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Abstract. The idea of Łukasiewicz t-norm is used to construct the concept of Łukasiewicz fuzzy sets based on a given fuzzy set. The Łukasiewicz fuzzy sets are applied to BCK-algebras and BCI-algebras. Moreover, the notion of Łukasiewicz fuzzy subalgebra is introduced and its various properties are investigated. Three types of subsets so called $\in$-set, q-set and O-set are constructed, and the conditions under which they can be subalgebras are explored.

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Corresponding Author: Y. B. Jun (skywine@gmail.com)

1. Introduction

A fuzzy concept, which is introduced by Zadeh [1], is understood as a concept which is "to an extent applicable" in a situation. That means the concept has gradations of significance or unsharp (variable) boundaries of application. Prior to the emergence of the fuzzy set, the very idea of inferring as an unclear concept faced considerable resistance from the elite in the academic world. They did not want to endorse the use of imprecise concepts in research or argumentation. Yet although people might not be aware of it, the use of fuzzy concepts has risen gigantically in all walks of life from the 1970s onward. That is mainly due to advances in electronic engineering, fuzzy mathematics and digital computer programming. The new technology allows very complex inferences about "variations on a theme" to be anticipated and fixed in a program. Łukasiewicz logic, which is the logic of the Łukasiewicz t-norm, is a non-classical and many-valued logic. It was originally defined in the early 20th century by Łukasiewicz as a three-valued logic. BCK/BCI-algebras originally defined by Iséki and Tanaka in [2] to generalize the set difference in set theory. In 1999, Jun et al. [3] studied fuzzy subalgebras and fuzzy ideals in BCK-algebras. Also, Jun [4] studied fuzzy subalgebras with thresholds in BCK/BCI-algebras.

In this paper, using the idea of Łukasiewicz t-norm, we construct the concept of Łukasiewicz fuzzy sets based on a given fuzzy set and apply it to BCK-algebras and BCI-algebras. We define the concepts of (strong) Łukasiewicz fuzzy subalgebras, and investigate several properties. We provide conditions for Łukasiewicz fuzzy set to be a Łukasiewicz fuzzy subalgebra. We explore the conditions under which Łukasiewicz fuzzy subalgebra becomes strong. We disuss characterizations of Łukasiewicz fuzzy subalgebras. We construct a three kind of subsets so called $\in$-set, q-set and O-set, and we find the conditions under which they can be subalgebras.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by Iséki (See [5] and [2]) and was extensively investigated by several researchers.

We recall the definitions and basic results required in this paper. See the books [6, 7] for further information regarding BCK-algebras and BCI-algebras.

If a set X has a special element 0 and a binary operation $*$ satisfying the conditions:

(I_1) $(\forall a, b, c \in X)\ (((a * b) * (a * c)) * (c * b) = 0)$,
(I_2) $(\forall a, b \in X)\ ((a * (a * b)) * b = 0)$,
(I_3) $(\forall a \in X)\ (a * a = 0)$,
(I_4) $(\forall a, b \in X)\ (a * b = 0,\ b * a = 0\ \Rightarrow\ a = b)$,

then we say that X is a *BCI-algebra*. If a BCI-algebra X satisfies the following identity:

(K) $(\forall a \in X)\ (0 * a = 0)$,

then X is called a *BCK-algebra*.

A BCI-algebra X is said to be *p-semisimple* (See [6]), if $0 * (0 * a) = a$ for all $a \in X$.

The order relation "$\leq$" in a BCK/BCI-algebra X is defined as follows:

$$(\forall a, b \in X)(a \leq b\ \Leftrightarrow\ a * b = 0). \tag{2.1}$$

Every BCK/BCI-algebra X satisfies the following conditions (See [6, 7]):

$$(\forall a \in X)\,(a * 0 = a)\,, \tag{2.2}$$

$$(\forall a, b, c \in X)\,(a \leq b\ \Rightarrow\ a * c \leq b * c,\ c * b \leq c * a)\,, \tag{2.3}$$

$$(\forall a, b, c \in X)\,((a * b) * c = (a * c) * b)\,. \tag{2.4}$$

Every BCI-algebra X satisfies (See [6]):

$$(\forall a, b \in X)\,(a * (a * (a * b)) = a * b)\,, \tag{2.5}$$

$$(\forall a, b \in X)\,(0 * (a * b) = (0 * a) * (0 * b))\,. \tag{2.6}$$

A subset A of a BCK/BCI-algebra X is called a *subalgebra* of X (See [6, 7]), if it satisfies:

$$(\forall a, b \in A)(a * b \in A), \tag{2.7}$$

A fuzzy set f in a set X of the form

$$f(b) := \begin{cases} t \in (0,1] & \text{if } b = a, \\ 0 & \text{if } b \neq a, \end{cases}$$

is said to be a *fuzzy point* with support a and value t and is denoted by $[a/t]$.

For a fuzzy set f in a set X, we say that a fuzzy point $[a/t]$ is

(i) *contained* in f, denoted by $[a/t] \in f$ ([8]), if $f(a) \geq t$,
(ii) *quasi-coincident* with f, denoted by $[a/t]\, q\, f$ ([8]), if $f(a) + t > 1$.

