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Abstract. We define the concept of rough limit set of a triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of fuzzy numbers and obtain the relation between the set of rough limit and the extreme limit points of a triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of fuzzy numbers. Finally, we investigate some properties of the rough limit set of Bernstein Stancu polynomials.

2020 AMS Classification: 40F05, 40J05, 40G05, 46S40

Keywords: Chlodowsky type (λ, q)-triple sequences, rough convergence, closed and convex, cluster points and rough limit points, fuzzy numbers, Bernstein Stancu polynomials.

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1. Introduction

The idea of rough convergence was first introduced by Phu [13, 14, 15] in finite dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r.

Aytar [1] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [2] studied that the r-limit set of the sequence is equal to intersection of these sets and that r-core of the sequence is equal to the union of these sets. Dundar and Cakan [12] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence The notion of I-convergence of a triple sequence spaces which is based on the structure of the ideal I of subsets of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

Let K be a subset of the set of positive integers $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and let us denote the set $K_{i,k,\ell}=\{(m,n,k)\in K: m\le i, n\le j, k\le \ell\}$. Then the natural density of K is given by

$$\delta(K)=\lim_{i,j,\ell\to\infty}\frac{|K_{i,j,\ell}|}{ij\ell},$$

where $|K_{i,j,\ell}|$ denotes the number of elements in $K_{i,j,\ell}$.

First applied the concept of (p,q)-calculus in approximation theory and introduced the (p,q)-analogue of Bernstein operators. Later, based on (p,q)-integers, some approximation results for Bernstein-Stancu operators, Bernstein-Kantorovich operators, (p,q)-Lorentz operators, Bleimann-Butzer and Hahn operators and Bernstein-Shurer operators etc.

Very recently, Khalid et al. have given a nice application in computer-aided geometric design and applied these Bernstein basis for construction of (p,q)-Bezier curves and surfaces based on (p,q)-integers which is further generalization of q-Bezier curves and surfaces.

Motivated by the above mentioned work on (p,q)-approximation and its application, in this paper we study statistical approximation properties of Bernstein-Stancu Operators based on (p,q)-integers.

Now we recall some basic definitions about (p,q)-integers. For any $(u,v,w)\in\mathbb{N}^3$, the (p,q)-integer $[uvw]_{p,q}$ is defined by

$$[0]_{p,q}:=0 \text{ and } [uvw]_{p,q}=\frac{p^{uvw}-q^{uvw}}{p-q} \text{ if } u,v,w\ge 1,$$

where $0<q<p\le 1$. The (p,q)-factorial is defined by

$$[0]_{p,q}!:=1 \text{ and } [uvw]!_{p,q}=[1]_{p,q}[2]_{p,q}\cdots[uvw]_{p,q} \text{ if } u,v,w\ge 1 \text{ and } u,v,w,m,n,k\in\mathbb{N}.$$

Also the (p,q)-binomial coefficient is defined by

$$\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q}=\frac{[u]!_{p,q}}{[m]!_{p,q}\,[u-m]!_{p,q}}\;\frac{[v]!_{p,q}}{[n]!_{p,q}\,[v-n]!_{p,q}}\;\frac{[w]!_{p,q}}{[k]!_{p,q}\,[w-k]!_{p,q}}$$

for all $u,v,w,m,n,k\in\mathbb{N}$ with $u\ge m, v\ge n, w\ge k$.

The formula for (p,q)-binomial expansion is as follows:

$$\begin{aligned}(ax+by)^{u,v,w}_{p,q}=&\sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}p^{\frac{(u-m)(u-m-1)+(v-n)(v-n-1)+(w-k)(w-k-1)}{2}}q^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}\\&\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q}a^{(u-m)+(v-n)+(w-k)}b^{m+n+k}x^{(u-m)+(v-n)+(w-k)}y^{m+n+k},\end{aligned}$$

$$(x+y)^{u,v,w}_{p,q}=(x+y)(px+qy)\left(p^2x+q^2y\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)}x+q^{(u-1)+(v-1)+(w-1)}y\right),$$

$$(1-x)^{u,v,w}_{p,q}=(1-x)\,(p-qx)\,\left(p^2-q^2x\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)}-q^{(u-1)+(v-1)+(w-1)}x\right),$$

and

$$(x)^{m,n,k}_{p,q}=x\,(px)\,\left(p^2x\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)}x\right)=p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}.$$

The Bernstein operator of order rst is given by

$$B_{r,s,t}\left(f,x\right)=\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k}x^{m+n+k}\left(1-x\right)^{(m-r)+(n-s)+(k-t)}$$

where f is a continuous (real or complex valued) function defined on $[0,1]$.

