Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 22, No. 1, (August 2021) pp. 65–76
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2021.22.1.65

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 22, No. 1, August 2021

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Received 30 March 2021; Revised 12 April 2021; Accepted 10 May 2021

Abstract. A grill-P_β-connectedness is more generalization of P_β- connectedness and connectedness and amounting to grill-preconnectedness. The properties of this motif are studied and its relationship with other forms of grill-connectedness. Grill locally preindiscrete spaces are defiened as the spaces in which grill-preopen sets are grill-closed. In these spaces grill connectedness becomes amounting to grill preconnectedness and hence to grill-P_β-connectedness, and grill-semi-connectedness becomes equivalent to grill-P_β-connectedness. The motif of locally grill-P_β-connected space is introduced.

2020 AMS Classification: 54G20, 54F50, 54D05, 54B40.

Keywords: G-preconnected, G-P_β-separated, G-P_β-connected.

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1. Introduction

The expression of semi-connectedness by Pipitone et al [20], preconnectedness [21], α-connectedness [15] and β-connectedness [15, 6, 22] in topological spaces are based on the motif of semi-open set [17], preopen set [18], α-open [19] and $\beta-$open set [1], respectively. The classes of these sets in a topological space contain the class of open sets. The classes of β-connected, semi-connected and pre-connected topological spaces shape subclasses of the class of connected topological spaces. Functions and Connectedness are studied in terms of grill theory as in [2, 3, 4]. In this paper we characterized a property called P_β-connectedness which is amounting to motif of pre-connectedness but stronger than the property of connectedness. The notion of P_β-connectedness is independent of the topic of hyper connectedness. It is shown that the property of P_β-connectedness similarly to the behavior connectedness.

This paper is promote the necessary concept of G-P_β-open set is introduced, and G-P_β-closure and its properties are obtained. The expression of G-locally preindiscrete topological space is introduced. The notion of G-P_β-connected space and its relationship with different other weaker and stronger forms of connectedness is realized. It is shown that G-P_β-connectedness is equivalent to G-preconnectedness. Sundry characterizations of G-P_β-connected spaces are received. Also, we study the properties of G-P_β-connected sets and the comprehensible of G-P_β-components. G-locally P_β-connected spaces are introduced.

2. Preliminaries

Definition 2.1 ([14]). Let (Ω, τ, G) be a grill topological space (GTS, for short) and $h \subset \Omega$. Then a set h is said to be *G-dense in* Ω, if $\Psi(h) = \Omega$.

Definition 2.2 ([8]). A nonempty subcollection G of a space S which carries topology τ is named *grill* on this space, if the following conditions are true:
(i) $\phi \notin G$,
(ii) $\xi \in G$ and $\xi \subseteq \mathcal{B} \subseteq S \Longrightarrow \mathcal{B} \in G$,
(iii) if $\xi \cup \mathcal{B} \in G$ for $\xi, \mathcal{B} \subseteq S$, then $\xi \in G$ or $\mathcal{B} \in G$.

Grill depends on the two mappings Φ and Ψ which are generated a unique GTS finer than τ on space S denoted by τ_G on S and discussed in [14, 23].

Definition 2.3 ([23]). Let (S, τ) be a topological space (TS, for short) and G be a grill on S. A mapping $\Phi : P(S) \to P(S)$ is defined as follows:

$$\Phi(\mathcal{A}) = \Phi_G(\mathcal{A}, \tau) = \{s \in S : \mathcal{A} \cap U \in G\}$$

for all $U \in \tau(s)$ and $\mathcal{A} \in P(S)$. Then the mapping Φ is called the *operator associated with* the grill G and the topology τ.

Proposition 2.4 ([23]). *Let (Υ, τ) be a TS and G be a grill on Υ. Then for all $h, j \subseteq \Upsilon$:*
(1) *$h \subseteq j$ implies that $\Phi(h) \subseteq \Phi(j)$,*
(2) *$\Phi(h \cup j) = \Phi(h) \cup \Phi(j)$,*
(3) *$\Phi(\Phi(h)) \subseteq \Phi(h) = Cl(\Phi(h)) \subseteq Cl(h)$.*

Proposition 2.5 ([23]). *Let (Ω, τ) be a TS and G be a grill on Ω. Then $\Psi : P(\Omega) \to P(\Omega)$ is defined by $\Psi(\eta) = \eta \cup \Phi(\eta)$ for all $\eta \in P(\Omega)$. The map Ψ is a Kuratowski [16] closure axiom corresponding to a grill G on a topological space (Ω, τ), if there exists a unique topology τ_G on Ω given by $\tau_G = \{U \subseteq \Omega : \Psi(\Omega \setminus U) = \Omega \setminus U\}$ where for any $\eta \subseteq \Omega$, $\Psi(\eta) = \eta \cup \Phi(\eta) = \tau_G$-$Cl(\eta)$. For any grill G on a TS (Ω, τ), $\tau \subseteq \tau_G$. By τ_G-$Int(\eta)$, we denote the interior of η with respect to τ_G. If (Ω, τ) is a TS with a grill G on Ω, then we call it a GTS and denote by (Ω, τ, G).*

Definition 2.6 ([14, 16, 5, 7]). A subset ζ of a space Ω which carries topology τ with grill G is said to be:
(i) *r-G-open* (resp. *r-G-closed*), if $\zeta = Int(\Psi(\zeta))$ (resp. $\zeta = \Psi(Int(\zeta))$,
(ii) *G-open* or *ϕ-open*, if $\zeta \subseteq int(\phi(\zeta))$,
(iii) *G-α-open*, if $\zeta \subseteq int(\Psi(int(\zeta)))$,
(iv) *G-preopen*, if $\zeta \subseteq int(\Psi(\zeta))$,
(v) *G-semiopen*, if $\zeta \subseteq \Psi(int(\zeta))$,
(vi) *G-β-open*, if $\zeta \subseteq cl(int(\Psi(\zeta)))$.

The family of all G-open (resp. G-α-open, G-preopen, G-semiopen, G-β-open) sets in a GTS (Ω, τ, G) is denoted by $GO(\Omega)$ (rep. $G\alpha O(\Omega)$, $GPO(\Omega)$, $GSO(\Omega)$, $G\beta O(\Omega)$).

3. GrillP_β-open(closed) sets in (χ, τ, G)

Definition 3.1. Let (χ, τ, G) be GTS. Then $\mathcal{A}$ is said to be:

(i) *G-β-closed*, if its complement $\chi \setminus \mathcal{A}$ is G-β-open,

(ii) *G-preclosed*, if its complement $\chi \setminus \mathcal{A}$ is G-preopen,

(iii) *G-P_β-open*, if $\mathcal{A}$ is G-preopen subset of χ such that for each $x \in \mathcal{A}$ there exists a G-β-closed set ξ and $x \in \xi \subseteq \mathcal{A}$,

(iv) *G-P_β-closed*, if its complement $\chi \setminus \mathcal{A}$ is G-P_β-open.

