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Abstract. As an extension of bipolar-valued fuzzy sets, the notion of $(\widetilde{\alpha}, \alpha)$- crossing cubic QS-ideals of a QS-algebra is introduced, and several related properties are investigated. Characterizations of $(\widetilde{\alpha}, \alpha)$-crossing cubic QS-ideal in QS-algebras are established. The relations between $(\widetilde{\alpha}, \alpha)$-crossing cubic subalgebras and $(\widetilde{\alpha}, \alpha)$ crossing cubic QS-ideals of a QS-algebras are studied. Moreover, the homomorphic image (pre-image) of $(\widetilde{\alpha}, \alpha)$-crossing cubic QS-ideal of a QS-algebra under a homomorprhism of QS-algebras is discussed.

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Keywords: QS-algebra, Crossing cubic set, $(\widetilde{a}, a)$-level set, $(\widetilde{\alpha}, \alpha)$-crossing cubic QS-ideal, The homomorphic images (pre-images) of $(\widetilde{\alpha}, \alpha)$-crossing cubic QS-ideal.

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1. Introduction

In 1966, Iséki and Tanaka introduced two classes of abstract algebras: BCK-algebras and BCI-algebras (See [1, 2, 3]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Neggers et al. [4] introduced a concept, called Q-*algebras*, which is a generalization of $BCH/BCI/BCK$-algebras and generalized some theorems discussed in BCI-algebras. Moreover, Ahn and Kim [5] introduced the notion of QS-algebras which is a proper subclass of Q-algebras. Kondo [6] proved that each theorem of QS-algebras is provable in the theory of Abelian groups and conversely each theorem of Abelian groups is provable in the theory of QS-algebras. QS-algebras in the fuzzy setting have also been considered by some authors (See [7, 8]). The concept of fuzzy sets was introduced by Zadeh [9]. In 1991, Xi [10] applied the concept of fuzzy sets to BCI-, BCK-, MV-algebras. Since its inception, the theory of fuzzy sets, ideal theory and its fuzzification has been developed in many directions and applied to a wide variety of fields. Jun et al. [11, 12, 13] introduced the notions of cubic sub-algebras and ideals in BCK/BCI-algebras, and then they investigated several properties. They discussed the relationships between a cubic sub-algebra and a cubic ideal. Also, they provided characterizations of a cubic sub-algebra/ideal and considered a method to produce a new cubic subalgebra from an old one. Zhang [14] proposed the concept of bipolar fuzzy sets as an extension of fuzzy sets in 1998. After then, Lee [15] defined some operations between bipolar fuzzy sets and studied some of their properties. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. Recently, Jun et al. [16, 17] introduced a new function which is called a *negative-valued function*, and constructed N-structures. They applied N-structures to BCK/BCI-algebras, and discussed N-subalgebras and N-ideals in BCK/BCI-algebras. Jun et al. [18, 19] established an extension of a bipolar-valued fuzzy set, which is introduced by Lee [15]. They called it a *crossing cubic structure*, and investigated several properties. They applied crossing cubic structures to BCK/BCI-algebras, and studied crossing cubic sub algebras. In this paper, we modify the ideas of Jun et al. [18], and Jun and Song [19] in order to introduce the notion $(\widetilde{\alpha}, \alpha)$-crossing cubic QS-ideal of QS-algebra. Moreover, the homomorphic image (pre-image) of $(\widetilde{\alpha}, \alpha)$-crossing cubic QS-ideal of a QS-algebra under homomorprhism of QS-algebras are discussed. Finally, many related results are derived.

2. Preliminaries

Now, we will recall some basic concepts related to QS-algebras from the literature, which will be helpful in further study of this paper.

Definition 2.1 ([5])**.** A QS-algebra $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms: for all $x,\ y \in X$,
(QS-1) $(x * y) * z = (x * z) * y$,
(QS-2) $x * 0 = x$,
(QS-3) $x * x = 0$,
(QS-4) $(x * y) * (x * z) = z * y$.

Example 2.2. (1) Let $X = \{0, 1, 2\}$ be a set with the operation $*$ on X defined as follows:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Table 2.1

Then $(X, *, 0)$ is a QS-algebra (See [6]).

(2) Let X be the set of all integers. Define a binary operation $*$ on X by

$$x * y := x - y \text{ for any } x,\ y \in X.$$

Then $(X; *, 0)$ is a QS-algebra.

Definition 2.3 ([5])**.** Let $(X, *, 0)$ be a QS-algebra. Then a binary relation $\leq$ on X defined as follows: for any $x,\ y \in X$,

$$x \leq y \text{ if and only if } x * y = 0.$$

It is obvious that $(X, \leq)$ forms a partially ordered set.

Definition 2.4 ([5])**.** Let $(X, *, 0)$ be a QS-algebra and let S be a nonempty subset of X. Then S is called a *subalgebra* of X, if $x * y \in S$ for any $x,\ y \in X$.

Definition 2.5 ([8])**.** Let X be a QS-algebra and let S be a nonempty subset of X. Then S is called a *subalgebra* of X, if $x * y \in S$ for any $x,\ y \in S$.

Definition 2.6 ([7])**.** Let X be a QS-algebra and let J be a nonempty subset of X. Then J is called a *QS-ideal* of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(QSI_1) $0 \in J$,
(QSI_2) $x * z \in J$ and $z * y \in J$ imply $x * y \in J$.

Definition 2.7 ([5])**.** Let $(X, *, 0)$ and $(Y, *^{'}, 0^{'})$ be QS-algebras. Then a mapping $f : X \to Y$ is called a *homomorphism*, if $f(x * y) = f(x) *^{'} f(y)$ for any $x,\ y \in X$.

For a set X, a mapping $A : X \to I$ is called a *fuzzy set* in X, where $I = [0, 1]$ (See [9]). We denote the set of all fuzzy sets as I^X.

Definition 2.8 ([10])**.** Let X be a BCI-algebra and let $J \in I^X$. Then J is called a *fuzzy BCI-ideal* of X, if it satisfies the following conditions: for any $x,\ y \in X$,
(FI_1) $J(0) \geq J(x)$,
(FI_2) $J(x) \geq J(x * y) \wedge J(y)$.

Definition 2.9 ([7])**.** Let X be a QS-algebra and let $J \in I^X$. Then J is called a *fuzzy QS-ideal* of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(FQSI_1) $J(0) \geq J(x)$,
(FQSI_2) $J(x * y) \geq J(x * z) \wedge J(z * y)$.

Lemma 2.10. *Every fuzzy QS-ideal of a QS-algebra X is a fuzzy BCI-ideal of X.*

Proof. Straightforward. □

Definition 2.11 ([16, 17])**.** Let X be a nonempty set. Then a mapping $A^N : X \to [-1, 0]$ is called a *negative-valued function on* X (briefly, N-function on X). In particular, $(-1)^N$ [resp. 0^N] is called the *whole* [resp. *empty*] *N-function* on X and defined as follows: for each $x \in X$

$$(-1)^N(x) = -1 \text{ [resp. } 0^N(x) = 0].$$

The pair (X, A^N) is called an *N-structure* and we denote the set of all N-functions on X as $NF(X)$.

Definition 2.12 ([16, 17])**.** Let X be a nonempty set and let $A^N,\ B^N \in NF(X)$.
(i) We say that A^N *is a subset of* B^N, denoted by $A^N \subset B^N$, if $A^N(x) \geq B^N(x)$ for each $x \in X$.
(ii) The *complement* of A^N, denoted by $c(A^N)$, is an N-function on X defined as follows: for each $x \in X$,

$$c(A^N)(x) = -1 - A^N(x).$$

(iii) The *intersection* of A^N and B^N, denoted by $A^N \cap B^N$, is an N-function on X defined as follows: for each $x \in X$,

$$(A^N \cap B^N)(x) = A^N(x) \vee B^N(x).$$

(iii) The *union* of A^N and B^N, denoted by $A^N \cup B^N$, is an N-function on X defined as follows: for each $x \in X$,

$$(A^N \cup B^N)(x) = A^N(x) \wedge B^N(x).$$

Let $[I] = \{\widetilde{a} = [a^-, a^+] \subset I : 0 \leq a^- \leq a^+ \leq 1\}$ be the set of all closed intervals of I. In particular, $\widetilde{1} = [1, 1]$ and $\widetilde{0} = [0, 0]$. We define $\leq$, $=$, $+$, the scalar product, the sup and the inf on $[I]$ as follows: for any $\widetilde{a}$, $\widetilde{b} \in [I]$ and for each $(\widetilde{a_k})_{k \in K} \subset [I]$ (K is an index set),

(i) $\widetilde{a} \leq \widetilde{b}$ if and only if $a^- \leq b^-, \ a^+ \leq b^+$,
(ii) $\widetilde{a} = \widetilde{b}$ if and only if $a^- = b^-, \ a^+ = b^+$,
(iii) $\widetilde{a} + \widetilde{b} = [a^- + b^-, a^+ + b^+]$ where $a^+ + b^+ \leq 1$,
(iv) $c\widetilde{a} = [ca^-, ca^+]$ for each $c \in I$,
(v) $\bigvee_{k \in K} \widetilde{a_k} = [\bigvee_{k \in K} a_k^-, \bigvee_{k \in K} a_k^+]$,
(vi) $\bigwedge_{k \in K} \widetilde{a_k} = [\bigwedge_{k \in K} a_k^-, \bigwedge_{k \in K} a_k^+]$.

