

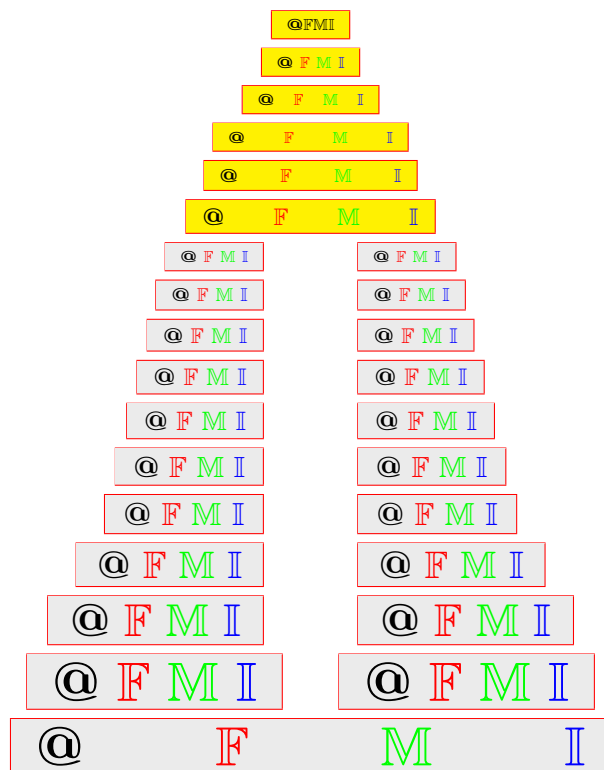
Annals of **F**uzzy **M**athematics and **I**nformatics
 Volume 22, No. 1, (August 2021) pp. 25–42
 ISSN: 2093–9310 (print version)
 ISSN: 2287–6235 (electronic version)
<http://www.afmi.or.kr>
<https://doi.org/10.30948/afmi.2021.22.1.25>

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Reprinted from the
 Annals of Fuzzy Mathematics and Informatics
 Vol. 22, No. 1, August 2021

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Commutative quantales and commutative quantale frame

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Received 9 February 2021; Revised 26 February 2021; Accepted 28 March 2021

ABSTRACT. We introduce the notion of commutative quantale frames as a viewpoint of relational semantics for a fuzzy logic. We investigate the relations between commutative quantale frames and commutative quantales. We study their properties and give their examples.

2020 AMS Classification: 03E72, 54A40, 54B10

Keywords: Fuzzy partially ordered sets, commutative quantale frames, commutative quantales, Alexandrov topologies.

Corresponding Author: Yong Chan Kim (yck@gwnu.ac.kr)

1. INTRODUCTION

Mulvey (See [1, 2]) introduced Quantales as the non-commutative generalization of the lattice of open sets in topological spaces. In fact, a commutative unital quantale coincides with a complete residuated lattice. Höhle (See [3, 4]) developed the algebraic structures and many valued topologies in a sense of quantales and cqm-lattices.

Discrete duality is a duality between a class of algebras and an associated class of relational systems (referred to as frames) without a topology. Dualities are developed between algebras and logical relational systems such as Boolean algebras and classical propositional logic; MV-algebra and Lukasiewicz logic; Heyting algebra and intuitionistic logic; BL-algebra and basic fuzzy logics; Monoidal t-norm logic and MTL-algebras (See [5, 6, 7, 8, 9, 10]). A duality leads in a natural way to relational semantics for a logic which supports part of foundation of theoretic computer science. As a duality between algebras and logical relational systems, Oh and Kim (See [11, 12]) introduced the notion of residuated connections and residuated frames in fuzzy logics.

