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Abstract. The aim of this paper is to introduce the basic concepts of the so-called *octahedron topological spaces*, for example, octahedron base and subbase, octahedron subspace, octahedron closure and interior, octahedron continuity, etc. We find some properties for each concept and give some necessary examples.

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Keywords: Octahedron set, Octahedron topology, Octahedron base and subbase, Octahedron relative topology, Octahedron closure (operator) and interior (operator), Octahedron continuity, Octahedron quotient space, Octahedron product space.

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1. Introduction

To solve uncertainties of the real world as it is, Zadeh [1] had introduced the concept of fuzzy sets as generalization of crisp sets in 1965. After then, the notions of interval-valued fuzzy sets, rough sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, vague sets, neutrosophic sets, bipoar fuzzy sets, soft sets and hesitant fuzzy sets was proposed by Zadeh [2], Pawlak [3] (1982), Atanassov [4] (1983), Atanassov and Gargov [5] (1989), Gau and Buchrer [6] (1993), Smarandache [7] (1998), Zhang [8] (1998), Molodtsov [9] (1999) and Torra [10] (2010), in turn in order to deal with various real-life problems.

Topology forms a general framework for the study of various notions in analysis (See the Historical Note to Chapter I [11] for the historical background of topology). It is not only a powerful tool in many branches of mathematics, but it also has a beauty of itself. So, many researchers have studied topological structures based on the various concepts mentioned above. For example, Chang [12] investigated topological structures for fuzzy sets (furthermore, refer to [13, 14, 15, 16, 17]), Lashin et al. [18] (furthermore, refer to [19, 20]), Çoker [21] for intuitionistic fuzzy sets (furthermore, refer to [22, 23]), Mondal and Samanta [24, 25] for interval valued fuzzy sets and interval-valued intuitionistic fuzzy sets, Salama and Alblowi [26] and Lupiáñez [27] for neutrosophic sets, Azhagappan and Kamaraj [28] and Kim et al. [29] for bipolar fuzzy sets, Shabir and Naz [30] for soft sets (furthermore refer to [31, 32, 33, 34, 35, 36, 37, 38]), and Deepak et al. [39], Lee and Hur [40] for hesitant fuzzy sets. In particular, Al-shami [41] dealt with decision-making problems by using soft separation axioms.

Recently, Lee et al. [42] proposed an octahedron set combined with an interval-valued fuzzy set, intuitionistic fuzzy set and fuzzy set as a tool to solve complex problems. After that time, Şenel et al. [43] discussed with MCGDM problems by using similarity measures for octahedron sets.

Our research's aim is to study topological structures based on octahedron sets. To accomplish it, this paper organized as follows: In Section 2, we recall basic notions related to octahedron sets. In Section 3, we define an octahedron topology, an octahedron base and subbase, and find some their properties. Also we give some examples to help one understand each concept. In Section 4, we deal with octahedron subspaces, octahedron closures and interiors. Section 5 is devoted to investigate octahedron continuities.

2. Preliminaries

In this section, we list some basic definitions needed in the next sections.

Let $I \oplus I = \{\bar{a} = (a^{\in}, a^{\notin}) \in I \times I : a^{\in} + a^{\notin} \leq 1\}$, where $I = [0, 1]$. Then each member $\bar{a}$ of $I \oplus I$ is called an *intuitionistic point* or *intuitionistic number*. In particuar, we denote $(0, 1)$ and $(1, 0)$ as $\bar{0}$ and $\bar{1}$, respectively. Refer to [44] for the definitions of $\leq$ and $=$ on $I \oplus I$, the complement of an intuitionistic number, and the infimum and the supremum of any intuitionistic numbers.

Definition 2.1 ([45])**.** For a nonempty set X, a mapping $\bar{A} : X \to I \oplus I$ is called an *intuitionistic fuzzy set* (briefly, IF set) in X, where for each $x \in X$, $\bar{A}(x) = (A^{\in}(x), A^{\notin}(x))$, and $A^{\in}(x)$ and $A^{\notin}(x)$ represent the degree of membership and the degree of nonmembership of an element x to $\bar{A}$, respectively. Let $(I \oplus I)^X$ denote the set of all IF sets in X and for each $\bar{A} \in (I \oplus I)^X$, we write $A = (A^{\in}, A^{\notin})$. In particular, $\bar{\mathbf{0}}$ and $\bar{\mathbf{1}}$ denote the IF empty set and the IF whole set in X defined by, respectively: for each $x \in X$,

$$\bar{\mathbf{0}}(x) = \bar{0} \text{ and } \bar{\mathbf{1}}(x) = \bar{1}.$$

The set of all closed subintervals of I is denoted by $[I]$, and members of $[I]$ are called *interval numbers* and are denoted by $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, etc., where $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$. In particular, if $a^- = a^+$, then we write as $\widetilde{a} = \mathbf{a}$. Refer to [24, 42] for the definitions of $\leq$ and $=$ on $I \oplus I$, the complement of an interval-valued number, and the infimum and the supremum of any interval-valued numbers.

Definition 2.2 ([2, 46])**.** For a nonempty set X, a mapping $\widetilde{A} : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an IVF set) in X. Let $[I]^X$ denote the set of all IVF sets in X. For each $\widetilde{A} \in [I]^X$ and $x \in X$, $\widetilde{A}(x) = [A^-(x), A^+(x)]$ is called the *degree of membership of an element x to $\widetilde{A}$*, where $A^-, A^+ \in I^X$ are called a *lower fuzzy set* and *an upper fuzzy set* in X, respectively. For each $\widetilde{A} \in [I]^X$, we write $\widetilde{A} = [A^-, A^+]$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy empty whole set in X defined by, respectively: for each $x \in X$,

$$\widetilde{0}(x) = \mathbf{0} \text{ and } \widetilde{1}(x) = \mathbf{1}.$$

Refer to [24, 42] for the definitions of $\subset$ and $=$ on $[I]^X$, the *complement* of an interval-valued set, and the *union* and the *intersection* of any interval-valued sets.

Definition 2.3 ([47]). Let X be a nonempty set. Then a complex mapping $\mathbf{A} = \left\langle \widetilde{A}, A \right\rangle : X \to [I] \times I$ is called a *cubic set* in X.

A cubic set $\mathbf{A} = \left\langle \widetilde{A}, A \right\rangle$ in which $\widetilde{A}(x) = \mathbf{0}$ and $A(x) = 1$ (resp. $\widetilde{A}(x) = \mathbf{1}$ and $A(x) = 0$) for each $x \in X$ is denoted by $\dot{0}$ (resp. $\dot{1}$).

A cubic set $\mathbf{B} = \left\langle \widetilde{B}, B \right\rangle$ in which $\widetilde{B}(x) = \mathbf{0}$ and $B(x) = 0$ (resp. $\widetilde{B}(x) = \mathbf{1}$ and $B(x) = 1$) for each $x \in X$ is denoted by $\hat{0}$ (resp. $\hat{1}$). In this case, $\hat{0}$ (resp. $\hat{1}$) will be called a *cubic empty* (resp. *whole*) *set* in X.

We denote the set of all cubic sets in X as $\mathcal{C}(X)$.

We denote members of $[I] \times (I \oplus I) \times I$ as $\widetilde{\widetilde{a}} =< \widetilde{a}, \bar{a}, a >=< [a^-, a^-], (a^{\in}, a^{\notin}), a >$, $\widetilde{\widetilde{b}} =< \widetilde{b}, \bar{b}, b >=< [b^-, b^-], (b^{\in}, b^{\notin}), b >$, etc. and they are called *octahedron numbers*. Furthermore, we define the following order relations between $\widetilde{\widetilde{a}}$ and $\widetilde{\widetilde{b}}$ (See [42]):

(i) (*Equality*) $\widetilde{\widetilde{a}} = \widetilde{\widetilde{b}} \Leftrightarrow \widetilde{a} = \widetilde{b},\ \bar{a} = \bar{b},\ a = b$,
(ii) (*Type 1-order*) $\widetilde{\widetilde{a}} \leq_1 \widetilde{\widetilde{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \leq b^+,\ a^{\in} \leq b^{\in},\ a^{\notin} \geq b^{\notin},\ a \leq b$,
(iii) (*Type 2-order*) $\widetilde{\widetilde{a}} \leq_2 \widetilde{\widetilde{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \leq b^+,\ a^{\in} \leq b^{\in},\ a^{\notin} \geq b^{\notin},\ a \geq b$,
(iv) (*Type 3-order*) $\widetilde{\widetilde{a}} \leq_3 \widetilde{\widetilde{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \geq b^+,\ a^{\in} \geq b^{\in},\ a^{\notin} \leq b^{\notin},\ a \leq b$,
(v) (*Type 4-order*) $\widetilde{\widetilde{a}} \leq_4 \widetilde{\widetilde{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \leq b^+,\ a^{\in} \geq b^{\in},\ a^{\notin} \leq b^{\notin},\ a \geq b$.

Definition 2.4 ([42]). Let X be a nonempty set and let $\widetilde{A} = [A^-, A^+] \in [I]^X,\ \bar{A} = (A^{\in}, A^{\notin}) \in (I \oplus I)^X,\ A \in I^X$. Then the triple $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A \right\rangle$ is called an *octahedron set* in X. In fact, $\mathcal{A} : X \to [I] \times (I \oplus I) \times I$ is a mapping.

We can consider following special octahedron sets in X:

$$\left\langle \widetilde{0}, \bar{\mathbf{0}}, 0 \right\rangle = \ddot{0},$$
$$\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle, \left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle,\ \left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle,$$
$$\left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle,\ \left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle,\ \left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle,$$
$$\left\langle \widetilde{1}, \bar{\mathbf{1}}, 1 \right\rangle = \ddot{1}.$$

In this case, $\ddot{0}$ (resp. $\ddot{1}$) is called an *octahedron empty set* (resp. *octahedron whole set*) in X. We denote the set of all octahedron sets as $\mathcal{O}(X)$.

Definition 2.5 ([42]). Let X be a nonempty set and let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A \right\rangle,\ \mathcal{B} = \left\langle \widetilde{B}, \bar{B}, B \right\rangle \in \mathcal{O}(X)$. Then we can define following order relations between $\mathcal{A}$ and $\mathcal{B}$:

(i) (*Equality*) $\mathcal{A} = \mathcal{B} \Leftrightarrow \widetilde{A} = \widetilde{B},\ \bar{A} = \bar{B},\ A = B$,

(ii) (*Type 1-inclusion*) $\mathcal{A} \subset_1 \mathcal{B} \Leftrightarrow \widetilde{A} \subset \widetilde{B},\ \bar{A} \subset \bar{B},\ A \subset B$,
(iii) (*Type 2-inclusion*) $\mathcal{A} \subset_2 \mathcal{B} \Leftrightarrow \widetilde{A} \subset \widetilde{B},\ \bar{A} \subset \bar{B},\ A \supset B$,
(iv) (*Type 3-inclusion*) $\mathcal{A} \subset_3 \mathcal{B} \Leftrightarrow \widetilde{A} \subset \widetilde{B},\ \bar{A} \supset \bar{B},\ A \subset B$,
(v) (*Type 4-inclusion*) $\mathcal{A} \subset_4 \mathcal{B} \Leftrightarrow \widetilde{A} \subset \widetilde{B},\ \bar{A} \supset \bar{B},\ A \supset B$.

Definition 2.6 ([42])**.** Let X be a nonempty set and let $(\mathcal{A}_j)_{j\in J} = (\left\langle \widetilde{A}_j, \bar{A}_j, A_j \right\rangle)_{j\in J}$ be a family of octahedron sets in X. Then the Type i-union $\cup^i$ and Type i-intersection $\cap^i$ of $(\mathcal{A}_j)_{j\in J}$, $(i = 1,\ 2,\ ,3, 4)$, are defined as follows, respectively:

(i) (Type i-union) $\bigcup^1_{j\in J} \mathcal{A}_j = \left\langle \bigcup_{j\in J} \widetilde{A}_j, \bigcup_{j\in J} \bar{A}_j, \bigcup_{j\in J} A_j \right\rangle$,
$\bigcup^2_{j\in J} \mathcal{A}_j = \left\langle \bigcup_{j\in J} \widetilde{A}_j, \bigcup_{j\in J} \bar{A}_j, \bigcap_{j\in J} A_j \right\rangle$,
$\bigcup^3_{j\in J} \mathcal{A}_j = \left\langle \bigcup_{j\in J} \widetilde{A}_j, \bigcap_{j\in J} \bar{A}_j, \bigcup_{j\in J} A_j \right\rangle$,
$\bigcup^4_{j\in J} \mathcal{A}_j = \left\langle \bigcup_{j\in J} \widetilde{A}_j, \bigcap_{j\in J} \bar{A}_j, \bigcap_{j\in J} A_j \right\rangle$,

(ii) (Type i-intersection) $\bigcap^1_{j\in J} \mathcal{A}_j = \left\langle \bigcap_{j\in J} \widetilde{A}_j, \bigcap_{j\in J} \bar{A}_j, \bigcap_{j\in J} A_j \right\rangle$,
$\bigcap^2_{j\in J} \mathcal{A}_j = \left\langle \bigcap_{j\in J} \widetilde{A}_j, \bigcap_{j\in J} \bar{A}_j, \bigcup_{j\in J} A_j \right\rangle$,
$\bigcap^3_{j\in J} \mathcal{A}_j = \left\langle \bigcap_{j\in J} \widetilde{A}_j, \bigcup_{j\in J} \bar{A}_j, \bigcap_{j\in J} A_j \right\rangle$,
$\bigcap^4_{j\in J} \mathcal{A}_j = \left\langle \bigcap_{j\in J} \widetilde{A}_j, \bigcup_{j\in J} \bar{A}_j, \bigcup_{j\in J} A_j \right\rangle$.