For a set X, a mapping $\widetilde{A} = [A^-, A^+] : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, IVFS) (See [20]). In particular, $\widetilde{\mathbf{1}}$ [resp. $\widetilde{\mathbf{0}}$] is called the *whole* [resp. *empty*] IVFS in X and defined as follows: for each $x \in X$,

$$\widetilde{\mathbf{1}}(x) = \widetilde{1} = [1, 1] \text{ [resp. } \widetilde{\mathbf{0}}(x) = \widetilde{0} = [0, 0].$$

We denote the set of all IVFSs in X as $IVFS(X)$.

For a nonempty set X and each $A \in 2^X$, χ_A denotes the characteristic function of A, where 2^X is the power set of X.

3. A crossing cubic QS-ideal of a QS-algebra

In this section, we shall introduce a new notion called $(\widetilde{\alpha}, \alpha)$-*crossing cubic QS-ideal* of a QS-algebra and study several its properties.

Each member $(\widetilde{a}, a)$ of $[I] \times [-1, 0]$ is called a *crossing cubic number* and the order, the equality, the sup and the inf of arbitrary crossing cubic numbers defined as follows: for any $(\widetilde{a}, a)$, $(\widetilde{b}, b) \in [I] \times [-1, 0]$ and each $(\widetilde{a}_k, a_k)_{k \in K} \subset [I] \times [-1, 0]$,

(i) $(\widetilde{a}, a) \leq (\widetilde{b}, b)$ if and only if $\widetilde{a} \leq \widetilde{b}$ and $a \geq b$,
(ii) $(\widetilde{a}, a) = (\widetilde{b}, b)$ if and only if $\widetilde{a} = \widetilde{b}$ and $a = b$,
(iii) $sup_{k \in K}(\widetilde{a}_k, a_k) = (\bigvee_{k \in K} \widetilde{a}_k, \bigwedge_{k \in K} a_k)$,
(iv) $inf_{k \in K}(\widetilde{a}_k, a_k) = (\bigwedge_{k \in K} \widetilde{a}_k, \bigvee_{k \in K} a_k)$.

Definition 3.1 ([16, 17])**.** Let X be a nonempty set. Then a mapping $A = (\widetilde{A}, A^N) : X \to [I] \times [-1, 0]$ is called a *crossing cubic set* in X. The pair $(\widetilde{\mathbf{1}}, (-1)^N)$ [resp. $(\widetilde{\mathbf{0}}, 0^N)$] is called the *whole* [resp. *empty*] *crossing empty set* in X and denoted by $\dot{\mathbf{1}}$ [resp. $\dot{\mathbf{0}}$]. We will denote the set of all crossing cubic sets in X as $CCS(X)$.

In what follows, let X denotes a QS-algebra unless otherwise specified.

Definition 3.2 ([16, 17])**.** Let $\dot{\mathbf{0}} \neq A = (\widetilde{A}, A^N) \in CCS(X)$. Then A is called a *crossing cubic subalgebra* of X if it satisfies the following conditions: for any $x,\ y \in X$,
(CCS$_1$) $\widetilde{A}(x * y) \geq \widetilde{A}(x) \wedge \widetilde{A}(y)$,
(CCS$_2$) $A^N(x * y) \leq A^N(x) \vee A^N(y)$.

Definition 3.3. Let X be a nonempty set, let $A = (\widetilde{A}, A^N) \in CCS(X)$ and let $(\widetilde{a}, a) \in [I] \times [-1, 0]$. Then the $(\widetilde{a}, a)$*-level set* of A, denoted by $[A]_{(\widetilde{a},a)}$, is a subset of X defined as follows:

$$[A]_{(\widetilde{a},a)} = [\widetilde{A}]_{\widetilde{a}} \cup [A^N]_a,$$

where $[A]_{(\widetilde{a},a)} = \{x \in X : \widetilde{A}(x) \geq \widetilde{a}\}$ is a subset of X called the $\widetilde{a}$*-level set* of $\widetilde{A}$ and $[A^N]_a = \{x \in X : A^N(x) \leq a\}]$ is a subset of X called the a*-level set* of A^N.

Remark 3.4. Let X be a nonempty set and let $\dot{\mathbf{0}} \neq A = (\widetilde{A}, A^N) \in CCS(X)$. Then for each $(\widetilde{a}, a) \in [I] \times [-1, 0]$ such that $\widetilde{a} \neq \widetilde{0}$ and $a \neq 0$, $[A]_{(\widetilde{a},a)} \neq \varnothing$, i.e.,

$$[\widetilde{A}]_{\widetilde{a}} \neq \varnothing \text{ and } [A^N]_a \neq \varnothing.$$

Example 3.5. Let $A = (\widetilde{A}, A^N)$ be the crossing cubic set in I defined by: for each $x \in I$,

$$A(x) = ([\frac{x}{2}, \frac{x+1}{2}], \frac{x}{3}).$$

Then it is clear that $[A]_{([\frac{1}{3},\frac{2}{3}],-\frac{1}{4})} = [\widetilde{A}]_{[\frac{1}{3},\frac{2}{3}]} \cup [A^N]_{-\frac{1}{4}} = [-1, -\frac{3}{4}] \cup [\frac{2}{3}, 1]$.

The following is an immediate consequence of Definition 3.3.

Lemma 3.6. *Let X be a nonempty set, let $A = (\widetilde{A}, A^N) \in CCS(X)$ and let $(\widetilde{a}, a),\ (\widetilde{b}, b) \in [I] \times [-1, 0]$. If $(\widetilde{a}, a) \leq (\widetilde{b}, b)$, then $[\widetilde{A}]_{\widetilde{b}} \subset [\widetilde{A}]_{\widetilde{a}}$ and $[A^N]_b \subset [A^N]_a$.*

Theorem 3.7. *Let $A = (\widetilde{A}, A^N) \in CCS(X)$. Then A is a crossing cubic subalgebra of X if and only if for each $(\widetilde{a}, a) \in [I] \times [-1, 0]$ such that $\widetilde{a} \neq \widetilde{0}$ and $a \neq 0$, $[\widetilde{A}]_{\widetilde{a}}$ and $[A^N]_a$ are subalgebras of X.*

In this case, $[A]_{(\widetilde{a},a)}$ is called the $(\widetilde{a}, a)$*-level subalgebra* of X.

Proof. Suppose A is a crossing cubic subalgebra of X and let $(\widetilde{a}, a) \in [I] \times [-1, 0]$ such that $\widetilde{a} \neq \widetilde{0}$ and $a \neq 0$. It is clear that $[\widetilde{A}]_{\widetilde{a}} \neq \varnothing$ and $[A^N]_a \neq \varnothing$ from Remark 3.4.

Let $x,\ y \in [\widetilde{A}]_{\widetilde{a}}$. Then $\widetilde{A}(x) \geq \widetilde{a},\ \widetilde{A}(y) \geq \widetilde{a}$. Thus $\widetilde{A}(x) \wedge \widetilde{A}(y) \geq \widetilde{a}$. By the conditions (CCS$_1$) and (CCS$_2$), $\widetilde{A}(x * y) \geq \widetilde{a}$. So $x * y \in [A]_{(\widetilde{a},a)}$. Hence $[A]_{(\widetilde{a},a)}$ is a subalgebra of X.

Now let $x,\ y \in [A^N]_a]_a$. Then $A^N(x) \leq a,\ (y) \leq a$. Thus $A^N(x) \vee A^N(y) \leq a$. By the conditions (CCS$_1$) and (CCS$_2$), $A^N(x * y) \leq a$. So $x * y \in [A^N]_a$. Hence $[A^N]_a$ is a subalgebra of X.

Conversely, suppose the necessary condition holds and for any $x,\ y \in X$, let $A(x) = (\widetilde{a}, a)$, $A(y) = (\widetilde{b}, b)$, where $(\widetilde{a}, a)$ and $(\widetilde{b}, b)$ are crossing cubic numbers such that $(\widetilde{a}, a) \leq (\widetilde{b}, b)$. Then clearly, $\widetilde{A}(x) = \widetilde{a},\ \widetilde{A}(y) = \widetilde{b}$ and $A^N(x) = a,\ A^N(y) = b$. By Definition 3.3, $x \in [\widetilde{A}]_{\widetilde{a}},\ y \in [\widetilde{A}]_{\widetilde{a}}$ and $x \in [A^N]_a,\ y \in [A^N]_a$. Since $(\widetilde{a}, a) \leq (\widetilde{b}, b)$, by Lemma 3.6, $[\widetilde{A}]_{\widetilde{b}} \subset [\widetilde{A}]_{\widetilde{a}}$ and $[A^N]_b \subset [A^N]_a$. Thus $x,\ y \in [\widetilde{A}]_{\widetilde{a}}$ and $x,\ y \in [A^N]_a$. By the hypothesis, $x * y \in [\widetilde{A}]_{\widetilde{a}}$ and $x * y \in [A^N]_a$. So we have

$$\widetilde{A}(x*y) \geq \widetilde{a} = \widetilde{a} \wedge \widetilde{b} = \widetilde{A}(x) \wedge \widetilde{A}(y) \text{ and } A^N(x*y) \leq a = a \vee b = A^N(x) \vee A^N(y).$$

Hence A is a crossing cubic subalgebra of X. □

Definition 3.8. Let X be a nonempty set, let $A = (\widetilde{A}, A^N) \in CCS(X)$ and let $(\widetilde{\alpha}, \alpha) \in [I] \times [-1, 0]$. Then the mapping $A_{(\widetilde{\alpha},\alpha)} = (\widetilde{A}_{\widetilde{\alpha}}, A^N_\alpha) : X \to [I] \times [-1, 0]$ defined as follows: for each $x \in X$,

$$A_{(\widetilde{\alpha},\alpha)}(x) = (\widetilde{A}_{\widetilde{\alpha}}(x) \wedge \widetilde{\alpha}, A^N_\alpha(x) \vee \alpha)$$

is a $(\widetilde{\alpha}, \alpha)$-*crossing cubic set* in X *with respect to* A.

