Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 23, No. 3, (June 2022) pp. 273–284
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2022.23.3.273

@FMI

Generalizations of Zadeh powerset operators in complete co-residuated lattices

Ju-mok Oh, Yong Chan Kim

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 23, No. 3, June 2022

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 23, No. 3, (June 2022) pp. 273–284
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2022.23.3.273

Generalizations of Zadeh powerset operators in complete co-residuated lattices

Ju-mok Oh, Yong Chan Kim

Received 2 February 2022; Revised 16 February 2022; Accepted 8 March 2022

Abstract. In this paper, using distance function based on complete co-residuated lattices, we investigate various operations as extensions of Zadeh powerset operations. We study the Alexandrov L-topologies and L-fuzzy rough sets determined by fuzzy closure and fuzzy interior operators in complete co-residuated lattice. We give their examples.

2020 AMS Classification: 03E72, 54A40, 54B10

Keywords: Complete co-residuated lattices, Distance functions, Fuzzy closure (interior) operators, Alexandrov topologies.

Corresponding Author: Yong Chan Kim (yck@gwnu.ac.kr)

1. Introduction

Alexandrov [1] introduced an Alexandrov topology in which the intersection and union of any family of open sets is open. Given a preordered set $(X, \leq)$, we can define Alexandrov topologies $\tau_{\leq}, \tau_{\leq^{-1}}$ on X by choosing the open sets to be the upper sets:

$$\tau_{\leq} = \{U \subseteq X \mid \forall x, y \in X, (x \in U) \wedge (x \leq y) \to y \in U\},$$

and by choosing the open sets to be the lower sets:

$$\tau_{\leq^{-1}} = \{L \subseteq X \mid \forall x, y \in X, (y \in L) \wedge (x \leq y) \to x \in L\}.$$

Bělohlávek [2, 3, 4] investigated the properties of fuzzy Galois connections and fuzzy closure operators on a residuated lattice which support information systems, decision rules and parts of foundation of theoretic computer science. Many researchers [5, 6, 7, 8, 9] developed fuzzy rough sets, L-lower and L-upper approximation operators in complete residuated lattices.

Pei et al. [10] investigated the Alexandrov L-topology and lattice structures on L-fuzzy rough sets determined by lower and upper sets in complete residuated lattice

$(L, \vee, \wedge, \odot, \rightarrow, \perp, \top)$. Given a fuzzy preordered set (X, e_X), Alexandrov topologies τ_{e_X} and $\tau_{e_X^{-1}}$ on X are defined by choosing the open sets to be the upper sets:

$$\tau_{e_X} = \{A \in L^X \mid \forall x, y \in X, A(x) \odot e_X(x, y) \leq A(y)\},$$

and by choosing the open sets to be the lower sets:

$$\tau_{e_X^{-1}} = \{A \in L^X \mid \forall x, y \in X, A(y) \odot e_X(x, y) \leq A(x)\}.$$

Fang [11], Fang and Yue [12] studied the relationship between L-fuzzy closure systems and L-fuzzy topological spaces from a category viewpoint on a complete residuated lattice L.

As a dual sense of complete residuated lattice, Zheng and Wang [13] introduced a complete co-residuated lattice as a generalization of t-conorm. Junsheng and Qing [14] investigated $(\odot, \&)$-generalized fuzzy rough set on $(L, \odot, \&)$ where $(L, \&)$ is a complete residuated lattice and $(L, \odot)$ is complete coresiduated lattice. Kim and Ko [15] introduced the concepts of fuzzy join and meet complete lattices using distance spaces instead of fuzzy partially ordered spaces in complete co-residuated lattices. Moreover, Oh and Kim [16, 17, 18, 19] investigated the properties of Alexandrov fuzzy topologies, distance functions, join preserving maps, join approximation maps, fuzzy complete lattices, various fuzzy connections and fuzzy concepts using distance functions instead of fuzzy partially orders in complete co-residuated lattices.

For a usual mapping $f : X \to Y$, the image of $f^{\rightarrow} : P(X) \to P(Y)$ and the preimage of $f^{\leftarrow} : P(Y) \to P(X)$ are defined as

$$f^{\rightarrow}(A) = \{f(x) \in Y \mid x \in A\},\ f^{\leftarrow}(B) = \{x \in X \mid f(x) \in B\}.$$

Höhle and Rodabaugh [20] show that $(f^{\rightarrow}, f^{\leftarrow})$ is an adjunction where Zadeh's powersets operators $f^{\rightarrow} : L^X \to L^Y, f^{\leftarrow} : L^Y \to L^X$ are defined as

$$f^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x),\ f^{\rightarrow}(B)(x) = B(f(x)).$$

Our aim in this paper, as extensions of Zadeh's powersets operators from fuzzy sets to fuzzy sets, is to study various operators in Definitions 2.8 and 3.1 from Alexandrov topologies to Alexandrov topologies using distance function based on co-residuated lattices.

Given a distance space (X, d_X) on the complete co-residuated lattice

$$(L, \vee, \wedge, \oplus, \ominus, \perp, \top),$$

we can define Alexandrov topologies $\tau_{d_X}, \tau_{d_X^{-1}}$ on X by

$$\tau_{d_X} = \{A \in L^X \mid \forall x, y \in X, A(x) \oplus d_X(x, y) \geq A(y)\}$$

and

$$\tau_{d_X^{-1}} = \{A \in L^X \mid \forall x, y \in X, A(y) \oplus d_X(x, y) \geq A(x)\}.$$

The notions of various operations facilitate to study topological structures, logic and lattices. As we all know, fuzzy partially ordered sets (resp. equivalence relations) plays an important role in fuzzy rough sets and fuzzy topological structures. Using distance functions instead of fuzzy partially ordered sets (resp. equivalence relations), we define fuzzy interior (fuzzy closure) operators in complete co-residuated lattices as senses of fuzzy Galois connections and adjunctions. Moreover, we investigate their properties and define a fuzzy rough set. From Oh and Kim [16], we will obtain a formal fuzzy concepts and an attribute-oriented fuzzy concepts.

2. Preliminaries

Definition 2.1 ([2, 3, 4, 15, 16, 17, 18, 19]). An algebra $(L,\wedge,\vee,\oplus,\perp,\top)$ is called a *complete co-residuated lattice*, if it satisfies the following conditions:

(C1) $L=(L,\vee,\wedge,\perp,\top)$ is a complete lattice, where $\perp$ is the bottom element and $\top$ is the top element,

(C2) $a=a\oplus\perp$, $a\oplus b=b\oplus a$ and $a\oplus(b\oplus c)=(a\oplus b)\oplus c$ for all $a,b,c\in L$,

(C3) $(\bigwedge_{i\in\Gamma}a_i)\oplus b=\bigwedge_{i\in\Gamma}(a_i\oplus b)$.

Let $(L,\wedge,\vee,\oplus,\perp,\top)$ be a complete co-residuated lattice. For each $x,y\in L$, we define

$$x\ominus y=\bigwedge\{z\in L\mid y\oplus z\geq x\}.$$

Then $(x\oplus y)\geq z$ iff $x\geq(z\ominus y)$.