Definition 2.7 ([42])**.** Let X be a nonempty set and let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A \right\rangle$ be an octahedron set in X. Then the *complement* $\mathcal{A}^c$, operators $[\,]$ and $\diamond$ of $\mathcal{A}$ are defined as follows, respectively: for each $x \in X$,

(i) $\mathcal{A}^c = \left\langle \widetilde{A}^c, \bar{A}^c, A^c \right\rangle$,
(ii) $[\,]\mathcal{A} = \left\langle \widetilde{A}, [\,]\bar{A}, A \right\rangle$,
(iii) $\diamond\mathcal{A} = \left\langle \widetilde{A}, \diamond\bar{A}, A \right\rangle$.

Let us denote the set of all fuzzy points (See [15]), intuitionistic fuzzy points (See [22]) and interval-valued fuzzy points (See [24]) in a set X as $F_P(X)$, $IF_P(X)$ and $IVF_P(X)$, respectively.

Definition 2.8 ([42])**.** Let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A \right\rangle \in \mathcal{O}(X)$, and let $\widetilde{\bar{a}} \in [I] \times (I \oplus I) \times I$ with $a^+ > 0$, $\bar{b} \neq \bar{0}$ and $a \neq 0$. Then $\mathcal{A}$ is called an *octahedron point* with the support $x \in X$ and the value $\widetilde{\bar{a}}$, denoted by $\mathcal{A} = x_{\widetilde{\bar{a}}}$, if for each $y \in X$,

$$x_{\widetilde{\bar{a}}}(y) = \begin{cases} \widetilde{\bar{a}} & \text{if } y = x \\ \langle \mathbf{0}, \bar{0}, 0 \rangle & \text{otherwise.} \end{cases}$$

The set of all octahedron points in X is denoted by $\mathcal{O}_P(X)$.

Definition 2.9 ([42])**.** Let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A \right\rangle \in \mathcal{O}(X)$ and let $x_{\widetilde{\bar{a}}} \in \mathcal{O}_P(X)$. Then $x_{\widetilde{\bar{a}}}$ is said to:

(i) *belong to* $\mathcal{A}$ *with respect to Type* 1*-order*, denoted by $x_{\widetilde{\bar{a}}} \in_1 \mathcal{A}$, if
$\widetilde{a} \leq \widetilde{A}(x)$, $\bar{a} \leq \bar{A}(x)$ and $a \leq A(x)$, i.e., $x_{\widetilde{a}} \in \widetilde{A}$, $x_{\bar{a}} \in \bar{A}$ and $x_a \in A$,

(ii) *belong to* $\mathcal{A}$ *with respect to Type* 2*-order*, denoted by $x_{\widetilde{a}} \in_2 \mathcal{A}$, if
$\widetilde{a} \leq \widetilde{A}(x)$, $\bar{a} \leq \bar{A}(x)$ and $a \geq A(x)$,
(iii) *belong to* $\mathcal{A}$ *with respect to Type* 3*-order*, denoted by $x_{\widetilde{a}} \in_3 \mathcal{A}$, if
$\widetilde{a} \leq \widetilde{A}(x)$, $\bar{a} \geq \bar{A}(x)$ and $a \leq A(x)$,
(iv) *belong to* $\mathcal{A}$ *with respect to Type* 4*-order*, denoted by $x_{\widetilde{a}} \in_4 \mathcal{A}$, if
$\widetilde{a} \leq \widetilde{A}(x)$, $\bar{a} \geq \bar{A}(x)$ and $a \geq A(x)$.
It is clear that $\mathcal{A} = \bigcup^1_{x_{\widetilde{a}} \in_1 \mathcal{A}} x_{\widetilde{a}}$, for each $\mathcal{A} \in \mathcal{O}(X)$.

3. Octahedron topological spaces

In this section, we define an octahedron topological space and find some of its properties. Also we introduce the concepts of octahedron bases and subbases and deal with some of their properties. In particular, we give the conditions for the family of octahedron sets to become an octahedron base (See Theorem 3.18).

Definition 3.1. Let $\tau \subset \mathcal{O}(X)$. Then τ is called a:
(i) Type 1-*octahedron topology* (briefly, *octahedron topology*) on X, if it satisfies the following axioms:
[1-OO$_1$] $\ddot{0}$, $\ddot{1} \in \tau$,
[1-OO$_2$] $\mathcal{A} \cap^1 \mathcal{B} \in \tau$ for any $\mathcal{A}$, $\mathcal{B} \in \tau$,
[1-OO$_3$] $\bigcup^1_{j \in J} \mathcal{A}_j \in \tau$ for any $(\mathcal{A}_j)_{j \in J} \subset \tau$,
(ii) Type 2-*octahedron topology* on X, if it satisfies the following axioms:
[2-OO$_1$] $\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle$, $\left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle \in \tau$,
[2-OO$_2$] $\mathcal{A} \cap^2 \mathcal{B} \in \tau$ for any $\mathcal{A}$, $\mathcal{B} \in \tau$,
[2-OO$_3$] $\bigcup^2_{j \in J} \mathcal{A}_j \in \tau$ for any $(\mathcal{A}_j)_{j \in J} \subset \tau$,
(iii) Type 3-*octahedron topology* on X, if it satisfies the following axioms:
[3-OO$_1$] $\left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle$, $\left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle \in \tau$,
[3-OO$_2$] $\mathcal{A} \cap^3 \mathcal{B} \in \tau$ for any $\mathcal{A}$, $\mathcal{B} \in \tau$,
[3-OO$_3$] $\bigcup^3_{j \in J} \mathcal{A}_j \in \tau$ for any $(\mathcal{A}_j)_{j \in J} \subset \tau$,
(iv) Type 4-*octahedron topology* on X, if it satisfies the following axioms:
[3-OO$_1$] $\left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle$, $\left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle \in \tau$,
[3-OO$_2$] $\mathcal{A} \cap^4 \mathcal{B} \in \tau$ for any $\mathcal{A}$, $\mathcal{B} \in \tau$,
[3-OO$_3$] $\bigcup^4_{j \in J} \mathcal{A}_j \in \tau$ for any $(\mathcal{A}_j)_{j \in J} \subset \tau$.

The pair (X, τ) is called a Type i-*octahedron topological space*. The members of τ are called i-*octahedron open set* (briefly, i-OOS) in X and we denote the set of all i-OOSs in X as $OOS_i(X)$. $\mathcal{A} \in \mathcal{O}(X)$ is called a i-*octahedron closed set* (briefly, i-OCS) in X, if $\mathcal{A}^c \in \tau$. We denote the collection of all Type i- octahedron topologies on X as $OT_i(X)$,

Example 3.2. (1) Let τ_i be the family of octahedron sets in I respectively given by:

$$\tau_1 = \{\ddot{0}, \ddot{1}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\},$$

$$\tau_2 = \left\{ \left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle, \left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_5, \mathcal{A}_6 \right\},$$

$$\tau_3 = \left\{ \left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle, \left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_7, \mathcal{A}_8 \right\},$$
$$\tau_4 = \left\{ \left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle, \left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_9, \mathcal{A}_{10} \right\},$$

where for each $x \in I$,

$$\mathcal{A}_1(x) = \langle [0.5, 0.7], (0.6, 0.3), 0.4 \rangle, \ \mathcal{A}_2(x) = \langle [0.4, 0.8], (0.5, 0.2), 0.6 \rangle,$$
$$\mathcal{A}_3(x) = \langle [0.4, 0.7], (0.5, 0.3), 0.4 \rangle, \ \mathcal{A}_4(x) = \langle [0.5, 0.8], (0.6, 0.2), 0.6 \rangle,$$
$$\mathcal{A}_5(x) = \langle [0.4, 0.7], (0.5, 0.3), 0.6 \rangle, \ \mathcal{A}_6(x) = \langle [0.5, 0.8], (0.6, 0.2), 0.4 \rangle,$$
$$\mathcal{A}_7(x) = \langle [0.4, 0.7], (0.6, 0.2), 0.4 \rangle, \ \mathcal{A}_8(x) = \langle [0.5, 0.8], (0.5, 0.4), 0.6 \rangle,$$
$$\mathcal{A}_9(x) = \langle [0.4, 0.7], (0.6, 0.2), 0.6 \rangle, \ \mathcal{A}_{10}(x) = \langle [0.5, 0.8], (0.5, 0.4), 0.4 \rangle.$$

Then we can easily check that $\tau_i \in OT_i(I)$ for $i = 1, 2, 3, 4$.

(2) Let X be a nonempty set. Then clearly, $\mathcal{O}(X)$ is a Type i-octahedron topology on X for each $i = 1, 2, 3, 4$. In this case, $\mathcal{O}(X)$ is called the *octahedron discrete topology* on X and will be denoted by $\tau_{O,1}$. The pair $(X, \tau_{O,1})$ is called the *octahedron discrete space.*

(3) Let X be a nonempty set. Then we can easily see that

$$\{\ddot{0}, \ddot{1}\} \in OT_1(X), \ \left\{ \left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle, \left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle \right\} \in OT_2(X),$$
$$\left\{ \left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle, \left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle \right\} \in OT_3(X), \ \left\{ \left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle, \left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle \right\} \in OT_4(X).$$

In this case, each collection in turn is called a Type 1-, 2-, 3-, 4-*octahedron indiscrete topology* on X and will be denoted by $\tau_{1,0}, \ \tau_{2,0}, \ \tau_{3,0}, \ \tau_{4,0}$. The pairs

$$(X, \tau_{1,0}), \ (X, \tau_{2,0}), \ (X, \tau_{3,0}), \ (X, \tau_{4,0})$$

are called the 1-, 2-, 3-, 4-*octahedron indiscrete spaces.*

(4) Let (X, T) be an ordinary topological space. Then we have

$$\tau_{1,O} = \{ \langle [\chi_U, \chi_U], (\chi_U, \chi_{U^c}), \chi_U \rangle \in \mathcal{O}(X) : U \in T \} \in OT_1(X),$$
$$\tau_{2,O} = \{ \langle [\chi_U, \chi_U], (\chi_U, \chi_{U^c}), \chi_{U^c} \rangle \in \mathcal{O}(X) : U \in T \} \in OT_2(X),$$
$$\tau_{3,O} = \{ \langle [\chi_U, \chi_U], (\chi_{U^c}, \chi_U), \chi_U \rangle \in \mathcal{O}(X) : U \in T \} \in OT_3(X),$$
$$\tau_{4,O} = \{ \langle [\chi_U, \chi_U], (\chi_{U^c}, \chi_U), \chi_{U^c} \rangle \in \mathcal{O}(X) : U \in T \} \in OT_4(X).$$

(5) Let (X, T) be an fuzzy topological space proposed by Chang [12]. Consider the following family of octahedron sets in X:

$$\tau_{1,F} = \left\{ \left\langle \widetilde{U}, (U, U^c), U \right\rangle \in \mathcal{O}(X) : U \in T \right\},$$
$$\tau_{2,F} = \left\{ \left\langle \widetilde{U}, (U, U^c), U^c \right\rangle \in \mathcal{O}(X) : U \in T \right\},$$
$$\tau_{3,F} = \left\{ \left\langle \widetilde{U}, (U^c, U), U \right\rangle \in \mathcal{O}(X) : U \in T \right\},$$
$$\tau_{4,F} = \left\{ \left\langle \widetilde{U}, (U^c, U), U^c \right\rangle \in \mathcal{O}(X) : U \in T \right\}.$$

Then clearly, $\tau_{i,F} \in OT_i(X)$ for each $i = 1, 2, 3, 4$.

(6) Let (X, T) be an interval-valued fuzzy topological space introduced by Mondal and Samanta [24]. Consider the following family of octahedron sets in X:

$$\tau_{1,IV} = \left\{ \left\langle \widetilde{U}, (U^-, {U^+}^c), U^- \right\rangle \in \mathcal{O}(X) : \widetilde{U} \in T \right\},$$

$$\tau_{2,IV} = \left\{ \left\langle \widetilde{U}, (U^{-}, U^{+^c}), U^{-^c} \right\rangle \in \mathcal{O}(X) : \widetilde{U} \in T \right\},$$

$$\tau_{3,IV} = \left\{ \left\langle \widetilde{U}, (U^{+^c}, U^{-}), U^{-} \right\rangle \in \mathcal{O}(X) : \widetilde{U} \in T \right\},$$

$$\tau_{4,IV} = \left\{ \left\langle \widetilde{U}, (U^{+^c}, U^{-}), U^{-^c} \right\rangle \in \mathcal{O}(X) : \widetilde{U} \in T \right\}.$$

Then clearly, $\tau_{i,IV} \in OT_i(X)$ for each $i = 1,\ 2,\ 3,\ 4$.

(7) Let (X, T) be an intuitionistic fuzzy topological space proposed by Çoker [21]. Consider the following family of octahedron sets in X:

$$\tau_{1,IF} = \left\{ \left\langle [U^{\in}, U^{\notin^c}], \bar{U}, U^{\in} \right\rangle \in \mathcal{O}(X) : \bar{U} \in T \right\},$$

$$\tau_{2,IF} = \left\{ \left\langle [U^{\in}, U^{\notin^c}], \bar{U}, U^{\in^c} \right\rangle \in \mathcal{O}(X) : \widetilde{U} \in T \right\},$$

$$\tau_{3,IF} = \left\{ \left\langle [U^{\in}, U^{\notin^c}], \bar{U}^c, U^{\in} \right\rangle \in \mathcal{O}(X) : \widetilde{U} \in T \right\},$$

$$\tau_{4,IF} = \left\{ \left\langle [U^{\in}, U^{\notin^c}], \bar{U}^c, U^{\in^c} \right\rangle \in \mathcal{O}(X) : \widetilde{U} \in T \right\}.$$

Then clearly, $\tau_{i,IF} \in OT_i(X)$ for each $i = 1,\ 2,\ 3,\ 4$.