A fuzzy set f in a BCK/BCI-algebra X is called a *fuzzy subalgebra* of X, if it satisfies:

$$(\forall a, b \in X)(f(a * b) \geq \min\{f(a), f(b)\}). \tag{2.8}$$

3. Łukasiewicz fuzzy subalgebras

Definition 3.1. Let f be a fuzzy set in a set X and let $\varepsilon \in [0,1]$. A function

$$Ł_f^\varepsilon : X \to [0,1],\ x \mapsto \max\{0, f(x) + \varepsilon - 1\}$$

is called an *ε-Łukasiewicz fuzzy set* of f in X.

Let $Ł_f^\varepsilon$ be an ε-Łukasiewicz fuzzy set of a fuzzy set f in X. If $\varepsilon = 1$, then $Ł_f^\varepsilon(x) = \max\{0, f(x)+1-1\} = \max\{0, f(x)\} = f(x)$ for all $x \in X$. This shows that if $\varepsilon = 1$, then the ε-Łukasiewicz fuzzy set of a fuzzy set f in X is the classisical fuzzy set f itself in X. If $\varepsilon = 0$, then $Ł_f^\varepsilon(x) = \max\{0, f(x)+0-1\} = \max\{0, f(x)-1\} = 0$ for all $x \in X$, that is, if $\varepsilon = 0$, then the ε-Łukasiewicz fuzzy set is the zero fuzzy set. Therefore, in handling the ε-Łukasiewicz fuzzy set, the value of ε can always be considered to be in $(0,1)$.

Let f be a fuzzy set in a set X and $\varepsilon \in (0,1)$. If $f(x) + \varepsilon \leq 1$ for all $x \in X$, then the ε-Łukasiewicz fuzzy set $Ł_f^\varepsilon$ of f in X is the 0-constant function, that is, $Ł_f^\varepsilon(x) = 0$ for all $x \in X$. Therefore, in order for the ε-Łukasiewicz fuzzy set to have a meaningful form, a fuzzy set f in X and $\varepsilon \in (0,1)$ must be set to satisfy the following condition:

$$(\exists x \in X)(f(x) + \varepsilon > 1). \tag{3.1}$$

Proposition 3.2. *If f is a fuzzy set in a set X and $\varepsilon \in (0,1)$, then its ε-Łukasiewicz fuzzy set $Ł_f^\varepsilon$ satisfies:*

$$(\forall x, y \in X)(f(x) \geq f(y) \;\Rightarrow\; Ł_f^\varepsilon(x) \geq Ł_f^\varepsilon(y)), \tag{3.2}$$

$$(\forall x \in X)([x/\varepsilon]\, q\, f \;\Rightarrow\; Ł_f^\varepsilon(x) = f(x) + \varepsilon - 1), \tag{3.3}$$

$$(\forall x \in X)(\forall \delta \in (0,1))(\varepsilon \geq \delta \;\Rightarrow\; Ł_f^\varepsilon(x) \geq Ł_f^\delta(x)). \tag{3.4}$$

Proof. Straightforward. □

Proposition 3.3. *If f and g are fuzzy sets in a set X, then*

$$(\forall \varepsilon \in (0,1))\left(Ł_{f\cap g}^\varepsilon = Ł_f^\varepsilon \cap Ł_g^\varepsilon\right). \tag{3.5}$$

Proof. For every $x \in X$, we have

$$\begin{aligned} Ł^{\varepsilon}_{f\cap g}(x) &= \max\{0, (f \cap g)(x) + \varepsilon - 1\} \\ &= \max\{0, \min\{f(x), g(x)\} + \varepsilon - 1\} \\ &= \max\{0, \min\{f(x) + \varepsilon - 1, g(x) + \varepsilon - 1\}\} \\ &= \min\{\max\{0, f(x) + \varepsilon - 1\}, \max\{0, g(x) + \varepsilon - 1\}\} \\ &= \min\{Ł^{\varepsilon}_f(x), Ł^{\varepsilon}_g(x)\} = (Ł^{\varepsilon}_f \cap Ł^{\varepsilon}_g)(x) \end{aligned}$$

which proves (3.5). □

In what follows, let X be a BCK-algebra or a BCI-algebra, and ε is an element of $(0, 1)$ unless otherwise specified.

Definition 3.4. Let f be a fuzzy set in X. Then its ε-Łukasiewicz fuzzy set $Ł^{\varepsilon}_f$ in X is called an *ε-Łukasiewicz fuzzy subalgebra* of X if it satisfies:

$$[x/t_a] \in Ł^{\varepsilon}_f,\ [y/t_b] \in Ł^{\varepsilon}_f \ \Rightarrow [(x * y)/\min\{t_a, t_b\}] \in Ł^{\varepsilon}_f \tag{3.6}$$

for all $x, y \in X$ and $t_a, t_b \in (0, 1]$.

Example 3.5. Consider a BCK-algebra $X = \{0, a_1, a_2, a_3, a_4\}$ (See [7]) with a binary operation "$*$" given by Table 1.