For $\alpha\in L, A\in L^X$, we denote $(\alpha\ominus A),(\alpha\oplus A),\alpha_X\in L^X$ as

$$(\alpha\ominus A)(x)=\alpha\ominus A(x),\ (\alpha\oplus A)(x)=\alpha\oplus A(x),\ \alpha_X(x)=\alpha.$$

Put $n(x)=\top\ominus x$. The condition $n(n(x))=x$ for each $x\in L$ is called a *double negative law*.

Lemma 2.2 ([15, 16, 17, 18, 19]). *Let $(L,\wedge,\vee,\oplus,\ominus,\perp,\top)$ be a complete co-residuated lattice. For each $x,y,z,x_i,y_i\in L$, we have the following properties.*

(1) *If $y\leq z$, $x\oplus y\leq x\oplus z$, $y\ominus x\leq z\ominus x$ and $x\ominus z\leq x\ominus y$.*

(2) $(\bigvee_{i\in\Gamma}x_i)\ominus y=\bigvee_{i\in\Gamma}(x_i\ominus y)$ *and* $x\ominus(\bigwedge_{i\in\Gamma}y_i)=\bigvee_{i\in\Gamma}(x\ominus y_i)$.

(3) $(\bigwedge_{i\in\Gamma}x_i)\ominus y\leq\bigwedge_{i\in\Gamma}(x_i\ominus y)$.

(4) $x\ominus(\bigvee_{i\in\Gamma}y_i)\leq\bigwedge_{i\in\Gamma}(x\ominus y_i)$.

(5) $x\ominus x=\perp$, $x\ominus\perp=x$ *and* $\perp\ominus x=\perp$. *Moreover,* $x\ominus y=\perp$ *iff* $x\leq y$.

(6) $y\oplus(x\ominus y)\geq x$, $y\geq x\ominus(x\ominus y)$ *and* $(x\ominus y)\oplus(y\ominus z)\geq x\ominus z$.

(7) $x\ominus(y\oplus z)=(x\ominus y)\ominus z=(x\ominus z)\ominus y$.

(8) $x\ominus y\geq(x\oplus z)\ominus(y\oplus z)$, $x\ominus y\geq(x\ominus z)\ominus(y\ominus z)$, $y\ominus x\geq(z\ominus x)\ominus(z\ominus y)$ *and* $(x\oplus y)\ominus(z\oplus w)\leq(x\ominus z)\oplus(y\ominus w)$.

(9) $x\oplus y=\perp$ *iff* $x=\perp$ *and* $y=\perp$.

(10) $(x\oplus y)\ominus z\leq x\oplus(y\ominus z)$ *and* $(x\ominus y)\oplus z\geq x\ominus(y\ominus z)$.

(11) $(\bigvee_{i\in\Gamma}x_i)\ominus(\bigvee_{i\in\Gamma}y_i)\leq\bigvee_{i\in\Gamma}(x_i\ominus y_i)$.

(12) $(\bigwedge_{i\in\Gamma}x_i)\ominus(\bigwedge_{i\in\Gamma}y_i)\leq\bigvee_{i\in\Gamma}(x_i\ominus y_i)$.

(13) *If L satisfies a double negative law and $n(x)=\top\ominus x$, then $n(x\oplus y)=n(x)\ominus y=n(y)\ominus x$ and $x\ominus y=n(y)\ominus n(x)$.*

Definition 2.3 ([15, 16, 17, 18, 19]). Let $(L,\wedge,\vee,\oplus,\ominus,\perp,\top)$ be a complete co-residuated lattice. Let X be a set. A function $d_X:X\times X\to L$ is called a *distance function*, if it satisfies the following conditions:

(M1) $d_X(x,x)=\perp$ for all $x\in X$,

(M2) $d_X(x,y)\oplus d_X(y,z)\geq d_X(x,z)$ for all $x,y,z\in X$,

(M3) If $d_X(x,y)=d_X(y,x)=\perp$, then $x=y$.

The pair (X,d_X) is called a *distance space*.

Remark 2.4 ([15, 16, 17, 18, 19])**.** (1) We define a distance function $d_X : X \times X \to [0, \infty]$. Then (X, d_X) is called a pseudo-quasi-metric space.

(2) Let $(L, \wedge, \vee, \oplus, \ominus, \perp, \top)$ be a complete co-residuated lattice. Define a function $d_L : L \times L \to L$ as $d_L(x, y) = x \ominus y$. By Lemma 2.3 (5) and (6), (L, d_L) is a distance space. For $\tau \subset L^X$, we define a function $d_\tau : \tau \times \tau \to L$ as

$$d_\tau(A, B) = \bigvee_{x \in X} (A(x) \ominus B(x)).$$

Then (τ, d_τ) is a distance space.

In this paper, we assume $(L, \wedge, \vee, \oplus, \ominus, \perp, \top)$ is a complete co-residuated lattice.

Definition 2.5 ([1, 15, 16, 17, 18, 19])**.** (1) A subset $\tau \subset L^X$ is called an *Alexandrov topology* on X if it satisfies the following conditions:

(A1) If $A_i \in \tau$ for all $i \in I$, then $\bigvee_{i \in I} A_i, \bigwedge_{i \in I} A_i \in \tau$.

(A2) If $A \in \tau$ and $\alpha \in L$, then $\alpha_X, A \ominus \alpha, A \oplus \alpha \in \tau$.

The pair (X, τ) is called an *Alexandrov topological space* on X.

Theorem 2.6 ([15, 16, 17, 18, 19])**.** *Let (X, d_X) be a distance space. We define*

$$\tau_{d_X} = \{A \in L^X \mid A(x) \oplus d_X(x, y) \geq A(y)\}$$
$$\tau_{d_X^{-1}} = \{A \in L^X \mid A(x) \oplus d_X(y, x) \geq A(y)\}.$$

(1) *τ_{d_X} and $\tau_{d_X^{-1}}$ are Alexandrov topologies.*

(2) *$(\tau_{d_X}, d_{\tau_{d_X}})$ and $(\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ are complete lattices.*

(3)

$$\tau_{d_X} = \{\bigvee_{x \in X} A(x) \oplus d_X(x, -) \mid A \in L^X\}$$

and

$$\tau_{d_X^{-1}} = \{\bigvee_{x \in X} A(x) \oplus d_X(-, x) \mid A \in L^X\}.$$

Definition 2.7 ([15, 16, 17, 18, 19])**.** Let (X, d_X) and (Y, d_Y) be distance spaces and $f : X \to Y$ be a map. Define $f^* : L^X \to L^Y$ as

$$f^*(A)(y) = \begin{cases} \top, & \text{if } f^{-1}(\{y\}) = \varnothing, \\ \bigwedge A(x), & \text{if } x \in f^{-1}(\{y\}). \end{cases}$$

Theorem 2.8 ([15, 16, 17, 18, 19])**.** *Let (X, d_X) and (Y, d_Y) be distance spaces. Define $f^\oplus, f^{s\oplus} : L^X \to L^Y$ and $f^\leftarrow_\oplus, f^{s\leftarrow}_\oplus : L^Y \to L^X$ as*

$$f^\oplus(A)(y) = \bigwedge_{x \in X}(A(x) \oplus d_Y(f(x), y)),$$
$$f^{s\oplus}(A)(y) = \bigwedge_{x \in X}(A(x) \oplus d_Y(y, f(x))),$$
$$f^\leftarrow_\oplus(B)(x) = \bigwedge_{z \in X}(B(f(z)) \oplus d_X(z, x)),$$
$$f^{s\leftarrow}_\oplus(B)(x) = \bigwedge_{z \in X}(B(f(z)) \oplus d_X(x, z)).$$

Then the following properties hold.