The family of all G-P_β-open, G-P_β-closed and G-preclosed subsets of a GTS (χ, τ, G) is denoted by $GP_\beta O(\chi)$, $GP_\beta C(\chi)$ and $GPC(\chi)$.

The following figure summarizes the relationship of G-P_β-open set with various types of G-open sets:

$$\begin{tikzcd}
& G\text{-}\alpha\text{-open sets} \arrow[r] & G\text{-semiopen sets} \arrow[dr] & \\
G\text{-open set} \arrow[ur] \arrow[dr] & & & G\text{-}\beta\text{-open set} \\
& G\text{-}P_\beta\text{-open sets} \arrow[r] & G\text{-preopen sets} \arrow[ur] &
\end{tikzcd}$$

Definition 3.2. Let $(\mathcal{X}, \Gamma, G)$ be a GTS and $\mathcal{A}$ subset of $\mathcal{X}$.

(i) The *set G-preclosure* of $\mathcal{A}$ is the intersection of all G-preclosed sets of $\mathcal{X}$ containing $\mathcal{A}$ and is denoted by $P_G Cl(\mathcal{A}) : P_G Cl(\mathcal{A}) = \bigcap \{\mu \supseteq \mathcal{A} : \mu$ is G-preclosed set of $\mathcal{X}\}$.

(ii) The *set G-P_β-closure* of $\mathcal{A}$ is the intersection of all G-P_β-closed sets of $\mathcal{X}$ containing $\mathcal{A}$ and is denoted by $P_{\beta_G} Cl(\mathcal{A}) : P_{\beta_G} Cl(\mathcal{A}) = \bigcap \{\mu \supseteq \mathcal{A} : \mu$ is G-P_β-closed set of $\mathcal{X}\}$.

Lemma 3.3. *Let $(\mathcal{X}, \tau, G)$ be a GTS and μ, h are any subsets of $\mathcal{X}$. Then*

(1) $P_{\beta_G} Cl(\phi) = \phi$ *and* $P_{\beta_G} Cl(\mathcal{X}) = \mathcal{X}$,

(2) $\mu \subseteq P_{\beta_G} Cl(\mu)$,

(3) $\mu \subseteq h \Rightarrow P_{\beta_G} Cl(\mu) \subseteq P_{\beta_G} Cl(h)$,

(4) $P_{\beta_G} Cl(P_{\beta_G} Cl(\mu)) = P_{\beta_G} Cl(\mu)$,

(5) $P_{\beta_G} Cl(\mu)$ *is a* $G - P_\beta$*closed set,*

(6) $P_{\beta_G} Cl(\mu) \cup P_{\beta_G} Cl(h) \subseteq P_{\beta_G} Cl(\mu \cup h)$,

(7) $P_{\beta_G} Cl(\mu \cap h) \subseteq P_{\beta_G} Cl(\mu) \cap P_{\beta_G} Cl(h)$.

In general may be $P_{\beta_G} Cl(\mu) \cup P_{\beta_G} Cl(h) \neq P_{\beta_G} Cl(\mu \cup h)$ and $P_{\beta_G} Cl(\mu \cap h) \neq P_{\beta_G} Cl(\mu) \cap P_{\beta_G} Cl(h)$, as shown in the following example.

Example 3.4. Let $\mathcal{X} = \{1, 2, 3\}$, and $\tau = \{\mathcal{X}, \phi, \{1\}, \{3\}, \{1, 3\}, \{2, 3\}\}$. If grill G on $\mathcal{X}$ such that $G = \{\mathcal{X}, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}\}$, then $GPO(\mathcal{X}) = G\beta O(\mathcal{X}) =$

$\{\mathcal{X}, \phi, \{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}\}$, $GO(\mathcal{X}) = \{\mathcal{X}, \phi, \{1\}, \{3\}, \{1,3\}, \{2,3\}\}$ and $G\beta C(\mathcal{X}) = \{\mathcal{X}, \phi, \{2,3\}, \{1,3\}, \{1,2\}, \{2\}, \{1\}\}$, implies that $GP_\beta O(\mathcal{X}) = \{\mathcal{X}, \phi, \{1\}, \{2\}, \{1,3\}, \{2,3\}\}$ and $GP_\beta C(\mathcal{X}) = \{\mathcal{X}, \phi, \{2,3\}, \{1,3\}, \{2\}, \{1\}\}$, if $\{1,2\}, \{1,3\}$ two sets in $\mathcal{X}$, then we have $P_{\beta_G}Cl(\{1,2\} \cap \{1,3\}) = P_{\beta_G}Cl(\{1\}) = \{1\}$ and $P_{\beta_G}Cl(\{1,2\}) = \mathcal{X}$, $P_{\beta_G}Cl(\{1,3\}) = \{1,3\}$ then $P_{\beta_G}Cl(\{1,2\} \cap \{1,3\}) \neq P_{\beta_G}Cl(\{1,2\})$ $\cap P_{\beta_G}Cl(\{1,3\})$. Also if $\{1\}, \{2\}$ two sets in $\mathcal{X}$, then we have $P_{\beta_G}Cl(\{1\} \cup \{2\}) = P_{\beta_G}Cl(\{1,2\}) = \mathcal{X}$ and $P_{\beta_G}Cl(\{1\}) = \{1\}$, $P_{\beta_G}Cl(\{2\}) = \{2\}$, then $P_{\beta_G}Cl(\{1\} \cup \{2\}) \neq P_{\beta_G}Cl(\{1\}) \cup P_{\beta_G}Cl(\{2\})$.

Remark 3.5. G-P_β-open sets are obtained from G-preopen sets but the collection of these sets is neither a sub-collection of G-open sets nor does it contain the collection of G-open sets.

Example 3.6. From Example 3.4, we find $\{2\} \in GP_\beta O(\mathcal{X})$ but $\{2\} \notin GO(\mathcal{X})$ and $\{3\} \in GO(\mathcal{X})$ but $\{3\} \notin GP_\beta O(\mathcal{X})$.

Theorem 3.7. *Let $(\mathcal{X}, \tau, G)$ be a GTS and μ subset of $\mathcal{X}$. Then*

$$P_GCl(\mu) \subseteq P_{\beta_G}Cl(\mu).$$

Proof. It is Obvious from Definition of $P_GCl(\mu)$ and $P_{\beta_G}Cl(\mu)$. □

Example 3.8. From Example 3.4, we find $P_GCl(\mathcal{A}) \subseteq P_{\beta_G}Cl(\mathcal{A}) \forall \mathcal{A} \in \mathcal{X}$.