Table 1. Cayley table for the binary operation "$*$"

$*$	0	a_1	a_2	a_3	a_4
0	0	0	0	0	0
a_1	a_1	0	a_1	0	0
a_2	a_2	a_2	0	0	0
a_3	a_3	a_3	a_3	0	0
a_4	a_4	a_3	a_4	a_1	0

Define a fuzzy set f in X as follows:

$$f : X \to [0, 1],\ x \mapsto \begin{cases} 0.76 & \text{if } x = 0, \\ 0.69 & \text{if } x = a_1, \\ 0.63 & \text{if } x = a_2, \\ 0.57 & \text{if } x = a_3, \\ 0.42 & \text{if } x = a_4. \end{cases}$$

Given $\varepsilon := 0.55$, the ε-Łukasiewicz fuzzy set $Ł^{\varepsilon}_f$ of f in X is given as follows:

$$Ł^{\varepsilon}_f : X \to [0, 1],\ x \mapsto \begin{cases} 0.31 & \text{if } x = 0, \\ 0.24 & \text{if } x = a_1, \\ 0.18 & \text{if } x = a_2, \\ 0.12 & \text{if } x = a_3, \\ 0 & \text{if } x = a_4. \end{cases}$$

It is routine to verify that $Ł^{\varepsilon}_f$ is an ε-Łukasiewicz fuzzy subalgebra of X.

Theorem 3.6. *If f is a fuzzy subalgebra of X, then its ε-Łukasiewicz fuzzy set $Ł^{\varepsilon}_f$ in X is an ε-Łukasiewicz fuzzy subalgebra of X.*

Proof. Assume that f is a fuzzy subalgebra of X. Let $x, y \in X$ and $t_a, t_b \in (0,1]$ be such that $[x/t_a] \in Ł_f^\varepsilon$ and $[y/t_b] \in Ł_f^\varepsilon$. Then $Ł_f^\varepsilon(x) \geq t_a$ and $Ł_f^\varepsilon(y) \geq t_b$. Thus

$$\begin{aligned} Ł_f^\varepsilon(x * y) &= \max\{0, f(x*y) + \varepsilon - 1\} \\ &\geq \max\{0, \min\{f(x), f(y)\} + \varepsilon - 1\} \\ &= \max\{0, \min\{f(x) + \varepsilon - 1, f(y) + \varepsilon - 1\}\} \\ &= \min\{\max\{0, f(x) + \varepsilon - 1\}, \max\{0, f(y) + \varepsilon - 1\}\} \\ &= \min\{Ł_f^\varepsilon(x), Ł_f^\varepsilon(y)\} \geq \min\{t_a, t_b\}. \end{aligned}$$

So $[(x*y)/\min\{t_a, t_b\}] \in Ł_f^\varepsilon$. Hence $Ł_f^\varepsilon$ is an ε-Lukasiewicz fuzzy subalgebra of X. □

The following example shows that the converse of Theorem 3.6 may not be true.

Example 3.7. Consider a BCI-algebra $X = \{0, a_1, a_2, a_3, a_4\}$ (See [6]) with a binary operation "$*$" given by Table 2.

Table 2. Cayley table for the binary operation "$*$"

$*$	0	a_1	a_2	a_3	a_4
0	0	0	a_2	a_3	a_4
a_1	a_1	0	a_2	a_3	a_4
a_2	a_2	a_2	0	a_4	a_3
a_3	a_3	a_3	a_4	0	a_2
a_4	a_4	a_4	a_3	a_2	0

Define a fuzzy set f in X as follows:

$$f : X \to [0,1],\ x \mapsto \begin{cases} 0.72 & \text{if } x = 0, \\ 0.68 & \text{if } x = a_1, \\ 0.61 & \text{if } x = a_2, \\ 0.57 & \text{if } x = a_3, \\ 0.39 & \text{if } x = a_4. \end{cases}$$

Given $\varepsilon := 0.42$, the ε-Lukasiewicz fuzzy set $Ł_f^\varepsilon$ of f in X is given as follows:

$$Ł_f^\varepsilon : X \to [0,1],\ x \mapsto \begin{cases} 0.14 & \text{if } x = 0, \\ 0.10 & \text{if } x = a_1, \\ 0.03 & \text{if } x = a_2, \\ 0 & \text{if } x = a_3, \\ 0 & \text{if } x = a_4. \end{cases}$$

It is routine to verify that $Ł_f^\varepsilon$ is an ε-Lukasiewicz fuzzy subalgebra of X. But f is not a fuzzy subalgebra of X because of

$$f(a_2 * a_3) = f(a_4) = 0.39 \ngeq 0.57 = \min\{f(a_2), f(a_3)\}.$$

We consider a characterization of ε-Lukasiewicz fuzzy subalgebra.