(8) Let (X, T) be a P-cubic topological space in the sense of Zeb et al. [48] and let us consider the following family of octahedron sets in X:

$$\tau_{1,PC} = \left\{ \left\langle \widetilde{U}, (U^{-}, U^{+^c}), U \right\rangle \in \mathcal{O}(X) : \mathbf{U} = \left\langle \widetilde{U}, U \right\rangle \in T \right\},$$

$$\tau_{2,PC} = \left\{ \left\langle \widetilde{U}, (U^{-}, U^{+^c}), U^c \right\rangle \in \mathcal{O}(X) : \mathbf{U} = \left\langle \widetilde{U}, U \right\rangle \in T \right\},$$

$$\tau_{3,PC} = \left\{ \left\langle \widetilde{U}, (U^{+^c}, U^{-}), U \right\rangle \in \mathcal{O}(X) : \mathbf{U} = \left\langle \widetilde{U}, U \right\rangle \in T \right\},$$

$$\tau_{4,PC} = \left\{ \left\langle \widetilde{U}, (U^{+^c}, U^{-}), U^c \right\rangle \in \mathcal{O}(X) : \mathbf{U} = \left\langle \widetilde{U}, U \right\rangle \in T \right\}.$$

Then clearly, $\tau_{i,PC} \in OT_i(X)$ for each $i = 1,\ 2,\ 3,\ 4$.

(9) Let (X, T) be an R-cubic topological space in the sense of Zeb et al. [48] and let us consider the following family of octahedron sets in X:

$$\tau_{1,RC} = \left\{ \left\langle \widetilde{U}, (U^{-}, U^{+^c}), U^c \right\rangle \in \mathcal{O}(X) : \mathbf{U} = \left\langle \widetilde{U}, U \right\rangle \in T \setminus \{\hat{0}, \hat{1}\} \right\},$$

$$\tau_{2,RC} = \left\{ \left\langle \widetilde{U}, (U^{-}, U^{+^c}), U \right\rangle \in \mathcal{O}(X) : \mathbf{U} = \left\langle \widetilde{U}, U \right\rangle \in T \setminus \{\hat{0}, \hat{1}\} \right\},$$

$$\tau_{3,RC} = \left\{ \left\langle \widetilde{U}, (U^{+^c}, U^{-}), U^c \right\rangle \in \mathcal{O}(X) : \mathbf{U} = \left\langle \widetilde{U}, U \right\rangle \in T \setminus \{\hat{0}, \hat{1}\} \right\},$$

$$\tau_{4,RC} = \left\{ \left\langle \widetilde{U}, (U^{+^c}, U^{-}), U \right\rangle \in \mathcal{O}(X) : \mathbf{U} = \left\langle \widetilde{U}, U \right\rangle \in T \setminus \{\hat{0}, \hat{1}\} \right\}.$$

Then clearly, $\tau_{i,RC} \in OT_i(X)$ for each $i = 1,\ 2,\ 3,\ 4$.

Now let us denote the family of all fuzzy [resp. interval-valued fuzzy, intuitionistic fuzzy, P-cubic, and R-cubic] topologies on X in the sense of Chang [12] [resp. Mondal and Samanta [24], Çoker [21] and Zeb et al. [48]] as $FT(X)$ [resp. $IVT(X)$, $IFT(X)$, $PCT(X)$ and $RCT(X)$]. Then from Definition 3.1, we can easily get the following.

Remark 3.3. (1) If $\tau \in OT_1(X)$, then $\tau_{IV} \in IVT(X)$, $\tau_{IF} \in IFT(X)$, $\tau_F \in FT(X)$, where $\tau_{IV} = \{\widetilde{U} \in IVS(X) : \mathcal{U} \in \tau\}$, $\tau_{IF} = \{\bar{U} \in (I \oplus I)^X : \mathcal{U} \in \tau\}$, $\tau_F = \{U \in I^X : \mathcal{U} \in \tau\}$. Furthermore, we have five fuzzy topologies on X in the sense of Chang:

$$\tau^-_{IV},\ \tau^+_{IV},\ \tau^{\in}_{IF},\ \tau^{\notin}_{IF},\ \tau_F,$$

where $\tau^-_{IV} = \{U^- \in I^X : \mathcal{U} \in \tau\}$, $\tau^+_{IV} = \{U^+ \in I^X : \mathcal{U} \in \tau\}$,
$\tau^{\in}_{IF} = \{U^{\in} \in I^X : \mathcal{U} \in \tau\}$, $\tau^{\notin}_{IF} = \{U^{\notin^c} \in I^X : \mathcal{U} \in \tau\}$ (See Remark 3.7 in [21]).

(2) If $\tau \in OT_2(X)$, then $\tau_{IV} \in IVT(X)$, $\tau_{IF} \in IFT(X)$, $\tau^c_F \in FT(X)$, where $\tau^c_F = \{U \in I^X : U^c \in \tau_F\}$.

(3) If $\tau \in OT_3(X)$, then $\tau_{IV} \in IVT(X)$, $\tau^c_{IF} \in IFT(X)$, $\tau_F \in FT(X)$, where $\tau^c_{IF} = \{\bar{U} \in (I \oplus I)^X : \bar{U}^c \in \tau_{IF}\}$.

(4) If $\tau \in OT_4(X)$, then $\tau_{IV} \in IVT(X)$, $\tau^c_{IF} \in IFT(X)$, $\tau^c_F \in FT(X)$.

(5) If $\tau \in OT_1(X)$, then $\tau_{PC} \in PCT(X)$, $\tau_{IF} \in IFT(X)$,
where $\tau_{PC} = \left\{ \left\langle \widetilde{U}, U \right\rangle \in \mathcal{C}(X) : \mathcal{U} \in \tau \right\}$.

From Example 3.2 and Remark 3.3, we get the following implications:

An ordinary topology $\Rightarrow$ an IVT (or IFT)
$\Rightarrow$ a P-cubic topology
$\Rightarrow$ a Type 1-octahedron topology.

Proposition 3.4. *Let X be a nonempty set, let $x_{\tilde{\tilde{a}}} \in \mathcal{O}_P(X)$ and let the family $\tau_{x_{\tilde{\tilde{a}}}}$ of octahedron sets in X given by:*

$$\tau_{x_{\tilde{\tilde{a}}}} = \{\mathcal{U} \in \mathcal{O}(X) : \mathcal{U} = \ddot{0} \text{ or } x_{\tilde{\tilde{a}}} \in_1 \mathcal{U}\}$$

Then $\tau_{x_{\tilde{\tilde{a}}}} \in OT_1(X)$.

In this case, $\tau_{x_{\tilde{\tilde{a}}}}$ is called the *included octahedron point $x_{\tilde{\tilde{a}}}$ topology on X.*

Proof. It is clear that $x_{\tilde{\tilde{a}}} \in_1 \ddot{1}$. Then $\ddot{0}, \ddot{1} \in \tau_{x_{\tilde{\tilde{a}}}}$. Thus the axiom [1-OO$_1$] is satisfied.

Let $\mathcal{U}, \mathcal{V} \in \tau_{x_{\tilde{\tilde{a}}}}$. If $\mathcal{U} = \ddot{0}$ or $\mathcal{V} = \ddot{0}$. Then clearly, $\mathcal{U} \cap^1 \mathcal{V} = \ddot{0}$. Thus $\mathcal{U} \cap^1 \mathcal{V} \in \tau_{x_{\tilde{\tilde{a}}}}$. If $\mathcal{U} \neq \ddot{0}$ and $\mathcal{V} \neq \ddot{0}$. Then clearly, $x_{\tilde{\tilde{a}}} \in_1 \mathcal{U}$ and $x_{\tilde{\tilde{a}}} \in_1 \mathcal{V}$. Thus $x_{\tilde{\tilde{a}}} \in_1 \mathcal{U} \cap^1 \mathcal{V}$. So $\mathcal{U} \cap^1 \mathcal{V} \in \tau_{x_{\tilde{\tilde{a}}}}$. Hence in either case, the axiom [1-OO$_2$] is satisfied. Now let $(\mathcal{U}_j)_{j \in J} \subset \tau_{x_{\tilde{\tilde{a}}}}$. If $\mathcal{U}_j = \ddot{0}$ for each $j \in J$, then $\bigcup^1_{j \in J} \mathcal{U}_j = \ddot{0}$. Thus $\bigcup^1_{j \in J} \mathcal{U}_j \in tau_{x_{\tilde{\tilde{a}}}}$. If there is $j \in J$ such that $\mathcal{U}_j \neq \ddot{0}$, then $x_{\tilde{\tilde{a}}} \in_1 \mathcal{U}_j$. Thus $x_{\tilde{\tilde{a}}} \in_1 \bigcup^1_{j \in J} \mathcal{U}_j$. So $\bigcup^1_{j \in J} \mathcal{U}_j \in tau_{x_{\tilde{\tilde{a}}}}$. Hence in either case, the axiom [1-OO$_3$] is satisfied. This completes the proof. □

Proposition 3.5. *Let X be a nonempty set, let $\mathcal{A} \in \mathcal{O}(X)$ and let the family $\tau_{\mathcal{A}}$ of octahedron sets in X given by:*

$$\tau_{\mathcal{A}} = \{\mathcal{U} \in \mathcal{O}(X) : \mathcal{U} = \ddot{0} \text{ or } \mathcal{A} \subset_1 \mathcal{U}\}$$

Then $\tau_{\mathcal{A}} \in OT_1(X)$.

In this case, $\tau_{\mathcal{A}}$ is called the *included octahedron set $\mathcal{A}$ topology on X.*

Proof. The proof is similar to Proposition 3.4. □

Proposition 3.6. *Let X be a nonempty set, let $x_{\widetilde{\bar{a}}} \in \mathcal{O}_P(X)$ and let the family $\tau_{\breve{x}_{\widetilde{\bar{a}}}}$ of octahedron sets in X given by:*

$$\tau_{\breve{x}_{\widetilde{\bar{a}}}} = \{\mathcal{U} \in \mathcal{O}(X) : \mathcal{U} = \ddot{1} \text{ or } x_{\widetilde{\bar{a}}} \in_1 \mathcal{U}^c\}$$

Then $\tau_{\breve{x}_{\widetilde{\bar{a}}}} \in OT_1(X)$.

In this case, $\tau_{\breve{x}_{\widetilde{\bar{a}}}}$ is called the *excluded octahedron point $x_{\widetilde{\bar{a}}}$ topology on X*.

Proof. It is obvious that $\tau_{\breve{x}_{\widetilde{\bar{a}}}}$ satisfies the axiom [1-OO$_1$]. Let $\mathcal{U},\ \mathcal{V} \in \tau_{\breve{x}_{\widetilde{\bar{a}}}}$. If $\mathcal{U} = \ddot{1}$ or $\mathcal{V} = \ddot{1}$. Then $\mathcal{U} \cap^1 \mathcal{V} = \mathcal{U}$ or $\mathcal{U} \cap^1 \mathcal{V} = \mathcal{V}$. Thus $\mathcal{U} \cap^1 \mathcal{V} \in \tau_{\breve{x}_{\widetilde{\bar{a}}}}$. If $\mathcal{U} \neq \ddot{1}$ and $\mathcal{V} \neq \ddot{1}$. Then $x_{\widetilde{\bar{a}}} \in_1 \mathcal{U}^c$ and $x_{\widetilde{\bar{a}}} \in_1 \mathcal{V}^c$. Thus $x_{\widetilde{\bar{a}}} \in_1 \mathcal{U}^c \cup^1 \mathcal{V}^c = (\mathcal{U} \cap^1 \mathcal{V})^c$. So $\mathcal{U} \cap^1 \mathcal{V} \in \tau_{\breve{x}_{\widetilde{\bar{a}}}}$. Hence in either case, $\tau_{\breve{x}_{\widetilde{\bar{a}}}}$ satisfies the axiom [1-OO$_2$]. Finally, let $(\mathcal{U}_j)_{j\in J} \subset \tau_{\breve{x}_{\widetilde{\bar{a}}}}$. If $\mathcal{U}_j \neq \ddot{1}$ for each $j \in J$, then $x_{\widetilde{\bar{a}}} \in_1 \mathcal{U}_j^c$ for each $j \in J$. Thus $x_{\widetilde{\bar{a}}} \in_1 \bigcap^1_{j\in J} \mathcal{U}_j^c = (\bigcup^1_{j\in J} \mathcal{U}_j)^c$. So $\bigcup^1_{j\in J} \mathcal{U}_j \in \tau_{\breve{x}_{\widetilde{\bar{a}}}}$. If there is $j \in J$ such that $\mathcal{U}_j = \ddot{1}$, then $\bigcup^1_{j\in J} \mathcal{U}_j = \ddot{1}$. Thus $\bigcup^1_{j\in J} \mathcal{U}_j \in \tau_{\breve{x}_{\widetilde{\bar{a}}}}$. So in either case, $\tau_{\breve{x}_{\widetilde{\bar{a}}}}$ satisfies the axiom [1-OO$_3$]. This completes the proof. □

Proposition 3.7. *Let X be a nonempty set, let $\mathcal{A} \in \mathcal{O}(X)$ and let the family $\tau_{\breve{\mathcal{A}}}$ of octahedron sets in X given by:*

$$\tau_{\breve{\mathcal{A}}} = \{\mathcal{U} \in \mathcal{O}(X) : \mathcal{U} = \ddot{1} \text{ or } \mathcal{A} \subset_1 \mathcal{U}^c\}$$

Then $\tau_{\breve{\mathcal{A}}} \in OT_1(X)$.

In this case, $\tau_{\breve{\mathcal{A}}}$ is called the *excluded octahedron set $\mathcal{A}$ topology on X*.