(1) *$d_{L^X}(B, A) \geq d_{L^Y}(f^\oplus(B), f^\oplus(A))$ and $d_{L^X}(B, A) \geq d_{L^Y}(f^{s\oplus}(B), f^{s\oplus}(A))$.*

(2) *$d_{L^Y}(C, D) \geq d_{L^X}(f^\leftarrow_\oplus(C), f^\leftarrow_\oplus(E))$ and $d_{L^Y}(C, D) \geq d_{L^X}(f^{s\leftarrow}_\oplus(C), f^{s\leftarrow}_\oplus(E))$.*

(3) *$f^\oplus(A) \in \tau_{d_Y}$ and $f^{s\oplus}(A) \in \tau_{d_Y^{-1}}$.*

(4) *$f^\leftarrow_\oplus(B) \in \tau_{d_X}$ and $f^{s\leftarrow}_\oplus(B) \in \tau_{d_X^{-1}}$.*

3. Various operations in complete co-residuated lattices

In this section, as generalizations of Zadeh powerset operators we define the following operations and investigate their properties.

Definition 3.1. Let (X, d_X) and (Y, d_Y) be distance spaces and $f : X \to Y$ be a map. Define $f^{\oslash}, f^{s\oslash} : L^X \to L^Y$ and $f^{\leftarrow}_{\oslash}, f^{s\leftarrow}_{\oslash} : L^Y \to L^X$ as

$$\begin{array}{l} f^{\oslash}(A)(y) = \bigvee_{x\in X}(A(x) \ominus d_Y(f(x), y)),\\ f^{s\oslash}(A)(y) = \bigvee_{x\in X}(A(x) \ominus d_Y(y, f(x))), \end{array}$$

$$\begin{array}{l} f^{\leftarrow}_{\oslash}(B)(x) = \bigvee_{z\in X}(B(f(z)) \ominus d_X(z, x)),\\ f^{s\leftarrow}_{\oslash}(B)(x) = \bigvee_{z\in X}(B(f(z)) \ominus d_X(x, z)). \end{array}$$

Remark 3.2. Let $(L, \leq, \vee, \wedge, \oplus, \ominus, \bot, \top)$ be a complete co-residuated lattice. Let X, Y be sets and $f : X \to Y$ a function. Define $d_X \in L^{X\times X}, d_Y \in L^{Y\times Y}$ as

$$d_X(x, z) = \begin{cases} \bot, & \text{if } z = x,\\ \top, & \text{if } z \neq x, \end{cases} \quad d_Y(y, w) = \begin{cases} \bot, & \text{if } y = w,\\ \top, & \text{if } y \neq w. \end{cases}$$

We easily show that d_X and d_Y are distance functions. We obtain

$$\begin{array}{l} f^{\oslash}(A)(y) = \bigvee_{x\in X}(A(x) \ominus d_Y(f(x), y))\\ = \bigvee_{f(x)=y} A(x) = f^{\rightarrow}(A)(y) = f^{s\oslash}(A)(y),\\ f^{\leftarrow}_{\oslash}(B)(x) = \bigvee_{z\in X}(B(f(z)) \ominus d_X(z, x))\\ = B(f(x)) = f^{\leftarrow}(B)(x) = f^{s\leftarrow}_{\oslash}(B)(x). \end{array}$$

Then $f^{\oslash}, f^{s\oslash}, f^{\leftarrow}_{\oslash}, f^{s\leftarrow}_{\oslash}$ are generalizations of Zadeh power operators $f^{\rightarrow}$ and $f^{\leftarrow}$.

Theorem 3.3. *Let (X, d_X) and (Y, d_Y) be distance spaces. For each $A, C \in L^X$ and $B, D \in L^Y$, the followings hold.*

(1) $d_{L^X}(A, C) \geq d_{L^Y}(f^{\oslash}(A), f^{\oslash}(C))$ and $d_{L^X}(A, C) \geq d_{L^Y}(f^{s\oslash}(A), f^{s\oslash}(C))$.

(2) $d_{L^Y}(B, D) \geq d_{L^X}(f^{\leftarrow}_{\oslash}(B), f^{\leftarrow}_{\oslash}(D))$ and $d_{L^Y}(B, D) \geq d_{L^X}(f^{s\leftarrow}_{\oslash}(B), f^{s\leftarrow}_{\oslash}(D))$.

(3) $f^{\oslash}(A) \in \tau_{d_Y^{-1}}$, $f^{s\oslash}(A) \in \tau_{d_Y}$, $f^{\oslash}(A) \geq f^{\rightarrow}(A)$ and $f^{s\oslash}(A) \geq f^{\rightarrow}(A)$ for each $A \in L^X$.

(4) $f^{\leftarrow}_{\oslash}(B) \in \tau_{d_X^{-1}}$, $f^{s\leftarrow}_{\oslash}(B) \in \tau_{d_X}$, $f^{\leftarrow}_{\oslash}(B) \geq f^{\leftarrow}(B)$ and $f^{s\leftarrow}_{\oslash}(B) \geq f^{\leftarrow}(B)$ for each $B \in L^Y$.

Proof. (1) For $A, C \in L^X$,

$$\begin{array}{l} \bigvee_{x\in X}(A(x) \ominus d_Y(f(x), y)) \ominus \bigvee_{x\in X}(C(x) \ominus d_Y(f(x), y))\\ \leq \bigvee_{x\in X}((A(x) \ominus d_Y(f(x), y)) \ominus (C(x) \ominus d_Y(f(x), y)))\\ \quad \text{[By Lemma 2.2 (8) and (11)]}\\ \leq \bigvee_{x\in X}(A(x) \ominus C(x)). \end{array}$$

Then $d_{L^X}(A, C) \geq d_{L^Y}(f^{\oslash}(A), f^{\oslash}(C))$. Similarly, $d_{L^X}(A, C) \geq d_{L^Y}(f^{s\oslash}(A), f^{s\oslash}(C))$.

(2) For $B, C \in L^Y$,

$$\begin{array}{l} \bigvee_{x\in X}(B(f(x)) \ominus d_X(x, z)) \ominus \bigvee_{x\in X}(D(f(x)) \ominus d_X(x, z))\\ \leq \bigvee_{x\in X}((B(f(x)) \ominus d_X(x, z)) \ominus (D(f(x)) \ominus d_X(x, z)))\\ \quad \text{[By Lemma 2.2 (8) and (11)]}\\ \leq \bigvee_{x\in X}(B(f(x)) \ominus D(f(x))) \leq d_{L^Y}(B, D). \end{array}$$

Then $d_{L^Y}(B, D) \geq d_{L^X}(f^{\leftarrow}_{\oslash}(B), f^{\leftarrow}_{\oslash}(D))$. Similarly, $d_{L^Y}(B, D) \geq d_{L^X}(f^{s\leftarrow}_{\oslash}(B), f^{s\leftarrow}_{\oslash}(D))$.