Theorem 3.8. *Let f be a fuzzy set in X. Then its ε-Lukasiewicz fuzzy set $Ł_f^\varepsilon$ in X is an ε-Lukasiewicz fuzzy subalgebra of X if and only if it satisfies:*

$$(3.7) \qquad (\forall x, y \in X)(Ł_f^\varepsilon(x * y) \geq \min\{Ł_f^\varepsilon(x), Ł_f^\varepsilon(y)\}).$$

Proof. Assume that $Ł_f^\varepsilon$ is an ε-Łukasiewicz fuzzy subalgebra of X. Let $x, y \in X$. It is clear that $[x/Ł_f^\varepsilon(x)] \in Ł_f^\varepsilon$ and $[y/Ł_f^\varepsilon(y)] \in Ł_f^\varepsilon$. Then

$$[(x * y)/\min\{Ł_f^\varepsilon(x), Ł_f^\varepsilon(y)\}] \in Ł_f^\varepsilon$$

by (3.6), which implies that $Ł_f^\varepsilon(x * y) \geq \min\{Ł_f^\varepsilon(x), Ł_f^\varepsilon(y)\}$.

Conversely, suppose that $Ł_f^\varepsilon$ satisfies the condition (3.7). Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $[x/t_a] \in Ł_f^\varepsilon$ and $[y/t_b] \in Ł_f^\varepsilon$. Then $Ł_f^\varepsilon(x) \geq t_a$ and $Ł_f^\varepsilon(y) \geq t_b$, which implies from (3.7) that

$$Ł_f^\varepsilon(x * y) \geq \min\{Ł_f^\varepsilon(x), Ł_f^\varepsilon(y)\} \geq \min\{t_a, t_b\}.$$

Thus $[(x * y)_{\min\{t_b, t_b\}}] \in Ł_f^\varepsilon$. So $Ł_f^\varepsilon$ is an ε-Łukasiewicz fuzzy subalgebra of X. □

Proposition 3.9. *If f is a fuzzy subalgebra of X, then its ε-Lukasiewicz fuzzy set $Ł_f^\varepsilon$ satisfies:*

$$(3.8) \qquad (\forall x \in X)(Ł_f^\varepsilon(0) \geq Ł_f^\varepsilon(x)).$$

Proof. If f is a fuzzy subalgebra of X, then $f(0) = f(x*x) \geq \min\{f(x), f(x)\} = f(x)$ for all $x \in X$. It follows from (3.2) that $Ł_f^\varepsilon(0) \geq Ł_f^\varepsilon(x)$ for all $x \in X$. □

Proposition 3.10. *If f is a fuzzy subalgebra of X, then its ε-Lukasiewicz fuzzy set $Ł_f^\varepsilon$ satisfies:*

$$(3.9) \qquad (\forall x, y \in X)\left(Ł_f^\varepsilon(x) = Ł_f^\varepsilon(0) \ \Leftrightarrow \ Ł_f^\varepsilon(x * y) \geq Ł_f^\varepsilon(y)\right).$$

Proof. Assume that $Ł_f^\varepsilon(x) = Ł_f^\varepsilon(0)$ for all $x \in X$. Then

$$Ł_f^\varepsilon(x * y) \geq \min\{Ł_f^\varepsilon(x), Ł_f^\varepsilon(y)\} = \min\{Ł_f^\varepsilon(0), Ł_f^\varepsilon(y)\} = Ł_f^\varepsilon(y)$$

for all $x, y \in X$ by the combination of Theorem 3.6 and Proposition 3.9.

Conversely, suppose that $Ł_f^\varepsilon(x * y) \geq Ł_f^\varepsilon(y)$ for all $x, y \in X$. Using (2.2) induces $Ł_f^\varepsilon(x) = Ł_f^\varepsilon(x * 0) \geq Ł_f^\varepsilon(0)$. The combination of this and Proposition 3.9 leads to $Ł_f^\varepsilon(x) = Ł_f^\varepsilon(0)$ for all $x \in X$. □

Proposition 3.11. *If f is a fuzzy subalgebra of a BCI-algebra X, then its ε-Lukasiewicz fuzzy set $Ł_f^\varepsilon$ satisfies:*

$$(3.10) \qquad (\forall x \in X)(Ł_f^\varepsilon(0 * x) \geq Ł_f^\varepsilon(x)).$$

Proof. If f is a fuzzy subalgebra of a BCI-algebra X, then

$$f(0 * x) \geq \min\{f(0), f(x)\} = f(x)$$

for all $x \in X$. It follows from (3.2) that $Ł_f^\varepsilon(0 * x) \geq Ł_f^\varepsilon(x)$ for all $x \in X$. □

Proposition 3.12. *If f is a fuzzy subalgebra of a BCI-algebra X, then its ε-Lukasiewicz fuzzy set $Ł_f^\varepsilon$ satisfies:*

$$(3.11) \qquad [x/t_a] \in Ł_f^\varepsilon,\ [y/t_b] \in Ł_f^\varepsilon \ \Rightarrow [(x * (0 * y))/\min\{t_a, t_b\}] \in Ł_f^\varepsilon$$

for all $x, y \in X$ and $t_a, t_b \in (0, 1]$.