The (p,q)-Bernstein operators are defined as follows:

$$\begin{aligned} B_{r,s,t,p,q}(f,x)=&\frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{2}}}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k}p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}x^{m+n+k}\\ &\prod_{u_1=0}^{(r-m-1)}\left(p^{u_1}-q^{u_1}x\right)\prod_{u_2=0}^{(s-n-1)}\left(p^{u_2}-q^{u_2}x\right)\prod_{u_3=0}^{(t-k-1)}\left(p^{u_3}-q^{u_3}x\right)\\ &f\left(\frac{[m]_{p,q}[n]_{p,q}[k]_{p,q}}{p^{(m-r)+(n-s)+(k-t)}[r]_{p,q}[s]_{p,q}[t]_{p,q}+\mu}\right),x\in[0,1] \end{aligned}\tag{1.1}$$

Also, we have

$$\begin{aligned}(1-x)_{p,q}^{r,s,t}=&\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}(-1)^{m+n+k}p^{\frac{(r-m)(r-m-1)+(s-n)(s-n-1)+(t-k)(t-k-1)}{6}}q^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}}\\ &\binom{r}{m}\binom{s}{n}\binom{t}{k}x^{m+n+k}.\end{aligned}$$

(p,q)-Bernstein-Stancu operators are defined as follows:

$$\begin{aligned} S_{r,s,t,p,q}\left(f,x\right)=&\frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{6}}}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k}p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}x^{m+n+k}\\ &\prod_{u_1=0}^{(r-m-1)}\left(p^{u_1}-q^{u_1}x\right)\prod_{u_2=0}^{(s-n-1)}\left(p^{u_2}-q^{u_2}x\right)\prod_{u_3=0}^{(t-k-1)}\left(p^{u_3}-q^{u_3}x\right)\\ &f\left(\frac{p^{(r-m)+(s-n)+(t-k)}[m]_{p,q}[n]_{p,q}[k]_{p,q}+\eta}{[r]_{p,q}[s]_{p,q}[t]_{p,q}+\mu}\right),x\in[0,1] \end{aligned}\tag{1.2}$$

Note that for $\eta=\mu=0$, (p,q)-Bernstein-Stancu operators given by (1.2) reduces into (p,q)-Bernstein-Stancu operators. Also for $p=1$, (p,q)-Bernstein-Stancu operators given by (1.1) turn out to be q-Bernstein-Stancu operators.

In this paper, we construct Chlodowsky type (λ,q)-Bernstein-Stancu operators of triple sequence space is defined as

$$B_{(r,s,t),\lambda,q}^{\alpha,\beta}\left(f;x\right)=\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\widehat{b}_{r,s,t,m,n,k}\left(x;q\right)f\left(\frac{[mnk]_q+\alpha}{[rst]_q+\beta}b_{r,s,t}\right),\tag{1.3}$$

where $r,s,t\in\mathbb{N}$, $0<q\leq 1, 0\leq x\leq b_{r,s,t}$ and $b_{r,s,t}$ is a sequence of positive numbers such that $\lim_{r,s,t\to\infty}b_{r,s,t}=\infty$, $\lim_{r,s,t\to\infty}\frac{b_{r,s,t}}{[rst]_q}=0$,

$$\widehat{b}_{r,s,t,m,n,k}\left(x;q\right)=\binom{r}{m}\binom{s}{n}\binom{t}{k}\left(\frac{x}{b_{r,s,t}}\right)^{m+n+k}\left(1-\frac{x}{b_{r,s,t}}\right)^{(r-m)+(s-n)+(t-k)}$$

and $\alpha, \beta \in \mathbb{R}$ and $0 \leq \alpha \leq \beta$. For $\alpha = \beta = 0$ we obtain the Chlodowsky type (λ, q)-Bernstein Stancu polynomials.

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu polynomials $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f; x)\right)$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st - \lim x = 0$, provided that the set

$$K_\epsilon := \left\{(m, n, k) \in \mathbb{N}^3 : \left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f; x) - (f, x)\right| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu polynomials. i.e., $\delta(K_\epsilon) = 0$. That is,

$$\lim_{r,s,t\to\infty} \frac{1}{pqj} \left|\left\{m \leq p, n \leq q, k \leq j : \left|B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f; x) - (f, x)\right| \geq \epsilon\right\}\right| = 0.$$

In this case, we write $\delta - \lim B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f; x) = (f, x)$ or $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f; x) \to^{S_S} (f, x)$.