In this paper, we introduce the notions of commutative quantales and commutative quantale frames as a duality. Let (X, e_X) be a fuzzy partially ordered set. In Theorem 3.7, $\tau_{e_X} = \{A \in L^X \mid A(x) \odot e_X(x, y) \leq A(y)\}$ is an Alexandrov topology. In Theorem 3.8, if (X, e_X, P) is a commutative quantale frame, we show that $(\tau_{e_X}, \vee, \wedge, \otimes, \nearrow, 0_X, \infty_X)$ is a commutative quantale where, $A, B \in \tau_{e_X}$,

$$(A \otimes B)(z) = \bigvee_{x, y \in X} P(x, y, z) \odot A(x) \odot B(y).$$

$$(A \Rightarrow B)(x) = \bigwedge_{y, z \in X} (P(x, y, z) \odot A(y) \rightarrow B(z)).$$

In Theorem 3.14, if $(X, \wedge, \vee, \oplus, \nearrow, 0, 1)$ is a commutative quantale and e_X is a r -fuzzy poset on X , we show that (L^X, e_{L^X}, P) is a commutative quantale frame where $P : L^X \times L^X \times L^X \rightarrow L$ as

$$P(A, B, C) = \bigwedge_{x \in X} ((A \otimes B)(x) \rightarrow C(x))$$

and $(A \otimes B)(x) = \bigvee_{y, z \in X} (A(y) \odot B(z) \odot e_X(y \odot z, x))$. We give their examples.

2. PRELIMINARIES

Definition 2.1 ([1, 2, 3]). A triple $(L, \leq, \odot)$ is called a *commutative quantale* (*c-quantale*, for short), if it satisfies the following conditions:

- (Q1) $L = (L, \leq, \vee, \wedge, 0, 1)$ is a complete lattice, where 1 is the universal upper bound and 0 denotes the universal lower bound,
- (Q2) $a \odot b = b \odot a$ and $a \odot (b \odot c) = (a \odot b) \odot c$ for all $a, b, c \in L$,
- (Q3) $\odot$ is distributive over arbitrary joins, i.e.,

$$\left(\bigvee_{i \in \Gamma} a_i\right) \odot b = \bigvee_{i \in \Gamma} (a_i \odot b).$$

A c-quantale is called *unital*, if $a = a \odot e$, for each $a \in L$. A unital c-quantale is called a *strictly two-sided, commutative quantale* (*sc-quantale*, for short), if $e = 1$.

Remark 2.2 ([1, 2, 3]). (1) A completely distributive lattice is an sc-quantale. In particular, the unit interval $([0, 1], \leq, \vee, \odot = \wedge, 0, 1)$ is an sc-quantale.

(2) The unit interval with a left-continuous t-norm $\odot$, $([0, 1], \leq, \odot)$, is an sc-quantale.

(3) Let $(L, \leq, \odot)$ be a c-quantale. For each $x, y \in L$, we define

$$x \rightarrow y = \bigvee \{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies Galois correspondence, i.e.,

$$(x \odot y) \leq z \text{ iff } x \leq (y \rightarrow z).$$

(4) $([0, \infty], \leq_{op}, \vee_{op}, +, \wedge_{op}, \infty, 0)$ is a sc-quantale, where $\leq_{op} = \geq, \vee_{op} = \wedge, \wedge_{op} = \vee$ and

$$\begin{aligned} x \rightarrow y &= \bigvee_{op} \{z \in [0, \infty] \mid x + z \leq_{op} y\} \\ &= \bigwedge \{z \in [0, \infty] \mid z \geq -x + y\} = (y - x) \vee 0. \end{aligned}$$

In this paper, we assume $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a c-quantale. For $\alpha \in L, A, B \in L^X$, we denote $(\alpha \rightarrow A), (A \rightarrow B), (\alpha \odot A), \alpha_X \in L^X$ as $(\alpha \rightarrow A)(x) = \alpha \rightarrow A(x), (A \rightarrow B)(x) = (A(x) \rightarrow B(x)), (\alpha \odot A)(x) = \alpha \odot A(x), (A \odot B)(x) = A(x) \odot B(x)$, and $\alpha_X(x) = \alpha$.

