Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 24, No. 2, (October 2022) pp. 129–136
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2022.24.2.129

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 24, No. 2, October 2022

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Received 25 February 2022; Revised 21 April 2022; Accepted 9 June 2022

Abstract. The paper presents the study of linear extensions on an ordered multiset structure. Results on linear extensions of a partially ordered set are generalized to multisets. A heuristic algorithm for generating these linear extensions is also presented.

2020 AMS Classification: 05E99, 06A07

Keywords: Partially ordered set (poset), Multiset order, Partially ordered multiset (pomset), Multiset linear extension, M-LIN algorithm.

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1. Introduction

The theory of partially ordered sets (or posets for short) has come to an age [1, 2, 3]. A linear extension of a finite poset , $P = (S, \preccurlyeq), S$ being a set (usually referred to as the *ground set*) and a $\preccurlyeq$ reflexive, antisymmetric, and transitive relation on S, is an order preserving bijection $\alpha : P \to \{1, 2, ...|p|\}$ such that $x \preccurlyeq y$ in P implies $\alpha(x) \preccurlyeq \alpha(y)$. Following Szpilrajn's fundamental result *on order extension principle* [4], numerous works have appeared on *linear extensions* (also known as *topological sorts*) of posets [5, 6, 7]. The concept of linear extensions is useful for solving problems like the *scheduling problem*, also known as the *jump number* problem [8]. The quest for optimization has prompted the study of different classes of linear extensions. This is usually achieved by placing certain restrictions on the methods by which the linear extensions are constructed (See [9, 10] for greedy linear extensions and [11] for super greedy linear extensions). A number of algorithms [12, 13, 14, 15] have been proposed for generating linear extensions of a poset.

This paper generalizes the notion of linear extensions using a partially ordered multiset structure (or pomset, for short). A multiset is an unordered collection having elements that are not necessarily distinct. The theory of multisets addresses the limitation of distinctness of membership poised by the Cantorian set (See for instance [16, 17]). The multiset theory in [16] assumes separate axioms for multiplicity arithmetic (i.e., the number of times an object occurs) since multiplicities are considered as different types of objects from the multisets they support; the author presented a first-order two-sorted theory. In [18], a different approach is proposed via a single-sorted multiset theory. In the single-sorted theory, objects and their multiplicities come from the same domain and satisfy the same axioms. To extend the theory proposed in [18] and deal with multiplicities separately, new axioms would be required. The theory MST developed in [16] will be a basis for this study. Multisets are studied as an extension of set theory, and have applications in different spheres of mathematics, computer science, economics, formal language, biosystems and philosophy ([19] gave an overview of these applications). Various notions in order theory are being generalized using multisets [20, 21, 22, 23, 24]. Here, analogous results on linear extensions are presented in the multiset setting via an ordering that is induced by the underlying generic set. In section 2, we recall basic definitions on multisets and outline a structure of the ordered multiset to be used in this work. In the next section, we present an analogous notion of linear extensions for ordered multisets and prove related results. Lastly, a heuristic algorithm for generating multiset linear extensions is proposed in section 4.

2. Multisets and partially ordered multisets

Basic definitions and terminologies on multisets (See [19] for details), and an outline of the proposed pomset structure are discussed in this section.

2.1 Multisets

A multiset M over a set S is a cardinal-valued function, i.e.,M:$S \to \aleph$, such that $x \in$ *Dom(M)* implies *M(x)* is a cardinal number and *M(x)* $= m_M(x) > 0$, where $m_M(x)$ is the *multiplicity* of x in *M*. A multiset is finite if it contains a finite number of objects having finite multiplicities, it is infinite if the number of objects or/and their multiplicities are infinite. In theoretical development infinite multiplicities of elements are studied, however we will restrict our attention to finite multisets since this is the case in most applications of multisets. For convenience we will denote an arbitrary multiset M by $[m_1x_1, m_2x_2, ...m_nx_n]$, where m_i denotes the multiplicity of an object x_i in M for $i \in \{1, 2, ..., n\}$. The notation m_ix_i reads x_i is an element of M with multiplicity m_i. All occurrences of an object in a multiset are usually assumed to be indistinguishable. The *root set* of M, denoted by M^*, will be the set $\{x \in S|M(x) > 0\}$ (See separation schema in [16] for details). For any two multisets M and N in M(S) the multiset M is a *submultiset* of N, denoted by $M \subseteq N$, if $M(x) \leq N(x)$ for all $x \in S$, and $M \subset N$ if and only if $M(x) < N(x)$ for at least one x. This work also assumes multiplicies to be non-negative integer valued. In different theories of multisets, multiplicities are assumed to take values from different sets in the number system. For instance, the multiset theory presented in [25] assumes that the multiplicity of an object is a positive real number. While [26] studies objects with multiplicities from the set of integers. In [27], a generalized theory is proposed by assuming multiplicities to be real number valued. These existing multiset theories are generalizations of the Zermelo-Fraenkel set theory (ZF) and hence, contain ZF.