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Sahiner et al.* [16, 17], *Esi et al.* [3, 4, 5, 6, 7, 8, 9], *Dutta et al.* [10], *Subramanian et al.* [18], *Debnath et al.* [11] and many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$\sup_{m,n,k} |x_{m,n,k}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3.

The set of fuzzy real numbers is denoted by $(f, X)(\mathbb{R})$, and d denotes the supremum metric on $(f, X)\left(\mathbb{R}^3\right)$. Now let r be nonnegative real number. A triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f; X)\right)$ of fuzzy numbers is r-convergent to a fuzzy number (f, X) and we write

$$B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f; X) \to^r (f, X)\ ,$$

provided that for every $\epsilon > 0$ there is a positive integer $m_\epsilon, n_\epsilon, k_\epsilon$ so that

$$d\left(\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f; X)\right), (f, X)\right) < r + \epsilon \text{ whenever } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon.$$

The set

$$\mathrm{LIM}^r\, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f; X) \to^r (f, X) := \Big\{(f, X) \in (f, X)\left(\mathbb{R}^3\right) : B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f; X) \to^r (f, X),\ \text{as } m, n, k \to \infty\Big\}$$

is called the r-limit set of the triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f; X) \to^r (f, X)\right)$.

A triple sequence space of Chlodowsky type (λ, q)-Bernstein-Stancu polynomials of fuzzy numbers which is divergent can be convergent with a certain roughness degree. For instance, let us define

$$B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \rightarrow^r (f,X) = \begin{cases} \eta(X), & \text{if } m,n,k \text{ are odd integers} \\ \mu(X), & \text{otherwise} \end{cases},$$

where

$$\eta(X) = \begin{cases} X, & \text{if } X \in [0,1], \\ -X+2, & \text{if } X \in [1,2], \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mu(X) = \begin{cases} X-3, & \text{if } X \in [3,4], \\ -X+5, & \text{if } X \in [4,5], \\ 0, & \text{otherwise} \end{cases}.$$

Then we have where

$$\mathrm{LIM}^r\, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \rightarrow^r (f,X) = \begin{cases} \phi, & \text{if } r < \frac{3}{2}, \\ [\mu - r_1, \eta + r_1], & \text{otherwise} \end{cases},$$

where r_1 is nonnegative real number with

$$[\mu - r_1, \eta + r_1] := \Big\{ B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \rightarrow^r (f,X) \in (f,X)\left(\mathbb{R}^3\right) : \\ \mu - r_1 \leq B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \rightarrow^r (f,X) \leq \eta + r_1 \Big\}.$$

The ideal of rough convergence of a triple sequence space of Bernstein Stancu polynomials can be interpreted as follows:

Let $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \rightarrow^r (f,Y)\right)$ be a convergent triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of fuzzy numbers. Assume that $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \rightarrow^r (f,Y)\right)$ cannot be determined exactly for every $(m,n,k) \in \mathbb{N}^3$. That is, $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \rightarrow^r (f,Y)\right)$ cannot be calculated so we can use approximate value of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \rightarrow^r (f,Y)\right)$ for simplicity of calculation. We only know that $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \rightarrow^r (f,Y)\right) \in [\mu_{m,n,k}, \lambda_{m,n,k}]$, where $d(\mu_{m,n,k}, \lambda_{m,n,k}) \leq r$ for every $(m,n,k) \in \mathbb{N}^3$. The triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \rightarrow^r (f,X)\right)$ satisfying $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \rightarrow^r (f,X)\right) \in [\mu_{m,n,k}, \lambda_{m,n,k}]$, for all m,n,k. Then the triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of

$\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \to^{r} (f,X)\right)$ may not be convergent, but the inequality

$$\begin{aligned} & d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \to^{r} (f,X),(f,X)\right) \\ \leq & d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \to^{r} (f,X), B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \to^{r} (f,Y)\right) \\ & + d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \to^{r} (f,Y),(f,Y)\right) \\ \leq & r + d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \to^{r} (f,YX),(f,Y)\right) \end{aligned}$$

implies that the triple sequence space of Bernstein Stancu polynomials of $(S_{r,s,t,p,q}(f,X))$ is r-convergent.

In this paper, we first define the concept of rough convergence of a triple sequence space of Chlodowsky type (λ,q)-Bernstein Stancu polynomials of fuzzy numbers. Also obtain the relation between the set of rough limit and the extreme limit points of a triple sequence space of Chlodowsky type (λ,q)-Bernstein Stancu polynomials of fuzzy numbers. We show that the rough limit set of a triple sequence space of Bernstein Stancu polynomials is closed, bounded and convex.