Theorem 3.9. *If (Ω, τ, G) be a GTS, then an arbitrary union of G-P_β-open sets in Ω is G-P_β-open.*

Example 3.10. Let $\Omega = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}$, and $\tau = \{\Omega, \phi, \{\hbar_1\}, \{\hbar_2\}, \{\hbar_1, \hbar_2\}, \{\hbar_3, \hbar_4\}, \{\hbar_1, \hbar_3, \hbar_4\}, \{\hbar_2\}, \{\hbar_3, \hbar_4\}\}$. If grill G on Ω such that $G = \{\Omega, \{\hbar_1\}, \{\hbar_3\}, \{\hbar_1, \hbar_3, \hbar_4\}\}$, then $GPO(\Omega) = G\beta O(\Omega) = \{\Omega, \phi, \{\hbar_1\}, \{\hbar_2\}, \{\hbar_3\}, \{\hbar_1, \hbar_2\}, \{\hbar_3, \hbar_4\}, \{\hbar_1, \hbar_3, \hbar_4\}, \{\hbar_2, \hbar_3, \hbar_4\}\}$ and $G\beta C(\Omega) = \{\Omega, \phi, \{\hbar_2, \hbar_3, \hbar_4\}, \{\hbar_1, \hbar_3, \hbar_4\}, \{\hbar_1, \hbar_2, \hbar_4\}, \{\hbar_3, \hbar_4\}, \{\hbar_1, \hbar_2\}, \{\hbar_2\}\}, \{\hbar_1\}\}$, then $GP_\beta O(\Omega) = \{\Omega, \phi, \{\hbar_1\}, \{\hbar_2\}, \{\hbar_1, \hbar_2\}, \{\hbar_3, \hbar_4\}, \{\hbar_1, \hbar_3, \hbar_4\}, \{\hbar_2, \hbar_3, \hbar_4\}\}$. Thus any arbitrary union of G-P_β-open sets in a (Ω, τ, G) is G-P_β-open.

Lemma 3.11. *In GTS $(\mathcal{X}, \tau, G)$, any G-clopen subset of $\mathcal{X}$ is both G-P_β-open and G-P_β-closed.*

Proof. Let μ be any G-clopen subset of $(\mathcal{X}, \tau, G)$. Then μ is G-open and G-closed in $\mathcal{X}$, implies $\mu = Int(\mu) \subseteq Int(\Psi(\mu)) \subseteq Cl(Int(\Psi(\mu)))$, μ is G-β-open it follows that μ is G-P_β-closed. Since μ is G-closed,

$$\mu = Cl(\mu) \supseteq Cl(Int_G(\mu)) \supseteq Int(Cl(Int_G(\mu))).$$

Then μ is G-β-closed implies that μ is G-P_β-open. Thus μ is both G-P_β-open and G-P_β-closed. □

The converse of the above Lemma need not be true in general.

Example 3.12. Let $\mathcal{X} = \{\imath, \jmath, \ell\}$, and $\tau = \{\mathcal{X}, \phi, \{\jmath\}, \{\imath, \ell\}\}$, $G = \{\mathcal{X}, \{\imath\}, \{\imath, \jmath\}, \{\ell\}, \{\jmath, \ell\}\}$, then $GPO(\mathcal{X}) = G\beta O(\mathcal{X}) = \{\mathcal{X}, \phi, \{\imath\}, \{\jmath\}, \{\ell\}, \{\imath, \ell\}, \{\imath, \jmath\}, \{\jmath, \ell\}\}$ and $G\beta C(\mathcal{X}) = \{\mathcal{X}, \phi, \{\jmath, \ell\}, \{\imath, \ell\}, \{\imath, \jmath\}, \{\jmath\}, \{\ell\}, \{\imath\}\}$, then $GP_\beta O(\mathcal{X}) = \{\mathcal{X}, \phi, \{\imath\}, \{\jmath\}, \{\ell\}, \{\imath, \ell\}, \{\imath, \jmath\}, \{\jmath, \ell\}\}$ and $GP_\beta C(\mathcal{X}) = \{\mathcal{X}, \phi, \{\jmath, \ell\}, \{\imath, \ell\}, \{\imath, \jmath\}, \{\jmath\}, \{\ell\}, \{\imath\}\}$. We find the set $\{\imath\}$ is both G-P_β-open and G-P_β-closed. But it is not clopen in $(\mathcal{X}, \tau, G)$.

Definition 3.13. Let (Ω, τ, G) be a GTS. Then it is said to be a:

(i) *grill locally indiscrete topological space* (GLITS, for short), if every G-open subset of Ω is G-closed set,

(ii) *grill hyperconnected topological space* (GHTS, for short), if every nonempty G-open subset of Ω is G-dense,

(iii) *grill extremally disconnected topological space* (GEDTS, for short), if $\Psi(\mu)$ is G-open for μ is G-open,

(iv) *grill locally pre-indiscrete topological space* (GLP_ITS, for short), if every G-preopen subset of Ω is G-closed.

Theorem 3.14. *If (χ, τ, G) is GLITS, then every G-preopen set is G-P_β-open set.*

Proof. Let $\mathcal{V}$ be a G-preopen set, that is, $\mathcal{V} \subseteq Int(\Psi(\mathcal{V})))$. Then $Cl(\mathcal{V}) \subseteq Cl(Int(\Psi(\mathcal{V})))$. Thus $Int(\mathcal{V}) \supseteq Int(Cl(Int_G(\mathcal{V})))$. Since χ is GLITS,

$$\mathcal{V} = Int_G(\mathcal{V}) = Cl(Int_G(\mathcal{V})) \supseteq Int(Cl(Int_G(\mathcal{V}))).$$

So $\mathcal{V}$ is G-preclosed and it is G-β-closed. Hence $\mathcal{V}$ is G-P_β-open. □

Theorem 3.15. *If (Ω, τ, G) is a GHTS, then every G-β-closed set $\mathcal{W}$ with $\mathcal{W} \neq \Omega$, $Int_G(\mathcal{W}) = \phi$.*

Proof. Let Ω is GHTS. Then $\Psi(Int_G(\mathcal{W})) = \Omega$, since $\mathcal{W}$ is a G-β-closed set, that is, $Int_G(\mathcal{W}) = \mathcal{W} \supseteq \Omega \setminus Cl(Int(\Psi(Int_G(\mathcal{W}))))$ implies that $\Omega \setminus (Int_G(\mathcal{W})) \subseteq Cl(Int(\Omega)) = \Omega$, it follows that $\Omega \setminus (Int_G(\mathcal{W})) \subseteq \Omega$. Thus $Int_G(\mathcal{W}) = \Omega$ is rejected solution. So $Int_G(\mathcal{W}) = \phi$. □

Example 3.16. Let $\Upsilon = \{\rho, \varrho, \sigma\}$, and $\tau = \{\Upsilon, \phi, \{\varrho\}, \{\varrho, \sigma\}\}$, $G = \{\Upsilon, \{\varrho\}, \{\rho, \varrho\}, \{\rho\}\}$. Then $GPO(\Upsilon) = G\beta O(\Upsilon) = \{\Upsilon, \phi, \{\varrho\}, \{\rho, \varrho\}, \{\varrho, \sigma\}\}$ and $G\beta C(\Upsilon) = \{\Upsilon, \phi, \{\rho, \sigma\}, \{\sigma\}, \{\rho\}\}$, $Int_G(\{\rho, \sigma\}) = Int_G(\{\sigma\}) = Int_G(\{\rho\}) = \phi$.