For $A \in L^X$,

$$f^{\oslash}(A)(y) \geq \bigvee_{f(x)=y} A(x) = f^{\rightarrow}(A),$$
$$f^{s\oslash}(A)(y) \geq \bigvee_{f(x)=y} A(x) = f^{\rightarrow}(A).$$

(3) For $A \in L^X$,

$$\begin{aligned} & f^{\oslash}(A)(y) \oplus d_Y(w,y) \oplus d_Y(f(x),w) \\ = & (\bigvee_{x\in X}(A(x) \ominus d_Y(f(x),y))) \oplus d_Y(f(x),y) \\ \geq & A(x). \end{aligned}$$

Then $f^{\oslash}(A)(y) \oplus d_Y(w,y) \geq f^{\oslash}(A)(w)$ and $f^{\oslash}(A) \in \tau_{d_Y^{-1}}$. Similarly, $f^{s\oslash}(A) \in \tau_{d_Y}$.

(4) For $B \in L^Y$,

$$\begin{aligned} & (B(f(z)) \ominus d_X(z,x)) \oplus d_X(w,x) \odot d_X(z,w) \\ \geq & (B(f(z)) \ominus d_X(z,x)) \oplus d_X(z,x) \\ \geq & B(f(z)). \end{aligned}$$

Then $f^{\leftarrow}_{\oslash}(B)(x) \oplus d_X(w,x) \geq f^{\leftarrow}_{\oslash}(B)(w)$ and $f^{\leftarrow}_{\oslash}(B) \in \tau_{d_X^{-1}}$. □

Definition 3.4. A map $C : L^X \to L^X$ is called a *fuzzy closure operator*, if it satisfies the following conditions:

(C1) $A \leq C(A)$, for all $A \in L^X$,

(C2) $d_{L^X}(A,B) \geq d_{L^X}(C(A),C(B))$ for each $A,B \in L^X$.

A map $I : L^X \to L^X$ is called a *fuzzy interior operator*, if it satisfies the following conditions:

(I1) $I(A) \leq A$, for all $A \in L^X$,

(I2) $d_{L^X}(A,B) \geq d_{L^X}(I(A),I(B))$ for each $A,B \in L^X$.

The pair $(I(A),C(A))$ is called a *fuzzy rough set* of A.

Remark 3.5. Let (X,d_X) be a distance space and $id_X : X \to Y$ be an identity map. Then $id^{\oplus}_X, id^{s\oplus}_X : L^X \to L^X$ are fuzzy interior operators and $id^{\oslash}_X, id^{s\oslash}_X : L^X \to L^X$ are fuzzy closure operators defined as

$$id^{\oplus}_X(A)(y) = \bigwedge_{x\in X}(A(x) \oplus d_X(x,y)),$$
$$id^{s\oplus}_X(A)(y) = \bigwedge_{x\in X}(A(x) \oplus d_X(y,x)),$$
$$id^{\oslash}_X(A)(y) = \bigvee_{x\in X}(A(x) \ominus d_X(x,y)),$$
$$id^{s\oslash}_X(A)(y) = \bigvee_{x\in X}(A(x) \ominus d_X(y,x)).$$

Moreover $(id^{\oplus}_X(A), id^{\oslash}_X)$,$(id^{s\oplus}_X(A), id^{\oslash}_X)$, $(id^{\oplus}_X(A), id^{s\oslash}_X)$ and $(id^{s\oplus}_X(A), id^{s\oslash}_X)$ are fuzzy rough sets of A.

Theorem 3.6. *Let (X,d_X) and (Y,d_Y) be distance spaces and $f : (X,d_X) \to (Y,d_Y)$ be a map with $d_X(x,y) \geq d_Y(f(x),f(y))$ for all $x,y \in X$.*

(1) *Two operations $f^{\oslash} : \tau_{d_X^{-1}} \to \tau_{d_Y^{-1}}$ and $f^{s\leftarrow}_{\oplus} : \tau_{d_Y^{-1}} \to \tau_{d_X^{-1}}$ satisfy the followings:*

$$d_{\tau_{d_Y^{-1}}}(f^{\oslash}(A),B) \geq d_{\tau_{d_X^{-1}}}(A, f^{s\leftarrow}_{\oplus}(B)) \text{ and } A \leq f^{s\leftarrow}_{\oplus}(f^{\oslash}(A)).$$

Moreover, if f is surjective and $d_X(x,y) = d_Y(f(x),f(y))$ for all $x,y \in X$, then

$$d_{\tau_{d_Y^{-1}}}(f^{\oslash}(A),B) = d_{\tau_{d_X^{-1}}}(A, f^{s\leftarrow}_{\oplus}(B)).$$

(2) *Two operations $f^{s\oslash} : \tau_{d_X} \to \tau_{d_Y}$ and $f^{\leftarrow}_{\oplus} : \tau_{d_Y} \to \tau_{d_X}$ satisfy the followings:*

$$d_{\tau_{d_Y}}(f^{s\oslash}(A),B) \geq d_{\tau_{d_X}}(A, f^{\leftarrow}_{\oplus}(B)) \text{ and } A \leq f^{\leftarrow}_{\oplus}(f^{s\oslash}(A)).$$

Moreover, if f is surjective and $d_X(x,y) = d_Y(f(x), f(y))$ for all $x, y \in X$, then

$$d_{\tau_{d_Y}}(f^{s\oslash}(A), B) = d_{\tau_{d_X}}(A, f^{\leftarrow}_{\oplus}(B)).$$

(3) *Two operations $f^{s\oplus} : \tau_{d_X^{-1}} \to \tau_{d_Y^{-1}}$ and $f^{\leftarrow}_{\oslash} : \tau_{d_Y^{-1}} \to \tau_{d_X^{-1}}$ satisfy the followings:*

$$d_{\tau_{d_Y^{-1}}}(B, f^{s\oplus}(A)) \geq d_{\tau_{d_X^{-1}}}(f^{\leftarrow}_{\oslash}(B), A) \text{ and } f^{\leftarrow}_{\oslash}(f^{s\oplus}(A)) \leq A.$$

Moreover, if f is surjective and $d_X(x,y) = d_Y(f(x), f(y))$ for all $x, y \in X$, then

$$d_{\tau_{d_Y^{-1}}}(B, f^{s\oplus}(A)) = d_{\tau_{d_X^{-1}}}(f^{\leftarrow}_{\oslash}(B), A).$$

(4) *Two operations $f^{\oplus} : \tau_{d_X} \to \tau_{d_Y}$ and $f^{s\leftarrow}_{\oslash} : \tau_{d_Y} \to \tau_{d_X}$ satisfy the followings:*

$$d_{\tau_{d_Y}}(B, f^{\oplus}(A)) \geq d_{\tau_{d_X}}(f^{s\leftarrow}_{\oslash}(B), A) \text{ and } f^{s\leftarrow}_{\oslash}(f^{\oplus}(A)) \leq A.$$

Moreover, if f is surjective and $d_X(x,y) = d_Y(f(x), f(y))$ for all $x, y \in X$, then

$$d_{\tau_{d_Y}}(B, f^{\oplus}(A)) = d_{\tau_{d_X}}(f^{s\leftarrow}_{\oslash}(B), A).$$

(5) *An operation $f^{\leftarrow}_{\oslash} \circ f^{s\oplus} : \tau_{d_X^{-1}} \to \tau_{d_X^{-1}}$ is a fuzzy interior operator. An operation $f^{s\leftarrow}_{\oplus} \circ f^{\oslash} : \tau_{d_X^{-1}} \to \tau_{d_X^{-1}}$ is a fuzzy closure operator. The pair $(f^{\leftarrow}_{\oslash}(f^{s\oplus}(A)), f^{s\leftarrow}_{\oplus}(f^{\oslash}(A)))$ is a fuzzy rough set for $A \in \tau_{d_X^{-1}}$.*