2. Preliminaries

A fuzzy number X is a fuzzy subset of the real $\mathbb{R}^3$, which is normal fuzzy convex, upper semi-continuous, and the X^0 is bounded where $X^0; = \mathrm{cl}\left\{x \in \mathbb{R}^3 : X(x) > 0\right\}$ and cl is the closure operator. These properties imply that for each $\alpha \in (0,1]$, the α-level set X^α defined by

$$X^\alpha = \left\{x \in \mathbb{R}^3 : X(x) \geq \alpha\right\} = \left[\underline{X}^\alpha, \overline{X}^\alpha\right]$$

is a non empty compact convex subset of $\mathbb{R}^3$.

The supremum metric d on the set $L\left(\mathbb{R}^3\right)$ is defined by

$$d(X,Y) = \sup_{\alpha \in [0,1]} \max\left(\left|\underline{X}^\alpha - \underline{Y}^\alpha\right|, \left|\overline{X}^\alpha - \overline{Y}^\alpha\right|\right).$$

Now, given $X, Y \in L\left(\mathbb{R}^3\right)$, we define $X \leq Y$ if $\underline{X}^\alpha \leq \underline{Y}^\alpha$ and $\overline{X}^\alpha \leq \overline{Y}^\alpha$ for each $\alpha \in [0,1]$. We write $X \leq Y$ if $X \leq Y$ and there exists an $\alpha_0 \in [0,1]$ such that $\underline{X}^{\alpha_0} \leq \underline{Y}^{\alpha_0}$ or $\overline{X}^{\alpha_0} \leq \overline{Y}^{\alpha_0}$.

A subset E of $L\left(\mathbb{R}^3\right)$ is said to be bounded above if there exists a fuzzy number μ, called an upper bound of E, such that $X \leq \mu$ for every $X \in E$. μ is called the least upper bound of E if μ is an upper bound and $\mu \leq \mu^{'}$ for all upper bounds $\mu^{'}$.

A lower bound and the greatest lower bound are defined similarly. E is said to be bounded if it is both bounded above and below.

The notions of least upper bound and the greatest lower bound have been defined only for bounded sets of fuzzy numbers. If the set $E \subset L\left(\mathbb{R}^3\right)$ is bounded then its supremum and infimum exist.

The limit infimum and limit supremum of a triple sequence spaces $(X_{m,n,k})$ is defined by

$$\lim_{m,n,k\to\infty} \inf X_{m,n,k} := \inf A_X.$$
$$\lim_{m,n,k\to\infty} \sup X_{m,n,k} := \inf B_X.$$

where

$$A_X := \left\{\mu \in L\left(\mathbb{R}^3\right) : \text{The set } \left\{(m,n,k) \in \mathbb{N}^3 : X_{m,n,k} < \mu\right\} \text{ is infinite}\right\}$$
$$B_X := \left\{\mu \in L\left(\mathbb{R}^3\right) : \text{The set } \left\{(m,n,k) \in \mathbb{N}^3 : X_{m,n,k} > \mu\right\} \text{ is infinite}\right\}.$$

Now, given two fuzzy numbers $X, Y \in L\left(\mathbb{R}^3\right)$, we define their sum as $Z = X + Y$, where $\underline{Z}^\alpha := \underline{X}^\alpha + \underline{Y}^\alpha$ and $\overline{Z}^\alpha := \overline{X}^\alpha + \overline{Y}^\alpha$ for all $\alpha \in [0,1]$.

To any real number $a \in \mathbb{R}^3$, we can assign a fuzzy number $a_1 \in L\left(\mathbb{R}^3\right)$, which is defined by

$$a_1(x) = \begin{cases} 1, & \text{if } x = a, \\ 0, & \text{otherwise} \end{cases}.$$

An order interval in $L\left(\mathbb{R}^3\right)$ is defined by $[X,Y] := \left\{Z \in L\left(\mathbb{R}^3\right) : X \le Z \le Y\right\}$, where $X, Y \in L\left(\mathbb{R}^3\right)$. A set E of fuzzy numbers is called convex if $\lambda\mu_1 + (1-\lambda)\mu_2 \in E$ for all $\lambda \in [0,1]$ and $\mu_1, \mu_2 \in E$.