It is obvious that $A_{(\widetilde{1},-1)} = A$.

Lemma 3.9. *Let $A = (\widetilde{A}, A^N) \in CCS(X)$ and let $(\widetilde{\alpha}, \alpha)$ be a crossing cubic number. If A is a crossing cubic subalgebra of X, then $A_{(\widetilde{\alpha},\alpha)}$ is a crossing cubic subalgebra of X.*

In this case, $A_{(\widetilde{\alpha},\alpha)}$ is called the $(\widetilde{\alpha}, \alpha)$-*crossing cubic subalgebra* of X.

Proof. Let $x,\ y \in X$. Then we have

$$\begin{aligned}
\widetilde{A}_{\widetilde{\alpha}}(x*y) &= \widetilde{A}(x*y) \wedge \widetilde{\alpha} \text{ [By Definition 3.8]}\\
&\geq (\widetilde{A}(x) \wedge \widetilde{A}(y)) \wedge \widetilde{\alpha} \text{ [By the condition (CCS}_1\text{)]}\\
&= (\widetilde{A}(x) \wedge \widetilde{a}) \wedge (\widetilde{A}(y)) \wedge \widetilde{\alpha})\\
&= \widetilde{A}_{\widetilde{\alpha}}(x) \wedge \widetilde{A}_{\widetilde{\alpha}}(y),\\
A^N_\alpha(x*y) &= A^N(x*y) \vee \alpha \text{ [By Definition 3.8]}\\
&\leq (A^N(x) \vee A^N(y)) \vee a \text{ [By the condition (CCS}_2\text{)]}\\
&= (A^N(x) \vee \alpha) \wedge (A^N(y)) \vee \alpha)\\
&= A^N_\alpha(x) \vee A^N_\alpha(y).
\end{aligned}$$

Thus the results hold. □

It is clear that $(\widetilde{\alpha}, \alpha)$-crossing cubic subalgebra of a QS-algebra X is a generalization of a crossing cubic subalgebra of X and a crossing cubic subalgebra of X is a special case when $\widetilde{\alpha} = \widetilde{1}$ and $\alpha = -1$.

Theorem 3.10. *Let $A = (\widetilde{A}, A^N) \in CCS(X)$ and let $(\widetilde{\alpha}, \alpha)$ be a crossing cubic number. Then $A_{(\widetilde{\alpha},\alpha)}$ is an $(\widetilde{a}, a)$-crossing cubic subalgebra of X if and only if for any crossing cubic number $(\widetilde{\varepsilon}, \varepsilon)$ such that $\widetilde{\varepsilon} \neq \widetilde{0}$ and $\varepsilon \neq 0$, $[\widetilde{A}_{\widetilde{\alpha}}]_{\widetilde{\varepsilon}}$ and $[A^N_\alpha]_\varepsilon$ are subalgebras of X.*

Proof. The proof is similar to Theorem 3.7. □

Definition 3.11. Let $A = (\widetilde{A}, A^N) \in CCS(X)$. Then A is a crossing cubic QS-ideal of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

(CCI_1) $\widetilde{A}(0) \geq \widetilde{A}(x)$ and $A^N(0) \leq A^N(x)$,

(CCI_2) $\widetilde{A}(x*y) \geq \widetilde{A}(x*z) \wedge \widetilde{A}(z*y)$ and $A^N(x*y) \leq A^N(x*z) \vee A^N(z*y)$.

Theorem 3.12. *Let $A = (\widetilde{A}, A^N) \in CCS(X)$. Then A is a crossing cubic QS-ideal of X if and only if for each $(\widetilde{a}, a) \in [I] \times [-1, 0]$ such that $\widetilde{a} \neq \widetilde{0}$ and $a \neq 0$, $[\widetilde{A}]_{\widetilde{a}}$ and $[A^N]_a$ are QS-ideals of X*

In this case, $[A]_{(\widetilde{a},a)}$ is called the $(\widetilde{a}, a)$*-level QS-ideal* of X.

Proof. Suppose A is a crossing cubic QS-ideal of X and let $(\widetilde{a}, a) \in [I] \times [-1, 0]$ such that $\widetilde{a} \neq \widetilde{0}$ and $a \neq 0$. It is clear that $[\widetilde{A}]_{\widetilde{a}} \neq \varnothing$ and $[A^N]_a \neq \varnothing$ from Remark 3.4. Let $x \in X$. Since A is a crossing cubic QS-ideal of X, by the condition (CCI_1), we have

$$\widetilde{A}(0) \geq \widetilde{A}(x) \geq \widetilde{a} \text{ and } A^N(0) \leq A^N(x) \leq a.$$

Then $[\widetilde{A}]_{\widetilde{a}}$ and $[A^N]_a$ satisfy the condition (QSI_1).

Now suppose $x * z,\ z * y \in [\widetilde{A}]_{\widetilde{a}}$ and $x * z,\ z * y \in [A^N]_a$. Then we get

$$\begin{aligned}
\widetilde{A}(x * y) &\geq \widetilde{A}(x * z) \wedge \widetilde{A}(z * y) \text{ [By the condition } (\text{CCI}_2)] \\
&\geq \widetilde{a} \wedge \widetilde{a} \text{ [By the hypothesis]} \\
&= \widetilde{a}, \\
A^N(x * y) &\leq A^N(x * z) \vee A^N(z * y) \text{ [By the condition } (\text{CCI}_2)] \\
&\leq a \vee a \text{ [By the hypothesis]} \\
&= a.
\end{aligned}$$

Thus $[\widetilde{A}]_{\widetilde{a}}$ and $[A^N]_a$ satisfy the condition (QSI_2). So $[\widetilde{A}]_{\widetilde{a}}$ and $[A^N]_a$ are QS-ideals of X.

Conversely, suppose the necessary condition holds and for any $x \in X$, let $A(x) = (\widetilde{a}, a)$. Then clearly, $x \in [\widetilde{A}]_{\widetilde{a}}$ and $x \in [A^N]_a$. Since $(\widetilde{a}, a) \leq (\widetilde{1}, -1)$, by Lemma 3.6, we get

$$[\widetilde{A}]_{\widetilde{1}} \subset [\widetilde{A}]_{\widetilde{a}} \text{ and } [A^N]_{-1} \subset [A^N]_{\widetilde{a}}.$$

Since $[A]_{(\widetilde{1},-1)}$ is a QS-ideal of X, $0 \in [A]_{(\widetilde{1},-1)}$. Thus $0 \in [\widetilde{A}]_{\widetilde{a}}$ and $0 \in [A^N]_a$. So we have

$$\widetilde{A}(0) \geq \widetilde{a} = \widetilde{A}(x) \text{ and } A^N(0) \leq A^N(x).$$

Hence $\widetilde{A}$ and A^N satisfy the condition (CCI_1).

Now for any $x,\ y,\ z \in X$, let $A(x*z) = (\widetilde{a}, a), A(z*y) = (\widetilde{b}, b)$, where $\widetilde{a}, a) \leq \widetilde{b}, b)$. Then clearly, $x * z \in [\widetilde{A}]_{\widetilde{a}},\ z * y \in [\widetilde{A}]_{\widetilde{b}}$ and $x * z \in [A^N]_a,\ z * y \in [A^N]_b$. Since $(\widetilde{a}, a) \leq \widetilde{b}, b)$, by Lemma 3.6,

$$[\widetilde{A}]_{\widetilde{b}} \subset [\widetilde{A}]_{\widetilde{a}} \text{ and } [A^N]_b \subset [A^N]_a.$$

Thus $x*z,\ z*y \in [\widetilde{A}]_{\widetilde{a}}$ and $x*z,\ z*y \in [A^N]_a$. By the condition (QSI_2), $x*y \in [\widetilde{A}]_{\widetilde{a}}$ and $x * y \in [A^N]_a$. So we have

$$\widetilde{A}(x*y) \geq \widetilde{a} = \widetilde{a} \wedge \widetilde{b} = \widetilde{A}(x*z) \wedge \widetilde{A}(z*y) \text{ and } A^N(x*y) \leq a = a \vee b = A^N(x*z) \vee A^N(z*y).$$

Hence $\widetilde{A}$ and A^N satisfy the condition (CCI_2). Therefore A is a crossing cubic QS-ideal of X. □

Lemma 3.13. *Let $A = (\widetilde{A}, A^N) \in CCS(X)$ and let $(\widetilde{\alpha}, \alpha)$ be a crossing cubic number. If A is a crossing cubic QS-ideal of X, then $A_{(\widetilde{\alpha},\alpha)}$ is a crossing cubic QS-ideal of X, i.e., the followings hold: for any $x,\ y,\ z \in X$,*

(1) $\widetilde{A}_{\widetilde{\alpha}}(0) \geq \widetilde{A}_{\widetilde{\alpha}}(x)$ *and* $A^N_\alpha(0) \leq A^N_\alpha(x)$,

(2) $\widetilde{A}_{\widetilde{\alpha}}(x*y) \geq \widetilde{A}_{\widetilde{\alpha}}(x*z) \wedge \widetilde{A}_{\widetilde{\alpha}}(z*y)$,
(2) $A^N_\alpha(x*y) \leq A^N_\alpha(x*z) \vee A^N_\alpha(z*y)$.