Lemma 2.3 ([3, 4, 13, 14]). *For each $x, y, z, x_i, y_i \in L$, we have the following properties.*

- (1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.
- (2) $x \odot y \leq x \wedge y \leq x \vee y$.
- (3) $x \rightarrow (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma} (x \rightarrow y_i)$ and $(\bigvee_{i \in \Gamma} x_i) \rightarrow y = \bigwedge_{i \in \Gamma} (x_i \rightarrow y)$.
- (4) $x \rightarrow (\bigvee_{i \in \Gamma} y_i) \geq \bigvee_{i \in \Gamma} (x \rightarrow y_i)$.
- (5) $(\bigwedge_{i \in \Gamma} x_i) \rightarrow y \geq \bigvee_{i \in \Gamma} (x_i \rightarrow y)$.
- (6) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.
- (7) $x \odot (x \rightarrow y) \leq y$ and $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.
- (8) $y \odot z \leq x \rightarrow (x \odot y \odot z)$ and $x \odot (x \odot y \rightarrow z) \leq y \rightarrow z$.
- (9) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$.
- (10) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$.
- (11) $x \rightarrow y = 1$ iff $x \leq y$.
- (12) $x = \bigwedge_{y \in L} ((x \rightarrow y) \rightarrow y)$.

Definition 2.4 ([3, 4, 13, 14]). Let X be a set. A function $e_X : X \times X \rightarrow L$ is called:

- (E1) *reflexive*, if $e_X(x, x) = 1$ for all $x \in X$,
- (E2) *transitive*, if $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$,
- (E3) *anti-symmetric*, if $e_X(x, y) = e_X(y, x) = \top$, then $x = y$.

If e_X satisfies (E1) and (E2), (X, e_X) is a fuzzy preordered set. If e_X satisfies (E1), (E2) and (E3), (X, e_X) is a fuzzy partially ordered set (simply, fuzzy poset).

Example 2.5. (1) We define a function $e_{L^X} : L^X \times L^X \rightarrow L$ as

$$e_{L^X}(A, B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x)).$$

Then by Lemma 2.3 (10) and (11), (L^X, e_{L^X}) is a fuzzy poset.

(2) If (X, e_X) is a fuzzy poset and we define a function $e_X^{-1}(x, y) = e_X(y, x)$, then (X, e_X^{-1}) is a fuzzy poset.

3. COMMUTATIVE QUANTALES AND COMMUTATIVE QUANTALE FRAME

As an extension of MTL-frames [9], we define a commutative quantale frame.

Definition 3.1. Let (X, e_X) be a fuzzy poset. The triple (X, e_X, P) is called a *commutative quantale frame* (cq-frame, for short), if $P : X \times X \times X \rightarrow L$ satisfies the following conditions:

- (P1) $P(x, y, z) \odot e_X(x', x) \odot e_X(y', y) \odot e_X(z, z') \leq P(x', y', z')$,
 - (P2) $P(x, y, z) \leq P(y, x, z)$,
 - (P3) $P(x, y, z) \odot P(z, y', z') \leq \bigvee_{u \in X} (P(y, y', u) \odot P(x, u, z'))$,
 - (P4) $P(x, y, z) \odot P(x', z, z') \leq \bigvee_{u \in X} (P(x', x, u) \odot P(u, y, z'))$.
- A cq-frame (X, e_X, P) is called an *scq-frame*, if satisfies
- (P5) $\bigvee_{u \in X} P(u, x, x) = 1$.

$$(P6) \quad P(x, y, z) \leq e_X(x, z) \wedge e_X(y, z).$$

Definition 3.2. Let $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be a c-quantale. A fuzzy poset (X, e_X) is an *r-fuzzy poset*, if (R) $e_X(a * b, c) = e_X(a, b \nearrow c)$ for each $a, b, c \in X$.

Theorem 3.3. Let $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be a c-quantale. A pair (X, e_X) is an *r-fuzzy poset*. We define

$$P(x, y, z) = e_X(x * y, z).$$

Then the following properties hold.

(1) (X, e_X, P) is a cq-frame.

(2) If it is an sc-quantale, $e_X(x * y, z) \leq e_X(x, z) \wedge e_X(y, z)$ for each $x, y, z \in X$, then (X, e_X, P) is an scq-frame.