2.2 Pomsets

For a multiset $M \in \boldsymbol{M(S)}$, we shall refer to an object in M together with its multiplicity, i.e., $m_i x_i$ as a point. We need the following definitions (See [20] for details).

Definition 2.1. For any pair of points $m_i x_i$ and $m_j x_j$ in M with $i, j \in \{1, 2, ...n\}$, $m_i x_i \preccurlyeq\leq m_j x_j$ if and only if $x_i \preccurlyeq x_j$ in S. The two points coincide if and only if $x_i = x_j$ (See principle of exact multiplicity in [16]). The points $m_i x_i$ and $m_j x_j$ are *comparable*, if $(m_i x_i \preccurlyeq\leq m_j x_j) \vee (m_j x_j \preccurlyeq\leq m_i x_i)$, and otherwise, they are *incomparable* and denoted by $(m_i x_i || m_j x_j)$.

Remark 2.2. By definition 2.1, it follows that $' \preccurlyeq\leq'$ is a partial order if and only if $' \preccurlyeq'$ is a partial order. The generic set $(S, \preccurlyeq)$ is assumed to be partially ordered, by implication $\mathcal{M} = (M, \preccurlyeq\leq)$ will be a pomset. The strict ordering on M will be denoted by $' \prec<'$.

Definition 2.3. The ordering $\preccurlyeq\leq$ is a *linear multiset order*, if it is a partial multiset order and any two points $m_i x_i$ and $m_j x_j$ in M are comparable under $\preccurlyeq\leq$.

Definition 2.4. A point $m_i x_i$ in M is said to be *minimal* (resp. *maximal*), if there is no point $m_j x_j$ in M for which $m_j x_j \prec< m_i x_i$ (resp. $m_i x_i \prec< m_j x_j$) holds. If a pomset has a unique minimal (resp. maximal) point, then it is called the *minimum* (resp. *maximum*) *point* of $\mathcal{M}$.

Definition 2.5. For a submultiset N of M, the *upset* $(\uparrow N)$ and *downset* $(\downarrow N)$ of N with respect to $\preccurlyeq\leq$ are defined as follows:

$$\uparrow N = \{m_j x_j \in M \mid \exists n_i x_i \in N, \text{ with } n_i x_i \preccurlyeq\leq m_j x_j\}$$

and

$$\uparrow N = \{m_j x_j \in M \mid \exists n_i x_i \in N, \text{ with } n_j x_j \preccurlyeq\leq m_i x_i\}.$$

3. Multiset linear extensions

In this section, an analogous definition of linear extensions is presented with some results.

Definition 3.1. Let $\mathcal{M} = (M, \preccurlyeq\leq_{\mathcal{M}})$ and $\mathcal{N} = (M, \preccurlyeq\leq_{\mathcal{N}})$ be two pomsets defined over partially ordered sets P and Q respectively. Then $\preccurlyeq\leq_{\mathcal{N}}$ is called a *multiset order extension* of $\preccurlyeq\leq_{\mathcal{M}}$, if $m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_j x_j$ implies that $m_i x_i \preccurlyeq\leq_{\mathcal{N}} m_j x_j$, i.e., the relation $\preccurlyeq\leq_{\mathcal{M}}$ is contained in $\preccurlyeq\leq_{\mathcal{N}}$. A multiset extension of $\preccurlyeq\leq_{\mathcal{M}}$ that is a linear multiset order is called a *multiset linear extension*.