In this case, $A_{(\widetilde{\alpha},\alpha)}$ is called the $(\widetilde{\alpha},\alpha)$-*crossing cubic* QS-*ideal* of X.

Proof. (1) let $x \in X$. Then we have

$$\widetilde{A}_{\widetilde{\alpha}}(0) = \widetilde{A}(0) \wedge \widetilde{\alpha} \geq \widetilde{A}(x) \wedge \widetilde{\alpha} = \widetilde{A}_{\widetilde{\alpha}}(x),$$
$$A^N_\alpha(0) = A^N(0) \vee \alpha \leq A^N(x) \vee \alpha = A^N_\alpha(x).$$

(2) Let $x,\ y,\ z \in X$. Then we get

$$\begin{aligned}\widetilde{A}_{\widetilde{\alpha}}(x*y) &= \widetilde{A}(x*y) \wedge \widetilde{\alpha}\\ &\geq (\widetilde{A}(x*z) \wedge \widetilde{A}(z*y)) \wedge \widetilde{a}\\ &= (\widetilde{A}(x*z) \wedge \widetilde{\alpha}) \wedge (\widetilde{A}(z*y) \wedge \widetilde{\alpha})\\ &= \widetilde{A}_{\widetilde{\alpha}}(x*z) \wedge \widetilde{A}_{\widetilde{\alpha}}(z*y).\end{aligned}$$

(3) The proof is similar to (2). □

It is clear that $(\widetilde{\alpha},\alpha)$-crossing cubic QS-ideal of a QS-algebra X is a generalization of a crossing cubic QS-ideal of X and a crossing cubic QS-ideal of X is special case, when $\widetilde{\alpha} = \widetilde{1}$ and $\alpha = -1$.

Example 3.14. Let $X = \{0,1,2,3\}$ be a set and let $(X,*,0)$ be the QS-algebra with the operation $*$ on X defined as follows (See [7]):

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
2	3	2	1	0

Table 3.1

(1) Let $A = (\widetilde{A}, A^N)$ be the crossing cubic set in defined by:

$$A(0) = ([0.3,0.9],-0.3) = A(1),\ A(2) = ([0.1,0.6],-0.7) = A(3).$$

Then we can easily check that A is a crossing cubic QS-ideal of X. Moreover, we can see that $[\widetilde{A}]_{[0.2,0.7]} = \{0,1\}$ and $[A^N]_{-0.2} = X$. Thus $[\widetilde{A}]_{[0.2,0.7]}$ and $[A^N]_{-0.2}$ are QS-ideals of X.

(2) Let A be the crossing cubic QS-ideal of X defined in (1) and let $(\widetilde{\alpha},\alpha)$ be any crossing cubic number. We define the $(\widetilde{\alpha},\alpha)$-crossing cubic set in X with respect to A as follows:

$$A_{(\widetilde{\alpha},\alpha)}(0) = (\widetilde{A}(0) \wedge \widetilde{\alpha}, A^N(0) \vee \alpha) = A_{(\widetilde{\alpha},\alpha)}(1),$$
$$A_{(\widetilde{\alpha},\alpha)}(2) = (\widetilde{A}(2) \wedge \widetilde{\alpha}, A^N(2) \vee \alpha) = A_{(\widetilde{\alpha},\alpha)}(3).$$

Then we can easily see that $A_{(\widetilde{\alpha},\alpha)}$ is a $(\widetilde{\alpha},\alpha)$-crossing cubic QS-ideal of X.

The following can be considered as Theorem 3.12.

Theorem 3.15. *Let $A_{(\widetilde{\alpha},\alpha)}) = (\widetilde{A}_{\widetilde{\alpha}}, A^N_\alpha)$ be an $(\widetilde{\alpha},\alpha)$-crossing cubic set X. Then $A_{(\widetilde{\alpha},\alpha)})$ is an $(\widetilde{\alpha},\alpha)$-crossing cubic QS-ideal of X if and only if for each $(\widetilde{\varepsilon},\varepsilon) \in [I] \times [-1,0]$ such that $\widetilde{\varepsilon} \neq \widetilde{0}$ and $\varepsilon \neq 0$, $[\widetilde{A}_{\widetilde{\alpha}}]_{\widetilde{\varepsilon}}$ and $[A^N_\alpha]_\varepsilon$ are QS-ideals of X.*

Proof. The proof is similar to Theorem 3.12 but we prove the converse by the different method from 3.12.

Suppose the necessary conditions hold and let $(\widetilde{\varepsilon}, \varepsilon)$ be any crossing cubic number such that $\widetilde{\varepsilon} \neq \widetilde{0}$ and $\varepsilon \neq 0$. It is clear that $[\widetilde{A}_{\widetilde{\alpha}}]_{\widetilde{\varepsilon}} \neq \varnothing$ and $[A^N_\alpha]_\varepsilon \neq \varnothing$ from Remark 3.4.

(i) Assume that $\widetilde{A}_{\widetilde{\alpha}}(0) < \widetilde{A}_{\widetilde{\alpha}}(x)$ for some $x \in X$. Let $\widetilde{\varepsilon}_0 = \frac{1}{2}(\widetilde{A}_{\widetilde{\alpha}}(0) + \widetilde{A}_{\widetilde{\alpha}}(x))$. Then clearly, $\widetilde{A}_{\widetilde{\alpha}}(0) < \widetilde{\varepsilon}_0 < \widetilde{A}_{\widetilde{\alpha}}(x)$. Thus $0 \notin [\widetilde{A}_{\widetilde{\alpha}}]_{\widetilde{\varepsilon}_0}$. This contradicts the fact which $[\widetilde{A}_{\widetilde{\alpha}}]_{\widetilde{\varepsilon}_0}$ is a QS-ideal of X. So $\widetilde{A}_{\widetilde{\alpha}}(0) \geq \widetilde{A}_{\widetilde{\alpha}}(x)$ for each $x \in X$.

Now assume that $A^N_\alpha(0) > A^N_\alpha(x)$ for some $x \in X$. Let $\varepsilon_0 = \frac{1}{2}(A^N_\alpha(0) + A^N_\alpha(x))$. Then clearly, $A^N_\alpha(0) > \varepsilon_0 > A^N_\alpha(x)$. Thus $0 \notin [A^N_\alpha]_{\varepsilon_0}$. This contradicts the fact which $[A^N_\alpha]_{\varepsilon_0}$ is a QS-ideal of X. So $A^N_\alpha(0) \leq A^N_\alpha(x)$ for each $x \in X$. Hence $A_{(\widetilde{\alpha},\alpha)}$ satisfies the condition (CCI_1).

(ii) Assume that $\widetilde{A}_{\widetilde{\alpha}}(x * y) < \widetilde{A}_{\widetilde{\alpha}}(x * z) \wedge \widetilde{A}_{\widetilde{\alpha}}(z * y)$ for some $x,\ y,\ z \in X$. Let $\widetilde{\varepsilon}_1 = \frac{1}{2}(\widetilde{A}_{\widetilde{\alpha}}(x * y) + \widetilde{A}_{\widetilde{\alpha}}(x * z) \wedge \widetilde{A}_{\widetilde{\alpha}}(z * y))$. Then clearly, we have

$$\widetilde{A}_{\widetilde{\alpha}}(x * y) < \widetilde{\varepsilon}_1 < \widetilde{A}_{\widetilde{\alpha}}(x * z) \wedge \widetilde{A}_{\widetilde{\varepsilon}}(z * y).$$

Thus $x * z,\ z * y \in [\widetilde{A}_{\widetilde{\alpha}}]_{\widetilde{\varepsilon}_1}$ but $x * y \notin [\widetilde{A}_{\widetilde{\alpha}}]_{\widetilde{\varepsilon}_1}$. This contradicts the fact which $[\widetilde{A}_{\widetilde{\alpha}}]_{\widetilde{\varepsilon}_1}$ is a QS-ideal of X. So $\widetilde{A}_{\widetilde{\alpha}}(x * y) \geq \widetilde{A}_{\widetilde{\alpha}}(x * z) \wedge \widetilde{A}_{\widetilde{\alpha}}(z * y)$ for any $x,\ y,\ z \in X$.