Proof. Let $x, y \in X$ and $t_a, t_b \in (0,1]$ be such that $[x/t_a] \in \text{Ł}_f^\varepsilon$ and $[y/t_b] \in \text{Ł}_f^\varepsilon$. Then $\text{Ł}_f^\varepsilon(x) \geq t_a$ and $\text{Ł}_f^\varepsilon(y) \geq t_b$. Thus

$$\begin{aligned}
\text{Ł}_f^\varepsilon(x*(0*y)) &= \max\{0, f(x*(0*y)) + \varepsilon - 1\} \\
&\geq \max\{0, \min\{f(x), f(0*y)\} + \varepsilon - 1\} \\
&\geq \max\{0, \min\{f(x), \min\{f(0), f(y)\}\} + \varepsilon - 1\} \\
&= \max\{0, \min\{f(x), f(y)\} + \varepsilon - 1\} \\
&= \max\{0, \min\{f(x) + \varepsilon - 1, f(y) + \varepsilon - 1\}\} \\
&= \min\{\max\{0, f(x) + \varepsilon - 1\}, \max\{0, f(y) + \varepsilon - 1\}\} \\
&= \min\{\text{Ł}_f^\varepsilon(x), \text{Ł}_f^\varepsilon(y)\} \\
&\geq \min\{t_a, t_b\}.
\end{aligned}$$

So $[(x*(0*y))/\min\{t_a, t_b\}] \in \text{Ł}_f^\varepsilon$. □

We provide conditions for a Łukasiewicz fuzzy set to be a Łukasiewicz fuzzy subalgebra.

Theorem 3.13. *Let f be a fuzzy set in X. If ε-Łukasiewicz fuzzy set Ł_f^ε of f in X satisfies:*

$$[y/t_b] \in \text{Ł}_f^\varepsilon,\ [z/t_c] \in \text{Ł}_f^\varepsilon \ \Rightarrow\ [(x*y)/\min\{t_b, t_c\}] \in \text{Ł}_f^\varepsilon \tag{3.12}$$

for all $t_b, t_c \in (0,1]$ and $x, y, z \in X$ with $z \leq x$, then Ł_f^ε is an ε-Łukasiewicz fuzzy subalgebra of X.

Proof. Let $x, y \in X$ and $t_a, t_b \in (0,1]$ be such that $[x/t_a] \in \text{Ł}_f^\varepsilon$ and $[y/t_b] \in \text{Ł}_f^\varepsilon$. Since $x \leq x$ for all $x \in X$, it follows from (3.12) that $[(x*y)/\min\{t_a, t_b\}] \in \text{Ł}_f^\varepsilon$. Hence Ł_f^ε is an ε-Łukasiewicz fuzzy subalgebra of X. □

Proposition 3.14. *Let f be a fuzzy set in a BCI-algebra X. Then every ε-Łukasiewicz fuzzy subalgebra Ł_f^ε of X satisfies:*

$$[x/t_a] \in \text{Ł}_f^\varepsilon,\ [y/t_b] \in \text{Ł}_f^\varepsilon \ \Rightarrow [(x*(0*y))/\min\{t_a, t_b\}] \in \text{Ł}_f^\varepsilon \tag{3.13}$$

for all $x, y \in X$ and $t_a, t_b \in (0,1]$.

Proof. Let $x, y \in X$ and $t_a, t_b \in (0,1]$ be such that $[x/t_a] \in \text{Ł}_f^\varepsilon$ and $[y/t_b] \in \text{Ł}_f^\varepsilon$. Then $\text{Ł}_f^\varepsilon(x) \geq t_a$ and $\text{Ł}_f^\varepsilon(y) \geq t_b$. It follows from Theorem 3.8 and Proposition 3.9 that

$$\begin{aligned}
\text{Ł}_f^\varepsilon(x*(0*y)) &\geq \min\{\text{Ł}_f^\varepsilon(x), \text{Ł}_f^\varepsilon(0*y)\} \\
&\geq \min\{\text{Ł}_f^\varepsilon(x), \min\{\text{Ł}_f^\varepsilon(0), \text{Ł}_f^\varepsilon(y)\}\} \\
&= \{\text{Ł}_f^\varepsilon(x), \text{Ł}_f^\varepsilon(y)\} \geq \min\{t_a, t_b\},
\end{aligned}$$

i.e., $[(x*(0*y))/\min\{t_a, t_b\}] \in \text{Ł}_f^\varepsilon$. □

Given a fuzzy set f in a BCI-algebra X, its ε-Łukasiewicz fuzzy subalgebra Ł_f^ε does not satisfiy the following equality:

$$(\forall x \in X)(\text{Ł}_f^\varepsilon(0*x) = \text{Ł}_f^\varepsilon(x)). \tag{3.14}$$

In fact, the $\varepsilon(=0.42)$-Lukasiewicz fuzzy subalgebra Ł_f^ε in Example 3.7 does not satisfiy (3.14) since $\text{Ł}_f^\varepsilon(0*a_1)=\text{Ł}_f^\varepsilon(0)=0.14\neq 0.10=\text{Ł}_f^\varepsilon(a_1)$.