We have the following result.

Theorem 3.2. *Let $\mathcal{M} = (M, \preccurlyeq\leq_{\mathcal{M}})$ be a pomset and $m_i x_i, m_j x_j$ be any two points in M with $m_i x_i || m_j x_j$ in M. Then there exists a pomset $\mathcal{N}$ extending $\mathcal{M}$ such that $m_i x_i \preccurlyeq\leq_{\mathcal{N}} m_j x_j$.*

To prove theorem 3.2, we will first outline the proofs of the following two complementary results.

Lemma 3.3. *Let P be a poset and let x, y be incomparable elements in P. Then there exists a poset Q extending P such that $x \preccurlyeq_Q y$*

Proof. This proof is a direct consequence of Szpilrajn's extension theorem [4]:

Every partial order is contained into a total order.

We include this proof for the sake of convenience for the reader.

Let $Q = P \cup (\downarrow x \times \uparrow y)$, i.e.,$Q$ is obtained from P by adjoining P with all pairs of points *(u,v)* for which $u \preccurlyeq_P x$ and $y \preccurlyeq_P v$. Since $x \preccurlyeq_Q y$, we need to show that the ordering on Q is a partial order. Now $\downarrow x \cap \uparrow y = \oslash$, since if z is in this intersection, then $y \preccurlyeq_P z$ and $z \preccurlyeq_P x$ would yield $y \preccurlyeq_P x$, a contradiction. The ordering on Q is reflexive since $P \subseteq Q$. To show that the order on Q is antisymmetric, suppose that $u \preccurlyeq_Q v$ and $v \preccurlyeq_Q u$ with $u \neq v$. By the definition of Q, either $u \preccurlyeq_P v$ or, $u \preccurlyeq_P x$ and $y \preccurlyeq_P v$. Similarly, either $v \preccurlyeq_P u$ or, $v \preccurlyeq_P x$ and $y \preccurlyeq_P u$. Suppose $u \preccurlyeq_P v$. Then $v \preccurlyeq_P u$ cannot hold. Otherwise, we will have $u = v$, which is a contradiction. Thus $v \preccurlyeq_P x$ and $y \preccurlyeq_P u$. However, $y \preccurlyeq_P u \preccurlyeq_P v \preccurlyeq_P x$ gives a contradiction, i.e., $y \preccurlyeq_P x$. Similarly, the case $v \preccurlyeq_P u$ gives a contradiction. Suppose, $u \preccurlyeq_P x$ and $y \preccurlyeq_P v$ also, $v \preccurlyeq_P x$ and $y \preccurlyeq_P u$, which contradicts the fact that $\downarrow x \cap \uparrow y = \oslash$. So $u \preccurlyeq_P v$ and $v \preccurlyeq_P u$ must imply that $u = v$.

To show transitivity, assume that $u \preccurlyeq_Q v$ and $v \preccurlyeq_Q w$. Now, either $u \preccurlyeq_P v$ or, $u \preccurlyeq_P x$ and $y \preccurlyeq_P v$. Also, either $v \preccurlyeq_P w$ or, $v \preccurlyeq_P x$ and $y \preccurlyeq_P w$. Again we have to consider the following cases. First assume that $u \preccurlyeq_P v$, if also $v \preccurlyeq_P w$, then the claim follows from transitivity of P. Now suppose that $v \preccurlyeq_P x$ and $y \preccurlyeq_P w$. Also $u \preccurlyeq_P x$, and this implies that $u \in \downarrow x$ and $w \in \uparrow y$ hence $u \preccurlyeq_Q w$ as required. Next, suppose that $u \preccurlyeq_p x$ and $y \preccurlyeq_P v$. Now, if $v \preccurlyeq_P w$, then also $y \preccurlyeq_P w$ and this implies that $u \in \downarrow x$ and $w \in \uparrow y$, hence $u \preccurlyeq_Q w$. The case $v \preccurlyeq_P x$ and $y \preccurlyeq_P w$ gives a contradiction since $\downarrow x \cap \uparrow y = \oslash$. □