(6) *An operation $f^{s\leftarrow}_{\oslash} \circ f^{\oplus} : \tau_{d_X} \to \tau_{d_X}$ is a fuzzy interior operator. An operation $f^{\leftarrow}_{\oplus} \circ f^{s\oslash} : \tau_{d_X} \to \tau_{d_X}$ is a fuzzy closure operator. The pair $(f^{s\leftarrow}_{\oslash}(f^{\oplus}(A)), f^{\leftarrow}_{\oplus}(f^{s\oslash}(A)))$ is a fuzzy rough set for $A \in \tau_{d_X}$.*

Proof. (1) For $A \in \tau_{d_X^{-1}}, B \in \tau_{d_Y^{-1}}$,

$$\begin{aligned}
& d_{\tau_{d_Y^{-1}}}(f^{\oslash}(A), B) = \bigvee_{y \in Y}(f^{\oslash}(A)(y) \ominus B(y)) \\
&= \bigvee_{y \in Y}(\bigvee_{x \in X}(A(x) \ominus d_Y(f(x), y)) \ominus B(y)) \\
&= \bigvee_{x \in X}(A(x) \ominus \bigwedge_{y \in Y}(d_Y(f(x), y) \oplus B(y))) \\
&\geq \bigvee_{x \in X}(A(x) \ominus \bigwedge_{z \in X}(d_Y(f(x), f(z)) \oplus B(f(z)))) \\
&\geq \bigvee_{x \in X}(A(x) \ominus \bigwedge_{z \in X}(d_X(x, z) \oplus B(f(z)))) \\
&= \bigvee_{x \in X}(A(x) \ominus f^{s\leftarrow}_{\oplus}(B)(x)) = d_{\tau_{d_X^{-1}}}(A, f^{s\leftarrow}_{\oplus}(B)).
\end{aligned}$$

Moreover, $A \leq f^{s\leftarrow}_{\oplus}(f^{\oslash}(A))$ from the following inequality:

$$d_{\tau_{d_X^{-1}}}(A, f^{s\leftarrow}_{\oplus}(f^{\oslash}(A))) \leq d_{\tau_{d_Y^{-1}}}(f^{\oslash}(A), f^{\oslash}(A)) = \bot.$$

If f is surjective and $d_X(x,y) = d_Y(f(x), f(y))$ for all $x, y \in X$, then we have

$$\begin{aligned}
& d_{\tau_{d_Y^{-1}}}(f^{\oslash}(A), B) = \bigvee_{y \in Y}(f^{\oslash}(A)(y) \ominus B(y)) \\
&= \bigvee_{x \in X}(A(x) \ominus \bigwedge_{y \in Y}(d_Y(f(x), y) \oplus B(y))) \\
&= \bigvee_{x \in X}(A(x) \ominus \bigwedge_{z \in X}(d_Y(f(x), f(z)) \oplus B(f(z)))) \\
&= \bigvee_{x \in X}(A(x) \ominus \bigwedge_{z \in X}(d_X(x, z) \oplus B(f(z)))) \\
&= \bigvee_{x \in X}(A(x) \ominus f^{s\leftarrow}_{\oplus}(B)(x)) = d_{\tau_{d_X^{-1}}}(A, f^{s\leftarrow}_{\oplus}(B)).
\end{aligned}$$

(3) For $A \in \tau_{d_X}, B \in \tau_{d_Y^{-1}}$,

$$\begin{aligned}
& d_{\tau_{d_Y^{-1}}}(B, f^{s\oplus}(A)) = \bigvee_{y \in Y}(B(y) \ominus f^{s\oplus}(A)(y)) \\
&= \bigvee_{y \in Y}(B(y) \ominus \bigwedge_{x \in X}(A(x) \oplus d_Y(y, f(x)))) \\
&\geq \bigvee_{z \in X}(B(f(z)) \ominus \bigwedge_{x \in X}(A(x) \oplus d_Y(f(z), f(x)))) \\
&\geq \bigvee_{z \in X}(B(f(z)) \ominus \bigwedge_{x \in X}(A(x) \oplus d_X(z, x))) \\
&= \bigvee_{x \in X}(\bigwedge_{z \in X}(B(f(z)) \ominus d_X(z, x)) \ominus A(x))
\end{aligned}$$

$= \bigvee_{x\in X}(f^{\leftarrow}_{\oslash}(B)(x) \ominus A(x)) = d_{\tau_{d_X^{-1}}}(f^{\leftarrow}_{\oslash}(B), A).$

Other case, (2) and (4) can be proved in a similar way to (1) and (3) respectively.

(5) For $A, C \in \tau_{d_X^{-1}}$,

$$d_{\tau_{d_X^{-1}}}(A, C) \geq d_{\tau_{d_Y^{-1}}}(f^{s\oplus}(A), f^{s\oplus}(C)) \geq d_{\tau_{d_X^{-1}}}(f^{\leftarrow}_{\oslash}(f^{s\oplus}(A)), f^{\leftarrow}_{\oslash}(f^{s\oplus}(C))),$$
$$d_{\tau_{d_X^{-1}}}(A, C) \geq d_{\tau_{d_Y^{-1}}}(f^{s\oslash}(A), f^{s\oslash}(C)) \geq d_{\tau_{d_X^{-1}}}(f^{\leftarrow}_{\oplus}(f^{s\oslash}(A)), f^{\leftarrow}_{\oplus}(f^{s\oslash}(C))).$$

(6) The proof can be proved in a similar way to (5). □

Remark 3.7. From Oh and Kim [16], we will obtain a formal fuzzy concept and an attribute-oriented fuzzy concept as follows.

(1) Let $F : L^X \to L^Y, G : L^Y \to L^X$ be maps where X is a set of objects and Y is a set of attributes. If $d_{L^Y}(F(A), B) = d_{L^X}(G(B), A)$, then a formal fuzzy concept is a pair $(A, B) \in L^X \times L^Y$ such that $F(A) = B, G(B) = A$ as a Bělohlávek's sense [2, 3, 4, 5].

If $d_{L^Y}(B, F(A)) = d_{L^X}(G(B), A)$, then an attribute-oriented fuzzy concept is a pair $(A, B) \in L^X \times L^Y$ such that $F(A) = B, G(B) = A$ as Ciobanu and Văideanu's sense [21].

