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Some results on operator semigroups of a Γ-semigroup in terms of fuzzy subsets

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Abstract. In this paper, we examine the fuzzy subsets of left and right operator semigroups via the modified definition of Γ-semigroup and prove some related results. In particular, we obtain a one-to-one correspondence between the set of fuzzy subsets of $S \times \Gamma$ ($\Gamma \times S$) and the set of fuzzy subsets of left (right) operator semigroup L (R). Also, we define equivalence relations $\mathcal{L}^F$ and $\mathcal{R}^F$ of Γ-semigroups in terms of fuzzy subsets and prove some relationships between each of the equivalence relations and its left (right) operator semigroup by means of fuzzy subsets.

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1. Introduction

In algebra, it is an interesting problem to embed an algebraic structure in another algebraic structure and study this structure with the help of the latter. Taking this as a motivating factor, Sen [1] introduced the notion of a Γ-semigroup as a generalization of a semigroup in 1981. Later Sen and Saha [2] redefined Γ-semigroup by weakening slightly the defining conditions of Γ-semigroup. Many classical notions and results of the theory of semgroups have been extended and generalized to Γ-semigroup in several papers viz [3, 4, 5, 6, 7, 8]. Dutta and Adhikari [9] have found operator semigroups of a Γ-semigroup to be a very effective tool in studying Γ-semigroups. In line with this concept, Dutta and Chattopadhyay [10] defined operator semigroups based on the modified definition of Γ-semigroup.

The concept of fuzzy set was introduced by Zadeh [11] while the idea of fuzzy algebraic structure began with the work of Rosenfeld [12], which introduced the notion of fuzzy groups and some related results in group theory. Since then, there has been tremendous progress in fuzzifying most of the algebraic structures. In 1991, Kuroki [13] introduced the theory of fuzzy semigroups. Muhiuddin and Alanazi [14] studied the notions of int-soft (m, n)-ideals, int-soft $(m, 0)$-ideals and int-soft $(0, n)$-ideals of semigroups by generalising the concept of int-soft bi-ideals, int-soft right ideals, and int-soft left ideals in semigroups. Muhiuddin and Mahboobb [15] iniated the notions of int-soft left (right) ideals, int-soft interior ideals and int-soft bi-ideals over the soft sets in ordered semigroups and investigated some related properties. In relation to the work of [15], Muhiuddin *et al.* [16] considered relations between a convex soft set and an int-soft l-ideal (int-soft r-ideal). Sardar and Majumder [17] studied the fuzzification of ideals in Γ-semigroups and obtained some relationships between fuzzy ideals of a Γ-semigroup and that of its operator semigroups. This fact together with Green's equivalence relations for Γ-semigroups as seen in [18, 19, 20] has motivated us to write this paper. In this paper, among other results we show some relationships between Green's relations of a Γ-semigroup and its left(right) operator semigroup in terms of fuzzy subsets.

2. Preliminaries

In this section, we recall some fundamental definitions which are necessary for this paper.

Definition 2.1 ([2])**.** Let $S = \{a, b, c...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two non-empty sets. Then S is called a Γ-*semigroup*, if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ which maps $(a, \alpha, b) \longrightarrow a\alpha b$ satisfying $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$. A non-empty subset A of a Γ-semigroup S is said to be a Γ-*subsemigroup* of S, if $A\Gamma A \subseteq A$.

Example 2.2. Let $\Gamma = \{1, 2\}$. Define a mapping $\mathbb{N} \times \Gamma \times \mathbb{N} \longrightarrow \mathbb{N}$ by $a\gamma b = a + \gamma + b$ for all $a, b \in \mathbb{N}$ and $\gamma \in \Gamma$, where $+$ is usual addition on $\mathbb{N}$. Then $\mathbb{N}$ is a Γ-semigroup.

Example 2.3. Let $S = [0, 1]$ and $\Gamma = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$. Then S is a Γ-semigroup under the usual multiplication. Let $T = [0, \frac{1}{2}]$. Now T is a non-empty subset of S and $a\gamma b \in T$ for all $a, b \in T$ and $\gamma \in \Gamma$. Then T is a Γ-subsemigroup of S.

Definition 2.4 ([2])**.** A non-empty subset I of a Γ-semigroup is called a *left* (*right*) Γ-*ideal* of S, if $S\Gamma I \subseteq I$ $(I\Gamma S \subseteq S)$.