Theorem 3.17. *If in a GTS $(\mathcal{X}, \tau, G)$, $GPO(\mathcal{X})$ is a grill topology, then $GP_\beta O(\mathcal{X})$ is a grill topology.*

Proof. If the sets μ and ω are G-P_β-open, then $\mu \cap \omega$ is G-preopen. For each $x \in \mu \cap \omega$, there are G-β-closed sets μ_1 and μ_2 such that $x \in \mu_1 \cap \mu_2 \subseteq \mu \cap \omega$. Since $\mu_1 \cap \mu_2$ is a G-β-closed set, $GP_\beta O(\mathcal{X})$ is a grill topology in $\mathcal{X}$. □

Example 3.18. In Example 3.16, we have $GPO(\Upsilon) = \{\Upsilon, \phi, \{\varrho\}, \{\rho, \varrho\}, \{\varrho, \sigma\}\}$ and $G\beta C(\Upsilon) = \{\Upsilon, \phi, \{\rho, \sigma\}, \{\sigma\}, \{\rho\}\}$, then $GP_\beta O(\Upsilon) = \{\Upsilon, \phi\}$ is form grill topology on Υ.

Theorem 3.19. *A GLP_ITS is a GLITS and GEDTS.*

Proof. Since $GO(\mathcal{X}) \subseteq GPO(\mathcal{X})$, every GL$P_I$TS is a GLITS. Also, every G-preopen set in GLP_ITS is G-closed set implies $\mathcal{A} = \Psi(\mathcal{A})$, $\mathcal{A} \subseteq Int(\Psi(\mathcal{A})) = Int(\mathcal{A})$. Then $\mathcal{X}$ is GEDTS. □

Theorem 3.20. *Every G-preopen set in a GLP_ITS is G-clopen.*

Proof. If $\mathcal{B}$ is G-preopen in GLP_ITS, then $\mathcal{B} = Cl(\mathcal{B})$. Also $\mathcal{B} \subseteq Int(Cl(\mathcal{B})) = Int(\mathcal{B})$. □

Corollary 3.21. *Every G-P_β-open set in a GLP_ITS is G-clopen.*

Theorem 3.22. *In a $GLP_I TS$, every G-preopen set is G-preclosed, G-semiopen, G-semiclosed, G-α-open, G-α-closed, G-β-closed, G-regular open, G-regular closed, G-b-closed, G-P_β-open and G-P_β-closed.*

Proof. Obvious, from Theorem 3.20. □

Theorem 3.23. *In a $GLP_I TS$, G-preopen sets form a topology on $\mathcal{X}$.*

Proof. The proof is clear from Theorem 3.20. □

Corollary 3.24. *In a $GLP_I TS$, G-P_β-open sets form a topology on $\mathcal{X}$.*

Proof. Obvious, from Theorems 3.17 and 3.23. □

4. Grill P_β-connected space

Definition 4.1. Let $(\mathcal{X}, \tau, G)$ be a GTS and $\mathcal{A}, \mathcal{B}$ nonempty subsets of $\mathcal{X}$. Then $\mathcal{A}$ and $\mathcal{B}$ are:

(i) *G-P_β-separated*, if $\mathcal{A} \cap P_{\beta_G}Cl(\mathcal{B}) = \phi = P_{\beta_G}Cl(\mathcal{A}) \cap \mathcal{B}$,

(ii) *G-preseparated*, if $\mathcal{A} \cap P_G Cl(\mathcal{B}) = \phi = P_G Cl(\mathcal{A}) \cap \mathcal{B}$.

Note that G-P_β-separated sets are G-preseparated. However, the converse need not be true in general:

Example 4.2. Let $(\mathcal{X}, \tau, G)$ such that $\mathcal{X} = \{h, j, k, l\}$, $\tau = \{\mathcal{X}, \phi, \{h\}, \{j\}, \{h, j\}, \{h, j, k\}\}$ $G = \{\mathcal{X}, \{h\}, \{j\}, \{h, j\}, \{h, j, k\}, \{h, j, l\}\}$. Then $GPO(\mathcal{X}) = \{\mathcal{X}, \phi, \{h\}, \{j\}, \{h, j\}, \{h, j, k\}, \{h, j, l\}\}$, $GPC(\mathcal{X}) = \{\mathcal{X}, \phi, \{j, k, l\}, \{h, k, l\}, \{k, l\}, \{l\}, \{k\}\}$, $G\beta O(\mathcal{X}) = \{\mathcal{X}, \phi, \{h\}, \{j\}, \{h, j\}, \{h, j, k\}, \{h, j, l\}, \{h, k, l\}, \{j, k, l\}, \{h, k\}, \{h, l\}, \{j, k\}, \{j, l\}\}$ and $G\beta C(\mathcal{X}) = \{\mathcal{X}, \phi, \{j, k, l\}, \{h, k, l\}, \{k, l\}, \{l\}, \{k\}, \{j\}, \{h\}, \{j, l\}, \{j, k\}, \{h, l\}, \{h, k\}\}$. Thus $GP_\beta O(\mathcal{X}) = \{\mathcal{X}, \phi, \{h\}, \{j\}\}$ implies $GP_\beta C(\mathcal{X}) = \{\mathcal{X}, \phi, \{j, k, l\}, \{h, k, l\}\}$ and we find $P_G Cl(\{k\}) = \{k\}$, $P_G Cl(\{l\}) = \{l\}$. So $\{k\} \cap P_G Cl(\{l\}) = P_{\beta_G}Cl(\{k\}) \cap \{l\} = \phi$. Hence sets $\{k\}$ and $\{l\}$ are G- preseparated but not G-P_β-separated. Since $P_{\beta_G}Cl(\{k\}) = \{k, l\}$, $P_{\beta_G}Cl(\{l\}) = \{k, l\}$, $\{k\} \cap P_{\beta_G}Cl(\{l\}) = \{k\} \neq \phi \neq P_{\beta_G}Cl(\{k\}) \cap \{l\} = \{l\}$. Therefore $\{k\}$ and $\{l\}$ not are two G-P_β-separated sets of $\mathcal{X}$.