(2) Let (X, d_X) and (Y, d_Y) be distance spaces and $f : (X, d_X) \to (Y, d_Y)$ be a surjective map with $d_X(x, y) = d_Y(f(x), f(y))$ for all $x, y \in X$. By Theorem 3.6 (1) and (2),

$$d_{\tau_{d_Y}}(f^{s\oslash}(A), B) = d_{\tau_{d_X}}(A, f^{\leftarrow}_{\oplus}(B)) \text{ and } d_{\tau_{d_Y^{-1}}}(f^{\oslash}(A), B) = d_{\tau_{d_X^{-1}}}(A, f^{s\leftarrow}_{\oplus}(B)).$$

We can obtain two formal fuzzy concepts as

$$\{(A, B) \in \tau_{d_X} \times \tau_{d_Y} \mid f^{s\oslash}(A) = B, f^{\leftarrow}_{\oplus}(B) = A\}$$

and

$$\{(A, B) \in \tau_{d_X^{-1}} \times \tau_{d_Y^{-1}} \mid f^{\oslash}(A) = B, f^{s\leftarrow}_{\oplus}(B) = A\}$$

respectively. Moreover, by 3.6 (1) and (2),

$$d_{\tau_{d_Y}}(B, f^{\oplus}(A)) = d_{\tau_{d_X}}(f^{s\leftarrow}_{\oslash}(B), A) \text{ and } d_{\tau_{d_Y^{-1}}}(B, f^{s\oplus}(A)) = d_{\tau_{d_X^{-1}}}(f^{\leftarrow}_{\oslash}(B), A).$$

We can obtain two attribute-oriented fuzzy concepts as

$$\{(A, B) \in \tau_{d_X} \times \tau_{d_Y} \mid f^{\oplus}(A) = B, f^{s\leftarrow}_{\oslash}(B) = A\}$$

and

$$\{(A, B) \in \tau_{d_X^{-1}} \times \tau_{d_Y^{-1}} \mid f^{s\oplus}(A) = B, f^{\leftarrow}_{\oslash}(B) = A\}$$

respectively.

Example 3.8. Let $(L = [0, 1], \leq, \vee, \wedge, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice defined as $n(x) = 1 - x$,

$$x \oplus y = (x + y) \wedge 1, \quad x \ominus y = (x - y) \vee 0.$$

(1) Let X, Y be sets and $f : X \to Y$ a function. Define $d_X \in L^{X\times X}, d_Y \in L^{Y\times Y}$ as Remark 3.2. Since f is a function, $d_X(x, z) \geq d_Y(f(x), f(z))$. Then we have

$$\tau_{d_X} = \{A \in L^X \mid A(x) \oplus d_X(x, y) \geq A(y)\} = L^X = \tau_{d_X^{-1}}.$$

Moreover, $\tau_{d_Y} = L^Y = \tau_{d_Y^{-1}}$. For f^* in Definition 2.7, we obtain

$$f^{\oplus}(A)(y) = \bigwedge_{x\in X}(A(x)\oplus d_Y(f(x),y)) = f^{s\oplus}(A)(y) = \bigwedge_{f(x)=y} A(x) = f^*(A)(y),$$

$$f^{\leftarrow}_{\oplus}(B)(x) = \bigwedge_{z\in X}(B(f(z))\oplus d_X(z,x)) = f^{s\leftarrow}_{\oplus}(B)(x) = B(f(x)) = f^{\leftarrow}(B)(x),$$

$$f^{\oslash}(A)(y) = \bigvee_{x\in X}(A(x)\ominus d_Y(f(x),y)) = f^{s\oslash}(A)(y) = \bigvee_{f(x)=y} A(x) = f^{\rightarrow}(A)(y),$$

$$f^{\leftarrow}_{\oslash}(B)(x) = \bigvee_{z\in X}(B(f(z))\ominus d_X(z,x)) = f^{s\leftarrow}_{\oslash}(B)(x) = B(f(x)) = f^{\leftarrow}(B)(x).$$

Let $A \in \tau_{d_X} = L^X$ and let $B \in \tau_{d_Y} = L^Y$ Then we get

$$\begin{aligned} d_{L^Y}(B, f^*(A)) &= \bigvee_{y\in Y}(B(y)\ominus f^*(A)(y)) \\ &= \bigvee_{y\in Y}(B(y)\ominus \bigwedge_{f(x)=y} A(x)) \\ &= \bigvee_{x\in X}(B(f(x))\ominus A(x)) \\ &= d_{L^X}(f^{\leftarrow}(B), A). \end{aligned}$$

Thus we have

$$\begin{aligned} d_{L^Y}(B, f^{\oplus}(A)) &= d_{L^Y}(B, f^{s\oplus}(A)) = d_{L^Y}(B, f^*(A)) \\ &= d_{L^X}(f^{\leftarrow}(B), A) = d_{L^X}(f^{\leftarrow}_{\oplus}(B), A) \\ &= d_{L^X}(f^{s\leftarrow}_{\oplus}(B), A) = d_{L^X}(f^{\leftarrow}_{\oslash}(B), A) \\ &= d_{L^X}(f^{s\leftarrow}_{\oslash}(B), A). \end{aligned}$$

Since

$$\begin{aligned} d_{L^Y}(f^{\rightarrow}(A), B) &= \bigvee_{y\in Y}(f^{\rightarrow}(A)(y)\ominus B(y)) \\ &= \bigvee_{y\in Y}(\bigvee_{f(x)=y} A(x)\ominus B(y)) \\ &= \bigvee_{x\in X}(A(x)\ominus B(f(x))) = d_{L^X}(A, f^{\leftarrow}(B)), \end{aligned}$$

we get

$$\begin{aligned} d_{L^Y}(f^{\oslash}(A), B) &= d_{L^Y}(f^{s\oslash}(A), B) = d_{L^Y}(f^{\rightarrow}(A), B) \\ &= d_{L^X}(A, f^{\leftarrow}(B)) = d_{L^X}(A, f^{\leftarrow}_{\oslash}(B)) \\ &= d_{L^X}(A, f^{s\leftarrow}_{\oslash}(B)) = d_{L^X}(A, f^{\leftarrow}_{\oplus}(B)) \\ &= d_{L^X}(A, f^{s\leftarrow}_{\oplus}(B)). \end{aligned}$$

Let $X = \{a, b, c\}$, $Y = \{x, y, z\}$ and let $f : X \to Y$ be the function defined as follows:

$$f(a) = f(b) = x,\ \ f(c) = y.$$

Then $d_X(a, b) \geq d_Y(f(a), f(b))$ for all $a, b \in X$. Thus the properties of Theorems 3.3 and 3.6 hold. For $A = (0.6, 0.3, 0.5)$,

$$\begin{aligned} &f^*(A) = (A(a)\wedge A(b), A(c), 1) = (0.3, 0.5, 1), \\ &f^{\rightarrow}(A) = (A(a)\vee A(b), A(c), 0) = (0.6, 0.5, 0), \\ &f^{\leftarrow}_{\oslash}(f^{s\oplus}(A)) = f^{s\leftarrow}_{\oslash}(f^{\oplus}(A)) = f^{\leftarrow}(f^*(A)) = (0.3, 0.3, 0.5), \\ &f^{s\leftarrow}_{\oplus}(f^{\oslash}(A)) = f^{\leftarrow}_{\oplus}(f^{s\oslash}(A)) = f^{\leftarrow}(f^{\rightarrow}(A)) = (0.6, 0.6, 0.5). \end{aligned}$$

So the pair $(f^{\leftarrow}_{\oslash}(f^{s\oplus}(A)), f^{s\leftarrow}_{\oplus}(f^{\oslash}(A))) = ((0.3, 0.3, 0.5), (0.6, 0.6, 0.5))$ is a fuzzy rough set of A. The pair $(f^{s\leftarrow}_{\oslash}(f^{\oplus}(A)), f^{\leftarrow}_{\oplus}(f^{s\oslash}(A))) = ((0.3, 0.3, 0.5), (0.6, 0.6, 0.5))$ is a fuzzy rough set of A. For $B = (0.4, 0.7, 0.1)$,

$$\begin{aligned} &d_{L^Y}(f^{\rightarrow}(A), B) = d_{L^X}(A, f^{\leftarrow}(B)) = 0.2, \\ &d_{L^Y}(B, f^*(A)) = d_{L^X}(f^{\leftarrow}(B), A) = 0.2. \end{aligned}$$

