (Y_f^\varepsilon, t_a)_q,\ y \in (Y_f^\varepsilon, t_b)_q \\ \Rightarrow x * y \in (Y_f^\varepsilon, \min\{t_a, t_b\})_\in \end{array}\right),$$

then the O*-subset* $O(Y_f^\varepsilon)$ *of* W *is a subalgebra of* $(W, 0)_*$.

Proof. Assume that (W, Y_f^ε) satisfies the condition (4.17). Let $x, y \in O(Y_f^\varepsilon)$. Then $f(x) + \varepsilon + 1 < 0$ and $f(y) + \varepsilon + 1 < 0$. Thus

$$Y_f^\varepsilon(x) - 1 = \min\{0, f(x) + \varepsilon + 1\} - 1 = f(x) + \varepsilon + 1 - 1 = f(x) + \varepsilon < -1$$

and

$$Y_f^\varepsilon(y) - 1 = \min\{0, f(y) + \varepsilon + 1\} - 1 = f(y) + \varepsilon + 1 - 1 = f(y) + \varepsilon < -1,$$

i.e.,, $x \in (Y_f^\varepsilon, -1)_q$ and $y \in (Y_f^\varepsilon, -1)_q$. It follows from (4.17) that

$$x * y \in (Y_f^\varepsilon, \min\{-1, -1\})_\in = (Y_f^\varepsilon, -1)_\in.$$

If $x * y \notin O(Y_f^\varepsilon)$, then $Y_f^\varepsilon(x * y) = 0 > -1$, a contradiction. Thus $x * y \in O(Y_f^\varepsilon)$. So $O(Y_f^\varepsilon)$ is a subalgebra of $(W, 0)_*$. □

Theorem 4.19. *Let* (W, Y_f^ε) *be a young structure based on* (W, f)*. If it satisfies:*

$$(4.18)\qquad (\forall x, y \in W)(\forall t_a, t_b \in [-1, 0))\left(\begin{array}{c} x \in (Y_f^\varepsilon, t_a)_q,\ y \in (Y_f^\varepsilon, t_b)_q \\ \Rightarrow x * y \in (Y_f^\varepsilon, \min\{t_a, t_b\})_q \end{array}\right),$$

then the O*-subset* $O(Y_f^\varepsilon)$ *of* W *is a subalgebra of* $(W, 0)_*$.

Proof. Assume that (W, Y_f^ε) satisfies the condition (4.18). Let $x, y \in O(Y_f^\varepsilon)$. Then $f(x) + \varepsilon + 1 < 0$ and $f(y) + \varepsilon + 1 < 0$. Thus

$$Y_f^\varepsilon(x) - 1 = \min\{0, f(x) + \varepsilon + 1\} - 1 = f(x) + \varepsilon + 1 - 1 = f(x) + \varepsilon < -1$$

and

$$Y_f^\varepsilon(y) - 1 = \min\{0, f(y) + \varepsilon + 1\} - 1 = f(y) + \varepsilon + 1 - 1 = f(y) + \varepsilon < -1,$$

that is, $x \in (Y_f^\varepsilon, -1)_q$ and $y \in (Y_f^\varepsilon, -1)_q$. So $x*y \in (Y_f^\varepsilon, \min\{-1, -1\})_q = (Y_f^\varepsilon, -1)_q$ by (4.18). If $Y_f^\varepsilon(x*y) = 0$, then $x*y \notin (Y_f^\varepsilon, -1)_q$, a contradiction. Hence $Y_f^\varepsilon(x*y) < 0$, i.e., $x * y \in O(Y_f^\varepsilon)$. Therefore $O(Y_f^\varepsilon)$ is a subalgebra of $(W, 0)_*$. □

5. Conclusion

Information on social phenomena has both negative and positive information. As a tool for handling positive information, the fuzzy set is very useful. However, it cannot deal with negative information. Thus, in response to the need for tools for negative information processing, Jun et al. [1] introduced $\mathcal{N}$-structure and applied it to BCK/BCI-algebras. Based on the $\mathcal{N}$-structure, a new structure called the young structure is introduced in this paper and applied to BCI/BCK-algebras. We have introduced $\in$-subset, q-subset, and O-subset and studied several properties. We have discussed the relationship between $\mathcal{N}$-subalgebra and young subalgebra. We have shown that every $\mathcal{N}$-subalgebra is a young subalgebra, and have considered the characterization of young subalgebra. We have provided conditions for a young structure (W, Y_f^ε) based on (W, f) to be a (strong) young subalgebra. We finally have explored the conditions under which $\in$-subset, q-subset, and O-subset can be subalgebra. In the future, we will conduct research on other algebra systems based on ideas and contents in this paper.

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