Proof. The proof is similar to Proposition 3.6. □

Definition 3.8. Let $\tau^c \subset \mathcal{O}(X)$. Then τ is called a:

(i) Type 1-*octahedron cotopology* on X, if it satisfies the following axioms:
[1-OC$_1$] $\ddot{0},\ \ddot{1} \in \tau^c$,
[1-OC$_2$] $\mathcal{A} \cup^1 \mathcal{B} \in \tau^c$ for any $\mathcal{A},\ \mathcal{B} \in \tau^c$,
[1-OO$_3$] $\bigcap^1_{j\in J} \mathcal{A}_j \in \tau^c$ for any $(\mathcal{A}_j)_{j\in J} \subset \tau^c$,

(ii) Type 2-*octahedron cotopology* on X, if it satisfies the following axioms:
[2-OC$_1$] $\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle,\ \left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle \in \tau^c$,
[2-OC$_2$] $\mathcal{A} \cup^2 \mathcal{B} \in \tau^c$ for any $\mathcal{A},\ \mathcal{B} \in \tau^c$,
[2-OC$_3$] $\bigcap^2_{j\in J} \mathcal{A}_j \in \tau$ for any $(\mathcal{A}_j)_{j\in J} \subset \tau^c$,

(iii) Type 3-*octahedron cotopology* on X, if it satisfies the following axioms:
[3-OC$_1$] $\left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle,\ \left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle \in \tau^c$,
[3-OC$_2$] $\mathcal{A} \cup^3 \mathcal{B} \in \tau^c$ for any $\mathcal{A},\ \mathcal{B} \in \tau^c$,
[3-OC$_3$] $\bigcap^3_{j\in J} \mathcal{A}_j \in \tau^c$ for any $(\mathcal{A}_j)_{j\in J} \subset \tau^c$,

(iv) Type 4-*octahedron cotopology* on X, if it satisfies the following axioms:
[3-OC$_1$] $\left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle,\ \left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle \in \tau^c$,
[3-OC$_2$] $\mathcal{A} \cup^4 \mathcal{B} \in \tau^c$ for any $\mathcal{A},\ \mathcal{B} \in \tau^c$,
[3-OC$_3$] $\bigcap^4_{j\in J} \mathcal{A}_j \in \tau^c$ for any $(\mathcal{A}_j)_{j\in J} \subset \tau^c$.

The pair (X, τ^c) is called a Type i-*octahedron cotopological space*. The members of τ^c are called i-*octahedron closed set* (briefly, i-OCS) in X. We will denote the set of all Type 1-octahedron cotopologies on X as $OCT_i(X)$ for $i = 1,\ 2,\ 3,\ 4$.

The following is an immediate result of Definitions 3.1 and 3.8.

Proposition 3.9. *Let* $\tau \in OT_i(X)$ *and let* $\tau^c = \{\mathcal{U}^c \in \mathcal{O}(X) : \mathcal{U} \in \tau\}$*, where* $i = 1,\ 2,\ 3,\ 4$*. Then* $\tau^c \in OCT_i(X)$ *for* $i = 1,\ 2,\ 3,\ 4$*.*

It is well-known [12, 24, 21] that for any arbitrary family $(\tau_j)_{j\in J}$ of fuzzy [resp. interval-valued fuzzy and intuitionistic fuzzy] topologies on X,

$$\bigcap_{j\in J} \tau_j \in FT(X) \text{ [resp. } IVT(X) \text{ and } IFT(X)\text{]}.$$

Then we have the following:

$$\bigcup_{j\in J} \tau_j^c \in FCT(X) \text{ [resp. } IVCT(X) \text{ and } IFCT(X)\text{]},$$

where $FCT(X)$ [$IVCT(X)$ and $IFCT(X)$] denotes the collection of all fuzzy [resp. interval-valued fuzzy and intuitionistic fuzzy] cotopologies on X.

Remark 3.10. From Remark 3.3, Definition 3.8 and Proposition 3.9, it is obvious that the followings hold.

(1) If $(\tau_j)_{j\in J} \subset OT_1(X)$, then $(\tau_{_{IV},j})_{j\in J} \subset IVT(X)$, $(\tau_{_{IF},j})_{j\in J} \subset IFT(X)$, $(\tau_{_F,j})_{j\in J} \subset FT(X)$. Thus we have:

$$\bigcap_{j\in J} \tau_{_{IV},j} \in IVT(X),\ \bigcap_{j\in J} \tau_{_{IF},j} \in IFT(X),\ \bigcap_{j\in J} \tau_{_F,j} \in FT(X).$$

(2) If $(\tau_j)_{j\in J} \subset OT_2(X)$, then $(\tau_{_{IV},j})_{j\in J} \subset IVT(X)$, $(\tau_{_{IF},j})_{j\in J} \subset IFT(X)$, $(\tau_{_F,j})_{j\in J} \subset FCT(X)$. Thus we have:

$$\bigcap_{j\in J} \tau_{_{IV},j} \in IVT(X),\ \bigcap_{j\in J} \tau_{_{IF},j} \in IFT(X),\ \bigcup_{j\in J} \tau_{_F,j} \in FCT(X).$$

(3) If $(\tau_j)_{j\in J} \subset OT_3(X)$, then $(\tau_{_{IV},j})_{j\in J} \subset IVT(X)$, $(\tau_{_{IF},j})_{j\in J} \subset IFCT(X)$, $(\tau_{_F,j})_{j\in J} \subset FT(X)$. Thus we have:

$$\bigcap_{j\in J} \tau_{_{IV},j} \in IVT(X),\ \bigcup_{j\in J} \tau_{_{IF},j} \in IFCT(X),\ \bigcap_{j\in J} \tau_{_F,j} \in FT(X).$$

(4) If $(\tau_j)_{j\in J} \subset OT_3(X)$, then $(\tau_{_{IV},j})_{j\in J} \subset IVT(X)$, $(\tau_{_{IF},j})_{j\in J} \subset IFCT(X)$, $(\tau_{_F,j})_{j\in J} \subset FCT(X)$. Thus we have:

$$\bigcap_{j\in J} \tau_{_{IV},j} \in IVT(X),\ \bigcup_{j\in J} \tau_{_{IF},j} \in IFCT(X),\ \bigcup_{j\in J} \tau_{_F,j} \in FCT(X).$$

The following is an immediate consequence of Remarks 3.3 and 3.10.

Proposition 3.11. *Let* $(\tau_j)_{j\in J} \subset OT_i(X)$*. Then* $\bigcap_{j\in J} \tau_j \in OT_i(X)$ *for* $i = 1,\ 2,\ 3,\ 4$*.*

Definition 3.12. Let $\tau_1,\ \tau_2 \in OT_i(X)$ for $i = 1,\ 2,\ 3,\ 4$. Then we say that τ_1 is *coarser than* τ_2 or τ_2 is *finer than* τ_1, if $\tau_1 \subset \tau_2$.

From Example 3.2 (1) and (2), and Definition 3.12, it is clear that for each $\tau \in OT_i(X)$, $\tau_{_{i,0}} \subset \tau \subset \tau_{_{O,1}}$ for $i = 1,\ 2,\ 3,\ 4$. Moreover, from Proposition 3.11, $(OT_i(X), \subset)$ forms a meet complete lattice with the least element $\tau_{_{i,0}}$ and the largest $\tau_{_{O,1}}$ for $i = 1,\ 2,\ 3,\ 4$.

Remark 3.13. $(OT_i(X), \subset)$ does not form a join complete lattice in general (See Example 3.14).

Example 3.14. Let $X = \{a, b\}$ and consider two families of octahedron sets in X given by:

$$\tau_1 = \{\ddot{0}, \ddot{1}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\},$$
$$\tau_2 = \{\ddot{0}, \ddot{1}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\},$$

where
$$\begin{aligned}
&\mathcal{A}_1(a) = \langle [0.4, 0.8], (0.6, 0.3), 0.7\rangle\,,\ \mathcal{A}_1(b) = \langle [0.5, 0.7], (0.5, 0.2), 0.8\rangle\,,\\
&\mathcal{A}_2(a) = \langle [0.3, 0.9], (0.4, 0.2), 0.6\rangle\,,\ \mathcal{A}_2(b) = \langle [0.6, 0.6], (0.6, 0.3), 0.6\rangle\,,\\
&\mathcal{A}_3(a) = \langle [0.3, 0.8], (0.4, 0.3), 0.6\rangle\,,\ \mathcal{A}_3(b) = \langle [0.5, 0.6], (0.5, 0.3), 0.6\rangle\,,\\
&\mathcal{A}_4(a) = \langle [0.4, 0.9], (0.6, 0.2), 0.7\rangle\,,\ \mathcal{A}_4(b) = \langle [0.6, 0.7], (0.6, 0.2), 0.8\rangle\,,\\
&\mathcal{B}_1(a) = \langle [0.5, 0.7], (0.5, 0.2), 0.4\rangle\,,\ \mathcal{B}_1(b) = \langle [0.4, 0.7], (0.4, 0.3), 0.6\rangle\,,\\
&\mathcal{B}_2(a) = \langle [0.4, 0.8], (0.4, 0.3), 0.6\rangle\,,\ \mathcal{B}_2(b) = \langle [0.5, 0.6], (0.5, 0.4), 0.3\rangle\,,\\
&\mathcal{B}_3(a) = \langle [0.4, 078], (0.4, 0.3), 0.4\rangle\,,\ \mathcal{A}_3(b) = \langle [0.4, 0.6], (0.4, 0.4), 0.3\rangle\,,\\
&\mathcal{B}_4(a) = \langle [0.5, 0.8], (0.5, 0.3), 0.6\rangle\,,\ \mathcal{B}_4(b) = \langle [0.5, 0.7], (0.5, 0.3), 0.6\rangle\,.
\end{aligned}$$

Then we can easily check that $\tau_1,\ \tau_2 \in OT_1(X)$. Furthermore, we have

$$\tau_1 \cup \tau_2 = \{\ddot{0}, \ddot{1}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\}.$$

Thus $(\mathcal{A}_1 \cup^1 \mathcal{B}_1)(a) = \langle [0.5, 0.8], (0.6, 0.2), 0.7\rangle$. So $\mathcal{A}_1 \cup^1 \mathcal{B}_1 \notin \tau_1 \cup \tau_2$. Hence $\tau_1 \cup \tau_2 \notin OT_1(X)$.

Now we define a base and a subbase in an octahedron topological space X and discuss with their some properties.

Definition 3.15. Let (X, τ) be an octahedron topological space and let $\beta,\ \sigma \subset \tau$. Then

(i) β is called an octahedron base for τ, if for each $\mathcal{U} \in \tau$, $\mathcal{U} = \ddot{0}$ or there is a $\beta^{'} \subset \beta$ such that $\mathcal{U} = \cup^1 \beta$,

(ii) σ is called an octahedron subbase for for τ, if the family $\beta = \{\cap^1 \sigma^{'} :$ $\sigma^{'}$ a finite subset of $\sigma\}$ is an octahedron base for τ.

From Remark 3.3 and Definition 3.15, we have the followings.

Remark 3.16. Let $\tau \in OT_1(X)$. Then

(1) β is an octahedron base for τ if and only if $\beta_{_{IV}}$ is an interval-valued fuzzy base for $\tau_{_{IV}}$, $\beta_{_{IF}}$ is an intuitionistic fuzzy base for $\tau_{_{IF}}$ and $\beta_{_F}$ is a fuzzy base for $\tau_{_F}$, where $\beta_{_{IV}} = \{\widetilde{B} \in IVS(X) : \mathcal{B} \in \beta\}$, $\beta_{_{IF}} = \{\bar{B} \in (I \oplus I)^X : \mathcal{B} \in \beta\}$ and $\beta_{_F} = \{B \in I^X : \mathcal{B} \in \beta\}$,

(2) σ is an octahedron subbase for τ if and only if $\sigma_{_{IV}}$ is an interval-valued fuzzy subbase for $\tau_{_{IV}}$, $\sigma_{_{IF}}$ is an intuitionistic fuzzy subbase for $\tau_{_{IF}}$ and $\sigma_{_F}$ is a fuzzy subbase for $\tau_{_F}$, where $\sigma_{_{IV}} = \{\widetilde{S} \in IVS(X) : \mathcal{S} \in \sigma\}$, $\sigma_{_{IF}} = \{\bar{S} \in (I \oplus I)^X : \mathcal{S} \in \sigma\}$ and $\sigma_{_F} = \{S \in I^X : \mathcal{S} \in \sigma\}$.

There can be a subset of $\mathcal{O}(X)$ that is not an octahedron base for an octahedron topology on a set X (See Example 3.17).

Example 3.17. Let X a nonempty set and consider the family β of octahedron sets in X given by:

$$\beta = \{\mathcal{A}, \mathcal{B}, \ddot{1}\},$$

where $\mathcal{A}(x) = \langle [0.6, 0.8], (0.6, 0.1), 0.8\rangle$, $\mathcal{B}(x) = \langle [0.5, 0.9], (0.7, 0.2), 0.7\rangle$ for each $x \in X$. Suppose β is an octahedron base for an octahedron topology τ on X. Then by Definition 3.15, $\beta \subset \tau$. Thus $\mathcal{A} \cap^1 \mathcal{B} = \mathcal{C} \in \tau$,, where $\mathcal{C}(x) = \langle [0.5, 0.8], (0.6, 0.2), 0.7\rangle$ for each $x \in X$. Clearly, $\mathcal{C} \neq \ddot{0}$. However, there is no subcollection $\beta^{'}$ of β such that $\mathcal{C} = \cup^1 \beta^{'}$. So β is not an octahedron base for τ.

The following gives a necessary and sufficient condition for a collection of octahedron sets in X to be an octahedron base for an octahedron topology on X.

Theorem 3.18. *Let $\beta \subset \mathcal{O}(X)$. Then β is an octahedron base for some octahedron topology on X if and only if it satisfies the following conditions:*
(1) $\ddot{1} = \cup^1 \beta$,
(2) *if* $\mathcal{B}_1,\ \mathcal{B}_2 \in \beta$ *and* $x_{\tilde{\tilde{a}}} \in_1 \mathcal{B}_1 \cap^1 \mathcal{B}_2$, *then there is* $\mathcal{B} \in \beta$ *such that*

$$x_{\tilde{\tilde{a}}} \in_1 \mathcal{B} \subset_1 \mathcal{B}_1 \cap^1 \mathcal{B}_2.$$

Proof. Let β be an octahedron base for some octahedron topology τ on X. Then clearly, $\ddot{1} \in \tau$. Thus by Definition 3.15, $\ddot{1} = \cup^1 \beta$. So the condition (1) is satisfied. Suppose $\mathcal{B}_1,\ \mathcal{B}_2 \in \beta$ and $x_{\tilde{\tilde{a}}} \in_1 \mathcal{B}_1 \cap^1 \mathcal{B}_2$. Then clearly, $\mathcal{B}_1,\ \mathcal{B}_2 \in \tau$. Thus $\mathcal{B}_1 \cap^1 \mathcal{B}_2 \in \tau$. So there is $\beta^{'} \subset \beta$ such that $\mathcal{B}_1 \cap^1 \mathcal{B}_2 = \cup^1 \beta^{'}$. Since $x_{\tilde{\tilde{a}}} \in_1 \mathcal{B}_1 \cap^1 \mathcal{B}_2$, $x_{\tilde{\tilde{a}}} \in_1 \beta^{'}$. By Proposition 4.6 in [42], there is $\mathcal{B} \in \beta$ such that $x_{\tilde{\tilde{a}}} \in_1 \mathcal{B} \subset_1 \mathcal{B}_1 \cap^1 \mathcal{B}_2$. Hence he condition (2) is satisfied.