(2) Let $X=\{a,b,c\}$, $Y=\{x,y,z\}$ with $a<b<c$ and $x<y<z$ be totally order sets. Let $f:X\to Y$ be the function defined as follows:

$$f(a)=f(b)=x,\ f(c)=y.$$

Define $d_X\in L^{X\times X}, d_Y\in L^{Y\times Y}$ as

$$d_X(x,z)=\begin{cases}0, & \text{if } x\le z\\ 1, & \text{if } x>y,\end{cases}\quad d_Y(y,w)=\begin{cases}0, & \text{if } y\le w\\ 1, & \text{if } y>w.\end{cases}$$

Then we easily show that d_X and d_Y are distance functions. Since $f(a)\le f(b)$ for each $a\le b$, $d_X(x,z)\ge d_Y(f(x),f(z))$. Thus we have

$$\tau_{d_X}=\{A\in L^X\mid A(x)\oplus d_X(x,y)\ge A(y)\}=\{A\in L^X\mid A(x)\ge A(y),\ \forall x\le y\}$$

and

$$\tau_{d_X^{-1}}=\{A\in L^X\mid A(x)\oplus d_X^{-1}(x,y)\ge A(y)\}=\{A\in L^X\mid A(x)\le A(y),\ \forall x\le y\}.$$

For $A=(0.6,0.5,0.3)\in\tau_{d_X}$ and $B=(0.8,0.6,0.1)\in\tau_{d_Y}$, we get

$$\begin{aligned}
&f^{\oplus}(A)(-)=\textstyle\bigwedge_{x\in X}(A(x)\oplus d_Y(f(x),-))=\bigwedge_{f(x)\le -}A(x)=(0.5,0.3,0.3)\in\tau_{d_Y},\\
&f^{\leftarrow}_{\oplus}(B)(-)=\textstyle\bigwedge_{z\in X}(B(f(z))\oplus d_X(z,-))=\bigwedge_{z\le -}B(f(z))=(0.8,0.8,0.6)\in\tau_{d_X},\\
&f^{s\oslash}(A)(-)=\textstyle\bigvee_{x\in X}(A(x)\ominus d_Y(y,f(x)))=\bigvee_{-\le f(x)}A(x)=(0.6,0.6,0.5)\in\tau_{d_Y},\\
&f^{\rightarrow}(A)=(0.6,0.5,0),\\
&f^{s\leftarrow}_{\oslash}(B)(-)=\textstyle\bigvee_{z\in X}(B(f(z))\ominus d_X(-,z))=\bigvee_{-\le z}B(f(z))=(0.8,0.8,0.6)\in\tau_{d_X},\\
&f^{\leftarrow}(B)=(0.8,0.8,0.6).
\end{aligned}$$

For two operations $f^{s\oslash}:\tau_{d_X}\to\tau_{d_Y}$, $f^{\leftarrow}_{\oplus}:\tau_{d_Y}\to\tau_{d_X}$,

$$\begin{aligned}
&0.4=d_{\tau_{d_Y}}(f^{s\oslash}(A),B)\ge d_{\tau_{d_X}}(A,f^{\leftarrow}_{\oplus}(B))=0,\\
&(0.6,0.5,0.3)=A\le f^{\leftarrow}_{\oplus}(f^{s\oslash}(A))=(0.6,0.6,0.6).
\end{aligned}$$

For two operations $f^{\oplus}:\tau_{d_X}\to\tau_{d_Y}$, $f^{s\leftarrow}_{\oslash}:\tau_{d_Y}\to\tau_{d_X}$,

$$\begin{aligned}
&0.3=d_{\tau_{d_Y}}(B,f^{\oplus}(A))\ge d_{\tau_{d_X}}(f^{s\leftarrow}_{\oslash}(B),A)=0.3,\\
&(0.5,0.5,0.3)=f^{s\leftarrow}_{\oslash}(f^{\oplus}(A))\le(0.6,0.5,0.3).
\end{aligned}$$

For $C=(0.2,0.5,0.6)\in\tau_{d_X^{-1}}$ and $D=(0.4,0.6,0.8)\in\tau_{d_Y^{-1}}$, we get

$$\begin{aligned}
&f^{s\oplus}(C)(-)=\textstyle\bigwedge_{x\in X}(C(x)\oplus d_Y(-,f(x)))=\bigwedge_{-\le f(x)}C(x)=(0.2,0.2,0.6)\in\tau_{d_Y^{-1}},\\
&f^{s\leftarrow}_{\oplus}(D)(-)=\textstyle\bigwedge_{z\in X}(D(f(z))\oplus d_X(z,-))=\bigwedge_{-\le z}D(f(z))=(0.4,0.4,0.6)\in\tau_{d_X^{-1}},\\
&f^{\oslash}(C)(-)=\textstyle\bigvee_{x\in X}(C(x)\ominus d_Y(f(x),y))=\bigvee_{f(x)\le -}C(x)=(0.5,0.6,0.6)\in\tau_{d_Y^{-1}},\\
&f^{\rightarrow}(C)=(0.5,0.6,0),\\
&f^{\leftarrow}_{\oslash}(D)(-)=\textstyle\bigvee_{z\in X}(D(f(z))\ominus d_X(z,x))=\bigvee_{z\le -}D(f(z))=(0.4,0.4,0.6)\in\tau_{d_X^{-1}},\\
&f^{\leftarrow}(D)=(0.4,0.4,0.6).
\end{aligned}$$

For two operations $f^{\oslash}:\tau_{d_X^{-1}}\to\tau_{d_Y^{-1}}$, $f^{s\leftarrow}_{\oplus}:\tau_{d_Y^{-1}}\to\tau_{d_X^{-1}}$,

$$\begin{aligned}
&0.1=d_{\tau_{d_Y^{-1}}}(f^{\oslash}(C),D)\ge d_{\tau_{d_X^{-1}}}(C,f^{s\leftarrow}_{\oplus}(D))=0.1,\\
&(0.2,0.5,0.6)=C\le f^{s\leftarrow}_{\oplus}(f^{\oslash}(C))=(0.5,0.5,0.6).
\end{aligned}$$

For two operations $f^{s\oplus}:\tau_{d_X^{-1}}\to\tau_{d_Y^{-1}}$, $f_{\oslash}^{\leftarrow}:\tau_{d_Y^{-1}}\to\tau_{d_X^{-1}}$,

$$0.4=d_{\tau_{d_Y^{-1}}}(D,f^{s\oplus}(C))\geq d_{\tau_{d_X^{-1}}}(f_{\oslash}^{\leftarrow}(D),C)=0.2,$$
$$(0.2,0.2,0.2)=f_{\oslash}^{\leftarrow}(f^{s\oplus}(C))\leq(0.2,0.5,0.6).$$

(3) Let $X=\{a,b,c\}$, $Y=\{x,y,z\}$ and let $f:X\to Y$ be the function defined as follows:

$$f(a)=f(b)=x,\ f(c)=y.$$

Define $d_X\in L^{X\times X}$ and $d_Y\in L^{Y\times Y}$ as

$$d_X=\begin{pmatrix}0&0.5&0.8\\0.7&0&0.6\\0.4&0.6&0\end{pmatrix}\quad d_Y=\begin{pmatrix}0&0.4&0.9\\0.3&0&0.5\\0.7&0.4&0\end{pmatrix}$$