Now assume that $A^N_\alpha(x * y) > A^N_\alpha(x * z) \vee A^N_\alpha(z * y)$ for some $x,\ y,\ z \in X$. Let $\varepsilon_1 = \frac{1}{2}(A^N_\alpha(x * y) + A^N_\alpha(x * z) \vee A^N_\alpha(z * y)$. Then clearly, we have

$$A^N_\alpha(x * y) > \varepsilon_1 > A^N_\alpha(x * z) \vee A^N_\alpha(z * y).$$

Thus $x * z,\ z * y \in [A^N_\alpha]_{\varepsilon_1}$ but $x * y \notin [A^N_\alpha]_{\varepsilon_1}$. This contradicts the fact which $[A^N_\alpha]_{\varepsilon_1}$ is a QS-ideal of X. So $A^N_\alpha(x * y) \geq A^N_\alpha(x * z) \vee A^N_\alpha(z * y)$ for any $x,\ y,\ z \in X$. Hence $A_{(\widetilde{\alpha},\alpha)}$ satisfies the condition (CCI_2). Therefore $A_{(\widetilde{\alpha},\alpha)}$ is a $(\widetilde{\alpha}, \alpha)$-crossing cubic QS-ideal of X. □

Proposition 3.16. *Let A be a crossing cubic QS-ideal of X. If the inequality $x \leq z$ holds in X, then $\widetilde{A}(x) \geq \widetilde{A}(z),\ A^N(x) \leq A^N(z)$, i.e., $A(x) \geq A(z)$.*

Proof. Suppose the inequality $x \leq z$ holds in X. Then by Definition 2.3,

$$x * z = 0. \tag{3.1}$$

Then we get

$$\begin{aligned} \widetilde{A}(x) &= \widetilde{A}(x * 0) \text{ [By the condition (QS-2)]} \\ &\geq \widetilde{A}(x * z) \wedge \widetilde{A}(z * 0) \text{ [By the condition (CCI}_2\text{)]} \\ &= \widetilde{A}(x * z) \wedge \widetilde{A}(z) \text{ [By the condition (QS-2)]} \\ &= \widetilde{A}(0) \wedge \widetilde{A}(z) \text{ [By (3.1)]} \\ &= \widetilde{A}(z), \text{ [By the condition (CCI}_1\text{)]} \end{aligned}$$

$$\begin{aligned} A^N(x) &= A^N(x * 0) \\ &\leq A^N(x * z) \vee A^N(z * 0) \\ &= A^N(x * z) \vee A^N(z) \\ &= A^N(0) \vee A^N(z) \\ &= A^N(z). \end{aligned}$$

Thus the results hold. □

Corollary 3.17. *Let A be a crossing cubic QS-ideal of X and let $\widetilde{\alpha},\alpha)$ be a crossing cubic number. If the inequality $x \leq z$ holds in X, then $\widetilde{A}_{\widetilde{\alpha}}(x) \geq \widetilde{A}_{\widetilde{\alpha}}(z)$, $A^N_\alpha(x) \leq A^N_\alpha(z)$, i.e., $A_{(\widetilde{\alpha},\alpha)}(x) \geq A_{(\widetilde{\alpha},\alpha)}(z)$.*

Proof. The proof follows from Lemma 3.13 and Proposition 3.16. □

Proposition 3.18. *Let A be a crossing cubic QS-ideal of X. If the inequality $x * y \leq z$ holds in X, then $\widetilde{A}(y) \geq \widetilde{A}(x) \wedge \widetilde{A}(z)$, $A^N(y) \leq A^N(x) \vee A^N(z)$, i.e.,*

$$A(y) \geq A(x) \wedge A(z).$$

Proof. Suppose the inequality $x * y \leq z$ holds in X. Then by Proposition 2.3 (3),

$$x * z \leq y. \tag{3.2}$$

Then we have

$$\begin{aligned}\widetilde{A}(x) &= \widetilde{A}(x * 0) \text{ [By the condition (QS-2)]}\\ &\geq \widetilde{A}(x * z) \wedge \widetilde{A}(z * 0) \text{ [By the condition (CCI}_2\text{)]}\\ &= \widetilde{A}(x * z) \wedge \widetilde{A}(z) \text{ [By the condition (QS-2)]}\\ &\geq \widetilde{A}(y) \wedge \widetilde{A}(z). \text{ [By (3.2) and Proposition 3.14]}\end{aligned}$$

Similarly, we get $A^N(x) \leq A^N(y) \vee A^N(z)$. Thus the results hold. □

Corollary 3.19. *Let A be a crossing cubic QS-ideal of X and let $\widetilde{\alpha},\alpha)$ be a crossing cubic number. If the inequality $x * y \leq z$ holds in X, then $\widetilde{A}_{\widetilde{\alpha}}(y) \geq \widetilde{A}_{\widetilde{\alpha}}(x) \wedge \widetilde{A}_{\widetilde{\alpha}}(z)$, $A^N_\alpha(y) \leq A^N_\alpha(x) \vee A^N_\alpha(z)$, i.e.,*

$$A_{(\widetilde{\alpha},\alpha)}(y) \geq A_{(\widetilde{\alpha},\alpha)}(x) \wedge A_{(\widetilde{\alpha},\alpha)}(z).$$

Proof. The proof follows from Lemma 3.13 and Proposition 3.18. □

Proposition 3.20. *Let $(A_k)_{k\in K}$ be a family of crossing cubic QS-ideals of X indexed by a class K, where $A_k = (\widetilde{A}_k, A^N_k)$ for each $k \in K$. Then $\bigcap_{k\in K} A_k$ is a crossing cubic QS-ideal of X.*

Proof. Let $A = \bigcap_{k\in K} A_k = (\bigcap_{k\in K} \widetilde{A}_k, \bigcup_{k\in K} A^N_{k\in K}) = (\widetilde{A}, A^N)$ and let $x \in X$. Then we get

$$\begin{aligned}\widetilde{A}(x) &= (\textstyle\bigcap_{k\in K} \widetilde{A}_k)(x)\\ &= \textstyle\bigwedge_{k\in K} \widetilde{A}_k(x)\\ &\leq \textstyle\bigwedge_{k\in K} \widetilde{A}_k(0) \text{ [By the condition (CCI}_1\text{)]}\\ &= (\textstyle\bigcap_{k\in K} \widetilde{A}_k)(0)\\ &= \widetilde{A}(0),\end{aligned}$$

$$\begin{aligned}A^N(x) &= (\textstyle\bigcup_{k\in K} A^N_k)(x)\\ &= \textstyle\bigvee_{k\in K} A^N_k(x)\\ &\geq \textstyle\bigvee_{k\in K} A^N_k(0) \text{ [By the condition (CCI}_1\text{)]}\\ &= (\textstyle\bigcup k \in K A^N_k)(0)\\ &= A^N(0).\end{aligned}$$

Thus the condition (CCI$_1$) is satisfied.

Now let $x,\ y,\ z \in X$. Then we have

$$\widetilde{A}(x * y) = (\textstyle\bigcap_{k\in K} \widetilde{A}_k)(x * y)$$

$= \bigwedge_{k\in K} \widetilde{A}_k(x*y)$
$\geq \bigwedge_{k\in K}[\widetilde{A}_k(x*z) \wedge \widetilde{A}_k(z*y)]$ [By the condition (CCI$_2$)
$= (\bigwedge_{k\in K} \widetilde{A}_k(x*z)) \wedge \bigwedge_{k\in K} \widetilde{A}_k(z*y))$
$= (\bigcap_{k\in K} \widetilde{A}_k)(x*z) \wedge (\bigcap_{k\in K} \widetilde{A}_k)(z*y)$
$= \widetilde{A}(x*z) \wedge \widetilde{A}(z*y)$.

Similarly, we get $A^N(x*y) \leq A^N(x*z) \vee A^N(z*y)$. Thus the condition (CCI$_2$) is satisfied. So $A = \bigcap_{k\in K} A_k$ is a crossing cubic QS-ideal of X. □

Definition 3.21. Let X be a nonempty set, let $A_{(\widetilde{\alpha},\alpha)}$ and $B_{(\widetilde{\alpha},\alpha)}$ be two $(\widetilde{\alpha},\alpha)$-crossing cubic sets in X. Then the *intersection* of $A_{(\widetilde{\alpha},\alpha)}$ and $B_{(\widetilde{\alpha},\alpha)}$, denoted by $A_{(\widetilde{\alpha},\alpha)} \cap B_{(\widetilde{\alpha},\alpha)}$, is a $(\widetilde{\alpha},\alpha)$-crossing cubic set in X defined as follows:

$$A_{(\widetilde{\alpha},\alpha)} \cap B_{(\widetilde{\alpha},\alpha)} = (\widetilde{A}_{\widetilde{\alpha}} \cap \widetilde{B}_{\widetilde{\alpha}}, A^N_\alpha \cup B^N_\alpha).$$

The following is an immediate consequence of Lemma 3.13, Proposition 3.20 and Definition 3.21.

Corollary 3.22. *Let $(A_k)_{k\in K}$ be a family of crossing cubic QS-ideals of X indexed by a class K and let $(\widetilde{\alpha},\alpha)$ be a crossing cubic number. Then $\bigcap_{k\in K} A_{(\widetilde{\alpha},\alpha),k}$ is a $(\widetilde{\alpha},\alpha)$-crossing cubic QS-ideal of X.*

4. The image (preimage) of a crossing cubic structure QS-ideal under a homomorphism of QS-algebras

Definition 4.1. Let $(X,*,0)$ and $(Y,*',0')$ be QS-algebras. Then a mapping $f: X \to Y$ is called a *homomorphism*, if $f(x*y) = f(x)*' f(y)$ for any $x,\ y \in X$.