Let f be a fuzzy set in a BCI-algebra X. If an ε-Łukasiewicz fuzzy subalgebra Ł_f^ε of X satisfies the condition (3.14), we say it is *strong*.

We provide a condition for a Łukasiewicz fuzzy subalgebra to be strong.

Theorem 3.15. *Let f be a fuzzy set in a BCI-algebra X. If X is p-semisimple, then every ε-Łukasiewicz fuzzy subalgebra Ł_f^ε of X is strong.*

Proof. Assume that X is a p-semisimple BCI-algebra and let Ł_f^ε be an ε-Łukasiewicz fuzzy subalgebra of X. Then it satisfies the condition (3.13) (See Proposition 3.14). In the same way as the proof of Theorem 3.8, it can be revealed that (3.13) is equivalent to the following.

$$(\forall x,y\in X)(\text{Ł}_f^\varepsilon(x*(0*y))\geq \min\{\text{Ł}_f^\varepsilon(x),\text{Ł}_f^\varepsilon(y)\}). \tag{3.15}$$

Using (2.5), Theorem 3.8, Proposition 3.9 and the p-semisimplicity of X, we have

$$\begin{aligned}\text{Ł}_f^\varepsilon(x)&=\text{Ł}_f^\varepsilon(0*(0*x))\geq\min\{\text{Ł}_f^\varepsilon(0),\text{Ł}_f^\varepsilon(0*x)\}\\&=\text{Ł}_f^\varepsilon(0*x)=\text{Ł}_f^\varepsilon(0*(0*(0*x)))\\&\geq\min\{\text{Ł}_f^\varepsilon(0),\text{Ł}_f^\varepsilon(0*(0*x))\}=\text{Ł}_f^\varepsilon(0*(0*x))\\&\geq\text{Ł}_f^\varepsilon(x).\end{aligned}$$

Thus $\text{Ł}_f^\varepsilon(x)=\text{Ł}_f^\varepsilon(0*x)$ for all $x\in X$. So Ł_f^ε is a strong ε-Łukasiewicz fuzzy subalgebra of X. □

Corollary 3.16. *Let X be a BCI-algebra which satisfies any one of the following conditions:*

$$X=\{0*x\mid x\in X\}, \tag{3.16}$$

$$(\forall x,y\in X)(x*(0*y)=y*(0*x)), \tag{3.17}$$

$$(\forall x\in X)(0*x=0\ \Rightarrow\ x=0), \tag{3.18}$$

$$(\forall x,y\in X)(0*(y*x)=x*y), \tag{3.19}$$

$$(\forall x,y,z\in X)(z*x=z*y\ \Rightarrow\ x=y), \tag{3.20}$$

$$(\forall x,y,z\in X)((x*y)*(x*z)=z*y). \tag{3.21}$$

If f is a fuzzy subalgebra of X, then Ł_f^ε is a strong ε-Łukasiewicz fuzzy subalgebra of X.

Proof. It is strateforward by Theorems 3.6 and 3.15 because the p-semisimplicity of BCI algebra is equivalent to each of conditions (3.16)–(3.21) (See [9]). □

Let f be a fuzzy set in X. For an ε-Łukasiewicz fuzzy set Ł_f^ε of f in X and $t\in(0,1]$, consider the sets

$$(\text{Ł}_f^\varepsilon,t)_\in:=\{x\in X\mid [x/t]\in\text{Ł}_f^\varepsilon\},$$
$$(\text{Ł}_f^\varepsilon,t)_q:=\{x\in X\mid [x/t]\,q\,\text{Ł}_f^\varepsilon\},$$

which are called the $\in$-*set* and q-*set*, respectively, of Ł_f^ε (with value t).

We explore the conditions under which the $\in$-set and q-set of Łukasiewicz fuzzy set can be subalgebras.

Theorem 3.17. *Let $Ł_f^\varepsilon$ be an ε-Łukasiewicz fuzzy set of a fuzzy set f in X. Then the $\in$-set $(Ł_f^\varepsilon, t)_\in$ of $Ł_f^\varepsilon$ with value $t \in (0.5, 1]$ is a subalgebra of X if and only if the following assertion is valid.*

$$(\forall x, y \in X)\left(\min\{Ł_f^\varepsilon(x), Ł_f^\varepsilon(y)\} \leq \max\{Ł_f^\varepsilon(x * y), 0.5\}\right). \tag{3.22}$$

Proof. Assume that the $\in$-set $(Ł_f^\varepsilon, t)_\in$ of $Ł_f^\varepsilon$ with value $t \in (0.5, 1]$ is a subalgebra of X. If the condition (3.22) is not valid, then there exist $a, b \in X$ such that

$$\min\{Ł_f^\varepsilon(a), Ł_f^\varepsilon(b)\} > \max\{Ł_f^\varepsilon(a * b), 0.5\}.$$