3. Main Results

Theorem 3.1. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Chlodowsky type (λ,q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)$ of real numbers. If $(f,X) \in \mathrm{LIM}^r\, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)$, then*

$$\operatorname{diam}\left(\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x), (f,X)\right) \le r$$

and

$$\operatorname{diam}\left(\liminf B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x), (f,X)\right) \le r.$$

Proof. We assume that $\operatorname{diam}\left(\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;x), (f,X)\right) > r$. Define $\tilde{\epsilon} := \frac{\left(\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X),(f,X)\right)-r}{2}$. By definition of limit supremum, we have that given $\left(m^{'}_{\tilde{\epsilon}}, n^{'}_{\tilde{\epsilon}}, k^{'}_{\tilde{\epsilon}}\right) \in \mathbb{N}^3$ there exists an $(m,n,k) \in \mathbb{N}^3$ with $m \ge m^{'}_{\tilde{\epsilon}}, n \ge n^{'}_{\tilde{\epsilon}}, k \ge k^{'}_{\tilde{\epsilon}}$ such that $\operatorname{diam}\left(\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X), (f,X)\right) \le \tilde{\epsilon}$. Also, since $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \to^r (f,X)$ as $m,n,k \to \infty$, there are positive integers $m^{''}_{\tilde{\epsilon}}, n^{''}_{\tilde{\epsilon}}, k^{''}_{\tilde{\epsilon}}$ so that

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X), (f,X)\right) < r + \tilde{\epsilon}$$

whenever $m \geq m''_{\tilde{\epsilon}}, n \geq n''_{\tilde{\epsilon}}, k \geq k''_{\tilde{\epsilon}}$. Let

$$m_{\tilde{\epsilon}} := \max\left\{m'_{\tilde{\epsilon}}, m''_{\tilde{\epsilon}}\right\}.$$
$$n_{\tilde{\epsilon}} := \max\left\{n'_{\tilde{\epsilon}}, n''_{\tilde{\epsilon}}\right\}.$$
$$k_{\tilde{\epsilon}} := \max\left\{k'_{\tilde{\epsilon}}, k''_{\tilde{\epsilon}}\right\}.$$

There exists $(m,n,k) \in \mathbb{N}^3$ such that $m \geq m_{\tilde{\epsilon}}, n \geq n_{\tilde{\epsilon}}, k \geq k_{\tilde{\epsilon}}$ and

$$\begin{aligned}
&\operatorname{diam}\left(\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X),(f,X)\right)\\
\leq &(f,X)\operatorname{diam}\left(\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X), B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)\\
&+\operatorname{diam}\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)\\
<&\tilde{\epsilon}+r+\tilde{\epsilon}\\
<&r+2\tilde{\epsilon}\\
=&r+\operatorname{diam}\left(\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X),(f,X)\right)-r\\
=&\operatorname{diam}\left(\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X),(f,X)\right).
\end{aligned}$$

The contradiction proves the theorem. Similarly,

$$\operatorname{diam}\left(\liminf B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X),(f,X)\right) \leq r$$

can be proved using definition of limit infimum. □

Theorem 3.2. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Chlodowsky type (λ,q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)$ of real numbers. If $\operatorname{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \neq \phi$, then we have*

$$\operatorname{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \subseteq \left[\left(\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)-r_1, \left(\liminf B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)+r_1\right].$$

Proof. To prove that

$$(f,X) \in \left[\left(\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)-r_1, \left(\liminf B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)+r_1\right]$$

for an arbitrary $(f,X) \in \operatorname{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)$, i.e.,

$$\left(\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)-r_1 \leq (f,X) \leq \left(\liminf B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)+r_1.$$

Let us assume that $\left(\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)-r_1 \leq (f,X)$ does not hold. Thus, there exists an $\alpha_0 \in [0,1]$ such that

$$\left(\underline{\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)}^{\alpha_0}\right)-r_1 > \underline{(f,X)}^{\alpha_0}$$

or

$$\left(\overline{\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)}^{\alpha_0}\right)-r_1 > \overline{(f,X)}^{\alpha_0}$$

holds, i.e.,

$$\left(\underline{\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)}^{\alpha_0}\right) - \underline{(f,X)}^{\alpha_0} > r_1$$

or

$$\left(\overline{\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)}^{\alpha_0}\right) - \overline{(f,X)}^{\alpha_0} > r_1.$$

On the other hand, by theorem 3.1 we have

$$\left|\left(\underline{\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)}^{\alpha_0}\right) - \underline{(f,X)}^{\alpha_0}\right| \leq r_1$$

and

$$\left|\left(\overline{\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)}^{\alpha_0}\right) - \overline{(f,X)}^{\alpha_0}\right| \leq r_1.$$

We obtain a contradiction. Hence we get $\left(\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right) - r_1 \leq (f,X)$. By using the similar arguments and get it for second part. ☐

Note 3.3. *The converse inclusion in this theorem holds for f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Chlodowsky type (λ,q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)$ of real numbers, but it may not hold for triple sequences of Chlodowsky type (λ,q)-Bernstein Stancu polynomials of fuzzy numbers as in the following example:*