Conversely, suppose β satisfies the conditions (1) and (2). Let τ be be the collection of octahedron sets in X given by:

$$\tau = \{\ddot{0}\} \cup \{\mathcal{U} : \mathcal{U} = \cup^1 \beta^{'} \text{ for some } \beta^{'} \subset \beta\}.$$

Then clearly, $\ddot{0}$, $\ddot{1} \in \tau$. Let $(\mathcal{U}_j)_{j \in J} \subset \tau$. Then by the definition of τ, we can easily see that $\bigcup^1_{j \in J} \mathcal{U}_j \in \tau$. Now suppose $\mathcal{U}_1,\ \mathcal{U}_2 \in \tau$ and $x_{\tilde{\tilde{a}}} \in_1 \mathcal{U}_1 \cap^1 \mathcal{U}_2$. Then there are $\mathcal{B}_1,\ \mathcal{B}_2 \in \beta$ such that $x_{\tilde{\tilde{a}}} \in_1 \mathcal{B}_1 \subset_1 \mathcal{U}_1$ and $x_{\tilde{\tilde{a}}} \in_1 \mathcal{B}_2 \subset_1 \mathcal{U}_2$. Thus $x_{\tilde{\tilde{a}}} \in_1 \mathcal{B}_1 \cap^1 \mathcal{B}_2$. By the condition (2), there is $\mathcal{B} \in \beta$ such that $x_{\tilde{\tilde{a}}} \in_1 \mathcal{B} \subset_1 \mathcal{B}_1 \cap^1 \mathcal{B}_2$. So there is $\beta^{'} \subset \beta$ such that $\mathcal{U}_1 \cap^1 \mathcal{U}_2 = \bigcup^1 \beta^{'}$, where $\beta^{'} = \{\mathcal{B} \in \beta : x_{\tilde{\tilde{a}}} \in_1 \mathcal{B} \subset_1 \mathcal{B}_1 \cap^1 \mathcal{B}_2\}$. Hence $\mathcal{U}_1 \cap^1 \mathcal{U}_2 \in \tau$. This completes the proof. □

Example 3.19. Let $X = \{a, b, c\}$ consider the family β of octahedron sets in X given by:

$$\beta = \{\mathcal{A}, \mathcal{B}, \mathcal{C}\},$$

where for each $x \in X$,

$$\mathcal{A}(x) = \langle [0.6, 0.8], (0.6, 0.3), 0.6\rangle\,,$$

$$\mathcal{B}(x) = \begin{cases} \langle [0.6, 0.8], (0.6, 0.3), 0.6\rangle & \text{if } x = a \\ \langle \mathbf{1}, \bar{1}, 1\rangle & \text{otherwise,} \end{cases}$$

$$\mathcal{C}(x) = \begin{cases} \langle \mathbf{1}, \bar{1}, 1\rangle & \text{if } x = a \\ \langle [0.6, 0.8], (0.6, 0.3), 0.6\rangle & \text{otherwise.} \end{cases}$$

Then we can easily check that β satisfies the conditions of Theorem 3.18. Thus β is an octahedron base for an octahedron topology τ. In fact, $\tau = \{\ddot{0}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \ddot{1}\}$.

The following is a characterization for an octahedron topology τ_2 to be finer than an octahedron topology τ_1 in terms of octahedron bases for τ_1 and τ_2.

Theorem 3.20. *Let $\tau_1,\ \tau_2 \in OT_1(X)$, and let β_1 and β_2 be octahedron bases for τ_1 and τ_2 respectively. Then the following are equivalent:*
(1) *τ_2 is finer than τ_1,*
(2) *for each $x_{\tilde{\bar{a}}} \in_1 \ddot{1}$ and $\mathcal{B}_1 \in \beta_1$ such that $x_{\tilde{\bar{a}}} \in_1 \mathcal{B}_1$, there is $\mathcal{B}_2 \in \beta_2$ such that $x_{\tilde{\bar{a}}} \in_1 \mathcal{B}_2 \subset_1 \mathcal{B}_1$.*

Proof. Suppose (1) holds, let $x_{\tilde{\bar{a}}} \in_1 \ddot{1}$ and let $\mathcal{B}_1 \in \beta_1$ such that $x_{\tilde{\bar{a}}} \in_1 \mathcal{B}_1$. Then clearly, $\mathcal{B}_1 \in \tau_1$. Thus by the hypothesis, $\mathcal{B}_1 \in \tau_2$. Since β_2 is an octahedron base for τ_2, there is $\beta_2^{'} \subset \beta_2$ such that $\mathcal{B}_1 = \cup^1 \beta_2^{'}$. Since $x_{\tilde{\bar{a}}} \in_1 \mathcal{B}_1$, there is $\mathcal{B}_2 \in \beta_2$ such that $x_{\tilde{\bar{a}}} \in_1 \mathcal{B}_2 \subset_1 \mathcal{B}_1$.

Conversely, suppose (2) holds. Let $\mathcal{U} \in \tau_1$ and let $x_{\tilde{\bar{a}}} \in_1 \mathcal{U}$. Since β_1 is an octahedron base for τ_1, there is $\mathcal{B}_1 \in \beta_1$ such that $x_{\tilde{\bar{a}}} \in_1 \mathcal{B}_1 \subset_1 \mathcal{U}$. Then by the condition (2), there is $\mathcal{B}_2 \in \beta_2$ such that $x_{\tilde{\bar{a}}} \in_1 \mathcal{B}_2 \subset_1 \mathcal{B}_1$. Since $\mathcal{B}_1 \subset_1 \mathcal{U}$, $x_{\tilde{\bar{a}}} \in_1 \mathcal{B}_2 \subset_1 \mathcal{U}$. Thus there is $\beta_2^{'} \subset \beta_2$ such that $\mathcal{U} = \bigcup^1 \beta_2^{'}$. So $\mathcal{U} \in \tau_2$. Hence τ_2 is finer than τ_1. □

We know that every topology has a base since the topology itself forms a base. The following gives a sufficient condition for a subcollection of an octahedron topology τ to be an octahedron base for τ.

Proposition 3.21. *Let (X, τ) be an octahedron topological space. Suppose $\beta \subset \tau$ such that for each $\tilde{\bar{a}} \in_1 \ddot{1}$ and each $\mathcal{U} \in \tau$ with $\tilde{\bar{a}} \in_1 \mathcal{U}$, there is $\mathcal{B} \in \beta$ such that $\tilde{\bar{a}} \in_1 \mathcal{B} \subset_1 \mathcal{U}$. Then β is an octahedron base for τ.*

Proof. Let $\tilde{\bar{a}} \in_1 \ddot{1}$. Since $\ddot{1} \in \tau$, there is $\mathcal{B} \in \beta$ such that $\tilde{\bar{a}} \in_1 \mathcal{B} \subset_1 \ddot{1}$ by the hypothesis. Then $\ddot{1} = \cup^1 \beta$. Thus β satisfies the the condition (1) of Theorem 3.18. Now suppose $\mathcal{B}_1,\ \mathcal{B}_2 \in \beta$ and $\tilde{\bar{a}} \in_1 \mathcal{B}_1 \cap^1 \mathcal{B}_2$. Then clearly, $\mathcal{B}_1 \cap^1 \mathcal{B}_2 \in \tau$. Thus there is $\mathcal{B} \in \beta$ such that $\tilde{\bar{a}} \in_1 \mathcal{B} \subset_1 \mathcal{B}_1 \cap^1 \mathcal{B}_2$. So β satisfies the the condition (2) of Theorem 3.18. Hence by Theorem 3.18, β is an octahedron base for an octahedron topology τ_2 on X.

On the other hand, from Theorem 3.20, it is obvious that τ_2 is finer than τ_1, i.e., $\tau_1 \subset \tau_2$. Let $\mathcal{U} \in \tau_2$. Then there is $\beta^{'} \subset \beta$ such that $\mathcal{U} = \cup^1 \beta^{'}$. Since $\beta \subset \tau_1, \mathcal{U} \in \tau_1$. Thus $\tau_2 \subset \tau_1$. So $\tau_1 = \tau_2$. This completes the proof. □

The following provides a sufficient condition for a family of octahedron sets in X to be an octahedron subbase for a unique octahedron topology τ on X.

Proposition 3.22. *Let X be a set and let $\sigma \subset \mathcal{O}(X)$ such that $\ddot{1} = \cup^1 \sigma$. Then there is a unique octahedron topology τ on X such that σ is an octahedron subbase for τ.*

Proof. Let $\beta = \{\mathcal{B} \in \mathcal{O}(X) : \mathcal{B} = \cap^1 \sigma^{'}$ and $\sigma^{'}$ is a finite subset of $\sigma\}$ and let

$$\tau = \{\mathcal{U} \in \mathcal{O}(X) : \mathcal{U} = \ddot{0} \text{ or } \mathcal{U} = \cup^1 \beta^{'} \text{ for some } \beta^{'} \subset \beta\}.$$

Then from the same proof of the classical case, we can check that τ is a unique octahedron topology on X such that σ is an octahedron subbase for τ. □

Example 3.23. Let $X = \{a, b, c, d, e\}$ and consider the family σ of octahedron sets in X given by:

$$\sigma = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\},$$

where for each $x \in X$,

$$\mathcal{A}_1(x) = \begin{cases} \langle [0.6, 0.8], (0.6, 0.3), 0.6 \rangle & \text{if } x = a \\ \langle \mathbf{1}, \bar{1}, 1 \rangle & \text{otherwise,} \end{cases}$$

$$\mathcal{A}_2(x) = \begin{cases} \langle [0.5, 0.9], (0.7, 0.2), 0.8 \rangle & \text{if } x = b \text{ } or \text{ } c \\ \langle \mathbf{0}, \bar{0}, 0 \rangle & \text{if } x = a \\ \langle \mathbf{1}, \bar{1}, 1 \rangle & \text{if } x = d \text{ } or \text{ } e, \end{cases}$$

$$\mathcal{A}_3(x) = \begin{cases} \langle [0.5, 0.9], (0.7, 0.2), 0.8 \rangle & \text{if } x = b \text{ } or \text{ } c \\ \langle [0.4, 0.7], (0.5, 0.4), 0.5 \rangle & \text{if } x = d \\ \langle \mathbf{0}, \bar{0}, 0 \rangle & \text{if } x = a \\ \langle \mathbf{1}, \bar{1}, 1 \rangle & \text{if } x = e, \end{cases}$$

$$\mathcal{A}_4(x) = \begin{cases} \langle [0.6, 0.7], (0.8, 0.1), 0.7 \rangle & \text{if } x = e \\ \langle \mathbf{1}, \bar{1}, 1 \rangle & \text{if } x = a \\ \langle \mathbf{0}, \bar{0}, 0 \rangle & \text{otherwise.} \end{cases}$$

Then clearly, $\cup^1 \sigma = \ddot{1}$. Thus by Proposition 3.21, σ is an octahedron subbase for a unique octahedron topology τ on X. Let β be the octahedron base for τ. Then we can easily get

$$\beta = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6\},$$

where for each $x \in X$,

$$\mathcal{A}_5(x) = \begin{cases} \langle [0.6, 0.8], (0.6, 0.3), 0.6 \rangle & \text{if } x = a \\ \langle [0.6, 0.7], (0.8, 0.1), 0.7 \rangle & \text{if } x = e \\ \langle \mathbf{0}, \bar{0}, 0 \rangle & \text{otherwise,} \end{cases}$$

$$\mathcal{A}_6(x) = \begin{cases} \langle [0.6, 0.7], (0.8, 0.1), 0.7 \rangle & \text{if } x = e \\ \langle \mathbf{0}, \bar{0}, 0 \rangle & \text{otherwise.} \end{cases}$$

So we get the octahedron topology τ generated by σ:

$$\tau = \{\ddot{0}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \mathcal{A}_8, \ddot{1}\},$$

where for each $x \in X$,

$$\mathcal{A}_7(x) = \begin{cases} \langle [0.5, 0.9], (0.7, 0.2), 0.8 \rangle & \text{if } x = b \text{ } or \text{ } c \\ \langle \mathbf{1}, \bar{1}, 1 \rangle & \text{otherwise,} \end{cases}$$

$$\mathcal{A}_8(x) = \begin{cases} \langle [0.5, 0.9], (0.7, 0.2), 0.8 \rangle & \text{if } x = b \text{ } or \text{ } c \\ \langle [0.4, 0.7], (0.5, 0.4), 0.5 \rangle & \text{if } x = d \\ \langle \mathbf{1}, \bar{1}, 1 \rangle & \text{otherwise.} \end{cases}$$

4. Octahedron subspaces, and Octahedron closures and interiors

In this section, we define an octahedron subspace, an octahedron closures and interiors, and discus with their some properties.

Proposition 4.1. *Let (X, τ) be an octahedron topological space and let $\mathcal{A} \in \mathcal{O}(X)$. Then the family $\tau_{\mathcal{A}}$ of octahedron sets in X given by:*

$$\tau_{\mathcal{A}} = \{\mathcal{A} \cap^1 \mathcal{U} : \mathcal{U} \in \tau\}$$

is an octahedron topology on $\mathcal{A}$.