If we take $s := \min\{Ł_f^\varepsilon(a), Ł_f^\varepsilon(b)\}$, then $s \in (0.5, 1]$ and $[a/s], [b/s] \in Ł_f^\varepsilon$, i.e., $a, b \in (Ł_f^\varepsilon, s)_\in$. Since $(Ł_f^\varepsilon, s)_\in$ is a subalgebra of X, we have $a * b \in (Ł_f^\varepsilon, s)_\in$. But $[(a * b)/s] \notin Ł_f^\varepsilon$ implies $a * b \notin (Ł_f^\varepsilon, s)_\in$, a contradiction. Thus we have

$$\min\{Ł_f^\varepsilon(x), Ł_f^\varepsilon(y)\} \leq \max\{Ł_f^\varepsilon(x * y), 0.5\} \text{ for all } x,\ y \in X.$$

Conversely, suppose that $Ł_f^\varepsilon$ satisfies (3.22). Let $t \in (0.5, 1]$ and $x, y \in X$ be such that $x \in (Ł_f^\varepsilon, t)_\in$ and $y \in (Ł_f^\varepsilon, t)_\in$. Then $Ł_f^\varepsilon(x) \geq t$ and $Ł_f^\varepsilon(y) \geq t$, which imply from (3.22) that

$$0.5 < t \leq \min\{Ł_f^\varepsilon(x), Ł_f^\varepsilon(y)\} \leq \max\{Ł_f^\varepsilon(x * y), 0.5\}.$$

Thus $[(x * y)/t] \in Ł_f^\varepsilon$, i.e., $x * y \in (Ł_f^\varepsilon, t)_\in$. So $(Ł_f^\varepsilon, t)_\in$ is a subalgebra of X for $t \in (0.5, 1]$. $\square$

Theorem 3.18. *Let $Ł_f^\varepsilon$ be an ε-Łukasiewicz fuzzy set of a fuzzy set f in X. If f is a fuzzy subalgebra of X, then the q-set $(Ł_f^\varepsilon, t)_q$ of $Ł_f^\varepsilon$ with value $t \in (0, 1]$ is a subalgebra of X.*

Proof. Let $t \in (0, 1]$ and $x, y \in (Ł_f^\varepsilon, t)_q$. Then $[x/t]\, q\, Ł_f^\varepsilon$ and $[y/t]\, q\, Ł_f^\varepsilon$, i.e., $Ł_f^\varepsilon(x) + t > 1$ and $Ł_f^\varepsilon(y) + t > 1$. It follows from Theorems 3.6 and 3.8 that

$$Ł_f^\varepsilon(x * y) + t \geq \min\{Ł_f^\varepsilon(x), Ł_f^\varepsilon(y)\} + t = \min\{Ł_f^\varepsilon(x) + t, Ł_f^\varepsilon(y) + t\} > 1.$$

Thus $[(x * y)/t]\, q\, Ł_f^\varepsilon$. So $x * y \in (Ł_f^\varepsilon, t)_q$. Hence $(Ł_f^\varepsilon, t)_q$ is a subalgebra of X. $\square$

Theorem 3.19. *Let f be a fuzzy set in X. For an ε-Łukasiewicz fuzzy set $Ł_f^\varepsilon$ of f in X, if the q-set $(Ł_f^\varepsilon, t)_q$ is a subalgebra of X, then $Ł_f^\varepsilon$ satisfies:*

$$[x/t_a]\, q\, Ł_f^\varepsilon,\ [y/t_b]\, q\, Ł_f^\varepsilon\ \Rightarrow [(x * y)/\max\{t_a, t_b\}] \in Ł_f^\varepsilon \tag{3.23}$$

for all $x, y \in X$ and $t_a, t_b \in (0, 0.5]$.

Proof. Let $x, y \in X$ and $t_a, t_b \in (0, 0.5]$ be such that $[x/t_a]\, q\, Ł_f^\varepsilon$ and $[y/t_b]\, q\, Ł_f^\varepsilon$. Then $x \in (Ł_f^\varepsilon, t_a)_q \subseteq (Ł_f^\varepsilon, \max\{t_a, t_b\})_q$ and $y \in (Ł_f^\varepsilon, t_b)_q \subseteq (Ł_f^\varepsilon, \max\{t_a, t_b\})_q$. Thus $x * y \in (Ł_f^\varepsilon, \max\{t_a, t_b\})_q$. Since $\max\{t_a, t_b\} \leq 0.5$,

$$Ł_f^\varepsilon(x * y) > 1 - \max\{t_a, t_b\} \geq \max\{t_a, t_b\}.$$

So $[(x * y)/\max\{t_a, t_b\}] \in Ł_f^\varepsilon$. $\square$

Let f be a fuzzy set in X. For an ε-Łukasiewicz fuzzy set $Ł_f^\varepsilon$ of f in X, consider a set:

$$O(Ł_f^\varepsilon) := \{x \in X \mid Ł_f^\varepsilon(x) > 0\} \tag{3.24}$$

which is called an *O-set* of $Ł_f^\varepsilon$. It is observed that

$$O(Ł_f^\varepsilon) = \{x \in X \mid f(x) + \varepsilon - 1 > 0\}.$$

Theorem 3.20. *Let $Ł_f^\varepsilon$ be an ε-Lukasiewicz fuzzy set of a fuzzy set f in X. If f is a fuzzy subalgebra of X, then the O-set $O(Ł_f^\varepsilon)$ of $Ł_f^\varepsilon$ is a subalgebra of X.*