Example 3.4. Define

$$B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) = \begin{cases} \frac{-1}{2(mnk)}X+1, & \text{if } X \in [0,1], \\ 0, & \text{otherwise} \end{cases}$$

and

$$(f,X) = \begin{cases} 1, & \text{if } X \in [0,1], \\ 0, & \text{otherwise} \end{cases}.$$

Then we have $\left|\overline{(f,X)}^{1} - \overline{B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)}^{1}\right| = |1-0| = 1$, i.e., $d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X),(f,X)\right) \geq 1$ for all $(m,n,k) \in \mathbb{N}^3$. Although the triple sequence spaces of Chlodowsky type (λ,q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)$ is not convergent to (f,X), $\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)$ and $\liminf B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)$ of this triple sequence space of Chlodowsky type (λ,q)-Bernstein Stancu polynomials are equal to (f,X). Hence we get

$$L \in \left[\limsup B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) - \left(\frac{1}{2}\right)_1, \liminf B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) + \left(\frac{1}{2}\right)_1\right],$$

but $(f,X) \notin \mathrm{LIM}^{\frac{1}{2}} B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)$.

Theorem 3.5. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Chlodowsky type (λ,q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)$ of real numbers converges to the fuzzy number (f,X), then*

$$\mathrm{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) = \bar{S}_r((f,X)) := \left\{\mu \in (f,X)\left(\mathbb{R}^3\right) : d(\mu,(f,X)) \leq r\right\}.$$

Proof. Let $\epsilon > 0$. Since the triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)$ is convergent to (f,X), there are positive integers $m_\epsilon, n_\epsilon, k_\epsilon$ so that

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X),(f,X)\right) < \epsilon \text{ whenever } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon.$$

Let $Y \in \bar{B}_r((f,X))$, we have

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X),Y\right) \leq d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X),(f,X)\right) + d((f,X),Y) < \epsilon + r$$

for every $m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon$.

Hence we have $Y \in \mathrm{LIM}^r\, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)$.

Now let $Y \in \mathrm{LIM}^r\, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)$. Hence there are positive integers $m^{'}_\epsilon, n^{'}_\epsilon, k^{'}_\epsilon$ so that

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X),Y\right) < r + \epsilon$$

whenever $m \geq m^{'}_\epsilon$, $n \geq n^{'}_\epsilon$, $k \geq k^{'}_\epsilon$. Let

$$\left(m^{''}_\epsilon, n^{''}_\epsilon, k^{''}_\epsilon\right) := \max\left\{(m_\epsilon, n_\epsilon, k_\epsilon), \left(m^{'}_\epsilon, n^{'}_\epsilon, k^{'}_\epsilon\right)\right\}$$

for all $m \geq m^{''}_\epsilon$, $n \geq n^{''}_\epsilon$, $k \geq k^{''}_\epsilon$, we obtain

$$\begin{aligned} d(Y, f(X)) &\leq d\left(Y, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right) + d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X),(f,X)\right) \\ &< r + \epsilon + \epsilon < r + 2\epsilon. \end{aligned}$$

Since ϵ is arbitrary, we have $d(Y,(f,X)) \leq r$. Hence we get $Y \in \bar{B}_r((f,X))$. Thus, if the triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right) \to^r (f,X)$, then $\mathrm{LIM}^r\, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) = \bar{B}_r((f,X))$. $\square$

Theorem 3.6. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of real numbers of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)$ and $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;Y)\right) \in (f,X)(\mathbb{R}^3)$. If $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \to^r (f,X)$ then $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;Y) \to^r (f,Y)$ and $d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X), B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;Y)\right) \leq r$ for every $(m,n,k) \in \mathbb{N}^3$.*

Proof. Assume that $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;Y) \to^r (f,Y)$, as $m,n,k \to \infty$ and $d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)), B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;Y)\right) \leq r$ for every $(m,n,k) \in \mathbb{N}^3$. We have $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;Y) \to^r (f,Y)$, as $m,n,k \to \infty$ means that for every $\epsilon > 0$ there exists an $m_\epsilon, n_\epsilon, k_\epsilon$ such that

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;Y),(f,Y)\right) < \epsilon \text{ for all } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon.$$

If the in equality $d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X), B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;Y)\right) \leq r$ yields then

$$\begin{aligned} d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X),(f,X)\right) &\leq d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X), B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;Y)\right) \\ &\quad + d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;Y),(f,Y)\right) \\ &< r+\epsilon \end{aligned}$$

for all $m \geq m_\epsilon$, $n \geq n_\epsilon$, $k \geq k_\epsilon$.