Proof. (1) (P1) Since $e_X(x', y \nearrow z) = e_X(y, x' \nearrow z)$, we have

$$\begin{aligned} & P(x, y, z) \odot e_X(x', x) \odot e_X(y', y) \odot e_X(z, z') \\ &= e_X(x * y, z) \odot e_X(x', x) \odot e_X(y', y) \odot e_X(z, z') \\ &= e_X(x, y \nearrow z) \odot e_X(x', x) \odot e_X(y', y) \odot e_X(z, z') \\ &\leq e_X(x', y \nearrow z) \odot e_X(y', y) \odot e_X(z, z') \\ &= e_X(y, x' \nearrow z) \odot e_X(y', y) \odot e_X(z, z') \\ &\leq e_X(y', x' \nearrow z) \odot e_X(z, z') \\ &\leq e_X(x' * y', z) \odot e_X(z, z') \\ &\leq e_X(x' * y', z') = P(x', y', z'). \end{aligned}$$

(P2) For each $x, y, z \in X$, $P(x, y, z) = e_X(x * y, z) = e_X(y * x, z) = P(y, x, z)$.

(P3)

$$\begin{aligned} & P(x, y, z) \odot P(z, y', z') = e_X(x * y, z) \odot e_X(z * y', z') \\ &\leq e_X(x * y, y' \nearrow z') = e_X(y * y', x \nearrow z') \\ &= e_X(y * y', x \nearrow z') \odot e_X(x * (x \nearrow z'), z') \\ &\leq \bigvee e_X(y * y', u) \odot e_X(x * u, z') \\ &= \bigvee P(y, y', u) \odot P(x * u, z'). \end{aligned}$$

(P4)

$$\begin{aligned} & P(x, y, z) \odot P(x', z, z') = e_X(x * y, z) \odot e_X(x' * z, z') \\ &\leq e_X(x * y, x' \nearrow z') = e_X(x * x', y \nearrow z') \\ &= e_X(x' * x, y \nearrow z') \odot e_X(y \nearrow z', y \nearrow z') \\ &= e_X(x' * x, y \nearrow z') \odot e_X((y \nearrow z') * y, z') \\ &\leq \bigvee_{u \in X} (e_X(x' * x, u) \odot e_X(u * y, z')) \\ &= \bigvee_{u \in X} (P(x', x, u) \odot P(u * y, z')). \end{aligned}$$

(2) (P5) $\bigvee_{u \in X} P(u, x, x) \geq e_X(1 * x, x) = e_X(x, x) = 1$.

(P6) $P(x, y, z) = e_X(x * y, z) \leq e_X(y, z)$ and $P(x, y, z) = e_X(x * y, z) \leq e_X(x, z)$. $\square$

Example 3.4. Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a c-quantale.

(1) Define $e_L : L \times L \rightarrow L$ as $e_L(x, y) = x \rightarrow y$ and $P(x, y, z) = (x \odot y) \rightarrow z = e_L(x \odot y, z)$. Since $P(x, y, z) = (x \odot y) \rightarrow z = e_L(x \odot y, z) = e_L(x, y \rightarrow z)$, (L, e_L) is an r-fuzzy poset. By Theorem 3.3 (1), (L, e_L, P) is a cq-frame.

(2) Let $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be a c-quantale. Define $e_X : X \times X \rightarrow L$ as

$$e_X(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{if } x \not\leq y, \end{cases}$$

Since $x * y \leq z$ iff $x \leq y \nearrow z$, $e_X(x * y, z) = e_X(x, y \nearrow z)$ for each $x, y, z \in X$. Hence (X, e_X) is an r-fuzzy poset. Furthermore, define $P(x, y, z) = e_X(x * y, z)$. Then (X, e_X, P) is an cq-frame.