Definition 2.5 ([21])**.** Let X be a semigroup and $a, b \in X$. Then *Green's equivalence relation* $\mathcal{L}$ and $\mathcal{R}$ are defined by the following rules:

(i) $(a, b) \in \mathcal{L}$ if and only if $L(a) = L(b)$,
(ii) $(a, b) \in \mathcal{R}$ if and only if $R(a) = R(b)$.

Green's relations of a Γ-semigroup are defined similarly. Let $a \in S$. Then

$$L(a) = S\Gamma a \cup \{a\} \text{ and } R(A) = a\Gamma S \cup \{a\}.$$

Thus $(a, b) \in \mathcal{L}$ if and only if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = x\alpha b$ and $b = y\beta a$. Similarly, $(a, b) \in \mathcal{R}$ if and only if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = b\alpha x$ and $b = a\beta y$.

Definition 2.6 ([2])**.** A Γ-semigroup S is said to be *left* (*right*) *zero*, if $a \in S$ satisfies $a\alpha s = a$ $(s\alpha a = a)$ for all $s \in S$ and $\alpha \in \Gamma$.

Definition 2.7 ([2])**.** Let S be a Γ-semigroup, $a \in S$ and $\alpha, \beta \in \Gamma$. Then $a \in S$ is said to be α-*idempotent*, if $a\alpha a = a$. Also, a is said to be (α, β)-*regular*, if there exists $x \in S$ such that $a = a\alpha x\beta a = a$.

Definition 2.8 ([10])**.** Let S be a Γ-semigroup. We define a relation ρ on $S \times \Gamma$ as follows:

$$(x, \alpha)\rho(y, \beta) \Longleftrightarrow x\alpha s = y\beta s, \ \forall \ s \in S.$$

Obviously, ρ is an equivalence relation. Let $[x, \alpha]$ denote the equivalence class containing (x, α). Let $L = \{[x, \alpha] : x \in S, \ \alpha \in \Gamma\}$. Then L is a semigroup with respect to multiplication defined by $[x, \alpha][y, \beta] = [x\alpha y, \beta]$. The semigroup L is called the *left operator semigroup* of S. Similarly, the *right operator semigroup* R of a Γ-semigroup S is defined as $R = \{[\alpha, x] : \alpha \in \Gamma, \ x \in S\}$, where $[\alpha, x][\beta, y] = [\alpha, x\beta y]$, for all $x, y \in S$ and $\alpha, \beta \in \alpha, \beta \in \Gamma$.

Throughout the paper unless otherwise stated, S will denote a Γ-semigroup.

Definition 2.9 ([11])**.** A mapping $\mu : X \longrightarrow [0, 1]$ is a fuzzy subset of S.

A fuzzy subset $\mu : S \longrightarrow [0, 1]$ is *non-empty*, if μ is not the constant map which assumes the value 0. For any two fuzzy subsets λ and μ of S, $\lambda \subseteq \mu$ means that $\lambda(a) \leq \mu(a)$ for all $a \in S$. The Γ-semigroup S can be considered a fuzzy subset of itself such that $S(x) = 1$ for all $x \in S$.

Definition 2.10 ([11])**.** Let λ and μ be any two fuzzy subsets of S. Then $\lambda \cap \mu$ and $\lambda \cup \mu$ are fuzzy subsets of S defined by

$$(\lambda \cap \mu)(x) = \lambda(x) \wedge \mu(x),$$
$$(\lambda \cup \mu)(x) = \lambda(x) \vee \mu(x)$$

Definition 2.11 ([11])**.** Let λ and μ be any two fuzzy subsets of S. Then the *product* $\lambda \circ_\Gamma \mu$ of λ and μ is a fuzzy subset of S defined by

$$\lambda \circ_\Gamma \mu\,(a) = \begin{cases} \bigvee_{a = x\gamma y}\{\lambda(x) \bigwedge \mu(y)\} & \text{if } a \text{ is expressible as } a = x\gamma y, \\ & \text{where } a, x, y \in S \text{ and } \gamma \in \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.12 ([22])**.** A fuzzy subset μ of a Γ-semigroup S is called a *fuzzy* Γ-*subsemigroup* of S, if $\mu(x\gamma y) \geq \mu(x) \bigwedge \mu(y) \ \forall \ x, y \in S, \ \gamma \in \Gamma$.

Definition 2.13 ([17])**.** A fuzzy subset μ of a Γ-semigroup S is called a *fuzzy left* (*right*) Γ-*ideal* of S, if $\mu(x\gamma y) \geq \mu(y)$ $(\mu(x\gamma y) \geq \mu(x))$ $\forall \ x, y \in S, \ \gamma \in \Gamma$. If μ is both a fuzzy left Γ-ideal and a fuzzy right Γ-ideal of S, then μ is called a *fuzzy* Γ-*ideal* of S.

We denote the set of fuzzy left Γ-ideal, fuzzy right Γ-ideal and fuzzy Γ-ideal of S by $FI_L(S)$, $FI_R(S)$ and $FI(S)$ respectively.

Example 2.14. Let $S = \{0, -1, -2, ...\}$ and $\Gamma = \{0, -2, -4, ...\}$. Clearly, S is a Γ-semigroup with the usual multiplication. Moreover, the fuzzy subset $\mu : S \longrightarrow [0, 1]$ defined as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.1 & \text{if } x = -1, -2, \\ 0.2 & \text{if } x < 2. \end{cases}$$

.

Then μ is a fuzzy Γ-ideal of S.

Remark 2.15. Let S be a right (left) zero Γ-semigroup. If μ is a fuzzy subset of S, then μ is a fuzzy left (right) Γ- ideal of S.

3. Main results

Definition 3.1 ([17])**.** For a fuzzy subset μ of L, we define a fuzzy subset μ^+ of S by $\mu^+(a) = \bigwedge_{\gamma\in\Gamma} \mu([a, \gamma])$, where $a \in S$. For a fuzzy subset μ of S, we define a fuzzy subset $\mu^{+'}$ of L by $\mu^{+'}([a, \alpha]) = \bigwedge_{s\in S} \mu(a\alpha s)$, where $[a, \alpha] \in L$.

For a fuzzy subset μ of R, we define a fuzzy subset μ^* of S by $\mu^*(a) = \bigwedge_{\gamma\in\Gamma} \mu([\gamma, a])$, where $a \in S$. For a fuzzy subset μ of S, we define a fuzzy subset $\mu^{*'}$ of R by $\mu^{*'}([\alpha, a]) = \bigwedge_{s\in S} \mu(s\alpha a)$, where $[\alpha, a] \in R$.

We denote the set of fuzzy subsets of left and right operator semigroups by $F(L)$ and $F(R)$ respectively.

Proposition 3.2. *If μ is a fuzzy Γ-subsemigroup of S, then $\mu^{+'}$ is a fuzzy Γ-subsemigroup of L.*

Proof. It follows from Definition 2.12. □

Following Definition 2.11 terminology, the product of $\lambda^{+'}$ and $\mu^{+'}$ can be defined as follows: for each $a \in S$, if there is $[x, \beta], [y, \gamma] \in L$ such that $[a, \alpha] = [x, \beta][y, \gamma]$, then

$$\lambda^{+'} \circ_\Gamma \mu^{+'}(a) = \bigvee_{[a,\alpha]=[x,\beta][y,\gamma]} [\lambda^{+'}(x) \bigwedge \mu^{+'}(y)]$$

and otherwise, $\lambda^{+'} \circ_\Gamma \mu^{+'}(a) = 0$.

Proposition 3.3. *Let S be a Γ-semigroup, let $a \in S$ and let $\mu^{+'}$ be a fuzzy subset of L. If a is α-idempotent and μ is a fuzzy Γ-subsemigroup of S, then $\mu^{+'} \circ_\Gamma \mu^{+'} \subseteq \mu^{+'}$.*

Proof. Since a is α-idempotent and μ is a fuzzy Γ-subsemigroup of S, $\mu(a\alpha a) \geq \mu(a)$. Then $\mu(a\alpha a) = \mu(a)$ for all $a \in S$ and for some $\alpha \in \Gamma$.
On the other hand,

$$\begin{aligned}
\mu^{+'} \circ_\Gamma \mu^{+'}([a, \alpha]) &= \bigvee_{[a,\alpha]=[a,\alpha][a,\alpha]} \{\mu^{+'}([a, \alpha]) \bigwedge \mu^{+'}([a, \alpha])\} \\
&= \bigvee_{[a,\alpha]=[a,\alpha][a,\alpha]} \{(\bigwedge_{a\in S} \mu(a\alpha a)) \bigwedge (\bigwedge_{a\in S} \mu(a\alpha a))\} \\
&\leq \bigvee_{[a,\alpha]=[a,\alpha][a,\alpha]} \{\bigwedge_{a\in S} \mu(a\alpha a)\} \\
&= \bigvee_{[a,\alpha]=[a,\alpha][a,\alpha]} \mu^{+'}([a, \alpha][a, \alpha]) \\
&= \mu^{+'}([a, \alpha]).
\end{aligned}$$

Thus $\mu^{+'} \circ_\Gamma \mu^{+'} \subseteq \mu^{+'}$. □

Proposition 3.4. *Let S be a Γ-semigroup, let $a \in S$ and let $\mu^{+'}$ be a fuzzy subset of L. If a is (α, β) regular and μ is a fuzzy Γ-subsemigroup of S, then $\mu^{+'}$ is idempotent.*

Proof. Let μ be a fuzzy Γ-subsemigroup of S and let $a \in S$. Since a is (α, β) regular, there exists $b \in S$ and $\alpha, \beta \in S$ such that $a = a\alpha b\beta a$. Then we have

$$\begin{aligned}
\mu^{+'} \circ_\Gamma \mu^{+'}([a,\alpha]) &= \bigvee_{[a,\alpha]=[a,\alpha][b,\beta][a,\alpha]} \{\mu^{+'}([a,\alpha][b,\beta]) \bigwedge \mu^{+'}([a,\alpha])\} \\
&= \bigvee_{[a,\alpha]=[a,\alpha][b,\beta],\alpha]} \{(\bigwedge_{a\in S} \mu(a\alpha b\beta a)) \bigwedge (\bigwedge_{a\in S} \mu(a\alpha a))\} \\
&\geq \bigvee_{[a,\alpha]=[a,\alpha][b,\beta][a,\alpha]} \{\left(\bigwedge_{a\in S} \left(\mu(a\alpha a) \bigwedge \mu(b\beta a)\right)\right) \bigwedge \left(\bigwedge_{a\in S} \mu(a\alpha a)\right)\} \\
&\geq \bigvee_{[a,\alpha]=[a,\alpha][b,\beta][a,\alpha]} \{\left(\bigwedge_{a\in S} \mu(a\alpha a)\right) \bigwedge \left(\bigwedge_{a\in S} \mu(a\alpha a)\right)\} \\
&\geq \left(\bigwedge_{a\in S} \mu(a\alpha a)\right) \bigwedge \left(\bigwedge_{a\in S} \mu(a\alpha a)\right) \\
&= \bigwedge_{a\in S} \mu(a\alpha a) \\
&= \mu^{+'}([a,\alpha]).
\end{aligned}$$

Thus $\mu^{+'} \subseteq \mu^{+'} \circ_\Gamma \mu^{+'}$. So by Proposition 3.3, $\mu^{+'} \circ_\Gamma \mu^{+'} \subseteq \mu^{+'}$. Hence $\mu^{+'} \circ_\Gamma \mu^{+'} = \mu^{+'}$. □

Proposition 3.5. *Let T be a left zero Γ-subsemigroup and let $\mu^{+'}$ be a fuzzy subset of L. If $\mu \in FI_L(T)$, then $\mu^{+'}$ is a constant function.*

Proof. Let $a,\ b \in S$. Since T is left zero, $a\alpha b = a$ and $b\alpha a = b$ for all $\alpha \in \Gamma$. Then we get

$$\begin{aligned}
\mu^{+'}([a,\alpha]) &= \mu^{+'}([a\alpha b,\alpha]) \geq \bigwedge_{s\in T} \mu(b\alpha s) \\
&= \mu^{+'}([b,\alpha]) = \mu^{+'}([b\alpha a,\alpha]) \\
&\geq \bigwedge_{s\in T} \mu(a\alpha s) = \mu^{+'}([a,\alpha]).
\end{aligned}$$

Thus $\mu^{+'}([a,\alpha]) = \mu^{+'}([b,\alpha])$ for all $a, b \in T$ and $\alpha \in \Gamma$. So $\mu^{+'}$ is a constant function. □

Proposition 3.6. *Let T be a right zero Γ-subsemigroup and let $\mu^{*'}$ be a fuzzy subset of R. If $\mu \in FI_R(T)$, then $\mu^{*'}$ is a constant function.*