Definition 4.3. A subset $\mathcal{U}$ of a (Ω, τ, G) is said to be *G-P_β-connected*, if $\mathcal{U}$ is not the union of two G-P_β-separated sets in Ω. Otherwise, it is said to be *G-P_β-disconnected*. If $(\mathcal{A}, \mathcal{B})$ is a G-P_β-separation of Ω, then $\mathcal{A}$ and $\mathcal{B}$ are G-P_β-closed.

Example 4.4. From Example 3.16, we find $\{\varrho, \sigma\}$ is G-P_β-connected subset of (Υ, τ, G).

Definition 4.5. Let (Ω, τ, G) be a GTS. Then Ω is said to be:

(i) *G-P_β-connected*, if Ω cannot be expressed as the union of two disjoint nonempty G-P_β-open subsets of Ω,

(ii) *G-preconnected*, if Ω cannot be expressed as the union of two disjoint nonempty G-preopen subsets of Ω,

(iii) *G-connected*, if Ω cannot be expressed as the union of two disjoint nonempty G-open subsets of Ω,

(iv) *G-α-connected*, if Ω cannot be expressed as the union of two disjoint nonempty G-α-open subsets of Ω,

(v) *G-semiconnected*, if Ω cannot be expressed as the union of two disjoint nonempty G-semiopen subsets of Ω,

(vi) *G-β-connected*, if Ω cannot be expressed as the union of two disjoint nonempty G-β-open subsets of Ω.

Theorem 4.6. *A $(\mathcal{X}, \tau, G)$ is G-P_β-connected iff $\mathcal{X}$ cannot be expressed as the union of two disjoint nonempty G-P_β-open subsets of $\mathcal{X}$.*

Proof. ($\Rightarrow$) Let $(\mathcal{X}, \tau, G)$ be G-P_β-connected space and $\mathcal{X} = \mathcal{U} \cup \mathcal{W}$, where $\mathcal{U}$ and $\mathcal{W}$ are disjoint nonempty G-P_β-open sets. Then $\mathcal{U} = \mathcal{X} \setminus \mathcal{W}$ and $\mathcal{W} = \mathcal{X} \setminus \mathcal{U}$ are G-P_β-closed in $\mathcal{X}$. Thus $\mathcal{U} \cap P_{\beta G}Cl(\mathcal{W}) = \phi = P_{\beta G}Cl(\mathcal{U}) \cap \mathcal{W}$. This is a contradiction.

($\Leftarrow$) Suppose that $\mathcal{X} = \mathcal{U} \cup \mathcal{W}, \mathcal{U} \neq \phi \neq \mathcal{W}$ and $\mathcal{U} \cap P_{\beta G}Cl(\mathcal{W}) = \phi = P_{\beta G}Cl(\mathcal{U}) \cap \mathcal{W}$. Then $P_{\beta G}Cl(\mathcal{U}) = \mathcal{U}$ and $P_{\beta G}Cl(\mathcal{W}) = \mathcal{W}$. Thus $\mathcal{U}$ and $\mathcal{W}$ are nonempty G-P_β-open subsets of $\mathcal{X}$. This a contradiction. □

Example 4.7. Let (Ω, τ, G), such that $\Omega = \{1, 2, 3, 4\}$, $\tau = \{\Omega, \phi, \{1\}, \{4\}, \{1, 2\}, \{1, 4\}, \{1, 2, 4\}, \{1, 3, 4\}\}$ and $G = \{\Omega, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$. Then $GPO(\Omega) = \{\Omega, \phi, \{1\}, \{4\}, \{1, 2\}, \{1, 4\}, \{1, 2, 4\}, \{1, 3, 4\}\}$, $G\beta O(\Omega) = \{\Omega, \phi, \{1\}, \{4\}, \{1, 2\}, \{1, 4\}, \{1, 3\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$, $G\beta C(\Omega) = \{\Omega, \phi, \{2, 3, 4\}, \{1, 2, 3\}, \{3, 4\}, \{2, 3\}, \{2, 4\}, \{4\}, \{1, 2\}, \{3\}, \{2\}\}$ implies $GP_\beta O(\Omega) = \{\Omega, \phi, \{4\}, \{1, 2\}\}$ and $GP_\beta C(\Omega) = \{\Omega, \phi, \{1, 2, 3\}, \{3, 4\}\}$. Thus we cannot express Ω as the union of two disjoint nonempty G-P_β-open subsets of Ω and so Ω is G-P_β-connected.

Lemma 4.8. *Any grill indiscrete topological space(GITS for short) $(\mathcal{X}, \tau, G)$ with more than one point is not G-P_β-connected, but it is G-connected.*

Example 4.9. Let $(\mathcal{X}, \tau, G)$ be GITS such that $\mathcal{X} = \{a, b, c\}$, $\tau = \{\mathcal{X}, \phi\}$ and $G = \{\mathcal{X}, \{a\}, \{b\}, \{b, c\}, \{a, b\}\}$. Then $GPO(\mathcal{X}) = G\beta O(\mathcal{X}) = \{\mathcal{X}, \phi, \{a\}, \{b\}, \{b, c\}, \{a, b\}\}$, $G\beta C(\mathcal{X}) = \{\mathcal{X}, \phi, \{b, c\}, \{a, c\}, \{a\}, \{c\}\}$ implies $GP_\beta O(\mathcal{X}) = \{\mathcal{X}, \phi, \{a\}, \{b, c\}\}$. Thus $\{a\}, \{b, c\}$ is a G-P_β-separation and G-P_β-open in $\mathcal{X}$, $\{a\} \cup \{b, c\} = \mathcal{X}$ implies $\mathcal{X}$ is not G-P_β-connected. But $\mathcal{X}$ is G-connected, since $GO(\mathcal{X}) = \{\mathcal{X}, \phi\}$.

Theorem 4.10. *Let (Ω, τ, G) be a GTS. Then the following statements hold:*

(1) *Ω is G-preconnected iff Ω is G-P_β-connected.*

(2) *If Ω is G-P_β-connected, then Ω is G-connected.*

Proof. (1) ($\Rightarrow$) Suppose that Ω is G-preconnected. Then Ω is G-P_β-connected , since $GP_\beta O(\Omega) \subseteq GPO(\Omega)$.

($\Leftarrow$) Suppose that Ω is not $G-$Preconnected. Then Ω is the union of two non empty disjoint G-preopen sets μ and υ. Since $GPO(\Omega) \subseteq G\beta O(\Omega)$, it follows that μ and υ are disjoint G-P_β-open sets. This is a contradiction.