Hence the triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)$ is r-convergent to the fuzzy number (f, X). □

Theorem 3.7. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)$ of real numbers and the diameter of an r-limit set is not greater than $3r$.*

Proof. We have to prove that

$$\sup\left\{d(W,Z) : W,Y,Z \in \mathrm{LIM}^r\, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right\} \leq 3r.$$

Assume on the contrary that

$$\sup\left\{d(W,Z) : W,Y,Z \in \mathrm{LIM}^r\, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right\} > 3r.$$

By this assumption, there exists, $W, Y, Z \in \mathrm{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)$ satisfying $\lambda := d(W,Z) > 3r$. For an arbitrary $\epsilon \in \left(0, \frac{\lambda}{3} - r\right)$, we have

$$\exists\left(m'_\epsilon, n'_\epsilon, k'_\epsilon\right) \in \mathbb{N}^3 : \forall m \geq m'_\epsilon, n \geq n'_\epsilon, k \geq k'_\epsilon \Rightarrow d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X), W\right) \leq r+\epsilon,$$

$$\exists\left(m''_\epsilon, n''_\epsilon, k''_\epsilon\right) \in \mathbb{N}^3 : \forall m \geq m''_\epsilon, n \geq n''_\epsilon, k \geq k''_\epsilon \Rightarrow d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X), Y\right) \leq r+\epsilon,$$

$$\exists\left(m'''_\epsilon, n'''_\epsilon, k'''_\epsilon\right) \in \mathbb{N}^3 : \forall m \geq m'''_\epsilon, n \geq n'''_\epsilon, k \geq k'''_\epsilon \Rightarrow d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X), Z\right) \leq r+\epsilon.$$

Define

$$m_{\tilde{\epsilon}} := \max\left\{m'_{\tilde{\epsilon}}, m''_{\tilde{\epsilon}}\right\}.$$
$$n_{\tilde{\epsilon}} := \max\left\{n'_{\tilde{\epsilon}}, n''_{\tilde{\epsilon}}\right\}.$$
$$k_{\tilde{\epsilon}} := \max\left\{k'_{\tilde{\epsilon}}, k''_{\tilde{\epsilon}}\right\}.$$

Thus we get

$$\begin{aligned} d(W,Z) &\leq d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X), W\right) + d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X), Y\right) + d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X), Z\right) \\ &< (r+\epsilon) + (r+\epsilon) + (r+\epsilon) \\ &< 3(r+\epsilon) \\ &< 3r + 3\left(\frac{\lambda}{3} - r\right) < 3r + \lambda - 3r \\ &= \lambda \text{ for all } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon, \end{aligned}$$

which contradicts to the fact that $\lambda = d(W, Z)$. □

Theorem 3.8. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of $(S_{r,s,t,p,q}(f,X))$ of real numbers is analytic if and only if there exists an $r \geq 0$ such that*
$\mathrm{LIM}^r\, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \neq \phi$.

Proof. (Necessity:) Let the triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)$ be a analytic sequence and

$$s := \sup\left\{d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)^{1/m+n+k}, 0\right) : (m,n,k) \in \mathbb{N}^3\right\} < \infty.$$

Then we have $0 \in \mathrm{LIM}^s\, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)$, i.e., $\mathrm{LIM}^r\, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \neq \phi$, where $r = s$.

(Sufficiency:) If $\mathrm{LIM}^r\, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \neq \phi$ for some $r \geq 0$, then there exists $(f,X) \in \mathrm{LIM}^r\, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)$. By definition, for every $\epsilon > 0$ there are positive integers $m_\epsilon, n_\epsilon, k_\epsilon$ so that

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X), (f,X)\right) < r + \epsilon \text{ whenever } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon.$$

Define

$$t = t(\epsilon) := \max\{d((f,X),0), d(B_{1,1,1,p,q}(f,X),0), \ldots, d(B_{r_\epsilon,s_\epsilon,t_\epsilon,p,q}(f,X),0), r+\epsilon\}.$$

Then we have

$$B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \in \left\{\mu \in (f,X)\left(\mathbb{R}^3\right) : d(\mu,0) \leq t + r + \epsilon\right\} \text{ for every } (m,n,k) \in \mathbb{N}^3,$$

which proves the boundedness of the triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)$. □

Theorem 3.9. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of $(B_{u_m,v_n,w_k,p,q}(f,X))$ of real numbers is a sub sequence of a triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)$, then*
$\mathrm{LIM}^r\, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X) \subset \mathrm{LIM}^r\, B_{u_m,v_n,w_k,p,q}(f,X)$.