Proof. Similar to the proof of Proposition 3.5. □

Proposition 3.7. *Let S be a Γ-semigroup and let $\mu^{+'}$ be a fuzzy subset of L. If μ is a left Γ-ideal of S and $E_\alpha(S)$ is the set of all α-idempotents of S such that $e\alpha e' = e$ and $e'\alpha e = e'$, then $\mu^{+'}([e,\alpha]) = \mu^{+'}([e',\alpha])$ for all $e, e' \in E_\alpha(S)$ and $\alpha \in \Gamma$.*

Proof. Suppose $e, e' \in E_\alpha(S)$. Then $e\alpha e' = e$ and $e'\alpha e = e'$ for all $\alpha \in \Gamma$. Since $\mu^{+'} \in F(L)$, we have

$$\begin{aligned} \mu^{+'}([e,\alpha]) &= \mu^{+'}([e\alpha e',\alpha]) \geq \bigwedge_{s\in S} \mu(e'\alpha s) \\ &= \mu^{+'}([e',\alpha]) = \mu^{+'}([e'\alpha e,\alpha]) \\ &\geq \bigwedge_{s\in S} \mu(e\alpha s) = \mu^{+'}([e,\alpha]). \end{aligned}$$

Thus $\mu^{+'}([e,\alpha]) = \mu^{+'}([e',\alpha])$ for all $e, e' \in E_\alpha(S)$ and $\alpha \in \Gamma$. This completes the proof. □

Proposition 3.8. *Let S be a Γ-semigroup and let $\mu^{*'}$ be a fuzzy subset of R. If μ is a right Γ-ideal of S and $E_\alpha(S)$ is the set of all α-idempotents of S such that $e\alpha e' = e$ and $e'\alpha e = e'$, then $\mu^{*'}([e,\alpha]) = \mu^{*'}([e',\alpha])$ for all $e, e' \in E_\alpha(S)$ and $\alpha \in \Gamma$.*

Proof. Similar to the proof of Proposition 3.7. □

Proposition 3.9. *There exists a bijective $f : F(S\times\Gamma) \longrightarrow F(L)$.*

Proof. Define a mapping $f : F(S\times\Gamma) \longrightarrow F(L)$ by: for each $(a,\alpha) \in S\times\Gamma$,

$$f(\mu(a,\alpha)) = \mu^{+'}([a,\alpha]).$$

If $\mu(a,\alpha) = \mu(b,\beta)$ for some $a, b \in S$ and $\alpha, \beta \in \Gamma$, then for every $s \in S$, we have

$$\bigwedge_{a\in S} \mu(a\alpha s) = \mu^{+'}([a,\alpha]) = \mu^{+'}([b,\beta]) = \bigwedge_{a\in S} \mu(b\beta a).$$

Thus $\mu^{+'}([a,\alpha]) = \mu^{+'}([b,\beta])$. So f is well-defined.

Suppose $f(\mu(a,\alpha)) = f(\mu(b,\beta))$. Then clearly, $\mu^{+'}([a,\alpha]) = \mu^{+'}([b,\beta])$. Thus $\bigwedge_{a\in S} \mu(a\alpha s) = \bigwedge_{a\in S} \mu(b\beta s)$ for each $s \in S$. So $\mu(a,\alpha) = \mu(b,\beta)$. Hence, f is one-to-one. By Definitions 2.8 and 3.1, it is obvious that f is onto. Therefore we have that f is a bijection from $F(S\times\Gamma)$ to $F(L)$. □

Definition 3.10. Let S be a Γ-semigroup. We define a fuzzy relation $\sim$ on $S\times\Gamma$ as follows: for any (x,α), $(y,\beta) \in S\times\Gamma$ and each $s \in S$,

$$\mu(x,\alpha) \sim \mu(y,\beta) \Longleftrightarrow \bigwedge_{s\in S} \mu(x\alpha s) = \bigwedge_{s\in S} \mu(y\beta s).$$

Then we can easily check that $\sim$ is a fuzzy equivalence relation.

Definition 3.11. Let S be a Γ-semigroup and and let $a \in S$. Then $L^F(a)$ $\left(R^F(a)\right)$ is called the *fuzzy left* (*right*) Γ-*ideal* of S *generated by* a.

Definition 3.12. Let S be a Γ-semigroup. We define relations $\mathcal{L}^F$ and $\mathcal{R}^F$ on S as follows:

(*i*) $a\mathcal{L}^F b$ if and if only $L^F(a) = L^F(b)$,
(*ii*) $a\mathcal{R}^F b$ if and if only $R^F(a) = R^F(b)$.

In the following, we prove some relationships between Green's relations of a Γ-semigroup and left (right) operator semigroup in terms of fuzzy subsets. The notations $\mathcal{L}^F$ and $\mathcal{R}^F$ are used for Green's relation of Γ-semigroup S in terms of fuzzy subsets. Similarly, $\mathcal{L}^F_{O_l}$ and $\mathcal{R}^F_{O_l}$ are used for Green's relations of the left operator semigroup of S in terms of fuzzy subsets. Also, $\mathcal{L}^F_{O_r}$ and $\mathcal{R}^F_{O_r}$ are used for Green's relations of the right operation semigroup of S in terms of fuzzy subsets.

Proposition 3.13. *Let $a,\ b \in S$. If $(a, b) \in \mathcal{L}^F$, then $([a, \alpha], [b, \alpha]) \in \mathcal{L}^F_{O_l}$ for every $\alpha \in \Gamma$.*

Proof. Suppose there exist $x, y \in S$ and $\gamma, \lambda \in \Gamma$ such that $a = x\gamma b$ and $b = y\lambda a$. If $(a, b) \in \mathcal{L}^F$, then $L^F(a) = \bigwedge_{\mu\in FI_L(S)}\{\mu(x\gamma b)\} \geq \bigwedge_{\mu\in FI_L(S)}\{\mu(b)\} = L^F(b)$ and $L^F(b) = \bigwedge_{\mu\in FI_L(S)}\{\mu(y\lambda a)\} \geq \bigwedge_{\mu\in FI_L(S)}\{\mu(a)\} = L^F(a)$. Thus $L^F(a) = L^F(b)$. On the other hand

$$\begin{aligned}
\bigwedge_{\mu^{+'}\in FI_L(L)}\{\mu^{+'}([a,\alpha])\} &= \bigwedge_{\mu^{+'}\in FI_L(L)}\{\mu^{+'}([x,\gamma][b,\alpha])\} \\
&= \bigwedge_{\mu^{+'}\in FI_L(L)}\{\mu^{+'}([x\gamma b,\alpha])\} \\
&= \bigwedge_{\mu^{+'}\in FI_L(L)}\{\bigwedge_{s\in S}\mu(x\gamma b\alpha s)\} \\
&\geq \bigwedge_{\mu^{+'}\in FI_L(L)}\{\bigwedge_{s\in S}\mu(b\alpha s)\} \\
&= \bigwedge_{\mu^{+'}\in FI_L(L)}\{\mu^{+'}([b,\alpha])\}.
\end{aligned}$$

Similarly, we can show for $\bigwedge_{\mu^{+'}\in FI_L(L)}\{\mu^{+'}([b,\alpha]) \geq \bigwedge_{\mu^{+'}\in FI_L(L)}\{\mu^{+'}([a,\alpha])\}$. So $([a,\alpha],[a,\alpha]) \in \mathcal{L}^F_{O_l}$. □

Proposition 3.14. *Let $a,\ b \in S$. If $(a, b) \in \mathcal{R}^F$, then $([a, \alpha], [b, \alpha]) \in \mathcal{R}^F_{O_r}$ for every $\alpha \in \Gamma$.*

Proof. Similar to the proof of Proposition 3.13. □

Proposition 3.15. *Let $a,\ b \in S$. If a is α-idempotent and b is β-idempotent for some $\alpha, \beta \in \Gamma$ such that $([a, \alpha], [b, \beta]) \in \mathcal{R}^F_{O_l}$, then $(a, b) \in \mathcal{R}^F$.*

Proof. Suppose that $([a,\alpha],[b,\beta]) \in \mathcal{R}^F_{O_l}$. If there exist $x, y \in S$ and $\gamma, \lambda \in \Gamma$ such that $a = b\beta x$ and $b = a\alpha y$, then

$$\begin{aligned} R^F([a,\alpha]) &= \bigwedge_{\mu^{+'} \in FI_R(L)} \{\mu^{+'}([a,\alpha])\} \\ &= \bigwedge_{\mu^{+'} \in FI_R(L)} \{\mu^{+'}([b\beta x,\alpha])\} \\ &= \bigwedge_{\mu^{+'} \in FI_R(L)} \{\bigwedge_{s\in S} \mu(b\beta x\alpha s)\} \\ &\geq \bigwedge_{\mu^{+'} \in FI_R(L)} \{\bigwedge_{s\in S} \mu(b\beta s)\} \\ &= \bigwedge_{\mu^{+'} \in FI_R(L)} \{\mu^{+'}([b,\beta])\} \\ &= L^F([b,\beta]). \end{aligned}$$

Similarly, we can show that $R^F([b,\beta]) \geq R^F([a,\alpha])$. Thus $R^F([a,\alpha]) = R^F([b,\beta])$.