(3) Let X be a set and $P(X) = \{A \mid A \subset X\}$. Then $(P(X), \cap, \cup, * = \cap, \nearrow, \emptyset, X)$ is an sc-quantale with

$$A \nearrow B = \bigcup \{C \in P(X) \mid A \cap C \subset B\} = A^c \cup B.$$

Define $e : P(X) \times P(X) \rightarrow L$ as

$$e(A, B) = \begin{cases} 1, & \text{if } A \subset B, \\ 0, & \text{if } A \not\subset B. \end{cases}$$

Since $A \cap B \subset C$ iff $A \subset B^c \cup C$, $e(A \cap B, C) = e(A, B \nearrow C)$ for each $A, B, C \in P(X)$. Then $(P(X), e)$ is an r-fuzzy poset. Thus by Theorem 3.3 (1), $(P(X), e, P)$ is a cq-frame with $P(A, B, C) = e(A \cap B, C)$ for each $A, B, C \in P(X)$. But it is not an scq-frame because $X = \{a, b, c\}$, $A = \{a, b\}$, $B = \{b, c\}$ and $P(A, B, A) = e(A \cap B, A) = 1 \not\leq e(A, A) \wedge e(B, A) = 0$.

(4) Let X be a set. Then $(L^X, \wedge, \vee, \odot, \rightarrow, 0_X, 1_X)$ is a c-quantale. Define $e_{L^X} : L^X \times L^X \rightarrow L$ as $e_{L^X}(A, B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x))$. Then (L^X, e_{L^X}) is an r-fuzzy poset. Furthermore, define $P(A, B, C) = \bigwedge_{x \in X} (A(x) \odot B(x) \rightarrow C(x))$. Then (L^X, e_{L^X}, P) is a cq-frame.

(5) Define on $L = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ as follows:

$$x \odot y = 0 \vee (x + y - 1), \quad x \rightarrow y = 1 \vee (1 - x + y).$$

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is an sc-quantale. Let $X = \{0, x, y, 1\}$ be a set such that $0 < x < y < 1$. Define $*$ and $\nearrow$ as follows:

*	0	x	y	1
0	0	0	0	0
x	0	0	x	x
y	0	x	y	y
1	0	x	y	1

$\nearrow$	0	x	y	1
0	1	1	1	1
x	x	1	1	1
y	0	x	1	1
1	0	x	y	1

Then $(X, \wedge, \vee, *, \nearrow, 0, 1)$ is a c-quantale.

(a) Define $e_X : X \times X \rightarrow L$ as follows

e_X	0	x	y	1
0	1	1	1	1
x	$\frac{1}{2}$	1	1	1
y	$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	1

Since

$$\begin{aligned}
 e_X(0, x) &= e_X(0 * 0, x) = e_X(0, 0 \nearrow x) \\
 &= e_X(0, 1) = e_X(0, 0 \nearrow 0) = e_X(0 * 0, 0) = e_X(0, 0) \\
 &= e_X(0 * x, x) = e_X(x, 0 \nearrow x) = e_X(x, 1) \\
 &= e_X(0 * y, x) = e_X(0, y \nearrow x) = e_X(0, x) = e_X(y, 1) \\
 &= e_X(0 * 1, x) = e_X(1, 0 \nearrow x) = e_X(1, 1) \\
 &= e_X(1, x \nearrow x) = e_X(1 * x, x) = e_X(x, x) \\
 &= e_X(1, y \nearrow y) = e_X(1 * y, y) = e_X(y, y) = 1, \\
 \\
 e_X(x, y) &= e_X(y * x, y) = e_X(x, y \nearrow y) = e_X(x, 1) \\
 &= e_X(x, 0 \nearrow x) = e_X(x * 0, x) = e_X(0, x) \\
 &= e_X(x * y, y) = e_X(y, x \nearrow y) = e_X(y, 1) \\
 &= e_X(y, 0 \nearrow y) = e_X(0 * y, y) = e_X(0, y) = 1, \\
 \\
 e_X(x, 0) &= e_X(x * y, 0) = e_X(x, y \nearrow 0) \\
 &= e_X(y * x, 0) = e_X(y, x \nearrow 0) = e_X(y, x) \\
 &= e_X(1 * x, 0) = e_X(1, x \nearrow 0) = e_X(1, x) \\
 &= e_X(x * 1, 0) = e_X(x, 1 \nearrow 0) = \frac{1}{2}.
 \end{aligned}$$

Similarly, $e_X(y, 0) = e_X(1, 0) = \frac{1}{2}$. Then we get for all $x, y, z \in X$,

$$e_X(x, y) \odot e_X(y, z) \leq e_X(x, z) \text{ and } e_X(x * y, z) = e_X(x, y \nearrow z) \leq e_X(x, z).$$

Thus (X, e_X) is an r-fuzzy poset. Put $P(x, y, z) = e_X(x * y, z)$. Then (X, e_X, P) is an sqc-frame.