Proof. Let $x, y \in O(Ł_f^\varepsilon)$. Then $f(x) + \varepsilon - 1 > 0$ and $f(y) + \varepsilon - 1 > 0$. If f is a fuzzy subalgebra of X, then $Ł_f^\varepsilon$ is an ε-Lukasiewicz fuzzy subalgebra of X (See Theoem 3.6). It follows from Theoem 3.8 that

$$Ł_f^\varepsilon(x * y) \geq \min\{Ł_f^\varepsilon(x), Ł_f^\varepsilon(y)\} = \min\{f(x) + \varepsilon - 1, f(y) + \varepsilon - 1\} > 0$$

Thus $x * y \in O(Ł_f^\varepsilon)$. So $O(Ł_f^\varepsilon)$ is a subalgebra of X. □

Theorem 3.21. *Let f be a fuzzy set in X. If an ε-Lukasiewicz fuzzy set $Ł_f^\varepsilon$ of f in X satisfies:*

$$[x/t_a] \in Ł_f^\varepsilon,\ [y/t_b] \in Ł_f^\varepsilon \ \Rightarrow [(x * y)/\max\{t_a, t_b\}]\, q\, Ł_f^\varepsilon \tag{3.25}$$

for all $x, y \in X$ and $t_a, t_b \in (0, 1]$, then the O-set $O(Ł_f^\varepsilon)$ of $Ł_f^\varepsilon$ is a subalgebra of X.

Proof. Assume that $Ł_f^\varepsilon$ satisfies the condition (3.25) for all $x, y \in X$ and $t_a, t_b \in (0, 1]$. Let $x, y \in O(Ł_f^\varepsilon)$. Then $f(x) + \varepsilon - 1 > 0$ and $f(y) + \varepsilon - 1 > 0$. Since $[x/Ł_f^\varepsilon(x)] \in Ł_f^\varepsilon$ and $[y/Ł_f^\varepsilon(y)] \in Ł_f^\varepsilon$. it follows from (3.25) that

$$[(x * y)/\max\{Ł_f^\varepsilon(x), Ł_f^\varepsilon(y)\}]\, q\, Ł_f^\varepsilon. \tag{3.26}$$

If $x * y \notin O(Ł_f^\varepsilon)$, then $Ł_f^\varepsilon(x * y) = 0$. Thus we get

$$\begin{aligned} &Ł_f^\varepsilon(x * y) + \max\{Ł_f^\varepsilon(x), Ł_f^\varepsilon(y)\} = \max\{Ł_f^\varepsilon(x), Ł_f^\varepsilon(y)\} \\ &= \max\{\max\{0, f(x) + \varepsilon - 1\}, \max\{0, f(y) + \varepsilon - 1\}\} \\ &= \max\{f(x) + \varepsilon - 1, f(y) + \varepsilon - 1\} \\ &= \max\{f(x), f(y)\} + \varepsilon - 1 \\ &\leq 1 + \varepsilon - 1 = \varepsilon \leq 1, \end{aligned}$$

which shows that (3.26) is not valid. This is a contradiction. So $x * y \in O(Ł_f^\varepsilon)$. Hence $O(Ł_f^\varepsilon)$ is a subalgebra of X. □

Theorem 3.22. *Let f be a fuzzy set in X. If an ε-Lukasiewicz fuzzy set $Ł_f^\varepsilon$ of f in X satisfies the condition (3.23) for all $x, y \in X$ and $t_a, t_b \in (0, 1]$, then the O-set $O(Ł_f^\varepsilon)$ of $Ł_f^\varepsilon$ is a subalgebra of X.*

Proof. Let $x, y \in O(Ł_f^\varepsilon)$. Then $f(x) + \varepsilon - 1 > 0$ and $f(y) + \varepsilon - 1 > 0$. Hence

$$Ł_f^\varepsilon(x) + 1 = \max\{0, f(x) + \varepsilon - 1\} + 1 = f(x) + \varepsilon - 1 + 1 = f(x) + \varepsilon > 1$$

and

$$Ł_f^\varepsilon(y) + 1 = \max\{0, f(y) + \varepsilon - 1\} + 1 = f(y) + \varepsilon - 1 + 1 = f(y) + \varepsilon > 1,$$

i.e., $[x/1]\, q\, Ł_f^\varepsilon$ and $[y/1]\, q\, Ł_f^\varepsilon$. It follows from (3.23) that

$$[(x * y)/1] = [(x * y)/\max\{1, 1\}] \in Ł_f^\varepsilon. \tag{3.27}$$

If $x * y \notin O(Ł_f^\varepsilon)$, then $Ł_f^\varepsilon(x * y) = 0 < 1$ and so (3.27) is not valid. This is a contradiction Thus $x * y \in O(Ł_f^\varepsilon)$. So $O(Ł_f^\varepsilon)$ is a subalgebra of X. □

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Young Bae Jun (skywine@gmail.com)
Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea