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Abstract. The concept crossing intuitionistic structures set is combination of intuitionistic fuzzy set and N- function. In this paper, the concept crossing intuitionistic KU-ideals are introduced and several properties are investigated. Also, the relations between crossing intuitionistic KU-ideals and crossing intuitionistic ideals are given. The image and the pre-image of crossing intuitionistic KU-ideals under homomorphism of KU-algebras are defined and how the image and the pre-image of crossing intuitionistic KU-ideals under homomorphism of KU-algebras become crossing intuitionistic KU-ideals are studied. Moreover, the Cartesian product of crossing intuitionistic KU-ideals in Cartesian product KU-algebras is established.

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1. Introduction

BCK-algebras form an important class of logical algebras introduced by Iseki [1], and Iseki and Tanaka [2] and was extensively investigated by several researchers. It is an important way to research the algebras by its ideals. The notions of ideals and positive implicative ideals in BCK-algebras (i.e., Iseki's implicative ideals) were introduced by Iseki and Tanaka [1, 2]. Zadeh [3] coined the term "degree of membership" and defined the concept of a fuzzy set in order to deal with uncertainty. Atanassov [4, 5] incorporated the "degree of non-membership" in the concept of a fuzzy set as an independent component and proposed an intuitionistic fuzzy set. At present, this concept has been applied to many mathematical branches such as group, functional analysis, probability theory, topology and so on. In 1991, Xi [6] applied applied the concept of fuzzy sets to BCI, BCK, MV-algebras, and he introduced the notion of fuzzy subalgebras (ideals) of BCK-algebras with respect to minimum. After then, Jun [7] studied fuzzy ideals of BCK/BCI-algebras, and Ahmed and Amhed [8] discussed with various properties of fuzzy BCK-algebras. Prabpayak and Leerawat [9, 10] introduced a new algebraic structure which is called a *KU-algebra*. They defined a homomorphism of KU-algebras and studied some its properties. Mostafa et al. [11] introduced the concept of fuzzy KU-ideals of KU-algebras and dealt with several basic properties which are related to fuzzy KU-ideals. Also, Mostafa et al. [12] investigated intuitionistic fuzzy KU-ideals in KU-algebras. Senapati [13, 14] introduced the notion of fuzzy KU-subalgebras and KU-ideals of KU-algebras with respect to a given t-norm. Recently, by using triangular norm and co-norm, Senapati and Shum [15, 16] proposed Atanassov's intuitionistic fuzzy bi-normed KU- subalgebras/ideals of a KU-algebra, and obtained some of their properties.

Lee [17] introduced an extension of fuzzy sets named "bipolar-valued fuzzy sets". Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. Moreover, Lee [18] obtained some of bipolar fuzzy subalgebras and ideals of BCK/BCI-algebras. Recently, Jun et al. [19, 20] introduced a new function which is called a *negative-valued function* and *constructed N-structures*. They applied N-structures to BCK/BCI-algebras, and discussed with N-subalgebras and N-ideals in BCK/BCI-algebras. Also, Jun et al. [21, 22] established an extension of a bipolar-valued fuzzy set, which is introduced by Lee [17], they called it a *crossing cubic structure* and investigated several properties.

In this paper, the concept of crossing intuitionistic KU-ideals is introduced as an extension of bipolar fuzzy sets and several properties are investigated. Also, the relations between crossing KU-ideals and crossing intuitionistic ideals are given. The image and the pre-image of crossing intuitionistic KU-ideals under homomorphism of KU-algebras are defined and how the image and the pre-image of crossing intuitionistic ku-ideals under homomorphism of KU-algebras become crossing intuitionistic KU-ideals are studied. Moreover, the Cartesian product of crossing intuitionistic KU-ideals in Cartesian product KU-algebras is established.

2. Preliminaries

Now we review some definitions and properties that will be useful in our results.

Definition 2.1 ([9, 10])**.** A *KU-algebra* is a triple $(X, *, 0)$, where X is a nonempty set, $*$ is a binary operation on X and 0 is a fixed element of X such that the following axioms are satisfied: for all $x,\ y,\ z \in X$,

(KU_1) $(x * y) * [(y * z) * (x * z)] = 0$,

(KU_2) $x * 0 = 0$,

(KU_3) $0 * x = x$,

(KU_4) $x * y = 0$ and $y * x = 0$ implies $x = y$,

(KU_5) $x * x = 0$.

On a KU-algebra $(X, *, 0)$, we can define a binary relation $\leq$ on X by: for any $x,\ y \in X$,

$$x \leq y \Leftrightarrow y * x = 0.$$

Theorem 2.2 ([11]). *Let $(X, *, 0)$ be a KU-algebra. Then for all $x,\ y,\ z \in X$,*
(1) $x * (y * x) = 0$.
(2) *if* $x \leq y$*, then* $y * z \leq x * z$,
(3) $x * (y * z) = y * (x * z)$,
(4) $[y * (y * x)] * x = 0$.

Example 2.3. Let $X = \{0, a, b, c, d\}$ be the set with Cayley Table 2.1:

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	d	c	d
b	0	a	0	c	d
c	0	0	0	0	d
d	0	0	0	0	0

Table 2.1

Then $(X, *, 0)$ is a KU-algebra.

Definition 2.4 ([11]). Let I be a non-empty subset of a KU-algebra $(X, *, 0)$. Then I is called an *ideal* of X, if for all $x,\ y \in X$,
(i) $0 \in I$,
(ii) $y * x \in I$ and $y \in I$ imply $x \in I$.

Throughout this paper, let I denote the unit closed interval $[0, 1]$ in the set of real numbers $\mathbb{R}$. For a non-empty set X, a mapping $A : X \to I$ is called a *fuzzy set* in X (See [3]). The set of all fuzzy sets in X is denoted by I^X.

Let $I \oplus I = \{\bar{a} = (a^{\in}, a^{\notin}) \in I \times I : a^{\in} + a^{\notin} \leq 1\}$. Then each member $\bar{a}$ of $I \oplus I$ is called an *intuitionistic point* or *intuitionistic number*(See [23]).

Definition 2.5 ([11]). Let X be a KU-algebra and let $A \in I^X$. Then A is called a *fuzzy KU-ideal* of X, if it satisfies the following conditions:
(FI_1) $A(0) \geq A(x)$ for each $x \in X$,
(FI_2) $A(z * x) \geq A(z * (y * x)) \wedge A(y)$ for all $x,\ y,\ z \in X$.

Definition 2.6 ([4]). For a non-empty set X, a mapping $\bar{A} : X \to I \oplus I$ is called an *intuitionistic fuzzy set* (briefly, IF set) in X, where for each $x \in X$, $\bar{A}(x) = (A^{\in}(x), A^{\notin}(x))$, and $A^{\in}(x)$ and $A^{\notin}(x)$ represent the degree of membership and the degree of nonmembership of an element x to $\bar{A}$, respectively. Let $(I \oplus I)^X$ or $IFS(X)$ denote the set of all IF sets in X and for each $\bar{A} \in IFS(X)$, we write $A = (A^{\in}, A^{\notin})$. In particular, $\bar{\mathbf{0}}$ and $\bar{\mathbf{1}}$ denote the IF empty set and the IF whole set in X defined by, respectively: for each $x \in X$,

$$\bar{\mathbf{0}}(x) = \bar{0} \text{ and } \bar{\mathbf{1}}(x) = \bar{1}.$$

Definition 2.7 ([19, 20]). Let X be a non-empty set. Then a mapping $A^N : X \to [-1, 0]$ is called a *negative-valued function* (briefly, N-function) on X. The pair (X, A^N) is called an *N-structure*. The set of all N-functions on X is denoted by $NF(X)$.

Definition 2.8 ([19, 20]). Let X be a non-empty set and let $A^N,\ B^N \in NF(X)$.

(i) We say that A^N is a *subset* of B^N, denoted by $A^N \subset B^N$, if $A^N(x) \geq B^N(x)$ for each $x \in X$.

(ii) The *complement* of A^N, denoted by $c(A^N)$, is an N-function on X defined as: for each $x \in X$,

$$c(A^N)(x) = -1 - A^N(x).$$

(iii) The *intersection* of A^N and B^N, denoted by $A^N \cap B^N$, is an N-function on X defined as: for each $x \in X$,

$$(A^N \cap B^N)(x) = A^N(x) \vee B^N(x)(x).$$

(iv) The *union* of A^N and B^N, denoted by $A^N \cup B^N$, is an N-function on X defined as: for each $x \in X$,

$$(A^N \cup B^N)(x) = A^N(x) \wedge B^N(x)(x).$$

3. Crossing intuitionistic KU-ideal

Each member of $(I \oplus I) \times [-1, 0]$ is called a *crossing intuionistic number* and write $\widetilde{a} = \left\langle (a^{\in}, a^{\notin}), a^N \right\rangle$.

Definition 3.1. Let X be a non-empty set. Then a mapping $\widetilde{A} = \left\langle \bar{A}, A^N \right\rangle : X \to (I \oplus I) \times [-1, 0]$ is called a *crossing intuitionistic set* (briefly, CIS) in X. In particular, the *crossing intuitionistic empty set* and the *crossing intuitionistic whole set* are denoted by $\hat{0}$ and $\hat{1}$ respectively and are defined as respectively: for each $x \in X$,

$$\hat{0}(x) = \langle (0, 1), -1 \rangle\,,\ \hat{1}(x) = \langle (1, 0), 0 \rangle\,.$$

The set of all CISs in X is denoted by $CIS(X)$.

Example 3.2. (1) Let $X = \{0, a, b\}$ and let $\widetilde{A}$ be given by:

$$\widetilde{A}(0) = \langle (0.8, 0.2), -0.7 \rangle\,,\ \widetilde{A}(a) = \langle (0.7, 0.3), -0.3 \rangle\,,\ \widetilde{A}(b) = \langle (0.4, 0.2), -0.2 \rangle\,.$$

Then clearly, $\widetilde{A} \in CIS(X)$.

(2) Let $\bar{A}$ be an intuitionistic fuzzy set in a set X. Then we can easily check that $\left\langle \bar{A}, \neg A^{\in} \right\rangle \in CIS(X)$, where $\neg A^{\in}(x) = -A^{\in}(x)$ for each $x \in X$. In fact, $\neg A^{\in} : X \to [-1, 0]$ is a mapping.

(3) Let $A = (A^P, A^N)$ be a bipolar fuzzy set in a set X. Then we can easily see that $\left\langle (A^P, 1 - A^P), A^N \right\rangle \in CIS(X)$.

From (2) and (3) in Example 3.2, it is obvious that a crossing intuitionistic set is a generalization of an intuitionistic fuzzy set and a bipolar fuzzy set.

Definition 3.3. Let X be a non-empty set and let $\widetilde{A},\ \widetilde{B} \in CIS(X)$.

(i) We say that $\widetilde{A}$ is a *subset* of $\widetilde{B}$, denoted by $\widetilde{A} \subset \widetilde{B}$, if $\bar{A} \subset \bar{B}$ and $A^N \subset A^N$,

i.e., $A^{\in}(x) \leq B^{\in}(x),\ A^{\notin}(x) \geq B^{\notin}(x),\ A^N(x) \geq B^N(x)$ for each $x \in X$.

(ii) We say that $\widetilde{A}$ is *equal to* $\widetilde{B}$, denoted by $\widetilde{A} = \widetilde{B}$, if $\widetilde{A} \subset \widetilde{B}$ and $\widetilde{B} \subset \widetilde{A}$.

(iii) The *complement* of $\widetilde{A}$, denoted by $\widetilde{A}^c$, is an CIS in X defined as: for each $x \in X$,

$$\widetilde{A}^c(x) = \left\langle (A^{\notin}(x), A^{\in}(x)), c(A^N)(x) \right\rangle.$$

(iv) The *intersection* of $\widetilde{A}$ and $\widetilde{B}$, denoted by $\widetilde{A} \cap \widetilde{B}$ is a CIS in X, defined as: for each $x \in X$,

$$(\widetilde{A} \cap \widetilde{B}) = \left\langle (A^{\in}(x) \wedge B^{\in}(x), A^{\notin}(x) \vee B^{\notin}(x)), A^{N}(x) \vee B^{N}(x) \right\rangle .$$

(v) The *union* of $\widetilde{A}$ and $\widetilde{B}$, denoted by $\widetilde{A} \cup \widetilde{B}$ is a CIS in X, defined as: for each $x \in X$,

$$(\widetilde{A} \cup \widetilde{B}) = \left\langle (A^{\in}(x) \vee B^{\in}(x), A^{\notin}(x) \wedge B^{\notin}(x)), A^{N}(x) \wedge B^{N}(x) \right\rangle .$$

From Definitions 3.1 and 3.3 (i), it is obvious that $\hat{0} \subset \widetilde{A} \subset \hat{1}$ for each $\widetilde{A} \in CIS(X)$.

Example 3.4. Let X be a non-empt set and Consider two CISs $\widetilde{A}$ and $\widetilde{B}$ given by respectively: for each $x \in X$,

$$\widetilde{A}(x) = \langle (0.4, 0.5), -0.2 \rangle , \ \widetilde{B}(x) = \langle (0.7, 0.3), -0.7 \rangle .$$

Then we can easily check that the followings hold: (1) $\widetilde{A} \subset \widetilde{B}$,
(2) $\widetilde{A}^c(x) = \langle (0.5, 0.4), -0.8 \rangle$,
(3) $(\widetilde{A} \cap \widetilde{B})(x) = \langle (0.4, 0.5), -0.2 \rangle$,
(4) $(\widetilde{A} \cup \widetilde{B})(x) = \langle (0.7, 0.3), -0.7 \rangle$.

From Definitions 3.1 and 3.3, we obtain easily the following consequences.

Proposition 3.5. *Let X be a non-empty set, let $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C} \in CIS(X)$. Then*
(1) (Idempotent laws) $\widetilde{A} \cap \widetilde{A} = \widetilde{A}$, $\widetilde{A} \cup \widetilde{A} = \widetilde{A}$,
(2) (Commutative laws) $\widetilde{A} \cap \widetilde{B} = \widetilde{B} \cap \widetilde{A}$, $\widetilde{A} \cup \widetilde{B} = \widetilde{B} \cup \widetilde{A}$,
(3) (Associative laws) $\widetilde{A} \cap (\widetilde{B} \cap \widetilde{C}) = (\widetilde{A} \cap \widetilde{B}) \cap C$,
$\widetilde{A} \cup (\widetilde{B} \cup \widetilde{C}) = (\widetilde{A} \cup \widetilde{B}) \cup \mathcal{C}$,
(4) (Distributive laws) $\widetilde{A} \cup (\widetilde{B} \cap \widetilde{C}) = (\widetilde{A} \cup \widetilde{B}) \cap (\widetilde{A} \cup \widetilde{C})$,
$\widetilde{A} \cap (\widetilde{B} \cup \widetilde{C}) = (\widetilde{A} \cap \widetilde{B}) \cup (\widetilde{A} \cap \widetilde{C})$,
(5) (Absorption laws) $\widetilde{A} \cup (\widetilde{A} \cap \widetilde{B}) = \widetilde{A}$, $\widetilde{A} \cap (\widetilde{A} \cup \widetilde{B}) = \widetilde{A}$,
(6) (DeMorgan's laws) $(\widetilde{A} \cap \widetilde{B})^c = \widetilde{A}^c \cup \widetilde{B}^c$, $(\widetilde{A} \cup \widetilde{B})^c = \widetilde{A}^c \cap \widetilde{B}^c$,
(7) $(\widetilde{A}^c)^c = \widetilde{A}$,
(8) $\widetilde{A} \cap \widetilde{B} \subset \widetilde{A}$, $\widetilde{A} \cap \widetilde{B} \subset \widetilde{B}$,
(9) $\widetilde{A} \subset \widetilde{A} \cup \widetilde{B}$, $\widetilde{B} \subset \widetilde{A} \cup \widetilde{B}$,
(10) *if* $\widetilde{A} \subset \widetilde{B}$ *and* $\widetilde{B} \subset \widetilde{C}$, *then* $\widetilde{A} \subset \widetilde{C}$,
(11) *if* $\widetilde{A} \subset \widetilde{B}$, *then* $\widetilde{A} \cap \widetilde{C} \subset \widetilde{B} \cap \widetilde{C}$, $\widetilde{A} \cup \widetilde{C} \subset \widetilde{B} \cup \widetilde{C}$,
(12) (12_a) $\widetilde{A} \cup \hat{0} = \widetilde{A}$, $\widetilde{A} \cap \hat{0} = \hat{0}$,
(12_b) $\widetilde{A} \cup \hat{1} = \hat{1}$, $\widetilde{A} \cap \hat{1} = \widetilde{A}$,
(12_c) $\hat{1}^c = \hat{0}$, $\hat{0}^c = \hat{1}$,
(12_d) $\widetilde{A} \cup \widetilde{A}^c \neq \hat{1}$, $\widetilde{A} \cap \widetilde{A}^c \neq \hat{0}$ *in general* (See Example 3.6).

Example 3.6. Let X be a non-empt set and Consider the CIS $\widetilde{A}$ given by respectively: for each $x \in X$,

$$\widetilde{A}(x) = \langle (0.5, 0.5), -0.5 \rangle .$$

Then clearly, $\widetilde{A} \cup \widetilde{A}^c \neq \hat{1}$ and $\widetilde{A} \cap \widetilde{A}^c \neq \hat{0}$.

From now on, let us X be a KU-algebra, unless otherwise is stated.

Definition 3.7. Let $\widetilde{A} \in CIS(X)$. Then $\widetilde{I}$ is called a *crossing intuitionistic ideal* (briefly, CII) of X, it satisfies the following conditions: for all $x,\ y \in X$,
(CII$_1$) $I^{\in}(0) \geq I^{\in}(x)$, $I^{\notin}(0) \leq I^{\notin}(x)$, $I^N(0) \leq I^N(x)$,
(CII$_2$) $I^{\in}(x) \geq I^{\in}(y * x) \wedge I^{\in}(y)$, $I^{\notin}(x) \leq I^{\notin}(y * x) \vee I^{\notin}(y)$, $I^N(x) \leq I^N(y * x) \vee I^N(y)$.

Definition 3.8. Let $\widetilde{A} \in CIS(X)$. Then $\widetilde{A}$ is called a *crossing intuitionistic KU-subalgebra* of X, it satisfies the following conditions: for all $x,\ y,\ z \in X$,
(CISA$_1$) $A^{\in}(y * x) \geq A^{\in}(x) \wedge A^{\in}(y)$, $A^{\notin}(y * x) \leq A^{\notin}(x) \vee A^{\notin}(y)$,
(CISA$_2$) $A^N(y * x) \leq A^N(x) \vee A^N(y)$.

Lemma 3.9. *If $\widetilde{A}$ is a crossing intuitionistic KU-subalgebra of X, then we have*

$$A^{\in}(0) \geq A^{\in}(x),\ \ A^{\notin}(0) \leq A^{\notin}(x),\ \ A^N(0) \leq A^N(x) \ \textit{for each}\ x \in X.$$

Proof. Put x = y in the Definition 3.8. Then from (KU$_5$) and Definition 3.8, we have

$$A^{\in}(x * x) = A^{\in}(0) \geq A^{\in}(x) \wedge A^{\in}(x) = A^{\in}(x),$$
$$A^{\notin}(x * x) = A^{\notin}(0) \leq A^{\notin}(x) \vee A^{\notin}(x) = A^{\notin}(x).$$

Similarly, we have $A^{\notin}(0) \leq A^{\notin}(x)$. □

Definition 3.10. Let $\widetilde{I} \in CIS(X)$. Then $\widetilde{I}$ is called a *crossing intuitionistic KU-ideal* of X, it satisfies the following conditions: for all $x,\ y,\ z \in X$,
(CIKUI$_1$) $I^{\in}(0) \geq I^{\in}(x)$, $I^{\in}(z * x) \geq I^{\in}(z * (y * x)) \wedge I^{\in}(y)$,
(CIKUI$_2$) $I^{\notin}(0) \leq I^{\notin}(x)$, $I^{\notin}(z * x) \leq I^{\notin}(z * (y * x)) \vee I^{\notin}(y)$,
(CIKUI$_3$) $I^N(0) \leq I^N(x)$, $I^N(z * x) \leq I^N(z * (y * x)) \vee I^N(y)$.

Example 3.11. Let $X = \{0, 1, 2, 3\}$ be the KU-algebra with Cayley Table 3.1:

$*$	0	1	2	3
0	0	1	2	2
1	0	0	0	c
2	0	2	0	1
3	0	0	0	0

Table 3.1

Consider the CIS $\widetilde{I}$ given by:

$$\widetilde{I}(0) = \langle (0.6, 0.1), -0.7 \rangle\,,\ \widetilde{I}(1) = \langle (0.5, 0.2), -0.7 \rangle\,,$$
$$\widetilde{I}(2) = \langle (0.3, 0.3), -0.6 \rangle\,,\ \widetilde{I}(4) = \langle (0.3, 0.4), -0.4 \rangle\,.$$

Then we can easily check that $\widetilde{I}$ is a crossing intuitionistic KU-ideal of X.

Proposition 3.12. *Every crossing intuitionistic KU-ideal of X is a crossing intuitionistic ideal of X.*

Proof. The proof is clear. □

Proposition 3.13. *Let $\widetilde{I}$ be a crossing intuitionistic KU-ideal of X and let $x,\ y \in X$ such that $x \leq y$. Then $I^{\in}(x) \geq I^{\in}(y)$, $I^{\notin}(x) \leq I^{\notin}(y)$ and $I^N(x) \leq I^N(y)$.*

Proof. Suppose $x \leq y$. Then by Definition 2.1, $y * x = 0$. Let $z = 0$ in Definition 3.10. Then we get

$$\begin{aligned} I^{\in}(0 * x) &= I^{\in}(x) \text{ [By (KU}_3\text{)]} \\ &\geq I^{\in}(0 * (y * x)) \wedge I^{\in}(y) \\ &\text{[Since } \widetilde{I} \text{ is a crossing intuitionistic } KU\text{-ideal of } X] \\ &= I^{\in}(0 * 0) \wedge I^{\in}(y) \text{ [Since } y * x = 0] \\ &= I^{\in}(0) \wedge I^{\in}(y) \\ &= I^{\in}(y), \\ I^{\notin}(0 * x) &= I^{\notin}(x) \\ &\leq I^{\notin}(0 * (y * x)) \vee I^{\in}(y) \\ &= I^{\notin}(0 * 0) \vee I^{\notin}(y) \\ &= I^{\notin}(0) \vee I^{\notin}(y) \\ &= I^{\notin}(y). \end{aligned}$$

Similarly, we have $I^N(x) \leq I^N(y)$. □

Proposition 3.14. *Every crossing intuitionistic KU-ideal of is a crossing intuitionistic subalgebra of X.*

Proof. Let $\widetilde{I}$ be a crossing intuitionistic KU-ideal of X and let $x,\ y,\ z \in X$. Then by Theorem 2.2 (4), $[y * (y * x)] * x = 0$, i.e., $y * (y * x) \leq x$. Thus by Proposition 3.13, $I^{\in}(y * (y * x)) \geq I^{\in}(x)$, $I^{\notin}(y * (y * x)) \leq I^{\notin}(x)$ and $I^N(y * (y * x)) \leq I^N(x)$. Let $z = y$ in Definition 3.10. Then we get

$$\begin{aligned} I^{\in}(y * x) &\geq I^{\in}(y * (y * x)) \wedge I^{\in}(y) \text{ [By Definition 3.10]} \\ &\geq I^{\in}(x) \wedge I^{\in}(y), \text{ [Since } I^{\in}(y * (y * x)) \geq I^{\in}(x)] \\ I^{\notin}(y * x) &\leq I^{\notin}(y * (y * x)) \vee I^{\notin}(y) \leq I^{\notin}(x) \vee I^{\notin}(y), \\ I^N(y * x) &\leq I^N(y * (y * x)) \vee I^N(y) \leq I^N(x) \vee I^N(y). \end{aligned}$$

Thus $\widetilde{I}$ is a crossing intuitionistic subalgebra of X. □

Proposition 3.15. *Let $\widetilde{I}$ be a crossing intuitionistic KU-ideal of X. For any $x,\ y,\ z \in X$, suppose $x * y \leq z$. Then we have*

$$I^{\in}(x) \geq I^{\in}(y) \wedge I^{\in}(z),\ \ I^{\notin}(x) \leq I^{\notin}(y) \wedge I^{\notin}(z)\ \ I^N(x) \leq I^N(y) \wedge I^N(z).$$

Proof. Let $x,\ y,\ z \in X$ such that $x * y \leq z$. Put $z = 0$ in Definition 3.10. Then we get

$$\begin{aligned} I^{\in}(0 * x) &= I^{\in}(x) \text{ [By (KU}_3\text{)]} \\ &\geq I^{\in}(0 * (y * x)) \wedge I^{\in}(y) \text{ [By Definition 3.10]} \\ &= I^{\in}(y * x) \wedge I^{\in}(y) \text{ [By (KU}_3\text{)]} \\ &\geq I^{\in}(z) \wedge I^{\in}(y), \\ &\quad \text{[Since } I^{\in}(0 * (y * x)) \geq I^{\in}(z) \text{ by Proposition 3.13]} \\ I^N(0 * x) &= I^N(x) \\ &\leq I^N(0 * (y * x)) \vee I^N(y) \\ &= I^N(y * x) \vee I^N(y) \\ &\leq I^N(z) \vee I^N(y). \end{aligned}$$

Similarly, we have $I^{\notin}(x) \leq I^{\notin}(y) \vee I^{\notin}(z)$. □

The following is the converse of Proposition 3.15.

Proposition 3.16. *Let $\widetilde{A}$ be a crossing intuitionistic subalgebra of X. Suppose for any $x,\ y,\ z \in X$ such that $x * y \leq z$, the following inequalities hold:*

$$I^{\in}(x) \geq I^{\in}(y) \wedge I^{\in}(z),\ \ I^{\notin}(x) \leq I^{\notin}(y) \wedge I^{\notin}(z)\ \ I^{N}(x) \leq I^{N}(y) \wedge I^{N}(z).$$

Then $\widetilde{A}$ is a crossing intuitionistic KU-ideal of X.

Proof. Let $\widetilde{A}$ be a crossing intuitionistic subalgebra of X. Recall that

$$A^{\in}(0) \geq A^{\in}(x),\ \ A^{\notin}(0) \leq A^{\notin}(x),\ \ A^{N}(0) \leq A^{N}(x) \text{ for each } x \in X.$$

Let $x,\ y,\ z \in X$ such that $x * y \leq z$. From Theorem 2.2 (1), it is obvious that $y * x \leq x$. Then by Proposition 3.13, we have

$$A^{\in}(y * x) \geq A^{\in}(x),\ \ A^{\notin}(y * x) \leq A^{\notin}(x),\ \ A^{N}(y * x) \leq A^{N}(x).$$

It follows from the hypothesis that

$$A^{\in}(x) \geq A^{\in}(y) \wedge A^{\in}(z) \geq A^{\in}(y) \wedge A^{\in}(y * x). \tag{3.1}$$

Substituting $z * x$ for x and $z * (y * x)$ for y in (3.1), we have

$$\begin{aligned} A^{\in}(z * x) &\geq A^{\in}(z * (y * x)) \wedge A^{\in}((z * (y * x)) * (z * x)) \\ &\geq A^{\in}(z * (y * x)) \wedge A^{\in}((y * x) * x)) \wedge A^{\in}(y). \end{aligned}$$

Since $y * ((y * x) * x) = 0$, i.e., $(y * x) * x \leq y$, $A^{\in}((y * x) * x) \geq A^{\in}(y)$. Thus

$$A^{\in}(z * x) \geq A^{\in}(z * (y * x)) \wedge A^{\in}(y).$$

Similarly, we can prove that the following inequalities hold:

$$A^{\notin}(z * x) \leq A^{\notin}(z * (y * x)) \vee A^{\notin}(y),\ \ A^{N}(z * x) \leq A^{N}(z * (y * x)) \vee A^{N}(y).$$

So $\widetilde{A}$ is a crossing intuitionistic KU-ideal of X. □

Let $\widetilde{a} = \left\langle (a^{\in}, a^{\notin}), a^{N} \right\rangle$ be a crossing intuitionistic number and let $\widetilde{A} \in CIS(X)$. Then the $\widetilde{a}$-level set of $\widetilde{A}$, denoted by $[\widetilde{A}]^{\widetilde{a}}$, is a subset of X defined by:

$$[\widetilde{A}]^{\widetilde{a}} = \{x \in X : A^{\in}(x) \geq a^{\in},\ \ A^{\notin}(x) \leq a^{\notin},\ \ A^{N}(x) \leq a^{N}\}.$$

Theorem 3.17. *$\widetilde{I}$ is a crossing intuitionistic KU-ideal of X if and only if $[\widetilde{I}]^{\widetilde{a}} \neq \varnothing$ is a KU-ideal of X, where $\widetilde{a}$ is a crossing intuitionistic number.*

Proof. Suppose $\widetilde{I}$ is a crossing intuitionistic KU-ideal of X and for each crossing intuitionistic number $\widetilde{a}$, let $[\widetilde{I}]^{\widetilde{a}} \neq \varnothing$. Then there is $x \in X$ such that

$$I^{\in}(x) \geq a^{\in},\ \ I^{\notin}(x) \leq a^{\notin},\ \ I^{N}(x) \leq a^{N}.$$

From Definition 3.10, it is clear that

$$I^{\in}(0) \geq I^{\in}(x),\ \ I^{\notin}(0) \leq I^{\notin}(x),\ \ I^{N}(0) \leq I^{N}(x).$$

Thus $I^{\in}(0) \geq a^{\in},\ I^{\notin}(0) \leq a^{\notin},\ I^{N}(0) \leq a^{N}$. So $0 \in [\widetilde{I}]^{\widetilde{a}}$.

Let $x,\ y,\ z \in X$ such that $z * (y * x),\ y \in [\widetilde{I}]^{\widetilde{a}}$. Then by Definition 3.10 and the definition of $[\widetilde{I}]^{\widetilde{a}}$, we get

$$\begin{aligned} I^{\in}(z * x) &\geq I^{\in}((z * (y * x)) \wedge I^{\in}(y) \geq a^{\in} \wedge a^{\in} = a^{\in}, \\ I^{\notin}(z * x) &\leq I^{\notin}((z * (y * x)) \vee I^{\notin}(y) \leq a^{\notin} \vee a^{\notin} = a^{\notin}, \\ I^{N}(z * x) &\leq I^{N}((z * (y * x)) \vee I^{N}(y) \leq a^{N} \vee a^{N} = a^{N}. \end{aligned}$$

Thus $z * x \in [\widetilde{I}]^{\widetilde{a}}$. So $[\widetilde{I}]^{\widetilde{a}}$ is a KU-ideal of X.

Conversely, suppose $[\widetilde{I}]^{\widetilde{a}}$ is a KU-ideal of X for any crossing intuitionistic number $\widetilde{a}$ and for each $x \in X$, let $I^{\in}(x) = a^{\in},\ I^{\notin}(x) = a^{\notin},\ I^N(x) = a^N$. Then clearly, $x \in [\widetilde{I}]^{\widetilde{a}}$. Since $0 \in [\widetilde{I}]^{\widetilde{a}}$, it follows that

$$I^{\in}(0) \geq I^{\in}(x),\ I^{\notin}(0) \leq I^{\notin}(x),\ I^N(0) \leq I^N(x) \text{ for each } x \in X.$$

Now we need to show that $\widetilde{I}$ satisfies the conditions in Definition 3.10. Assume that $\widetilde{I}$ does not satisfy the conditions in Definition 3.10. Then there are $a,\ b,\ c \in X$ such that the following inequalities:

$$I^{\in}(c * a) < I^{\in}(c * (b * a)) \wedge I^{\in}(b), \tag{3.2}$$

$$I^{\notin}(c * a) > I^{\notin}(c * (b * a)) \vee I^{\notin}(b), \tag{3.3}$$

$$I^N(c * a) > I^N(c * (b * a)) \vee I^N(b). \tag{3.4}$$

Let $\widetilde{a}_0 = \left\langle (a_0^{\in}, a_0^{\notin}), a_0^N \right\rangle$ be the crossing intuitionistic number given by:

$$a_0^{\in} = \frac{1}{2}[I^{\in}(c * a) + I^{\in}(c * (b * a)) \wedge I^{\in}(b)], \tag{3.5}$$

$$a_0^{\notin} = \frac{1}{2}[I^{\notin}(c * a) + I^{\notin}(c * (b * a)) \vee I^{\notin}(b)], \tag{3.6}$$

$$a_0^N = \frac{1}{2}[I^N(c * a) + I^N(c * (b * a)) \vee I^N(b)]. \tag{3.7}$$

Then we get the following inequalities:

$$I^{\in}(c * a) < a_0^{\in} < I^{\in}(c * (b * a)) \wedge I^{\in}(b), \tag{3.8}$$

$$I^{\notin}(c * a) > a_0^{\in} < I^{\notin}(c * (b * a)) \vee I^{\notin}(b), \tag{3.9}$$

$$I^N(c * a) > a_0^N > I^N(c * (b * a)) \vee I^N(b). \tag{3.10}$$

Thus in all cases, $c * (b * a),\ b \in [\widetilde{I}]^{\widetilde{a}_0}$ but $c * a \notin [\widetilde{I}]^{\widetilde{a}_0}$. So $[\widetilde{I}]^{\widetilde{a}_0}$ is not a KU-ideal of X. This is a contradiction. Hence $[\widetilde{I}]^{\widetilde{a}}$ is a crossing intuitionistic KLU-ideal of X. This completes the proof. □

4. The pre-image of a crossing intuitionistic KU-ideal

Definition 4.1. Let $(X, *, 0)$ and $(Y, *', 0')$ be two KU-algebras. Then a mapping $f : X \to Y$ is called a *homomorphism*, if $f(x * y) = f(x) *' f(y)$ for any $x,\ y \in X$.

Note that if $f : X \to Y$ is a homomorphism of $KU-$algebras, then $f(0) = 0'$.

Definition 4.2. Let $(X, *0)$ and $(Y, *', 0')$ be two non-empty sets, let $\widetilde{B} \in CIS(Y)$ and let $f : X \to Y$ be a mapping. Then the *pre-image of $\widetilde{B}$ under f*, denoted by $f^{-1}(\widetilde{B}) = \langle (f^{-1}(B^{\in}), f^{-1}(B^{\notin}), f^{-1}(B^N) \rangle$, is a crossing intuitionistic set in X defined as: for each $x \in X$,

$$f^{-1}(B^{\in})(x) = B^{\in}(f(x)),\ f^{-1}(B^{\notin})(x) = B^{\notin}(f(x)),\ f^{-1}(B^N)(x) = B^N(f(x)).$$

Proposition 4.3. *$f : X \to Y$ be a homomorphism of $KU-$algebras. If $\widetilde{I}$ is a crossing intuitionistic KU-ideal of Y, then $f^{-1}(\widetilde{I})$ is a crossing intuitionistic KU-ideal of X.*

Proof. Suppose $\widetilde{I}$ is a crossing intuitionistic KU-ideal of Y and let $x \in X$. Then

$$f^{-1}(I^{\in})(x) = I^{\in}(f(x)) \leq I^{\in}(0^{'}) = I^{\in}(f(0)) = f^{-1}(I^{\in})(0).$$

Similarly, we have $f^{-1}(I^{\in})(x) \geq f^{-1}(I^{\in})(0),\ f^{-1}(I^{N})(x) \geq f^{-1}(I^{N})(0)$.

Now let $x,\ y,\ z \in X$. Then

$$\begin{aligned} f^{-1}(I^{\in})(z*x) &= I^{\in}(f(z*x)) \text{ [By Definition 4.2]} \\ &= I^{\in}(f(x) *^{'} f(y)) \\ &\quad \text{[Since } f \text{ is a homomorphism of } KU-\text{algebras]} \\ &\geq I^{\in}(f(z) *^{'} (f(y) *^{'} f(x))) \wedge I^{\in}(f(y)) \text{ [By the hypothesis]} \\ &= I^{\in}(f(z*(y*x))) \wedge I^{\in}(f(y)) \\ &\quad \text{[Since } f \text{ is a homomorphism of } KU-\text{algebras]} \\ &= f^{-1}(I^{\in})(z*(y*x)) \wedge f^{-1}(I^{\in})(y), \\ f^{-1}(I^{\notin})(z*x) &= I^{\notin}(f(z*x)) \\ &= I^{\notin}(f(x) *^{'} f(y)) \\ &\leq I^{\notin}(f(z) *^{'} (f(y) *^{'} f(x))) \vee I^{\notin}(f(y)) \\ &= I^{\notin}(f(z*(y*x))) \vee I^{\notin}(f(y)) \\ &= f^{-1}(I^{\notin})(z*(y*x)) \vee f^{-1}(I^{\notin})(y). \end{aligned}$$

Similarly, we can show that the following inequality holds:

$$f^{-1}(I^{N})(z*x) \leq f^{-1}(I^{N})(z*(y*x)) \vee f^{-1}(I^{N})(y).$$

Thus $f^{-1}(\widetilde{I})$ is a crossing intuitionistic KU-ideal of X. □

The following provides a sufficient condition which the converse of Proposition 4.3 holds.

Proposition 4.4. *$f : X \to Y$ be an epimorphism of $KU-$algebras and let $\widetilde{I} \in CIS(Y)$. If $f^{-1}(\widetilde{I})$ is a crossing intuitionistic KU-ideal of X, then $\widetilde{I}$ is a crossing intuitionistic KU-ideal of Y.*

Proof. Suppose $f^{-1}(\widetilde{I})$ is a crossing intuitionistic KU-ideal of X and let $a \in Y$. Since f is surjective, there is $x \in X$ such that $a = f(x)$. Then

$$\begin{aligned} I^{\in}(a) &= I^{\in}(f(x)) \\ &= f^{-1}(I^{\in})(x) \text{ [By Definition 4.2]} \\ &\leq f^{-1}(I^{\in})(0) \text{ [By the hypothesis]} \\ &= I^{\in}(f(0)) \\ &= I^{\in}(0^{'}). \text{ [Since } f \text{ is a homomorphism]} \end{aligned}$$

Similarly, we have

$$I^{\notin}(a) \geq I^{\notin}(0^{'}),\ I^{N}(a) \geq I^{N}(0^{'}).$$

Now let $a,\ b,\ c \in Y$. Then there are $x,\ y,\ z \in X$ such that

$$a = f(x),\ b = f(y),\ c = f(z).$$

Thus we get

$$\begin{aligned} I^{\in}(c *^{'} a) &= I^{\in}(f(z) *^{'} f(x)) \\ &= I^{\in}(f(z*x)) \text{ [Since } f \text{ is a homomorphism]} \end{aligned}$$

$$
\begin{aligned}
&= f^{-1}(I^{\in})(z*x) \text{ [By Definition 4.2]}\\
&\geq f^{-1}(I^{\in})(z*(y*x)) \wedge f^{-1}(I^{\in})(y) \text{ [By the hypothesis]}\\
&= I^{\in}(f(z*(y*x))) \wedge I^{\in}(f(y))\\
&= I^{\in}(f(z)*^{'}(f(y)*^{'}f(x))) \wedge I^{\in}(f(y))\\
&= I^{\in}(c*^{'}(b*^{'}a)) \wedge I^{\in}(b),\\
I^{N}(c*^{'}a) &= I^{N}(f(z)*^{'}f(x))\\
&= I^{N}(f(z*x))\\
&= f^{-1}(I^{N})(z*x)\\
&\leq f^{-1}(I^{N})(z*(y*x)) \vee f^{-1}(I^{N})(y)\\
&= I^{N}(f(z*(y*x))) \vee I^{N}(f(y))\\
&= I^{N}(f(z)*^{'}(f(y)*^{'}f(x))) \vee I^{N}(f(y))\\
&= I^{N}(c*^{'}(b*^{'}a)) \vee I^{N}(b).
\end{aligned}
$$