Then we easily show that d_X and d_Y are distance functions with $d_X(a,b)\geq d_Y(f(a),f(b))$ for all $a,b\in X$. Moreover, for $A\in[0,1]^X$ as $A(a)=0.3, A(b)=0.2, A(c)=0.5$,

$$A=\bigwedge_{x\in X}(A(x)\oplus d_X(x,-))=\bigwedge_{x\in X}(A(x)\oplus d_X(-,x)).$$

Thus by Theorem 2.6 (3), $A\in\tau_{d_X}, A\in\tau_{d_X^{-1}}$. Moreover, $f_{\oslash}^{\leftarrow}\circ f^{s\oplus}:\tau_{d_X^{-1}}\to\tau_{d_X^{-1}}$ and $f_{\oplus}^{s\leftarrow}\circ f^{\oslash}:\tau_{d_X^{-1}}\to\tau_{d_X^{-1}}$ are fuzzy interior and fuzzy closure operators respectively. Since $f^{s\oplus}(A)=(0.2,0.5,0.9)$,

$$f_{\oslash}^{\leftarrow}(f^{s\oplus}(A))=(0.1,0.2,0.5)\text{ and }f^{\oslash}(A)=(0.3,0.5,0),f_{\oplus}^{s\leftarrow}(f^{\oslash}(A))=(0.3,0.3,0.5).$$

So the pair $(f_{\oslash}^{\leftarrow}(f^{s\oplus}(A)),f_{\oplus}^{s\leftarrow}(f^{\oslash}(A)))=((0.1,0.2,0.5),(0.3,0.3,0.5))$ is a fuzzy rough set for A.

Also, $f_{\oslash}^{s\leftarrow}\circ f^{\oplus}:\tau_{d_X}\to\tau_{d_X}$ and $f_{\oplus}^{\leftarrow}\circ f^{s\oslash}:\tau_{d_X}\to\tau_{d_X}$ are fuzzy interior and fuzzy closure operators respectively. Since $f^{\oplus}(A)=(0.2,0.5,0.9)$,

$$f_{\oslash}^{s\leftarrow}(f^{\oplus}(A))=(0.2,0.2,0.5)\text{ and }f^{s\oslash}(A)=(0.3,0.5,0.1),f_{\oplus}^{\leftarrow}(f^{s\oslash}(A))=(0.3,0.3,0.5).$$

Hence the pair $(f_{\oslash}^{s\leftarrow}(f^{\oplus}(A)),f_{\oplus}^{\leftarrow}(f^{s\oslash}(A)))=((0.2,0.2,0.5),(0.3,0.3,0.5))$ is a fuzzy rough set for A.

4. Conclusion

Using distance functions, we have investigated generalizations of Zadeh powerset operators in co-residuated lattices. The notions of various operations facilitate to study topological structures ,logic and lattices.

In particular, we study fuzzy closure (interior) operators and fuzzy rough sets based on co-residuated lattices as senses of fuzzy Galois connections and adjunctions.

In the future, we suggest information systems and decision rules in co-residuated lattices.

Funding: This work was supported by the Research Institute of Natural Science of Gangneung-Wonju National University.

References

[1] P. Alexandroff, Diskrete Raume, *Mat. Sb. (N.S.) (in German)* 2 (1937) 501–518.

[2] R. Bělohlávek, Similarity relations in concept lattices, J. Logic and Computation 10 (6) (2000) 823–845.

[3] R. Bělohlávek, Fuzzy Relational Systems, Kluwer Academic Publishers, New York 2002.

[4] R. Bělohlávek, Concept lattices and order in fuzzy logic, Ann. Pure Appl. Logic 128 (2004) 277–298.

[5] J. Qiao and B. Q. Hu, A short note on L-fuzzy approximation spaces and L-fuzzy pretopological spaces, Fuzzy Sets and Systems 312 (2017) 126–134.

[6] C. Y. Wang and B. Q. Hu, Granular variable precision fuzzy rough sets with general fuzzy relations, Fuzzy Sets and Systems, 275 (2015) 39–57.

[7] C. Y. Wang, Topological structures of L-fuzzy rough sets and similarity sets of L-fuzzy relations, Int. J. Approx. Reason. 83 (2017) 160–175.

[8] C. Y. Wang, Topological characterizations of generalized fuzzy rough sets, Fuzzy Sets and Systems 312 (2017) 109–125.

[9] Z. Wang, Y. Wang and K. Tang, Some properties of L-fuzzy approximation spaces based on bounded integral residuated lattices Inf. Sci. 278 (2014) 110–126.

[10] Z. Pei, D. Pei and L. Zheng, Topology vs generalized rough sets Int. J. Approx. Reason. 52 (2011) 231–239.

[11] J. Fang, I-fuzzy Alexandrov topologies and specialization orders, Fuzzy Sets and Systems 128 (2007) 2359–2374.

[12] J. Fang and Y. Yue, L-fuzzy closure systems Fuzzy Sets and Systems 161 (2010) 1242–1252.

[13] M. C. Zheng and G.J . Wang, Coresiduated lattice with applications, Fuzzy systems and Mathematics 19 (2005) 1–6.

[14] Q. Junsheng and Hu Bao Qing, On $(\odot, \&)$-fuzzy rough sets based on residuated and co-residuated lattices, Fuzzy Sets and Systems 336 (2018) 54–86.

[15] Y. C. Kim and J. M. Ko, Preserving maps and approximation operators in complete co-residuated lattices, Journal of the Korean Insitutute of Intelligent Systems 30 (5) (2020) 389–398.

[16] J. M. Oh and Y. C. Kim, Various fuzzy connections and fuzzy concepts in complete co-residuated lattices, International Journal of Approximate Reasoning 142 (2022) 451–468.

[17] J. M. Oh and Y. C. Kim, Distance functions, upper approximation operators and Alexandrov fuzzy topologies, Journal of Intelligent and Fuzzy Systems 40 (6) (2021) 11915–11925.

[18] J. M. Oh and Y. C. Kim, Fuzzy completeness and various operations in co-residuated lattices, Ann. Fuzzy Math. Inform. 21 (3) (2021) 295–305.

[19] J. M. Oh and Y. C. Kim, Adjunctions, Galois connectins and various operations in co-residuated lattices, Ann. Fuzzy Math. Inform. 22 (3) (2021) 297–310.

[20] U. Höhle and S. E. Rodabaugh, Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, The Handbooks of Fuzzy Sets Series 3, Kluwer Academic Publishers, Boston 1999.

[21] G. Ciobanu and Văideanu, Similarity relations in fuzzy attribute-oriented concept lattices, Fuzzy Sets and Systems 275 (2015) 88–109.

Ju-mok Oh (jumokoh@gwnu.ac.kr)
Department of mathematics, faculty of natural science, university of Gangneung-Wonju, Gangneung, Gangwondo, postal code 25457, Korea

Yong Chan Kim (yck@gwnu.ac.kr)
Department of mathemathics, faculty of natural science, university of Gangneung-Wonju, Gangneung, Gangwondo, postal code 25457, Korea