(2) Suppose Ω be G-P_β-connnected and Ω is not G-connected. Then there exist two a nonempty disjoint subsets μ, υ of Ω which are both G-open and G-closed in Ω. Thus by Lemma 3.11, μ, υ are also both G-P_β-open and G-P_β-closed in Ω. This is a contradiction. □

Example 4.11. From Example 4.9, $\mathcal{X}$ is G-connected. But not G-P_β-connected.

The following figure summarizes the relationship of G-P_β-connnectedness with various types of $G-$connectedness properties:

$$\begin{tikzcd}
& G\text{-semiconnected} \arrow[r] & G\text{-}\alpha\text{-connected} \arrow[dr, shift left] & \\
G\text{-}\beta\text{-connected} \arrow[ur] \arrow[dr] & & & G\text{-connected} \arrow[ul, shift left] \\
& G\text{-preconnected} \arrow[r, leftrightarrow] & G\text{-}P_\beta\text{-connnected} \arrow[uur] &
\end{tikzcd}$$

Example 4.12. Let (Ω, τ, G) be GTS such that $\Omega = \{\hbar, \jmath, \ell\}$, $\tau = \{\Omega, \phi, \{\hbar\}, \{\hbar, \ell\}\}$, $G = \{\Omega, \{\hbar\}, \{\ell\}, \{\hbar, \jmath\}, \{\hbar, \ell\}, \{\jmath, \ell\}\}$. Then $GO(\Omega) = \{\Omega, \phi\}$, $G\alpha O(\Omega) = \{\Omega, \phi, \{\hbar\}, \{\hbar, \jmath\}, \{\hbar, \ell\}\}$. Thus Ω is G-connected and G-α-connected. Also, $GPO(\Omega) = \{\Omega, \phi, \{\hbar\}, \{\ell\}, \{\hbar, \ell\}, \{\jmath, \ell\}\}$, $G\beta O(\Omega) = \{\Omega, \phi, \{\hbar\}, \{\ell\}, \{\hbar, \jmath\}, \{\hbar, \ell\}, \{\jmath, \ell\}\}$ and $G\beta C(\Omega) = \{\Omega, \phi, \{\jmath, \ell\}, \{\hbar, \jmath\}, \{\ell\}, \{\jmath\}, \{\hbar\}\}$ implies $P_\beta O(\Omega) = \{\Omega, \phi, \{\hbar\}, \{\ell\}, \{\hbar, \jmath\}, \{\jmath, \ell\}\}$ and $\{\hbar\} \cap \{\jmath, \ell\} = \phi$, $\{\hbar\} \cup \{\jmath, \ell\} = \Omega$ also $\{\ell\} \cap \{\hbar, \jmath\} = \phi$, $\{\ell\} \cup \{\hbar, \jmath\} = \Omega$. So Ω is not G-P_β-connected.

Example 4.13. Let $(\mathcal{X}, \tau, G)$ be GTS such that $\mathcal{X} = \{a, b, c\}$, $\tau = \{\mathcal{X}, \phi, \{a, b\}\}$ and $G = \{\mathcal{X}, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Then $GSO(\mathcal{X}) = \{\mathcal{X}, \phi, \{a, b\}\}$, $\mathcal{X}$ is G-semi-connected. But $\mathcal{X}$ is not G-β-connected, since $G\beta O(\mathcal{X}) = \{\mathcal{X}, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

Example 4.14. Let $(\mathcal{X}, \tau, G)$ be GTS such that $\mathcal{X} = \{h_1, h_2, h_3\}$, $\tau = \{\mathcal{X}, \phi, \{h_1\}, \{h_2\}, \{h_1, h_2\}\}$, $G = \{X, \{h_1\}, \{h_1, h_3\}\}$ and $G\alpha O(\mathcal{X}) = \{\mathcal{X}, \phi, \{h_1\}, \{h_2\}, \{h_1, h_2\}\}\}$. Then $\mathcal{X}$ is G-α-connected. But $\mathcal{X}$ is not G-semi-connected, since $GSO(\mathcal{X}) = \{\mathcal{X}, \phi, \{h_1\}, \{h_2\}, \{h_1, h_2\}, \{h_1, h_3\}\}$.

Example 4.15. Let (Ω, τ, G) be GTS such that $\Omega = \{\imath, \jmath, \ell, \wp\}$, $\tau = \{\Omega, \phi, \{\imath, \wp\}, \{\imath\}, \{\wp\}, \{\imath, \jmath, \wp\}\}$, $G = \{\Omega, \{\imath\}, \{\imath, \jmath\}, \{\imath, \jmath, \ell\}\}$ and $GPO(\Omega) = \{\Omega, \phi, \{\imath\}, \{\wp\}, \{\imath, \wp\}, \{\imath, \jmath, \wp\}\}$. Then Ω is G-preconnected. But Ω is not G-β-connected, since $G\beta O(\Omega) = \{\Omega, \phi, \{\imath\}, \{\wp\}, \{\imath, \jmath\}, \{\imath, \ell\}, \{\imath, \wp\}, \{\jmath, \wp\}, \{\ell, \wp\}, \{\imath, \jmath, \ell\}, \{\imath, \jmath, \wp\}, \{\imath, \ell, \wp\}, \{\jmath, \ell, \wp\}\}$.

Example 4.16. Let GTS $(\mathcal{X}, \tau, G)$ such that $\mathcal{X} = \{\imath, \jmath, \ell\}$, $\tau = \{\mathcal{X}, \phi, \{\imath, \ell\}\}$, $G = \{X, \{\imath\}, \{\ell\}, \{\imath, \ell\}, \{\jmath, \ell\}\}$ implies $GO(\mathcal{X}) = \{\mathcal{X}, \phi\}$ and $G\alpha O(\mathcal{X}) = \{\mathcal{X}, \phi, \{\imath\}, \{\imath, \jmath\}, \{\imath, \ell\}\}$. Then $\mathcal{X}$ is G-connected and G-α-connected. But $\mathcal{X}$ is not G-P_β-connected, since $GPO(\mathcal{X}) = \{\mathcal{X}, \phi, \{\imath\}, \{\ell\}, \{\imath, \ell\}, \{\jmath, \ell\}\}$, $G\beta O(\mathcal{X}) = \{\mathcal{X}, \phi, \{\imath\}, \{\ell\}, \{\imath, \ell\}, \{\jmath, \ell\}\}$, $G\beta C(\mathcal{X}) = \{\mathcal{X}, \phi, \{\jmath, \ell\}, \{\imath, \jmath\}, \{\jmath\}, \{\imath\}\}$ and $GP_\beta O(\mathcal{X}) = \{\mathcal{X}, \phi, \{\imath\}, \{\jmath, \ell\}\}$.