In this case, $\tau_{\mathcal{A}}$ is called a *relative octahedron topology on $\mathcal{A}$ determined by τ* and the pair $(\mathcal{A}, \tau_{\mathcal{A}})$ is called an *octahedron subspace* of (X, τ). The members of $\tau_{\mathcal{A}}$ is called *relatively octahedron open sets* or simply *octahedron open sets in $\mathcal{A}$*. $\mathcal{B} \in \mathcal{O}(X)$ is said to be *octahedron closed in* $(\mathcal{A}, \tau_{\mathcal{A}})$, if $\mathcal{A} - \mathcal{B} \in \tau_{\mathcal{A}}$, where $\mathcal{A} - \mathcal{B} = \mathcal{A} \cap^1 \mathcal{B}^c$ and $\mathcal{B} \subset_1 \mathcal{A}$.

Proof. Clearly, $\ddot{0}$, $\ddot{1} \in \tau$ and $\mathcal{A} \cap^1 \ddot{0} = \ddot{0}$, $\mathcal{A} \cap^1 \ddot{1} = \mathcal{A}$. Then $\ddot{0}$, $\mathcal{A} \in \tau_{\mathcal{A}}$. Thus $\tau_{\mathcal{A}}$ satisfies the axiom [1-OO_1]. Let $\mathcal{B}$, $\mathcal{C} \in \tau_{\mathcal{A}}$. Then there are $\mathcal{U}$, $\mathcal{V} \in \tau$ such that $\mathcal{B} = \mathcal{A} \cap^1 \mathcal{U}$ and $\mathcal{C} = \mathcal{A} \cap^1 \mathcal{V}$. Thus $\mathcal{B} \cap^1 \mathcal{C} = \mathcal{A} \cap^1 (\mathcal{U} \cap^1 \mathcal{V})$ and $\mathcal{U} \cap^1 \mathcal{V} \in \tau$. So $\mathcal{B} \cap^1 \mathcal{C} \in \tau_{\mathcal{A}}$. Hence $\tau_{\mathcal{A}}$ satisfies the axiom [1-OO_2]. Now let $(\mathcal{B}_j)_{j \in J} \subset \tau_{\mathcal{A}}$. Then clearly, for each $j \in J$, there is $\mathcal{U}_j \in \tau$ such that $\mathcal{B}_j = \mathcal{A} \cap^1 \mathcal{U}_j$. Thus $\bigcup^1_{j \in J} \mathcal{B}_j = \mathcal{A} \cap^1 (\bigcup^1_{j \in J} \mathcal{U}_j)$ and $\bigcup^1_{j \in J} \mathcal{U}_j \in \tau$. So $\bigcup^1_{j \in J} \mathcal{B}_j \in \tau_{\mathcal{A}}$. Hence $\tau_{\mathcal{A}}$ satisfies the axiom [1-OO_3]. This completes the proof. □

Example 4.2. (1) If τ is the octahedron discrete topology on a set X and $\mathcal{A} \in \mathcal{O}(X)$, then $\tau_{\mathcal{A}}$ is the octahedron discrete topology on $\mathcal{A}$.

(2) If τ is the 1-octahedron indiscrete topology on a set X and $\mathcal{A} \in \mathcal{O}(X)$, then $\tau_{\mathcal{A}}$ is the 1-octahedron discrete topology on $\mathcal{A}$.

Remark 4.3. Let (X, τ) be an octahedron topological space and let $\mathcal{A} \in \mathcal{O}(X)$. Then $(\tau_F)_A$ [resp. $(\tau_{IV})_{\widetilde{A}}$ and $(\tau_{IF})_{\bar{A}}$] is a relative fuzzy [resp. interval-valued fuzzy and intuitionistic fuzzy] topology on A [resp. $\widetilde{A}$ and $\bar{A}$] in the sense of [17] [resp. [24] and [23]] (See Remark 3.3).

The following is an immediate consequence of Proposition 4.1.

Proposition 4.4. *Let (X, τ) be an octahedron topological space and let $\mathcal{A}$, $\mathcal{B} \in \mathcal{O}(X)$ such that $\mathcal{A} \subset_1 \mathcal{B}$. Then $\tau_{\mathcal{A}} = (\tau_{\mathcal{B}})_{\mathcal{A}}$.*

Proposition 4.5. *Let (X, τ) be an octahedron topological space, let $\mathcal{A} \in \mathcal{O}(X)$ and let β be an octahedron base for τ. Then $\beta_{\mathcal{A}} = \{\mathcal{B} \cap^1 \mathcal{A} : \mathcal{B} \in \beta\}$ is an octahedron base for $\tau_{\mathcal{A}}$.*

Proof. Let $\mathcal{U} \in \tau_{\mathcal{A}}$ such that $\widetilde{\widetilde{a}} \in_1 \mathcal{U}$. Then there is $\mathcal{V} \in \tau$ such that $\mathcal{U} = \mathcal{V} \cap^1 \mathcal{A}$. Since $\widetilde{\widetilde{a}} \in_1 \mathcal{U}$, $\widetilde{\widetilde{a}} \in_1 \mathcal{V} \cap^1 \mathcal{A}$. Thus $\widetilde{\widetilde{a}} \in_1 \mathcal{V}$. Since $\mathcal{V} \in \tau$ and β is an octahedron base for τ, there is $\mathcal{B} \in \beta$ such that $\widetilde{\widetilde{a}} \in_1 \mathcal{B} \subset_1 \mathcal{V}$. So $\widetilde{\widetilde{a}} \in_1 \mathcal{B} \cap^1 \mathcal{A} \subset_1 \mathcal{V} \cap^1 \mathcal{A} = \mathcal{U}$. Hence by Proposition 3.21, $\beta_{\mathcal{A}}$ is an octahedron base for $\tau_{\mathcal{A}}$. □

The following provides a special situation in which every member of the relative octahedron topology is also a member of the octahedron topology.

Proposition 4.6. *Let (X, τ) be an octahedron topological space and let $\mathcal{A} \in \tau$. If $\mathcal{U} \in \tau_{\mathcal{A}}$, then $\mathcal{U} \in \tau$.*

Proof. The proof is straightforward. $\square$

Theorem 4.7. *Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be an octahedron subspace of an octahedron topological space (X, τ) and let $\mathcal{B} \in \mathcal{O}(X)$ such that $\mathcal{B} \subset_1 \mathcal{A}$. Then $\mathcal{B}$ is closed in $(\mathcal{A}, \tau_{\mathcal{A}})$ if and only if there is $\mathcal{C} \in \tau^c$ such that $\mathcal{B} = \mathcal{C} \cap^1 \mathcal{A}$.*

Proof. Suppose $\mathcal{B}$ is closed in $(\mathcal{A}, \tau_{\mathcal{A}})$. Then $\mathcal{A} - \mathcal{B} = \mathcal{A} \cap^1 \mathcal{B}^c \in \tau_{\mathcal{A}}$. Thus there is $\mathcal{U} \in \tau$ such that $\mathcal{A} - \mathcal{B} = \mathcal{U} \cap^1 \mathcal{A}$. So we get $\mathcal{B} = \mathcal{A} \cap^1 \mathcal{U}^c$. It is clear that $\mathcal{U}^c \in \tau^c$. Hence the result holds.

The converse is easily proved. $\square$

The following is an immediate consequence of Theorem 4.7.

Corollary 4.8. *Let (X, τ) be an octahedron topological space and let $\mathcal{A} \in \tau^c$. If $\mathcal{B}$ is closed in $(\mathcal{A}, \tau_{\mathcal{A}})$, then $\mathcal{B} \in \tau^c$.*

Definition 4.9. Let (X, τ) be an octahedron topological space and let $\mathcal{A} \in \mathcal{O}(X)$.

(i) The octahedron closure of $\mathcal{A}$ with respect to τ, denoted by $Ocl_\tau(\mathcal{A})$ or $Ocl(\mathcal{A})$, is an octahedron set in X defined as:

$$Ocl(\mathcal{A}) = \bigcap^{1} \{\mathcal{F} \in \tau^c : \mathcal{A} \subset_1 \mathcal{F}\}.$$

(ii) The octahedron interior of $\mathcal{A}$ with respect to τ, denoted by $Oint_\tau(\mathcal{A})$ or $Oint(\mathcal{A})$, is an octahedron set in X defined as:

$$Oint(\mathcal{A}) = \bigcup^{1} \{\mathcal{U} \in \tau : \mathcal{U} \subset_1 \mathcal{A}\}.$$

We can easily see that for each $\mathcal{A} \in \mathcal{O}(X)$, $Ocl(\mathcal{A})$ is the smallest octahedron closed set in X such that $\mathcal{A} \subset_1 Ocl(\mathcal{A})$ and $Oint(\mathcal{A})$ is the largest octahedron open set in X such that $Oint(\mathcal{A}) \subset_1 \mathcal{A}$.

Example 4.10. Let (X, τ) be the octahedron topological space given in Example 3.23. Then we have

$$\tau = \{\ddot{0}, \mathcal{A}_1^c, \mathcal{A}_2^c, \mathcal{A}_3^c, \mathcal{A}_4^c, \mathcal{A}_5^c, \mathcal{A}_6^c, \mathcal{A}_7^c, \mathcal{A}_8^c, \ddot{1}\},$$

where for each $x \in X$,

$$\mathcal{A}_1^c(x) = \begin{cases} \langle [0.2, 0.4], (0.3, 0.6), 0.4 \rangle & \text{if } x = a \\ \langle \mathbf{0}, \bar{0}, 0 \rangle & \text{otherwise,} \end{cases}$$

$$\mathcal{A}_2^c(x) = \begin{cases} \langle [0.1, 0.5], (0.2, 0.7), 0.2 \rangle & \text{if } x = b \text{ or } c \\ \langle \mathbf{1}, \bar{1}, 1 \rangle & \text{if } x = a \\ \langle \mathbf{0}, \bar{0}, 0 \rangle & \text{if } x = d \text{ or } e, \end{cases}$$

$$\mathcal{A}_3^c(x) = \begin{cases} \langle [0.1, 0.5], (0.2, 0.7), 0.2 \rangle & \text{if } x = b \text{ or } c \\ \langle [0.3, 0.6], (0.4, 0.5), 0.5 \rangle & \text{if } x = d \\ \langle \mathbf{1}, \bar{1}, 1 \rangle & \text{if } x = a \\ \langle \mathbf{0}, \bar{0}, 0 \rangle & \text{if } x = e, \end{cases}$$

$$\mathcal{A}_4^c(x) = \begin{cases} \langle [0.3, 0.4], (0.1, 0.8), 0.3 \rangle & \text{if } x = e \\ \langle \mathbf{0}, \bar{0}, 0 \rangle & \text{if } x = a \\ \langle \mathbf{1}, \bar{1}, 1 \rangle & \text{otherwise,} \end{cases}$$

$$\mathcal{A}_5^c(x) = \begin{cases} \langle [0.2, 0.4], (0.3, 0.6), 0.4 \rangle & \text{if } x = a \\ \langle [0.3, 0.4], (0.1, 0.8), 0.3 \rangle & \text{if } x = e \\ \langle \mathbf{1}, \bar{1}, 1 \rangle & \text{otherwise,} \end{cases}$$

$$\mathcal{A}_6^c(x) = \begin{cases} \langle [0.3, 0.4], (0.1, 0.8), 0.3 \rangle & \text{if } x = e \\ \langle \mathbf{1}, \bar{1}, 1 \rangle & \text{otherwise,} \end{cases}$$

$$\mathcal{A}_7^c(x) = \begin{cases} \langle [0.1, 0.5], (0.2, 0.7), 0.2 \rangle & \text{if } x = b \text{ or } c \\ \langle \mathbf{0}, \bar{0}, 0 \rangle & \text{otherwise,} \end{cases}$$

$$\mathcal{A}_8^c(x) = \begin{cases} \langle [0.1, 0.5], (0.2, 0.7), 0.2 \rangle & \text{if } x = b \text{ or } c \\ \langle [0.3, 0.6], (0.4, 0.5), 0.5 \rangle & \text{if } x = d \\ \langle \mathbf{0}, \bar{0}, 0 \rangle & \text{otherwise.} \end{cases}$$

Consider two octahedron sets $\mathcal{A}$ and $\mathcal{B}$ in X given by:

$\mathcal{A}(a) = \langle [0.7, 0.9], (0.7, 0.2), 0.7 \rangle$, $\mathcal{A}(b) = \mathcal{A}(c) = \langle [0.6, 0.9], (0.8, 0.1), 0.9 \rangle$,
$\mathcal{A}(d) = \langle [0.3, 0.5], (0.4, 0.5), 0.3 \rangle$, $\mathcal{A}(e) = \langle [0.7, 0.8], (0.9, 0.1), 0.8 \rangle$,
$\mathcal{B}(a) = \langle [0.1, 0.3], (0.2, 0.7), 0.3 \rangle$, $\mathcal{B}(b) = \mathcal{A}(c) = \langle [0.1, 0.4], (0.3, 0.6), 0.1 \rangle$,
$\mathcal{B}(d) = \langle [0.3, 0.5], (0.4, 0.5), 0.3 \rangle$, $\mathcal{B}(e) = \langle [0.2, 0.3], (0.1, 0.7), 0.2 \rangle$.

Then we can easily check that

$$\mathcal{A}_5,\ \mathcal{A}_6 \subset_1 \mathcal{A} \text{ and } \mathcal{A} \subset_1 \mathcal{A}_5^c,\ \mathcal{A}_6^c.$$

Thus we have $Oint(\mathcal{A}) = \mathcal{A}_5 \cup_1 \mathcal{A}_6$ and $Ocl(\mathcal{B}) = \mathcal{A}_5^c \cap_1 \mathcal{A}_6^c$.