Similarly, we can prove that the following inequality holds:

$$I^{\notin}(c*^{'}a) \leq I^{\notin}(c*^{'}(b*^{'}a)) \vee I^{\notin}(b).$$

So $\widetilde{I}$ is a crossing intuitionistic KU-ideal of Y. □

5. The product of crossing intuitionistic KU-ideals

Definition 5.1. Let $\widetilde{A}$, $\widetilde{B} \in CIS(X)$. Then the *Cartesian product* of $\widetilde{A}$ and $\widetilde{B}$, denoted by $\widetilde{A} \times \widetilde{B} = \langle (A^{\in} \times B^{\in}, A^{\notin} \times B^{\notin}), A^{N} \times B^{N} \rangle$, is a CIS in $X \times X$ defined as: for each $(x,y) \in X \times X$,

$$A^{\in} \times B^{\in}(x,y) = A^{\in}(x) \wedge B^{\in}(x),\ A^{\notin} \times B^{\notin}(x,y) = A^{\notin}(x) \vee B^{\notin}(x),$$
$$A^{N} \times B^{N}(x,y) = A^{N}(x) \vee B^{N}(x).$$

Remark 5.2. Let X and Y be two KU-algebras. We define an binary operation $*$ on $X \times Y$ as follows: for any $(x_1,y_1),\ (x_2,y_2) \in X \times Y$,

$$(x_1,y_1)*(x_2,y_2) = (x_1*x_2, y_1*y_2).$$

Then we can easily see that $(X \times Y, *, (0,0))$ is a KU-algebra.

Proposition 5.3. *Let $\widetilde{I}_1$ and $\widetilde{I}_2$ be two crossing intuitionistic KU-ideals of X. Then $\widetilde{I}_1 \times \widetilde{I}_2$ is a crossing intuitionistic KU-ideal of $X \times X$.*

Proof. Let $(x,y) \in X \times X$. Then we get

$$I_1^{\in} \times I_2^{\in}(0,0) = I_1^{\in}(0) \wedge I_2^{\in}(0) \geq I_1^{\in}(x) \wedge I_1^{\in}(y) = I_1^{\in} \times I_2^{\in}(x,y).$$

Similarly, we get the following inequalities:

$$I_1^{\notin} \times I_2^{\notin}(0,0) \leq I_1^{\notin} \times I_2^{\notin}(x,y),\ I_1^{N} \times I_2^{N}(0,0) \leq I_1^{N} \times I_2^{N}(x,y).$$

Now let $(x_1,x_2),\ (y_1,y_2),\ (z_1,z_2) \in X \times X$. Then

$$
\begin{aligned}
&I_1^{\in} \times I_2^{\in}[(z_1,z_2)*((y_1,y_2)*(x_1,x_2))] \wedge I_1^{\in} \times I_2^{\in}(y_1,y_2)\\
&= I_1^{\in} \times I_2^{\in}[(z_1,z_2)*(y_1*x_1, y_2*x_2)] \wedge I_1^{\in} \times I_2^{\in}(y_1,y_2)\\
&= I_1^{\in} \times I_2^{\in}[(z_1*(y_1*x_1), z_2*(y_2*x_2))] \wedge I_1^{\in} \times I_2^{\in}(y_1,y_2)\\
&= [I_1^{\in}(z_1*(y_1*x_1) \wedge I_2^{\in}(z_2*(y_2*x_2))] \wedge [I_1^{\in}(y_1) \wedge I_2^{\in}(y_2)]\\
&= [I_1^{\in}(z_1*(y_1*x_1) \wedge I_1^{\in}(y_1)] \wedge [I_2^{\in}(z_2*(y_2*x_2)) \wedge I_2^{\in}(y_2)]\\
&\leq I_1^{\in}(z_1*x_1) \wedge \leq I_2^{\in}(z_2*x_2)
\end{aligned}
$$

$$= I_1^{\in} \times I_2^{\in}(z_1 * x_1, z_2 * x_2),$$

$$\begin{aligned}
& I_1^{\notin} \times I_2^{\notin}[(z_1, z_2) * ((y_1, y_2) * (x_1, x_2))] \vee I_1^{\notin} \times I_2^{\notin}(y_1, y_2) \\
&= I_1^{\notin} \times I_2^{\notin}[(z_1, z_2) * (y_1 * x_1, y_2 * x_2)] \vee I_1^{\notin} \times I_2^{\notin}(y_1, y_2) \\
&= I_1^{\notin} \times I_2^{\notin}[(z_1 * (y_1 * x_1), z_2 * (y_2 * x_2))] \vee I_1^{\notin} \times I_2^{\notin}(y_1, y_2) \\
&= [I_1^{\notin}(z_1 * (y_1 * x_1) \vee I_2^{\notin}(z_2 * (y_2 * x_2))] \vee [I_1^{\notin}(y_1) \vee I_2^{\notin}(y_2)] \\
&= [I_1^{\notin}(z_1 * (y_1 * x_1) \vee I_1^{\notin}(y_1)] \vee [I_2^{\notin}(z_2 * (y_2 * x_2)) \vee I_2^{\notin}(y_2)] \\
&\leq I_1^{\notin}(z_1 * x_1) \vee \leq I_2^{\notin}(z_2 * x_2) \\
&= I_1^{\notin} \times I_2^{\notin}(z_1 * x_1, z_2 * x_2).
\end{aligned}$$

Similarly, we can show that the following inequality holds:

$$I_1^N \times I_2^N[(z_1, z_2) * ((y_1, y_2) * (x_1, x_2))] \wedge I_1^N \times I_2^N(y_1, y_2) \geq I_1^N \times I_2^N(z_1 * x_1, z_2 * x_2).$$

Thus $\widetilde{I}_1 \times \widetilde{I}_2$ is a crossing intuitionistic KU-ideal of $X \times X$. □

6. Conclusions

We studied some properties of crossing intuitionistic of KU-ideals in a KU-algebras as an extension of bipolar fuzzy sets. Also, how the pre-image of a crossing intuitionistic KU-ideal under a homomorphism of KU-algebras become a crossing intuitionistic KU-ideal. Moreover, the Cartesian product of crossing intuitionistic KU-ideals in Cartesian product KU-algebras was established.

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