(b) Define $e_X : X \times X \rightarrow L$ as follows as

$$e(x, y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$$

Since $1 = e_X(x, x) = e_X(y * x, x) \neq e_X(y, x \rightarrow x) = e_X(y, 1) = 0$, (X, e_X) is not an r-fuzzy poset.

Definition 3.5. (1) A subset $\tau_X \subset L^X$ is called an *Alexandrov topology* on X , if it satisfies the following conditions:

- (O1) $\alpha_X \in \tau_X$,
- (O2) if $A_i \in \tau_X$ for all $i \in I$, then $\bigvee_{i \in I} A_i, \bigwedge_{i \in I} A_i \in \tau_X$,
- (O3) if $A \in \tau_X$ and $\alpha \in L$, then $\alpha \odot A, \alpha \rightarrow A \in \tau_X$.

The pair (X, τ_X) is called an *Alexandrov topological space*.

Remark 3.6. Let $\tau_X \subset L^X$. Define $e_{\tau_X} : \tau_X \times \tau_X \rightarrow L$ as $e_{\tau_X}(A, B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x))$. Then (τ_X, e_{τ_X}) is a fuzzy poset.

Theorem 3.7. Let (X, e_X) be a fuzzy poset. Define

$$\tau_{e_X} = \{A \in L^X \mid A(x) \odot e_X(x, y) \leq A(y)\}.$$

Then τ_{e_X} is an Alexandrov topology on X such that

$$\tau_{e_X} = \left\{ \bigvee_{x \in X} (A(x) \odot e_X(x, -)) \mid A \in L^X \right\}.$$

Proof. Since $\alpha_X(x) \odot e_X(x, y) \leq \alpha_X(y)$, we have $\alpha_X \in \tau_{e_X}$.

If $A_i \in \tau_{e_X}$ for all $i \in I$, then

$$\begin{aligned} (\bigwedge_{i \in I} A_i)(x) \odot e_X(x, y) &\leq \bigwedge_{i \in I} (A_i \odot e_X(x, y)) \leq \bigwedge_{i \in I} A_i(y), \\ (\bigvee_{i \in I} A_i)(x) \odot e_X(x, y) &= \bigvee_{i \in I} (A_i(x) \odot e_X(x, y)) \leq \bigvee_{i \in I} A_i(y). \end{aligned}$$

Thus $\bigwedge_{i \in I} A_i, \bigvee_{i \in I} A_i \in \tau_{e_X}$.

If $A \in \tau_{e_X}$ and $\alpha \in L$, then $\alpha \odot (\alpha \rightarrow A(x)) \odot e_X(x, y) \leq A(x) \odot e_X(x, y) \leq A(y)$. Thus $(\alpha \rightarrow A(x)) \odot e_X(x, y) \leq (\alpha \rightarrow A(y))$. So $\alpha \rightarrow A \in \tau_{e_X}$. Easily, $\alpha \odot A \in \tau_{e_X}$. Hence τ_{e_X} is an Alexandrov topology on X .

Put $\tau = \{\bigvee_{x \in X} (A(x) \odot e_X(x, -)) \mid A \in L^X\}$. Let $A \in \tau_{e_X}$. Then we have

$$\bigvee_{x \in X} (A(x) \odot e_X(x, -)) = A \in \tau.$$

If $\bigvee_{x \in X} (A(x) \odot e_X(x, -)) \in \tau$, then we get

$$\bigvee_{x \in X} (A(x) \odot e_X(x, z)) \odot e_X(z, y) \leq \bigvee_{x \in X} (A(x) \odot e_X(x, y)).$$

Thus $\bigvee_{x \in X} (A(x) \odot e_X(x, -)) \in \tau_{e_X}$. So $\tau_{e_X} = \tau$. $\square$

Theorem 3.8. Let (X, e_X, P) be a cq-frame. For $A, B \in \tau_{e_X}$, we define

$$\begin{aligned} (A \otimes B)(z) &= \bigvee_{x, y \in X} (P(x, y, z) \odot A(x) \odot B(y)), \\ (A \Rightarrow B)(x) &= \bigwedge_{y, z \in X} ((P(x, y, z) \odot A(y)) \rightarrow B(z)). \end{aligned}$$

Then we have :

- (1) $(\tau_{e_X}, \vee, \wedge, \otimes, \Rightarrow, 0_X, 1_X)$ is a c-quantale,
- (2) if (X, e_X, P) be an scq-frame, then it is an sc-quantale.

Proof. (1) For $A, B \in \tau_{e_X}$, by (P1),

$$\begin{aligned} (A \otimes B)(z) \odot e_X(z, y) &= \bigvee_{x_0, y_0} (P(x_0, y_0, z) \odot A(x_0) \odot B(y_0) \odot e_X(z, y)) \\ &\leq \bigvee_{x_0, y_0} (P(x_0, y_0, y) \odot A(x_0) \odot B(y_0)) \\ &= (A \otimes B)(y). \end{aligned}$$

Then $A \otimes B \in \tau_{e_X}$.

For $A, B \in \tau_{e_X}$, also by (P1),

$$\begin{aligned} (A \Rightarrow B)(x) &= \bigwedge_{y, z \in X} (P(x, y, z) \odot A(y) \rightarrow B(z)) \\ &\leq \bigwedge_{y, z \in X} (e_X(x, x_0) \odot P(x_0, y, z) \odot A(y) \rightarrow B(z)) \\ &= e_X(x, x_0) \rightarrow \bigwedge_{y, z \in X} (P(x_0, y, z) \odot A(y) \rightarrow B(z)) \\ &= e_X(x, x_0) \rightarrow (A \Rightarrow B)(x_0). \end{aligned}$$

Thus $A \Rightarrow B \in \tau_{e_X}$.

For $A, B, C \in \tau_{e_X}$, we have

$$\begin{aligned} &((A \otimes B) \otimes C)(w) \\ &= \bigvee_{p, z \in X} (P(z, p, w) \odot (A \otimes B)(z) \odot C(p)) \\ &= \bigvee_{p, z \in X} (P(z, p, w) \odot (\bigvee_{x, y \in X} (A(x) \odot B(y) \odot P(x, y, z)) \odot C(p)) \\ &= \bigvee_{p, z \in X} \bigvee_{x, y \in X} (P(z, p, w) \odot P(x, y, z) \odot A(x) \odot B(y) \odot C(p)) \end{aligned}$$

$$\begin{aligned}
 &\leq \bigvee_{p \in X} \bigvee_{x, y \in X} \bigvee_{u \in X} \left(P(y, p, u) \odot P(x, u, w) \odot A(x) \odot B(y) \odot C(p) \right) \text{ [By (P3)]} \\
 &= \bigvee_{p, y \in X} \bigvee_{x, u \in X} \left(P(y, p, u) \odot P(x, u, w) \odot A(x) \odot B(y) \odot C(p) \right) \\
 &= \bigvee_{x, u \in X} \left(P(x, u, w) \odot A(x) \odot \bigvee_{p, y \in X} (P(y, p, u) \odot B(y) \odot C(p)) \right) \\
 &= \bigvee_{x, u \in X} \left(P(x, u, w) \odot A(x) \odot (B \otimes C)(u) \right) \\
 &= (A \otimes (B \otimes C))(w), \\
 &\quad (A \otimes (B \otimes C))(w) \\
 &= \bigvee_{p, z \in X} \left(P(z, p, w) \odot A(z) \odot (B \otimes C)(p) \right) \\
 &= \bigvee_{p, z \in X} \left(P(z, p, w) \odot A(z) \odot \bigvee_{x, y \in X} (P(x, y, p) \odot B(x) \odot C(y)) \right) \\
 &= \bigvee_{p, z \in X} \bigvee_{x, y \in X} \left(P(z, p, w) \odot P(x, y, p) \odot A(z) \odot B(x) \odot C(y) \right) \\
 &\leq \bigvee_{z \in X} \bigvee_{x, y \in X} \bigvee_{u \in X} \left(P(z, x, u) \odot P(u, y, w) \odot A(z) \odot B(x) \odot C(y) \right) \text{ [By (P4)]} \\
 &= \bigvee_{y \in X} \bigvee_{u \in X} \left(\left(\bigvee_{z, x \in X} (P(z, x, u) \odot A(z) \odot B(x)) \right) \odot P(u, y, w) \odot C(y) \right) \\
 &= \bigvee_{y \in X} \bigvee_{u \in X} \left((A \otimes B)(u) \odot P(u, y, w) \odot C(y) \right) \\
 &= ((A \otimes B) \otimes C)(w),
 \end{aligned}$$