Theorem 4.17. *In a GLP_I TS (Ω, τ, G), G-connectedness is equivalent to G-preconnectedness.*

Proof. ($\Rightarrow$) Suppose that Ω is not G-preconnected. Then there exist a two disjoint nonempty G-preopen μ, ω such that $\mu \cup \omega = \Omega$. Then by Theorem 3.20, μ, ω are G-clopen. Thus Ω is not G-connected.

($\Leftarrow$) Follows from Theorem 4.10. □

Corollary 4.18. *Let GLP_I TS (Ω, τ, G). Then $G-$connectedness is equivalent to G-P_β-connectedness.*

Proof. Obvious from Theorems 4.10 and 4.17. □

Theorem 4.19. *In GLP_I TS (Ω, τ, G), G-semiconnectedness is equivalent to G-preconnectedness.*

Proof. ($\Rightarrow$) Suppose that (Ω, τ, G) is not G-preconnected. Then there exist two a disjoint nonempty G-preopen subsets μ, ω of Ω such that $\mu \cup \omega = \Omega$. Thus from Theorem 3.22, μ, ω are G-semiopen. So Ω is not G-semiconnected.

($\Leftarrow$) Suppose that (Ω, τ, G) is not G-semiconnected. Then $\Omega = \mu \cup \omega$, where μ and ω are disjoint nonempty G-semiopen subsets of Ω, $\mu = \Omega \setminus \omega$, $\omega = \Omega \setminus \mu$ are disjoint nonempty G-semiclosed, it follows that μ and ω are disjoint nonempty G-preopen sets. Thus Ω is not G-preconnected. □

Corollary 4.20. *In GLP_I TS $(\mathcal{X}, \tau, G)$, G-semiconnectedness is equivalent to G-P_β-connectedness.*

Proof. Obvious from Theorems 4.10 and 4.19. □

5. Properties of grill P_β-connected spaces

Theorem 5.1. *If $\mathcal{B}$ is a G-P_β-connected set of a GTS $(\mathcal{X}, \tau, G)$ and μ, ω are G-P_β-separated sets of $\mathcal{X}$ such that $\mathcal{B} \subseteq \mu \cup \omega$, then $\mathcal{B} \subseteq \mu$ or $\mathcal{B} \subseteq \omega$.*

Proof. Assume that $\mathcal{B}$ is not a G-p_β-connected set and μ, ω is G-P_β-separated sets. Then $\mathcal{B} = \mu \cup \omega$ and $P_{\beta_G}Cl(\mu) \cap \omega = \mu \cap P_{\beta_G}Cl(\omega) = \phi$. Thus $\mu \subseteq \mathcal{X} \setminus P_{\beta_G}Cl(\omega)$, $\omega \subseteq \mathcal{X} \setminus P_{\beta_G}Cl(\mu)$. Since $\mu \cup \omega \subseteq \mathcal{X} \setminus (P_{\beta_G}Cl(\omega) \cup \mathcal{X} \setminus P_{\beta_G}Cl(\mu)) \subseteq \mathcal{X} \setminus (\mu \cup \omega)$, $\mathcal{B} \subseteq \mathcal{X} \setminus (\mu \cup \omega)$. But $\mathcal{B} \subseteq \mu \cup \omega$. So $\mathcal{B} \subseteq \mathcal{X} \setminus \mathcal{B}$. It is a contradiction. Thus $\mathcal{B}$ is a G-P_β-connected set. Hence $\mu \cup \omega \neq \mathcal{B}$ and $\mathcal{B} \subseteq \mu$ or $\mathcal{B} \subseteq \omega$. □

Theorem 5.2. *Let (Ω, τ, G), be a GTS and $\Omega = \mu \cup \omega$ be a G-P_β-separation of Ω. If $\mathcal{A}$ is a G-P_β-connected subset of Ω, then $\mathcal{A}$ is completely contained in either μ or ω.*

Proof. Let $\Omega = \mu \cup \omega$ be a G-P_β-separation of Ω. Suppose $\mathcal{A}$ intersects both μ and ω. Then $\mathcal{A} = (\mu \cap \mathcal{A}) \cup (\omega \cap \mathcal{A})$ is a G-P_β-separation of $\mathcal{A}$. This is a contradiction. □

Theorem 5.3. *If $\mathcal{U}$ is a G-P_β-connected set of a GTS $(\mathcal{X}, \tau, G)$ and $\mathcal{U} \subseteq \lambda \subseteq P_{\beta_G}Cl(\mathcal{U})$, then λ is G-P_β-connected.*

Proof. Assume that λ is not G-P_β-connected. Then there exist G-P_β-separated sets $\mathcal{A}$ and $\mathcal{B}$ such that $\lambda = \mathcal{A} \cup \mathcal{B}$. Thus $\mathcal{A}$ and $\mathcal{B}$ are nonempty and $\mathcal{A} \cap P_{\beta_G}Cl(\mathcal{B}) = \phi = P_{\beta_G}Cl(\mathcal{A}) \cap \mathcal{B}$. By Theorem 5.1, we obtain either $\mathcal{U} \subseteq \mathcal{A}$ or $\mathcal{U} \subseteq \mathcal{B}$. Suppose that $\mathcal{U} \subseteq \mathcal{A}$. Then $P_{\beta_G}Cl(\mathcal{U}) \subseteq P_{\beta_G}Cl(\mathcal{A})$ and $\mathcal{B} \cap P_{\beta_G}Cl(\mathcal{U}) = \phi$. But by hypothesis, $\mathcal{B} \subseteq \lambda \subseteq P_{\beta_G}Cl(\mathcal{U})$ and $\mathcal{B} = P_{\beta_G}Cl(\mathcal{U}) \cap \mathcal{B} = \phi$. This is a contradiction, since $\mathcal{B}$ is nonempty. □

Corollary 5.4. *If (Ω, τ, G) be a GTS and μ is a G-P_β-connected subset of Ω. Then $P_{\beta_G}Cl(\mu)$ is G-P_β-connected.*

Proof. It is obvious from the above theorem. □

Theorem 5.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of a GTS (Ω, τ, G). If $\mathcal{A}$ and $\mathcal{B}$ are G-P_β-connected and not G-P_β-separated in Ω, then $\mathcal{A} \cup \mathcal{B}$ is G-P_β-connected.*