Definition 4.2. Let $X,\ Y$ be two sets, let $f: X \to Y$ be a mapping and let $A = (\widetilde{A}, A^N) \in CCS(X)$, $B = (\widetilde{B}, B^N) \in CCS(Y)$.

(i) The *preimage of B under f*, denoted by $f^{-1}(B) = (f^{-1}(\widetilde{B}), f^{-1}(B^N))$, is a crossing cubic set in X defined as follows: for each $x \in X$,

$$f^{-1}(B)(x) = (\widetilde{B} \circ f)(x), (B^N \circ f)(x)) = ([B^-(f(x)), B^+(f(x)), B^N(f(x))).$$

(ii) The *image of A under f*, denoted by $f(A) = (f(\widetilde{A}), f(A^N))$, is a crossing cubic set in Y defined as follows: for each $y \in Y$,

$$f(\widetilde{A})(y) = \begin{cases} [\bigvee_{x\in f^{-1}(y)} A^-(x), \bigvee_{x\in f^{-1}(y)} A^+(x)] & \text{if } f^{-1}(y) \neq \phi \\ \widetilde{0} & \text{otherwise,} \end{cases}$$

$$f(A^N)(y) = \begin{cases} \bigwedge_{x\in f^{-1}(y)} A^N(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise,} \end{cases}$$

Proposition 4.3. *Let $f: X \to Y$ is a homomorphism of QS-algebras. if $B = (\widetilde{B}, B^N)$ is a crossing cubic QS-ideal of Y, then $f^{-1}(B)$ is a crossing cubic QS-ideal of X.*

Proof. Let $x \in X$. Then we get
$f^{-1}(\widetilde{B})(x) = \widetilde{B}(f(x))$
$\leq \widetilde{B}(0)$ [Since B is a crossing cubic QS-ideal of Y]
$= \widetilde{B}(f(0))$ [Since f is a homomorphism]

$$= f^{-1}(\widetilde{B})(0),$$

$$\begin{aligned} f^{-1}(B^N)(x) &= B^N(f(x)) \\ &\geq B^N(0) \\ &= B^N(f(0)) \\ &= f^{-1}(B^N)(0). \end{aligned}$$

Thus the condition (CCI$_1$) holds.

Now let $x,\ y,\ z \in X$. Then we get

$$\begin{aligned} f^{-1}(\widetilde{B})(x*y) &= \widetilde{B}(f(x*y)) \\ &= \widetilde{B}(f(x)*f(y)) \text{ [Since } f \text{ is a homomorphism]} \\ &\geq \widetilde{B}(f(x)*f(z)) \wedge \widetilde{B}(f(z)*f(y)) \\ &\quad \text{[Since } B \text{ is a crossing cubic } QS\text{-ideal of } Y] \\ &= \widetilde{B}(f(x*z)) \wedge \widetilde{B}(f(z*y)) \text{ [Since } f \text{ is a homomorphism]} \\ &= f^{-1}(\widetilde{B})(x*z) \wedge f^{-1}(\widetilde{B})(z*y), \end{aligned}$$

$$\begin{aligned} f^{-1}(B^N)(x*y) &= B^N(f(x*y)) \\ &= B^N(f(x)*f(y)) \\ &\leq B^N(f(x)*f(z)) \vee B^N(f(z)*f(y)) \\ &= B^N(f(x*z)) \vee B^N(f(z*y)) \\ &= f^{-1}(B^N)(x*z) \wedge f^{-1}(B^N)(z*y). \end{aligned}$$

Thus the condition (CCI$_2$) holds. So $f^{-1}(B)$ is a crossing cubic QS-ideal of X. □

Corollary 4.4. *Let $f : X \to Y$ is a homomorphism of QS-algebras and let a $(\widetilde{\alpha}, \alpha)$ be a crossing cubic number. If $B_{(\widetilde{\alpha},\alpha)} = (\widetilde{B}_{\widetilde{\alpha}}, A^N_\alpha)$ is an $(\widetilde{\alpha}, \alpha)$-crossing cubic QS-ideal of Y, then $f^{-1}(B_{(\widetilde{\epsilon},\alpha)})$ is a $(\widetilde{\alpha}, \alpha)$-crossing cubic QS-ideal of X.*

Proof. The proof is similar to Proposition 4.3. However, we prove it directly.

Let $x \in X$. Then we get

$$\begin{aligned} f^{-1}(\widetilde{B}_{\widetilde{\alpha}})(x) &= \widetilde{B}_{\widetilde{\alpha}}(f(x)) \\ &\leq \widetilde{B}_{\widetilde{\alpha}}(0) \text{ [By Lemma 3.13 (1)]} \\ &= \widetilde{B}_{\widetilde{\alpha}}(f(0)) \text{ [Since } f \text{ is a homomorphism]} \\ &= f^{-1}(\widetilde{B}_{\widetilde{\alpha}})(0), \end{aligned}$$

$$\begin{aligned} f^{-1}(B^N_\alpha)(x) &= B^N_\alpha(f(x)) \\ &\geq B^N_\alpha(0) \\ &= B^N_\alpha(f(0)) \\ &= f^{-1}(B^N_\alpha)(0). \end{aligned}$$

Thus the condition (CCI$_1$) holds.

Now let $x,\ y,\ z \in X$. Then we have

$$\begin{aligned} f^{-1}(\widetilde{B}_{\widetilde{\alpha}})(x*y) &= \widetilde{B}_{\widetilde{\alpha}}(f(x*y)) \\ &= \widetilde{B}_{\widetilde{\alpha}}(f(x)*f(y)) \text{ [Since } f \text{ is a homomorphism]} \\ &\geq \widetilde{B}_{\widetilde{\alpha}}(f(x)*f(x)) \wedge \widetilde{B}_{\widetilde{\alpha}}(f(z)*f(y)) \\ &\quad \text{[By Lemma 3.13 (2)]} \\ &= \widetilde{B}_{\widetilde{\alpha}}(f(x*z)) \wedge \widetilde{B}_{\widetilde{\alpha}}(f(z*y)) \\ &\quad \text{[Since } f \text{ is a homomorphism]} \end{aligned}$$

$$= f^{-1}(\widetilde{B}_{\widetilde{\alpha}})(x * z) \wedge f^{-1}(\widetilde{B}_{\widetilde{\alpha}})(z * y),$$

$$\begin{aligned}
f^{-1}(B^N_\alpha)(x * y) &= B^N_\alpha(f(x * y)) \\
&= B^N_\alpha(f(x) * f(y)) \\
&\leq B^N_\alpha(f(x) * f(x)) \vee B^N_\alpha(f(z) * f(y)) \\
&\quad \text{[By Lemma 3.13 (3)]} \\
&= B^N_\alpha(f(x * z)) \vee B^N_\alpha(f(z * y)) \\
&= f^{-1}(B^N_\alpha)(x * z) \vee f^{-1}(B^N_\alpha)(z * y).
\end{aligned}$$

Thus the condition (CCI_2) holds. So $f^{-1}(B_{(\widetilde{\alpha},\alpha)})$ is a $(\widetilde{\alpha},\alpha)$-crossing cubic QS-ideal of X. □

Proposition 4.5. *Let $f : X \to Y$ is an epimorphism of QS-algebras and let $B = (\widetilde{B}, B^N) \in CCS(Y)$. If $f^{-1}(B)$ is a crossing cubic QS-ideal of X, then B is a crossing cubic QS-ideal of Y.*

Proof. Let $a \in Y$. Since f is surjective, there is $x \in X$ such that $f(x) = a$. Then we have

$$\begin{aligned}
\widetilde{B}(a) &= \widetilde{B}(f(x)) \\
&= f^{-1}(\widetilde{B})(x) \\
&\leq f^{-1}(\widetilde{B})(0) \text{ [Since } f^{-1}(B) \text{ is a crossing cubic } QS\text{-ideal of } X] \\
&= \widetilde{B}(f(0)) \\
&= \widetilde{B}(0) \text{ [Since } f(0) = 0]
\end{aligned}$$

Similarly, we get $B^N(a) \geq B^N(0)$. Thus the condition (CCI_1) holds.

Now let $a,\ b,\ c \in Y$. Then clearly, there are $x,\ y,\ z \in X$ such that

$$f(x) = a,\ f(y) = b,\ f(z) = c.$$

Thus we have

$$\begin{aligned}
\widetilde{B}(a * b) &= \widetilde{B}(f(x) * f(y)) \\
&= \widetilde{B}(f(x * y)) \text{ [Since } f \text{ is a homomorphism]} \\
&= f^{-1}(\widetilde{B})(x * y) \\
&\geq f^{-1}(\widetilde{B})(x * z) \wedge f^{-1}(\widetilde{B})(z * y) \\
&\quad \text{[Since } f^{-1}(B) \text{ is a crossing cubic } QS\text{-ideal of } X] \\
&= \widetilde{B}(f(x * z)) \wedge \widetilde{B}(f(z * y)) \\
&= \widetilde{B}(f(x) * f(z)) \wedge \widetilde{B}(f(z) * f(y)) \text{ [Since } f \text{ is a homomorphism]} \\
&= \widetilde{B}(a * c) \wedge \widetilde{B}(c * b).
\end{aligned}$$

Similarly, we get $B^N(a * b) \leq B^N(a * c) \vee B^N(c * b)$. So the condition ($\text{CCI}_2$) holds. Hence B is a crossing cubic QS-ideal of Y. □

Corollary 4.6. *Let $f : X \to Y$ is an epimorphism of QS-algebras, let $(\widetilde{\alpha}, \alpha)$ be a crossing cubic number and let $B_{(\widetilde{\alpha},\alpha)} = (\widetilde{B}_{\widetilde{\alpha}}, A^N_\alpha)$ be an $(\widetilde{\alpha},\alpha)$-crossing cubic set in Y. If $f^{-1}B_{(\widetilde{\alpha},\alpha)})$ is an $(\widetilde{\alpha},\alpha)$-crossing cubic QS-ideal of Y, then $B_{(\widetilde{\alpha},\alpha)}$ is an $(\widetilde{\alpha},\alpha)$-crossing cubic QS-ideal of X.*

Proof. The proof is similar to Proposition 4.5. However, we prove it directly.

Let $a \in Y$. Then clearly, there is $x \in X$ such that $f(x) = a$. Thus we get

$$\begin{aligned}
\widetilde{B}_{\widetilde{\alpha}}(a) &= \widetilde{B}_{\widetilde{\alpha}}(f(x)) \\
&= f^{-1}(\widetilde{B}_{\widetilde{\alpha}})(x)
\end{aligned}$$

$$\begin{aligned}
&\leq f^{-1}(\widetilde{B}_{\widetilde{\alpha}})(0) \text{ [By the hypothesis]}\\
&= \widetilde{B}_{\widetilde{\epsilon}}(f(0))\\
&= \widetilde{B}_{\widetilde{\alpha}}(0). \text{ [Since } f(0)=0]
\end{aligned}$$

Similarly, $B^N_\alpha(a) \geq B^N_\alpha(0)$. So the condition (CCI$_1$) holds.

Now let $a,\ b,\ c \in Y$. Then clearly, there are $x,\ y,\ z \in X$ such that

$$f(x)=a,\ f(y)=b,\ f(z)=c.$$

Thus we have

$$\begin{aligned}
\widetilde{B}_{\widetilde{\alpha}}(a*b) &= \widetilde{B}_{\widetilde{\alpha}}(f(x)*f(y))\\
&= \widetilde{B}_{\widetilde{\alpha}}(f(x*y)) \text{ [Since } f \text{ is a homomorphism]}\\
&= f^{-1}(\widetilde{B}_{\widetilde{\alpha}})(x*y)\\
&\geq f^{-1}(\widetilde{B}_{\widetilde{\alpha}})(x*z) \wedge f^{-1}(\widetilde{B}_{\widetilde{\alpha}})(z*y)\\
&\quad \text{[Since } f^{-1}(B) \text{ is a crossing cubic } QS\text{-ideal of } X]\\
&= \widetilde{B}_{\widetilde{\alpha}}(f(x*z)) \wedge \widetilde{B}_{\widetilde{\alpha}}(f(z*y))\\
&= \widetilde{B}_{\widetilde{\alpha}}(f(x)*f(z)) \wedge \widetilde{B}_{\widetilde{\alpha}}(f(z)*f(y))\\
&\quad \text{[Since } f \text{ is a homomorphism]}\\
&= \widetilde{B}_{\widetilde{\alpha}}(a*c) \wedge \widetilde{B}_{\widetilde{\alpha}}(c*b).
\end{aligned}$$

Similarly, we get $B^N_\alpha(a*b) \leq B^N_\alpha(a*c) \vee B^N_\alpha(c*b)$. So the condition (CCI$_2$) holds. Hence B is a crossing cubic QS-ideal of Y. □

5. The product of crossing cubic QS-ideals

Definition 5.1. Let X be a nonempty set and le Let $A,\ B \in CCS(X)$. Then the *Cartesian product* of A and B, denoted by $A \times B = (\widetilde{A} \times \widetilde{B}, A^N \times B^N)$, is a crossing cubic set in $X \times X$ defined as follows: for each $(x,y) \in X \times X$,

$$(A \times B)(x,y) = (\widetilde{A}(x) \wedge \widetilde{B}(y), A^N(x) \vee B^N(y)).$$

Remark 5.2. Let X and Y be QS-algebras. We define $*$ on $X \times Y$ as follows: for any $(x,y),\ (u,v) \in X \times Y$,

$$(x,y)*(u,v) = (x*u, y*v).$$

Then it is obvious that $(x \times Y, *(0,0))$ is a QS-algebra.

Proposition 5.3. *Let $A,\ B \in CCS(X)$. If A and B are crossing cubic QS-ideals of X, then $A \times B$ is a crossing cubic QS-ideal of $X \times X$.*

Proof. Let $(x,y) \in X \times X$. Then we have

$$\begin{aligned}
(\widetilde{A} \times \widetilde{B})(x,y) &= \widetilde{A}(x) \wedge \widetilde{B}(y)\\
&\leq \widetilde{A}(0) \wedge \widetilde{B}(0) \text{ [By the hypothesis]}\\
&= (\widetilde{A} \times \widetilde{B})(0,0),
\end{aligned}$$

$$\begin{aligned}
(A^N \times B^N)(x,y) &= A^N(x) \vee B^N(y)\\
&\geq A^N(0) \vee B^N(0)\\
&= (A^N \times A^N)(0,0).
\end{aligned}$$

Thus the condition (CCI$_1$) holds.

Now let $(x_1,x_2),\ (y_1,y_2),\ (z_1,z_2) \in X \times X$. Then we get

$$(\widetilde{A} \times \widetilde{B})((x_1,x_2)*(y_1,y_2)) = \widetilde{A}(x_1*y_1) \wedge \widetilde{B}(x_2*y_2)$$

$$\geq (\widetilde{A}(x_1 * z_1) \wedge \widetilde{A}(z_1 * y_1)) \wedge (\widetilde{B}(x_2 * z_2) \wedge \widetilde{B}(z_2 * y_2))$$
$$= (\widetilde{A}(x_1 * z_1) \wedge \widetilde{B}(x_2 * z_2) \wedge (\widetilde{A}(z_1 * y_1) \wedge \widetilde{B}(z_2 * y_2))$$
$$= (\widetilde{A} \times \widetilde{B})(x_1 * z_1, x_2 * z_2) \wedge (\widetilde{A} \times \widetilde{B})(z_1 * y_1, z_2 * y_2))$$
$$= (\widetilde{A} \times \widetilde{B})((x_1, x_2) * (z_1, z_2)) \wedge (\widetilde{A} \times \widetilde{B})((z_1, z_2) * (y_1, y_2)),$$

$$(A^N \times B^N)((x_1, x_2) * (y_1, y_2)) = A^N(x_1 * y_1) \vee B^N(x_2 * y_2)$$
$$\leq (A^N(x_1 * z_1) \vee A^N(z_1 * y_1)) \vee (B^N(x_2 * z_2) \vee B^N(z_2 * y_2))$$
$$= (A^N(x_1 * z_1) \vee B^N(x_2 * z_2) \vee (A^N(z_1 * y_1) \vee B^N(z_2 * y_2))$$
$$= (A^N \times B^N)(x_1 * z_1, x_2 * z_2) \vee (A^N \times B^N)(z_1 * y_1, z_2 * y_2))$$
$$= (A^N \times B^N)((x_1, x_2) * (z_1, z_2)) \vee (A^N \times B^N)((z_1, z_2) * (y_1, y_2)).$$

Thus the condition (CCI_2) holds. So $A \times B$ is a crossing cubic QS-ideal of $X \times X$. □

The Cartesian product of two $(\widetilde{\alpha}, \alpha)$-crossing cubic sets can be defined similarly to Definition 5.1.

Definition 5.4. Let $A_{(\widetilde{\alpha},\alpha)}$ and $B_{(\widetilde{\alpha},\alpha)}$ be two $(\widetilde{\alpha}, \alpha)$-crossing cubic sets in a nonempty set X. Then the *Cartesian product* of $A_{(\widetilde{\alpha},\alpha)}$ and $B_{(\widetilde{\alpha},\alpha)}$, denoted by $A_{(\widetilde{\alpha},\alpha)} \times B_{(\widetilde{\alpha},\alpha)}$, is an $(\widetilde{\alpha}, \alpha)$-crossing cubic set in X defined as follows: for each $(x, y) \in X \times X$,

$$(A_{(\widetilde{\alpha},\alpha)} \times B_{(\widetilde{\alpha},\alpha)})(x, y) = (\widetilde{A}_{\widetilde{\alpha}}(x) \wedge \widetilde{B}_{\widetilde{\alpha}}(y), A^N_\alpha(x) \vee B^N_\alpha(y)).$$

In fact, $A_{(\widetilde{\alpha},\alpha)} \times B_{(\widetilde{\alpha},\alpha)} = (\widetilde{A}_{\widetilde{\alpha}} \times \widetilde{B}_{\widetilde{\alpha}}, A^N_\alpha \times B^N_\alpha)$, where $(\widetilde{A}_{\widetilde{\alpha}} \times \widetilde{B}_{\widetilde{\alpha}}) = \widetilde{A}_{\widetilde{\alpha}}(x) \wedge \widetilde{B}_{\widetilde{\alpha}}(y)$ and $(A^N_\alpha \times B^N_\alpha)(x, y) = A^N_\alpha(x) \vee B^N_\alpha(y))$ for each $(x, y) \in X \times X$.