Proposition 4.11. *Let (X, τ) be an octahedron topological space and let $\mathcal{A} \in \mathcal{O}(X)$. Then*

$$Oint(\mathcal{A}^c) = (Ocl(\mathcal{A}))^c \text{ and } Ocl(\mathcal{A}^c) = (Oint(\mathcal{A}))^c.$$

Proof.
$$\begin{aligned} Oint(\mathcal{A}^c) &= \bigcup\nolimits^1 \{\mathcal{U} \in \tau : \mathcal{U} \subset_1 \mathcal{A}^c\} \\ &= \bigcup \{\mathcal{U} \in \tau : \widetilde{U} \subset \widetilde{A}^c,\ \bar{U} \subset \bar{A}^c,\ U \subset A^c\} \\ &= \bigcup \{\mathcal{U} \in \tau : \widetilde{A} \subset \widetilde{U}^c,\ \bar{A} \subset \bar{U}^c,\ A \subset U^c\} \\ &= \bigcap \{\mathcal{U}^c \in \tau^c : \mathcal{A} \subset_1 \mathcal{U}^c\} \\ &= Ocl(\mathcal{A}). \end{aligned}$$

Similarly, we can show that $Ocl(\mathcal{A}^c) = (Oint(\mathcal{A}))^c$. □

Proposition 4.12. *Let (X, τ) be an octahedron topological space and let $\mathcal{A},\ \mathcal{A} \in \mathcal{O}(X)$.*

(1) *$\mathcal{A} \in \tau^c$ if and only if $\mathcal{A} = Ocl(\mathcal{A})$.* (2) *$Ocl(Ocl(\mathcal{A})) = \mathcal{A}$.*
(3) *$Ocl(\ddot{0}) = \ddot{0}$.* (4) *If $\mathcal{A} \subset_1 \mathcal{B}$, then $Ocl(\mathcal{A}) \subset_1 Ocl(\mathcal{B})$.*
(5) *$Ocl(\mathcal{A} \cup^1 \mathcal{B}) = Ocl(\mathcal{A}) \cup^1 Ocl(\mathcal{B})$.* (6) *$Ocl(\mathcal{A} \cap^1 \mathcal{B}) \subset_1 Ocl(\mathcal{A}) \cap^1 Ocl(\mathcal{B})$.*

Proof. The proofs of (1)–(4) are obvious from Definition 4.9.

(5) It is clear that $\mathcal{A} \cup_1 \mathcal{B} \subset_1 \mathcal{A}$ and $\mathcal{A} \cup_1 \mathcal{B} \subset_1 \mathcal{B}$. Then by (4), we have

$$Ocl(\mathcal{A} \cup^1 \mathcal{B}) \subset_1 Ocl(\mathcal{A}) \text{ and } Ocl(\mathcal{A} \cup^1 \mathcal{B}) \subset_1 Ocl(\mathcal{B}).$$

Thus $Ocl(\mathcal{A} \cup^1 \mathcal{B}) \subset_1 Ocl(\mathcal{A}) \cup^1 Ocl(\mathcal{B})$. On the other hand, since $\mathcal{A} \subset_1 Ocl(\mathcal{A})$ and $\mathcal{B} \subset_1 Ocl(\mathcal{B}\mathcal{G})$, $\mathcal{A} \cup^1 \mathcal{B} \subset_1 Ocl(\mathcal{A}) \cup^1 Ocl(\mathcal{B})$. Since $Ocl(\mathcal{A}) \cup^1 Ocl(\mathcal{B}) \in \tau^c$, $Ocl(\mathcal{A} \cup^1 \mathcal{B}) \subset_1 Ocl(\mathcal{A}) \cup^1 Ocl(\mathcal{B})$ by Definition 4.9. So (5) holds.

(6) It is obvious that $\mathcal{A} \cap^1 \mathcal{B} \subset_1 \mathcal{A}$ and $\mathcal{A} \cap^1 \mathcal{B} \subset_1 \mathcal{B}$. Then by by (4), we have

$$Ocl(\mathcal{A} \cap^1 \mathcal{B}) \subset_1 Ocl(\mathcal{A}) \text{ and } Ocl(\mathcal{A} \cap^1 \mathcal{B}) \subset_1 Ocl(\mathcal{B}).$$

Thus $Ocl(\mathcal{A} \cap^1 \mathcal{B}) \subset_1 Ocl(\mathcal{A}) \cap^1 Ocl(\mathcal{B})$. □

Definition 4.13. Let X be a set. Then a mapping $Ocl^* : \mathcal{O}(X) \to \mathcal{O}(X)$ is called an *octahedron closure operator* on X, if it satisfies the following axioms (called the *octahedron Kuratowski closure axioms*): for any $\mathcal{A},\ \mathcal{B} \in \mathcal{B}(X)$,
[OK1] $Ocl^*(\ddot{0}) = \ddot{0}$,
[OK2] $\mathcal{A} \subset_1 Ocl^*(\mathcal{A})$,
[OK3] $Ocl^*(Ocl^*(\mathcal{A})) = Ocl^*(\mathcal{A})$,
[OK3] $Ocl^*(\mathcal{A} \cup^1 \mathcal{B}) = Ocl^*(\mathcal{A}) \cup^1 Ocl^*(\mathcal{B})$.

The following shows that an octahedron closure operator completely determines an octahedron topology and that the octahedron operator is the octahedron closure in this octahedron topology.

Proposition 4.14. *Let Ocl^* be an octahedron closure operator on a set X and let τ be the family of octahedron sets in given by:*

$$\tau = \{\mathcal{A}^c \in \mathcal{O}(X) : Ocl^*(\mathcal{A}) = \mathcal{A}\}.$$

Then τ is an octahedron topology on X. Furthermore, if Ocl is the octahedron closure operator defined by τ, then $Ocl^*(\mathcal{A}) = Ocl(\mathcal{A})$ for each $\mathcal{A} \in \mathcal{O}(X)$.

Proof. The proof is almost similar to one of classical topological spaces. □

Proposition 4.15. *Let (X, τ) be an octahedron topological space and let $\mathcal{A},\ \mathcal{A} \in \mathcal{O}(X)$.*
(1) *$\mathcal{A} \in \tau$ if and only if $\mathcal{A} = Oint(\mathcal{A})$.* (2) *$Oint(Oint(\mathcal{A})) = \mathcal{A}$.*
(3) *$Oint(\ddot{1}) = \ddot{1}$.* (4) *If $\mathcal{A} \subset_1 \mathcal{B}$, then $Oint(\mathcal{A}) \subset_1 Oint(\mathcal{B})$.*
(5) *$Oint(\mathcal{A} \cap^1 \mathcal{B}) = Oint(\mathcal{A}) \cap^1 Oint(\mathcal{B})$.* (6) *$Oint(\mathcal{A}) \cup^1 Oint(\mathcal{B}) \subset_1 Ocl(\mathcal{A} \cup^1 \mathcal{B})$.*

Proof. The proof can be easily deduced from Propositions 4.11 and 4.12. □

Definition 4.16. Let X be a set. Then a mapping $Oint^* : \mathcal{O}(X) \to \mathcal{O}(X)$ is called an *octahedron interior operator* on X, if it satisfies the following axioms (called the *octahedron interior axioms*): for any $\mathcal{A},\ \mathcal{B} \in \mathcal{B}(X)$,
[OI1] $Oint^*(\ddot{1}) = \ddot{1}$,
[OI2] $Oint^*(\mathcal{A}) \subset_1 \mathcal{A}$,
[OI3] $Oint^*(Oint^*(\mathcal{A})) = Oint^*(\mathcal{A})$,
[OI3] $Oint^*(\mathcal{A} \cap^1 \mathcal{B}) = Oint^*(\mathcal{A}) \cap^1 Oint^*(\mathcal{B})$.

As one might expect, a property analogous to Proposition 4.14 holds for the octahedron interior operator. Then an octahedron interior operator completely determines an octahedron topology. In fact, the following result is the dual of Proposition 4.14.

Proposition 4.17. *Let $Oint^*$ be an octahedron interior operator on a set X and let τ be the family of octahedron sets in given by:*

$$\tau = \{\mathcal{A} \in \mathcal{O}(X) : Oint^*(\mathcal{A}) = \mathcal{A}\}.$$

Then τ is an octahedron topology on X. Furthermore, if $Oint$ is the octahedron interior operator defined by τ, then $Oint^*(\mathcal{A}) = Oint(\mathcal{A})$ for each $\mathcal{A} \in \mathcal{O}(X)$.

5. Octahedron continuities

In this section, we define an octahedron continuity and obtain its various properties.

Definition 5.1 ([42])**.** Let X, Y be two sets, let $f : X \to Y$ be a mapping and let $\mathcal{A} \in \mathcal{O}(X)$, $\mathcal{B} \in \mathcal{O}(Y)$.

(i) The preimage of $\mathcal{B}$ under f, denoted by $f^{-1}(\mathcal{B}) = \left\langle f^{-1}(\widetilde{B}), f^{-1}(\bar{B}), f^{-1}(B)\right\rangle$, is the octahedron set in X defined as follows: for each $x \in X$,

$$f^{-1}(\mathcal{B})(x) = \left\langle [(B^- \circ f)(x), (B^+ \circ f)(x))], ((B^{\in} \circ f)(x), (B^{\notin} \circ f)(x)), (B \circ f)(x) \right\rangle.$$

(ii) The image of $\mathcal{A}$ under f, denoted by $f(\mathcal{A}) = \left\langle f(\widetilde{A}), f(\bar{A}), f(A)\right\rangle$, is the octahedron set in Y defined as follows: for each $y \in Y$,

$$f(\widetilde{A})(y) = \begin{cases} [\bigvee_{x \in f^{-1}(y)} A^-(x), \bigvee_{x \in f^{-1}(y)} A^+(x)] & \text{if } f^{-1}(y) \neq \phi \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

$$f(\bar{A})(y) = \begin{cases} (\bigvee_{x \in f^{-1}(y)} A^{\in}(x), \bigwedge_{x \in f^{-1}(y)} A^{\notin}(x)) & \text{if } f^{-1}(y) \neq \phi \\ \bar{0} & \text{otherwise,} \end{cases}$$

$$f(A)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} A(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that $f(x_{\tilde{\bar{a}}}) = [f(x)]_{\tilde{\bar{a}}}$, for each $x_{\tilde{\bar{a}}} \in \mathcal{O}_P(X)$.

Result 5.2 ([42], Proposition 5.5)**.** *Let $\mathcal{A}$, $\mathcal{A}_1$, $\mathcal{A}_2 \in \mathcal{O}(X)$, $(\mathcal{A}_j)_{j\in J} \subset \mathcal{O}(X)$, let $\mathcal{B}$, $\mathcal{B}_1$, $\mathcal{B}_2 \in \mathcal{O}(Y)$, $(\mathcal{B}_j)_{j\in J} \subset \mathcal{O}(Y)$ and let $f : X \to Y$ be a mapping. Then for each $i = 1, 2, 3, 4$,*

(1) *if $\mathcal{A}_1 \subset_i \mathcal{A}_2$, then $f(\mathcal{A}_1) \subset_i f(\mathcal{A}_2)$,*
(2) *if $\mathcal{B}_1 \subset_i \mathcal{B}_2$, then $f^{-1}(\mathcal{B}_1) \subset_i f^{-1}(\mathcal{B}_2)$,*
(3) *$\mathcal{A} \subset_1 f^{-1}(f(\mathcal{A}))$ and if f is injective, then $\mathcal{A} = f^{-1}(f(\mathcal{A}))$,*
(4) *$f(f^{-1}(\mathcal{B})) \subset_1 \mathcal{B}$ and if f is surjective, $f(f^{-1}(\mathcal{B})) = \mathcal{B}$,*
(5) *$f^{-1}(\bigcup^i_{j\in J} \mathcal{B}_j) = \bigcup^i_{j\in J} f^{-1}(\mathcal{B}_j)$,*
(6) *$f^{-1}(\bigcap^i_{j\in J} \mathcal{B}_j) = \bigcap^i_{j\in J} f^{-1}(\mathcal{B}_j)$,*
(7) *$f(\bigcup^1_{j\in J} \mathcal{A}_j) = \bigcup^1_{j\in J} f(\mathcal{A}_j)$,*
(8) *$f(\bigcap^i_{j\in J} \mathcal{A}_j) \subset_i \bigcap^i_{j\in J} f(\mathcal{A}_j)$ and if f is injective, then $f(\bigcap^i_{j\in J} \mathcal{A}_j) = \bigcap^i_{j\in J} f(\mathcal{A}_j)$,*
(9) *if f is surjective, then $f(\mathcal{A})^c \subset_1 f(\mathcal{A}^c)$.*
(10) *$f^{-1}(\mathcal{B}^c) = f^{-1}(\mathcal{B})^c$.*
(11) *$f^{-1}(\ddot{0}) = \ddot{0}$, $f^{-1}(\ddot{1}) = \ddot{1}$, $f^{-1}(\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1\right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{0}}, 1\right\rangle$,*
$f^{-1}(\left\langle \widetilde{0}, \bar{\mathbf{1}}, 0\right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{1}}, 0\right\rangle$, $f^{-1}(\left\langle \widetilde{1}, \bar{\mathbf{0}}, 0\right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{0}}, 0\right\rangle$,
$f^{-1}(\left\langle \widetilde{0}, \bar{\mathbf{1}}, 1\right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{1}}, 1\right\rangle$, $f^{-1}(\left\langle \widetilde{1}, \bar{\mathbf{0}}, 1\right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{0}}, 1\right\rangle$,
$f^{-1}(\left\langle \widetilde{1}, \bar{\mathbf{1}}, 0\right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{1}}, 0\right\rangle$.

(12) $f(\ddot{0}) = \ddot{0}$ *and if f is surjective, then the following hold:*

$$f(\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle, \ f(\left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle,$$

$$f(\left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle, \ f(\left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle,$$

$$f(\left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle, \ f(\left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle, \ f(\ddot{1}) = \ddot{1}.$$

Definition 5.3. Let (X, τ) and (Y, γ) be two octahedron topological spaces. Then a mapping $f : X \to Y$ is called an *octahedron continuous*, if $f^{-1}(\mathcal{V}) \in \tau$ for each $\mathcal{V} \in \gamma$.

From Proposition 5.9 in [42] and Definition 5.3, we obtain the following properties.

Proposition 5.4. *Let (X, τ), (Y, γ), (Z, ζ) be octahedron topological spaces. The we have*
(1) *the identity mapping $id : (X, \tau) \to (X, \tau)$ is octahedron continuous,*
(2) *if $f : (X, \tau) \to (Y, \gamma)$ and $g : (Y, \gamma) \to (Z, \zeta)$ are octahedron continuous, then $g \circ f : (X, \tau) \to (Z, \zeta)$ is octahedron continuous.*

Remark 5.5. (1) If $f : (X, \tau) \to (Y, \gamma)$ is octahedron continuous, then $f : (X, \tau_{IV}) \to (Y, \gamma_{IV})$ [resp. $f : (X, \tau_{IF}) \to (Y, \gamma_{IF})$ and $f : (X, \tau_F) \to (Y, \gamma_F)$] is interval-valued fuzzy [resp. intuitionistic fuzzy and fuzzy] continuous in the sense of Mondal and Samanta [24] [resp. Çoker [21] and Chang [12]] (See Remark 3.3 (1)).
(2) Let $OTop$ be the collection of all sets and all octahedron continuous mappings between them. Then from Proposition 5.4, we can easily see that $OTop$ forms a concrete category.

The following is an characterization of octahedron continuity.

Theorem 5.6. *Let (X, τ), (Y, γ) be two octahedron topological spaces and let $f : X \to Y$ be a mapping. Let β be an octahedron base for γ and let σ be an octahedron subbase for γ. Then the followings are equivalent:*
(1) *f is octahedron continuous,*
(2) *$f^{-1}(\mathcal{B}) \in \tau^c$ for each $\mathcal{B} \in \gamma^c$,*
(3) *$f(Ocl(\mathcal{A})) \subset_1 Ocl(f(\mathcal{A}))$ for each $\mathcal{A} \in \mathcal{O}(X)$,*
(4) *$Ocl(f^{-1}(\mathcal{B})) \subset_1 f^{-1}(Ocl(\mathcal{B}))$ for each $\mathcal{B} \in \mathcal{O}(Y)$,*
(5) *$f^{-1}(\mathcal{B}) \in \tau$ for each $\mathcal{B} \in \beta$,*
(6) *$f^{-1}(\mathcal{S}) \in \tau$ for each $\mathcal{S} \in \sigma$.*

The following shows that there is an octahedron topology (usually called the *final octahedron topology*) on a set Y which for an octahedron topological space (X, τ), a mapping $f : (X, \tau) \to Y$ is an octahedron continuous as in the classical topology (See Definition 5.9).

Proposition 5.7. *Let (X, τ) be an octahedron topological space, let Y be a set and let $f : X \to Y$ be a mapping. Let γ be the family of octahedron sets in Y defined by:*

$$\gamma = \{\mathcal{V} \in \mathcal{O}(Y) : f^{-1}(\mathcal{V} \in \tau\}.$$

Then we have
(1) *$\gamma \in OT(Y)$,*

(2) *$f : (X, \tau) \to (Y, \gamma)$ is octahedron continuous,*

(3) *if $\zeta \in OT(Y)$ such that $f : (X, \tau) \to (Y, \zeta)$ is octahedron continuous, then γ is finer than ζ.*

Proof. (1) From the definition of γ, we can easily check that γ satisfies the axioms [1-OO_1], [1-OO_2] and [1-OO_3]

(2) The proof is obvious from Definition 5.3 and the definition of γ.

(3) The proof is straightforward from Definition 3.12 and the definition of γ. □

Definition 5.8. Let (X, τ), (Y, γ) be two octahedron topological spaces and let $f : X \to Y$ be a mapping. Then f is said to be *octahedron closed* [resp. *octahedron open*], if $f(\mathcal{A}) \in \gamma^c$ [resp. $f(\mathcal{A}) \in \gamma$] for each $\mathcal{A} \in \tau^c$ [resp. $\mathcal{A} \in \tau$].

Definition 5.9. Let (X, τ) be an octahedron topological space, let Y be a set and let $f : X \to Y$ be a surjection. Then $\gamma = \{\mathcal{V} \in \mathcal{O}(Y) : f^{-1}(\mathcal{V}) \in \tau\}$ is an octahedron topology on Y (See Proposition 5.7) and it is called the *octahedron quotient topology on Y induced by f*, and will be denoted by τ_f. The pair (Y, τ_f) is called an *octahedron quotient space* of X and f is called an *octahedron quotient mapping*.

From Proposition 5.7, it is obvious that the octahedron quotient mapping f is not only octahedron continuous, but tau_f is the finest octahedron topology on Y for which f is octahedron continuous. Furthermore, we can easily see that if (Y, γ) is an octahedron quotient space with quotient mapping f, then $\mathcal{C} \in \gamma^c$ if and only if $f^{-1}(\mathcal{C}) \in \tau^c$ for each $\mathcal{C} \in \mathcal{O}(Y)$.

The following provides conditions on f that make an octahedron topology on Y equal to τ_f.

Proposition 5.10. *Let (X, τ), (Y, γ) be two octahedron topological spaces and let $f : (X, \tau) \to (Y, \gamma)$ be octahedron continuous and surjective. If f is octahedron closed or octahedron open, then $\gamma = \tau_f$.*

Proof. Suppose f is octahedron open. Then by Proposition 5.7 and Definition 5.9, $\gamma \subset \tau_f$. Let $\mathcal{V} \in \tau_f$. Then by the definition of τ_f, $f^{-1}(\mathcal{V}) \in \tau$. Thus by the hypothesis and Result 5.2 (4), $\mathcal{V} = f(f^{-1}(\mathcal{V}) \in \gamma$. So $\tau_f \subset \gamma$. Hence $\gamma = \tau_f$. The remainder's proof is similar. □

The following is an immediate consequence of Proposition 5.7 and Definition 5.9.

Proposition 5.11. *The composition of two octahedron quotient mappings is an octahedron octahedron quotient mapping.*

Theorem 5.12. *Let (X, τ) be an octahedron topological space, let $f : X \to Y$ be a surjection, let (Y, τ_f) be an octahedron quotient space of X and let (Z, ζ) be an octahedron topological space. Then $g : (Y, \tau_f) \to (Z, \zeta)$ is octahedron continuous if and only if $g \circ f : (X, \tau) \to (Z, \zeta)$ is octahedron continuous.*

Proof. Suppose g is octahedron continuous. Then by Proposition 5.4 (2), $g \circ f$ is octahedron continuous.

Conversely, suppose $g \circ f$ is octahedron continuous and let $\mathcal{V} \in \zeta$. Then $(g \circ f)^{-1}(\mathcal{V}) \in \tau$ and $(g \circ f)^{-1}(\mathcal{V}) = f^{-1}(g^{-1}(\mathcal{V}))$. Thus by the definition of the octahedron quotient topology, $g^{-1}(\mathcal{V}) \in \tau_f$. So g is octahedron continuous. □

Let us consider the dual cases of Proposition 5.7.

Proposition 5.13. *Let X be a set, let (Y,γ) be an octahedron topological space and let $f : X \to Y$ be a mapping. Then there is a coarsest octahedron topology τ on X such that $f : (X,\tau) \to (Y,\gamma)$ is octahedron continuous.*

In this case, τ is called the *initial octahedron topology* on X.

Proof. Let $\tau = \{f^{-1}((\mathcal{V}) \in (\mathcal{O}(X) : \mathcal{V} \in \gamma\}$. Then clearly, $\ddot{0}$, $\ddot{1} \in \gamma$ and $f^{-1}(\ddot{0}) = \ddot{0}$, $f^{-1}(\ddot{1}) = \ddot{1}$ by Result 5.2 (11). Thus $\ddot{0}$, $\ddot{1} \in \tau$. So the axiom [1-OO_1] is satisfied. Suppose $f^{-1}(\mathcal{U})$, $f^{-1}(\mathcal{V}) \in \tau$. Then $\mathcal{U} \cap^1 \mathcal{V} \in \gamma$ and $f^{-1}(\mathcal{U}) \cap^1 f^{-1}(\mathcal{V}) = f^{-1}(\mathcal{U} \cap^1 \mathcal{V})$ by Result 5.2 (6). Thus $f^{-1}(\mathcal{U}) \cap^1 f^{-1}(\mathcal{V}) \in \tau$. So the axiom [1-$OO_2$] is satisfied. Now let $(f^{-1}(\mathcal{V}_j))_{j\in J} \subset \tau$. Then clearly, $(\mathcal{V}_j)_{j\in J} \subset \gamma$. Thus $\bigcup^1_{j\in J} \mathcal{V}_j \in \gamma$ and $\bigcup^1_{j\in J} f^{-1}(\mathcal{V}_j) = f^{-1}(\bigcup^1_{j\in J} \mathcal{V}_j)$ by Result 5.2 (7). So $\bigcup^1_{j\in J} f^{-1}(\mathcal{V}_j) \in \tau$. Hence the axiom [1-$OO_3$] is satisfied. This completes the proof. □

Definition 5.14. Let X be a set, $(Y_j,\gamma_j)_{j\in J}$ be a family of octahedron topological spaces and let $(f_j : X \to ((Y_j,\gamma_j)_{j\in J}$ be a family of mappings (usually called an *initial source in OTop*). Let $\sigma = \{f_j^{-1}(\mathcal{V}_j) \in \mathcal{O}(X) : \mathcal{V}_j \in \gamma_j,\ j \in J\}$. Then the octahedron topology τ on X with an octahedron subbase σ is called the *initial octahedron topology* (briefly, *octahedron topology*) induced by $(f_j)_{j\in J}$.

The following is an generalization of Proposition 5.13.

Proposition 5.15. *The octahedron initial topology on X induced by $(f_j)_{j\in J}$ is the coarsest octahedron topology on X for which $f_j : (X,\tau) \to (Y,\gamma_j)$ is octahedron continuous for each $j \in J$.*

Proof. The proof is straightforward. □

Now let us find an initial octahedron topology.

Definition 5.16. Let $(X_j,\tau_j)_{j\in J}$ be a family of octahedron topological spaces, let $X = \Pi_{j\in J} X_j$ and let $(\pi_j : X \to (X_j,\tau_j))_{j\in J}$ be a family of mappings, where π_j is the projection mapping. Then the initial octahedron topology τ on X induced by $(\pi_j)_{j\in J}$ is called the *octahedron product topology* on X and denoted by $\tau = \Pi_{j\in J}\tau_j$.

Proposition 5.17. *Let $(X_j,\tau_j)_{j\in J}$ be a family of octahedron topological spaces. Then $\pi_j : (\Pi_{j\in J}X_j, \Pi_{j\in J}\tau_j) \to (X_j,\tau_j)$ is octahedron continuous for each $j \in J$.*

Proof. The proof is straightforward. □

Proposition 5.18. *Let $(X_j,\tau_j)_{j\in J}$ be a family of octahedron topological spaces. Then $\Pi_{j\in J}\tau_j$ is the coarsest octahedron topology for which $\pi_j : (\Pi_{j\in J}X_j, \Pi_{j\in J}\tau_j) \to (X_j,\tau_j)$ is octahedron continuous for each $j \in J$.*

Proof. The proof is similar to one of classical case. □

Theorem 5.19. *Let (X,τ) be an octahedron topological space, let $(Y_j,\gamma_j)_{j\in J}$ be a family of octahedron topological spaces and let $f : X \to \Pi_{j\in J}Y_j$ be a mapping. Then $f : (X,\tau) \to (\Pi_{j\in J}Y_j, \Pi_{j\in J}\gamma_j)$ is octahedron continuous if and only if $\pi_j \circ f : (X,\tau) \to (Y_j,\gamma_j)$ is octahedron continuous for each $j \in J$.*

Proof. Suppose f is octahedron continuous. Then by Proposition 5.17, for each $j \in J$, $\pi_j : (\Pi_{j\in J}Y_j, \Pi_{j\in J}\gamma_j) \to (Y_j, \gamma j)$ is octahedron continuous. Thus by Proposition 5.4 (2), $\pi_j \circ f$ is octahedron continuous for each $j \in J$.

Conversely, suppose $\pi_j \circ f$ is octahedron continuous for each $j \in J$. Let σ be the octahedron subbase for the octahedron product topology $\Pi_{j\in J}\gamma_j$ on $\Pi_{j\in J}Y_j$ and let $\pi_j^{-1}(\mathcal{U}) \in \sigma$. Then $f^{-1}(\pi_j^{-1}(\mathcal{U})) = (\pi_j \circ f)^{-1}(\mathcal{U})$. Since $\pi_j \circ f$ is octahedron continuous, $(\pi_j \circ f)^{-1}(\mathcal{U}) \in \tau$. Thus $f^{-1}(\pi_j^{-1}(\mathcal{U})) \in \tau$. So by Theorem 5.6, f is octahedron continuous. □

The following is an immediate consequence of Theorem 5.19.

Corollary 5.20. *Let (X, τ) be an octahedron topological space, let $(Y_j, \gamma_j)_{j\in J}$ be a family of octahedron topological spaces and let $f_j : X \to Y_j$ be a mapping for each $j \in J$. Let $f : X \to \Pi_{j\in J}Y_j$ be the mapping defined as follows: for each $x \in X$,*

$$f(x) = (f_j(x))_{j\in J} \in \Pi_{j\in J}Y_j.$$

Then $f : (X, \tau) \to (\Pi_{j\in J}Y_j, \Pi_{j\in J}\gamma_j)$ is octahedron continuous if and only if $f_j : (X, \tau) \to (Y_j, \gamma_j)$ is octahedron continuous for each $j \in J$.

6. Conclusions

We defined a Type i-octahedron topology on a set X, $i = 1, 2, 3, 4$, and obtained some of its properties. Also we introduced the notions of octahedron base and subbase, octahedron subspace, octahedron closure and interior, and obtained some properties related to them, and gave some examples. Finally, we define an octahedron continuity and obtained its various properties. Furthermore, we found a final topological structure (called an octahedron quotient topology) on the codomain of a mapping (See Proposition 5.7). Also we obtained an initial topological structure (called an initial octahedron topology) on the domain of a mapping (See Proposition 5.13). Moreover, we constructed octahedron product topology on the product of any family of octahedron topologies as in classical topology.

In the future, we expect that one can deal with separation axioms, compactness and connectedness in an octahedron topological space. Also, we hope that one can the notion of octahedron sets apply to a semigroup, a group, a ring theory and decision-making problems. Furthermore, we expect that one can extend octahedron sets to an octahedron soft sets, octahedron neutrosophic sets, octahedron plithogenic sets, etc. and can study topological structures based on them respectively.

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