Proof. Omitted. □

Theorem 3.10. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)\right)$ of real numbers, for all $r \geq 0$, the r-limit set $\mathrm{LIM}^r\, S_{r,s,t,p,q}(f,X)$ of an arbitrary triple sequence space of Chlodowsky type (λ, q)-Bernstein Stancu polynomials of $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)$ is closed.*

Proof. Let $(Y_{mnk}) \subset \mathrm{LIM}^r B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)$ and $B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;Y) \to (f,Y)$ as $m,n,k \to \infty$. Let $\epsilon > 0$. Since the triple sequence space of Chlodowsky type (λ, q)-Bernstein

Stancu polynomials of $\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;Y)\right) \to^r (f,Y)$, there are positive integers $i_\epsilon, j_\epsilon, \ell_\epsilon$ so that

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;Y),(f,Y)\right) < \frac{\epsilon}{2} \text{ whenever } m \geq i_\epsilon, n \geq j_\epsilon, k \geq \ell_\epsilon.$$

Since $B_{i_\epsilon,j_\epsilon,\ell_\epsilon,p,q}(f,Y) \in \mathrm{LIM}^r\, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)$, there are positive integers $m_\epsilon, n_\epsilon, k_\epsilon$ so that

$$d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X), S_{i_\epsilon,j_\epsilon,\ell_\epsilon,p,q}(f,Y)\right) < r + \frac{\epsilon}{2} \text{ whenever } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon.$$

Therefore, we have

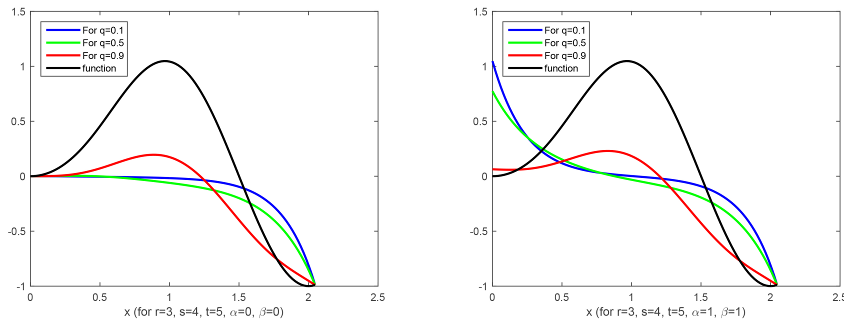
$$\begin{aligned} d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X),(f,X)\right) &\leq d\left(B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X), S_{i_\epsilon,j_\epsilon,\ell_\epsilon,p,q}(f,Y)\right) \\ &< r + \frac{\epsilon}{2} + \frac{\epsilon}{2} = r + \epsilon \end{aligned}$$

for every $m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon$.

Hence $L \in \mathrm{LIM}^r\, B^{\alpha,\beta}_{(r,s,t),\lambda,q}(f;X)$ implies that the set $\mathrm{LIM}^r\, S_{r,s,t,p,q}(f,X)$ is closed. □

Example 3.11. With the help of Matlab, we show comparisons and some illustrative graphics for the convergence of operators (1.3) to the function $f(x) = e^{-\pi x^2} - \sin\left(\pi\left(x + \frac{1}{2}\right)\right)$ under different parameters.

From Figure 1(a), it can be observed that as the value the q approaches towards 1 provided $0 < q \leq 1$, Chlodowsky type (λ, q)-Bernstein-Stancu operators given by (1.3) converges towards the function $f(x) = e^{-\pi x^2} - \sin\left(\pi\left(x + \frac{1}{2}\right)\right)$. From Figure 1(a), it can be observed that for $\alpha = \beta = 0$, as the value the (r,s,t) increases, Chlodowsky type (λ, q)-Bernstein-Stancu operators given by (1.3) converges towards the function. Similarly from Figure 1(b), it can be observed that for $\alpha = \beta = 1$, as the value the q approaches towards 1 or some thing else provided $0 < q \leq 1$, Chlodowsky type (λ, q)-Bernstein-Stancu operators given by (1.3) converges towards the function. From Figure 1(b), it can be observed that as the value the $[r,s,t]$ increases, Chlodowsky type (λ, q)-Bernstein-Stancu operators given by $f(x) = e^{-\pi x^2} - \sin\left(\pi\left(x + \frac{1}{2}\right)\right)$ converges towards the function.



(a) (b)

Figure 1. Chlodowsky type (λ, q)-Bernstein-Stancu operators

4. Conclusions

We introduced triple sequence space of Chlodowsky type (λ, q)-Bernstein-Stancu polynomials of rough convergence of fuzzy numbers. For the reference sections, consider the following introduction described the main results are motivating the research.

Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

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