The following can be thought of as a corollary of Proposition 5.3, so the proof is similar to the proof of property 5.3. However We prove it directly here.

Proposition 5.5. *Let $A_{(\widetilde{\alpha},\alpha)}$ and $B_{(\widetilde{\alpha},\alpha)}$ be two $(\widetilde{\alpha}, \alpha)$-crossing cubic QS-ideals of X. Then $A_{(\widetilde{\alpha},\alpha)} \times B_{(\widetilde{\alpha},\alpha)}$ is an $(\widetilde{\alpha}, \alpha)$-crossing cubic QS-ideal of $X \times X$.*

Proof. Let $(x, y) \in X \times X$. Then we get

$$\begin{aligned}(\widetilde{A}_{\widetilde{\alpha}} \times \widetilde{A}_{\widetilde{\alpha}})(x, y) &= \widetilde{A}_{\widetilde{\alpha}}(x) \wedge \widetilde{B}_{\widetilde{\alpha}}(y) \\ &\leq \widetilde{A}_{\widetilde{\alpha}}(0) \wedge \widetilde{B}_{\widetilde{\alpha}}(0) \text{ [By the hypothesis]} \\ &= (\widetilde{A}_{\widetilde{\alpha}} \times \widetilde{B}_{\widetilde{\alpha}})(0, 0),\end{aligned}$$

$$\begin{aligned}(A^N_\alpha \times B^N_\alpha)(x, y) &= A^N_\alpha(x) \vee B^N_\alpha(y) \\ &\geq A^N_\alpha(0) \vee B^N_\alpha(0) \\ &= (A^N_\alpha \times B^N_\alpha)(0, 0).\end{aligned}$$

Thus the condition (CCI_1) holds.

Now let $(x_1, x_2),\ (y_1, y_2),\ (z_1, z_2) \in X \times X$. Then we get

$$(\widetilde{A}_{\widetilde{\alpha}} \times \widetilde{B}_{\widetilde{\alpha}})((x_1, x_2) * (y_1, y_2)) = \widetilde{A}_{\widetilde{\alpha}}(x_1 * y_1) \wedge \widetilde{B}_{\widetilde{\alpha}}(x_2 * y_2)$$
$$\geq (\widetilde{A}_{\widetilde{\alpha}}(x_1 * z_1) \wedge \widetilde{A}_{\widetilde{\alpha}}(z_1 * y_1)) \wedge (\widetilde{B}_{\widetilde{\alpha}}(x_2 * z_2) \wedge \widetilde{B}_{\widetilde{\alpha}}(z_2 * y_2))$$
$$= (\widetilde{A}_{\widetilde{\alpha}}(x_1 * z_1) \wedge \widetilde{B}_{\widetilde{\alpha}}(x_2 * z_2) \wedge (\widetilde{A}_{\widetilde{\alpha}}(z_1 * y_1) \wedge \widetilde{B}_{\widetilde{\alpha}}(z_2 * y_2))$$
$$= (\widetilde{A}_{\widetilde{\alpha}} \times \widetilde{B}_{\widetilde{\alpha}})(x_1 * z_1, x_2 * z_2) \wedge (\widetilde{A}_{\widetilde{\alpha}} \times \widetilde{B}_{\widetilde{\alpha}})(z_1 * y_1, z_2 * y_2))$$
$$= (\widetilde{A}_{\widetilde{\alpha}} \times \widetilde{B}_{\widetilde{\alpha}})((x_1, x_2) * (z_1, z_2)) \wedge (\widetilde{A}_{\widetilde{\alpha}} \times \widetilde{B}_{\widetilde{\alpha}})((z_1, z_2) * (y_1, y_2)),$$

$$(A^N_\alpha \times B^N_\alpha)((x_1, x_2) * (y_1, y_2)) = A^N_\alpha(x_1 * y_1) \vee B^N_\alpha(x_2 * y_2)$$
$$\leq (A^N_\alpha(x_1 * z_1) \vee A^N_\alpha(z_1 * y_1)) \vee (B^N_\alpha(x_2 * z_2) \vee B^N_\alpha(z_2 * y_2))$$

$$= (A^N_\alpha(x_1 * z_1) \vee B^N_\alpha(x_2 * z_2) \vee (A^N_\alpha(z_1 * y_1) \vee B^N_\alpha(z_2 * y_2))$$
$$= (A^N_\alpha \times B^N_\alpha)(x_1 * z_1, x_2 * z_2) \vee (A^N_\alpha \times B^N_\alpha)(z_1 * y_1, z_2 * y_2))$$
$$= (A^N_\alpha \times B^N_\alpha)((x_1, x_2) * (z_1, z_2)) \vee (A^N_\alpha \times B^N_\alpha)((z_1, z_2) * (y_1, y_2)).$$

Thus the condition (CCI_2) holds. So $A_{(\widetilde{\alpha},\alpha)} \times B_{(\widetilde{\alpha},\alpha)}$ is an $(\widetilde{\alpha}, \alpha)$-crossing cubic QS-ideal of $X \times X$. □

6. Conclusions and future works

As generalization extension of bipolar-valued fuzzy sets, we have studied the $(\widetilde{\alpha}, \alpha)$-crossing cubic structure QS-ideal in QS-algebras. Also we discussed few results of the $(\widetilde{\alpha}, \alpha)$-crossing cubic structure QS-ideal in QS-algebras. The image and the pre-images of the $(\widetilde{\alpha}, \alpha)$-crossing cubic structure QS-ideal in QS-algebras under homomorphism are defined and how the image and the pre-image of the $(\widetilde{\alpha}, \alpha)$-crossing cubic structure QS-ideal in QS-algebras become the crossing cubic structure QS-ideal are studied. Moreover, the product of the $(\widetilde{\alpha}, \alpha)$-crossing cubic structure QS-ideal is established. In the future, we will focus on crossing cubic structure soft set and applications in artificial intelligence and general systems. this method is very effective for data analysis and one may apply for medical diagnosis.

7. Some algorithms

Algorithm for QS-algebras

Input (X: set with 0 element, $*$: Binary operation)
Output ("X is a QS-algebra or no")
If $X = \varnothing$ then;
Go to (1)
End if
If $0 \notin X$, then go to (1);
End if
Stop: = false
$i = 1$;
While $i \leq | X |$ and not (Stop) do
Stop: = true
End if
$j = 1$;
While $j \leq | X |$ and not (Stop) do
$k = 1$;
While $k \leq | X |$ and not (stop) do
If $(x * y) * (x * z) \neq z * y$, then
Stop: = true
End if
End while
End if
End while
If stop then
Output "X is a QS-algebra")
Else
(1) Output ("X is not a QS-algebra")

End if
End.

Algorithm for QS-ideals in a QS-algebra
Input (X: QS-algebra, J: subset of X)
Output ("J is a QS-ideal of X or not")
If $J = \varnothing$, then
Go to (1);
End if
If $0 \notin J$, then
Go to (1);
End if
Stop: = false
$i = 1$;
While $i \leq \mid X \mid$ and not (stop) do
$j = 1$ While $j \leq \mid X \mid$ and not (stop) do
$k = 1$
While $k \leq \mid X \mid$ and not (stop) do
If $x_i * x_k \in J$ and $x_k * x_j \in J$, then
If $x_i * x_j \notin J$, then
Stop: = false
End if
End while
End while
End while
If stop then
Output ("J is a QS-ideal of X")
Else
(1) Output ("J is not ("J is a QS-ideal of X")
End if
End.

Algorithm for fuzzy subsets
Input (X : QS-algebra, $A : X \to I$);
Output (" A is fuzzy subset of X or not")
Begin
Stop: =false;
$i = 1$;
While $i \leq \mid X \mid$ and not (Stop) do
If $A(x_i < 0$ or $A(x_i > 1$, then
Stop: = true;
End If
End While
If Stop then
Output ("A is an fuzzy subset of X")
Else

Output (" A is not an anti fuzzy subset of X")
End If
End

Algorithm for QS-ideals in a QS-algebra
Input (X : QS-algebra, J :subset of X); Output ("J is an QS-ideal of X or not");
Begin
If $J \neq \varnothing$, then go to (1);
End If
If $0 \notin J$, then go to (1);
End If
Stop: =false;
$i = 1$
While $i \leq\mid X \mid$ and not (Stop) do
$j = 1$
While $j \leq\mid X \mid$ and not (Stop) do
$j \leq\mid X \mid$
$k = 1$
While $k \leq\mid X \mid$ and not (Stop) do
If $x_i,\ x_j \in J$, then
If $x_i,\ x_j \notin J$, then
Stop: = true;
End If
End If
End While
End While
End While
If Stop then
Output ("I is is QS-ideal of X")
Else
(1) Output ("J is not is QS-ideal of X")
End If
End.

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