Proof. Assume that $\mathcal{A}\cup\mathcal{B}$ be not G-P_β-connected. Then there exist G-P_β-separated sets $\mathcal{U},\mathcal{W}$ in Ω such that $\mathcal{A}\cup\mathcal{B}=\mathcal{U}\cup\mathcal{W}$. Thus $\mathcal{A}\subseteq\mathcal{U}\cup\mathcal{W}$. by Theorem 5.1, we have $\mathcal{A}\subseteq\mathcal{U}$ or $\mathcal{A}\subseteq\mathcal{W}$. Furthermore, $\mathcal{B}\subseteq\mathcal{U}$ or $\mathcal{B}\subseteq\mathcal{W}$. If $\mathcal{A}\subseteq\mathcal{U}$ and $\mathcal{B}\subseteq\mathcal{U}$, then $\mathcal{A}\cup\mathcal{B}\subseteq\mathcal{U}$ and $\mathcal{W}=\phi$ a contradiction. Thus $\mathcal{A}\subseteq\mathcal{U}$ and $\mathcal{B}\subseteq\mathcal{W}$ or $\mathcal{A}\subseteq\mathcal{W}$ and $\mathcal{B}\subseteq\mathcal{U}$. In the first case, $P_{\beta_G}Cl(\mathcal{A})\cap\mathcal{B}\subseteq P_{\beta_G}Cl(\mathcal{U})\cap\mathcal{W}=\phi$ and $P_{\beta_G}Cl(\mathcal{B})\cap\mathcal{A}\subseteq P_{\beta_G}Cl(\mathcal{W})\cap\mathcal{U}=\phi$. So $\mathcal{A}$ and $\mathcal{B}$ are G-P_β-separated in Ω, a contradiction. Hence $\mathcal{A}\cup\mathcal{B}$ is G-P_β-connected. □

Theorem 5.6. *Let $\mathcal{A}_{\varepsilon:\varepsilon\in\Gamma}$ be a nonempty family of G-P_β-connected subsets of (Ω,τ,G) such that $\cap_{\varepsilon\in\Gamma}\mathcal{A}_\varepsilon\neq\phi$, then $\cup_{\varepsilon\in\Gamma}\mathcal{A}_\varepsilon$ is G-P_β-connected.*

Proof. Assume that $\cup_{\varepsilon\in\Gamma}\mathcal{A}_\varepsilon$ is not G-P_β-connected. Then there exist μ and ω are G-P_β-separated sets in Ω, where $\cup_{\varepsilon\in\Gamma}\mathcal{A}_\varepsilon=\mu\cup\omega$. If $x\in\cap_{\varepsilon\in\Gamma}\mathcal{A}_\varepsilon\neq\phi$, since $x\in\cup_{\varepsilon\in\Gamma}\mathcal{A}_\varepsilon$, $x\in\mu$ or $x\in\omega$. Suppose that $x\in\mu$. Since $x\in\mathcal{A}_\varepsilon$ for each $\varepsilon\in\Gamma$, $\mathcal{A}_\varepsilon$ and μ intersect for each $\varepsilon\in\Gamma$. Then by Theorem 5.1, we find $\mathcal{A}_\varepsilon\subseteq\mu\cup\omega$, $\mathcal{A}_\varepsilon\subseteq\mu$ or $\mathcal{A}_\varepsilon\subseteq\omega$, since μ and ω are G-P_β-separated sets in Ω. Thus $\mathcal{A}_\varepsilon\subseteq\mu$ $\forall\varepsilon\in\Gamma$. So $\cup_{\varepsilon\in\Gamma}\mathcal{A}_\varepsilon\subseteq\mu$ implies $\omega=\phi$ a contradiction. □

Definition 5.7. A space (Ω,τ,G) is said to be *totally G-P_β-disconnected* ($TGP_\beta D$, for short), if its only G-P_β-connected subsets are one point sets.

Definition 5.8. Let $(\mathcal{X},\tau,G)$ be a GTS and $x\in\mathcal{X}$. Then the following statements are equivalent:

(i) the G-P_β-component of $\mathcal{X}$ containing x is the union of all G-P_β-connected subsets of $\mathcal{X}$ containing x,

(ii) a G-P_β-component of $\mathcal{X}$ is $G-P_\beta$-connected.

Theorem 5.9. *Let (Ω,τ,G) be a GTS. Then the following properties hold:*

(1) *each G-P_β-component of Ω is a maximal G-P_β-connected (MGP_β-connected for short) subset of Ω,*

(2) *the set of all distinct G-P_β-components of Ω forms a partition of Ω,*

(3) *each G-P_β-component of Ω is G-P_β-closed in Ω.*

Proof. (1) Obvious.

(2) Since singletons are G-P_β-connected sets, each point x of Ω is contained in the G-P_β-component of Ω containing x. Let μ and υ are two distinct G-P_β-components of Ω. If μ and υ intersect, then $\mu\cup\upsilon$ is G-P_β-connected, by Theorem 5.6. Thus either μ is not maximal or υ is not maximal, a contradiction. So μ and υ are disjoint.

(3) Let μ be any G-P_β-component of Ω containing x. Then by Corollary 5.4, $P_{\beta_G}Cl(\mu)$ is G-P_β-connected set containing x. Since μ is MGP_β-connected set containing x, $P_{\beta_G}Cl(\mu)\subseteq\mu$. Thus μ is G-P_β-closed in Ω. □

Definition 5.10. A space (Ω,τ,G) is said to be *locally G-P_β-connected* (*LGP_β-connected*) at $x\in\Omega$, if for each G-P_β-open set $\mathcal{U}$ containing x, there is a G-P_β-connected G-P_β-open set $\mathcal{W}$ such that $x\in\mathcal{W}\subseteq\mathcal{U}$.

The space Ω is said to be *LGP_β-connected*, if it is LGP_β-connected at each of its points.

Theorem 5.11. *A space (Ω,τ,G) is LGP_β-connected iff the G-P_β-components of each G-P_β-open subset of Ω are G-P_β-open.*

Proof. ($\Rightarrow$) Let Ω be LGP_β-connected and λ be an G-P_β-open subset of Ω and υ be a G-P_β-component of λ. If $x \in \upsilon$, then there is a G-P_β-connected G-P_β-open set $\omega \subseteq \Omega$ such that $x \in \omega \subseteq \lambda$. Since υ is a G-P_β-component of λ and ω is a G-P_β-connected subset of λ containing x, $\omega \subseteq \upsilon$. Thus υ is a G-P_β-open set.

($\Leftarrow$) let $\mu \subseteq \Omega$ be a G-P_β-open set, and $x \in \lambda$. Then by our hypothesis, the G-P_β-component ω of λ containing x is G-P_β-open. Thus Ω is LGP_β-connected at x. □

6. Conclusions

The present paper represents a starting point for framework to modify and generalize P_β-connectedness. In fact, we have introduced new generalized connectedness called a grill-P_β-connectedness. In addition, rill locally preindiscrete spaces are obtained. Several examples are given to indicate the connections between types of grill connectedness. The introduced techniques are very useful in application because it opens the way for more topological applications from real life problems. In future, we will apply our results on topological structures induced by graphs and the results in [9, 10, 11, 12, 13].

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