Since a is α-idempotent and b is β-idempotent, we have

$$\begin{aligned} \bigwedge_{\mu\in FI_R(S)} \{\mu(a)\} = \bigwedge_{\mu\in FI_R(S)} \{\mu(a\alpha a)\} &= \bigwedge_{\mu\in FI_R(S)} \{\mu([a,\alpha]a)\} \\ &= \bigwedge_{\mu\in FI_R(S)} \{\mu(b\beta x\alpha a)\} \\ &\geq \bigwedge_{\mu\in FI_R(S)} \{\mu(b)\} \end{aligned}$$

and

$$\begin{aligned} \bigwedge_{\mu\in FI_R(S)} \{\mu(b)\} = \bigwedge_{\mu\in FI_R(S)} \{\mu(b\beta b)\} &= \bigwedge_{\mu\in FI_R(S)} \{\mu([b,\beta]b)\} \\ &= \bigwedge_{\mu\in FI_R(S)} \{\mu(a\alpha y\beta b)\} \\ &\geq \bigwedge_{\mu\in FI_R(S)} \{\mu(a)\} \end{aligned}$$

So $\bigwedge_{\mu\in FI_R(S)}\{\mu(a)\} = R^F(a) = R^F(b) = \bigwedge_{\mu\in FI_R(S)}\{\mu(b)\}$. Hence $a\mathcal{R}^F b$. □

Let ρ^F be an equivalence relation on a set A and $a \in A$. Then the ρ^F-class containing a is denoted by $[\rho^F]_a$.

Proposition 3.16. *Let α be an element of Γ and a be an α-idempotent element of S. Then for every $b \in S$ the following assertions hold:*

(1) $b \in [\mathcal{L}^F]_a$ *if and only if* $\mu^{+'}[b,\alpha] \in [\mathcal{L}^F_{O_l}]_{\mu^{+'}[a,\alpha]}$,

(2) $b \in [\mathcal{R}^F]_a$ *if and only if* $\mu^{*'}[\alpha, b] \in [\mathcal{L}^F_{O_r}]_{\mu^{*'}[\alpha,a]}$.

Proof. (1) $b \in [\mathcal{L}^F]_a$ if and only if $L^F(a) = L^F(b)$. Suppose there exist $x, y \in S$ and $\gamma, \lambda \in \Gamma$ such that $a = x\gamma b$ and $b = y\lambda a$ if and only if

$$\mu^{+'}([a,\alpha]) = \bigwedge_{s\in S} \mu(a\alpha s) = \bigwedge_{s\in S} \mu(x\gamma b\alpha s) = \mu^{+'}([x,\gamma][b,\alpha])$$

and

$$\mu^{+'}([b,\alpha]) = \bigwedge_{s\in S} \mu(b\alpha s) = \bigwedge_{s\in S} \mu(y\lambda a\alpha s = \mu^{+'}([y,\lambda][a,\alpha])$$

if and only if

$$\mu^{+'}([a,\alpha]) = \mu^{+'}([x,\gamma][b,\alpha]) \in L^F([b,\alpha])$$

and

$$\mu^{+'}([b,\alpha]) = \mu^{+'}([y,\lambda][a,\alpha]) \in L^F([a,\alpha])$$

if and only if $\mu^{+'}[b,\alpha] \in [\mathcal{L}^F_{O_l}]_{\mu^{+'}[a,\alpha]}$.

□

The other case can be proved similarly.

It is essential to note that all facts that mentioned and proved for the left operator semigroup of a Γ-semigroup in terms of fuzzy subsets can also be done in a similar manner for the right operator semigroup of a Γ-semigroup in terms of fuzzy subsets.

4. Conclusion

Based on the revised definition of Γ-semigroup, we have presented some results related to left (right) operator semigroup and also shown some relationships between equivalence relations $\mathcal{L}^F$ and $\mathcal{R}^F$ of a Γ-semigroup and its left (right) operator semigroup in terms of fuzzy subsets. For further studies, it will be likely feasible to extend this idea to other Green's relations to see whether similar results hold.

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