Lemma 3.4 (Necessary and sufficient condition for a pomset $\mathcal{N}$ to be a multiset extension of $\mathcal{M}$)**.** *Suppose $\mathcal{M}$ and $\mathcal{N}$ are two pomsets defined over posets P and Q, respectively, where P and Q have a common ground set, then $\mathcal{N}$ is a multiset extension of $\mathcal{M}$ if and only if Q is an extension of P*

Proof. Let $\mathcal{M} = (M, \preccurlyeq\leq_{\mathcal{M}})$, and $\mathcal{N} = (M, \preccurlyeq\leq_{\mathcal{N}})$ be pomsets over P and Q, respectively. Suppose $\mathcal{N}$ is a multiset extension of $\mathcal{M}$. Then by definition, we have

$$m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_j x_j \implies m_i x_i \preccurlyeq\leq_{\mathcal{N}} m_j x_j. \tag{3.1}$$

Also,

$$m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_j x_j \Longleftrightarrow x_i \preccurlyeq_P x_j (\text{Since M is defined over the poset P}). \tag{3.2}$$

Similarly,

$$m_i x_i \preccurlyeq\leq_{\mathcal{N}} m_j x_j \Longleftrightarrow x_i \preccurlyeq_Q x_j. \tag{3.3}$$

From (3.1) to (3.3), we have

$$x_i \preccurlyeq_P x_j \Longleftrightarrow m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_j x_j \implies m_i x_i \preccurlyeq\leq_{\mathcal{N}} m_j x_j \Longleftrightarrow x_i \preccurlyeq_Q x_j.$$

Thus $x_i \preccurlyeq_Q x_j$ *whenever* $x_i \preccurlyeq_P x_j$ So Q is an extension of P.

Next, suppose Q is an extension of P. Then

$$x_i \preccurlyeq_Q x_j \text{ whenever } x_i \preccurlyeq_P x_j. \tag{3.4}$$

Since x_i, x_j are also elements of M, we must have $m_i, m_j \in \mathbb{N}$ with $M(x_i) = m_i$ and $M(x_j) = m_j$.
Thus by definition,

(3.5) $\quad x_i \preccurlyeq_P x_j \implies m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_j x_j$ *for any pair of points* $m_i x_i, m_j x_j \in M$.

Similarly,

(3.6) $\quad x_i \preccurlyeq_Q x_j \implies m_i x_i \preccurlyeq\leq_{\mathcal{N}} m_j x_j,$ *for some pomset* $\mathcal{N}$ *induced by* $\preccurlyeq_Q$.

From (3.4) to (3.6), we have

$$m_i x_i \preccurlyeq\leq_{\mathcal{N}} m_j x_j \text{ whenever } m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_j x_j.$$

So $\mathcal{N}$ is a multiset extension of $\mathcal{M}$. □

We now give the proof of Theorem 3.2.

Proof. Let $\mathcal{M} = (M, \preccurlyeq\leq_{\mathcal{M}})$ be a pomset defined over a partially ordered set P. Suppose there exist points $m_i x_i$ and $m_j x_j$ in M with $m_i x_i || m_j x_j$ in M. Thus by definition, neither $x_i \preccurlyeq_P x_j$ nor $x_j \preccurlyeq_Q x_i$ holds in P. By Lemma 3.3, there exists a poset, say Q, extending P such that $x_i \preccurlyeq_Q x_j$. Let x_i and x_j be objects in M, with multiplicities m_i and m_j, respectively. By definition $x_i \preccurlyeq_Q x_j$ implies $m_i x_i \preccurlyeq\leq_{\mathcal{N}} m_j x_j$ for some pomset $\mathcal{N}$ induced by $\preccurlyeq_Q$. Suppose $\mathcal{N}$ is the pomset $(M, \preccurlyeq\leq_{\mathcal{N}})$. It follows from Lemma 3.4 that $\mathcal{N}$ is a multiset extension of $\mathcal{M}$. □

Theorem 3.5 (A generalization of Theorem 3.2)**.** *Every finite pomset has a multiset linear extension.*

Proof. For a pomset $\mathcal{M} = (M, \preccurlyeq\leq_{\mathcal{M}})$, let $\mathcal{U}$ denote the multiset containing all pairs $m_i x_i, m_j x_j$ in M with $m_i x_i || m_j x_j$ in M. By applying the construction in Theorem 3.2 inductively on all pairs $m_i x_i, m_j x_j \in \cup$, a pomset $\mathcal{N}$ with a multiset linear order $\preccurlyeq\leq_{\mathcal{N}}$ is obtained. The ordering $\preccurlyeq\leq_{\mathcal{N}}$ contains all extensions of the original multiset order $\preccurlyeq\leq_{\mathcal{M}}$. □

4. The M-LIN Algorithm

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset defined over a poset P. Since $\preccurlyeq\leq$ is induced by $\preccurlyeq$, the subposet $(M^*, \preccurlyeq)$ of P will be a generator for $(M, \preccurlyeq\leq)$. A multiset linear extension of $\preccurlyeq\leq$ will be $\preccurlyeq\leq_L$, with $m_i x_i \preccurlyeq\leq_L m_j x_j$ whenever $m_i x_i \preccurlyeq\leq m_j x_j$ in M, and $(m_i x_i \preccurlyeq\leq_L m_j x_j) \vee (m_j x_j \preccurlyeq\leq_L m_i x_i)$ holds for all $m_i x_i, m_j x_j$ in M. The multiset linear extension can be obtained via the following steps:
Given a multiset M

(1) Choose a minimal element x_1 in M^*
(2) Let $m_1 = M(x_1)$
(3) Choose $m_1 x_1$ to be minimal in $\mathcal{M}$
(4) Given that $m_1 x_1, m_2 x_2, ..., m_i x_i$ have been chosen, choose a minimal element x_{i+1} in $M^* \setminus \{x_1, x_2, ..., x_i\}$
(5) Let $m_{i+1} = M_(x_{i+1})$

The multiset linear order $m_1x_1 \preccurlyeq\leq_L m_2x_2 \preccurlyeq\leq_L ... \preccurlyeq\leq_L m_nx_n$ obtained through this process is a multiset linear extension of $\preccurlyeq\leq$.
The proposed algorithm M-LIN is a heuristic algorithm that constructs multiset linear extensions $[m_1x_1, m_2x_2, ...m_nx_n]$ of a pomset $\mathcal{M}$. The algorithm is split into two parts, viz., M-LIN 1 and M-LIN 2.

4.1 M-LIN Algorithm

M-LIN 1 ($M[x_1, ...x_n]$)
Create Array $M^*, M(x)$
For $i \leftarrow 0$ to len(M)-1
temp$\leftarrow [i]$
if temp is not in M^*
$M^*[i] \leftarrow$ temp
freq$\leftarrow$ 0
for j$\leftarrow$ 0 to len(M)-1
if temp=$M[j]$
freq=freq + 1
end if
end for
multiplicity[] $\leftarrow$ freq
end if
end for

M-LIN 2
Create Array *CM*
T=M^*
q=multiplicity
While (*len(T)*> 0)
x=Min(T)
i=T.index of (x)
Y=q[i]
If *(Y* > 1*)*
CM[] $\leftarrow$ *Y.X*
Else
CM[] $\leftarrow$ *x*
T.Remove(x)
end while
Return *CM*

The first part of the algorithm i.e., M-LIN 1 creates the root set and multiplicity arrays. Algorithm M-LIN 2 sorts the root set using a defined ordering and obtains the multiplicity of the minimal element at each stage. It returns a point m_ix_i. The final output is a multiset linear extension of the original multiset order.

5. Conclusions

The proposed heuristic algorithm for generating multiset linear extensions is decidable. It can be adopted for modeling and solving problems on ordered structures for which repetition is significant. Ordered multisets are useful for representing concurrent behaviours (e.g. [28]). For instance, events with a number of repeated tasks can be modeled using this algorithm, and implemented in order to obtain a linearization. An algorithm for generating the class of multiset linear extensions that realizes the pomset $\mathcal{M}$(i.e., the minimum number of multiset linear extensions whose intersection is $\mathcal{M}$) could also be investigated.

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