$$\begin{aligned}
 &(A \otimes B)(z) \leq C(z) \\
 &\Leftrightarrow \bigvee_{x, y \in X} (P(x, y, z) \odot A(x) \odot B(y)) \leq C(z) \\
 &\Leftrightarrow A(x) \leq \bigwedge_{y, z \in X} (P(x, y, z) \odot B(y)) \rightarrow C(z) \\
 &\Leftrightarrow A(x) \leq (B \Rightarrow C)(x).
 \end{aligned}$$

Since $A \otimes B_i \leq A \otimes \bigvee_{i \in \Gamma} B_i$, we have $\bigvee_{i \in \Gamma} (A \otimes B_i) \leq A \otimes \bigvee_{i \in \Gamma} B_i$. Since $A \otimes B_i \leq A \otimes B_i$ iff $B_i \leq A \Rightarrow (A \otimes B_i)$, then

$$\bigvee_{i \in \Gamma} B_i \leq \bigvee_{i \in \Gamma} (A \Rightarrow (A \otimes B_i)) \leq A \Rightarrow \bigvee_{i \in \Gamma} (A \otimes B_i).$$

Thus $\bigvee_{i \in \Gamma} (A \otimes B_i) \geq A \otimes \bigvee_{i \in \Gamma} B_i$. So $\bigvee_{i \in \Gamma} (A \otimes B_i) = A \otimes \bigvee_{i \in \Gamma} B_i$. Hence $(\tau_{e_X}, \vee, \wedge, \otimes, \Rightarrow, 0_X, 1_X)$ is a c-quantale.

$$\begin{aligned}
 (2) \text{ By (P6), } P(x, y, z) &\leq e_X(y, z). \text{ For } A \in \tau_{e_X}, \text{ since } A(x) \odot e_X(x, y) \leq A(y), \\
 (1_X \otimes A)(z) &= \bigvee_{x, y \in X} (P(x, y, z) \odot 1_X(x) \odot A(y)) \\
 &= \bigvee_{x, y \in X} (P(x, y, z) \odot A(y)) \\
 &\leq \bigvee_{y \in X} (e_X(y, z) \odot A(y)) \\
 &\leq A(z),
 \end{aligned}$$

$$\begin{aligned}
 A(z) &= \bigvee_{u \in X} (P(u, z, z) \odot A(z)) \text{ [By (P5)]} \\
 &\leq \bigvee_{u, y \in X} (P(u, y, z) \odot 1_X(u) \odot A(y)) \\
 &= (1_X \otimes A)(z).
 \end{aligned}$$

Then $(\tau_{e_X}, \vee, \wedge, \otimes, \Rightarrow, 0_X, 1_X)$ is an sc-quantale. $\square$

Theorem 3.9. Let $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be a c-quantale with an r -fuzzy poset (X, e_X) . For $A, B \in \tau_{e_X}$, we define

$$\begin{aligned}
 (A \otimes B)(z) &= \bigvee_{x, y \in X} (e_X(x * y, z) \odot A(x) \odot B(y)), \\
 (A \Rightarrow B)(x) &= \bigwedge_{y, z \in X} ((e_X(x * y, z) \odot A(y)) \rightarrow B(z)).
 \end{aligned}$$

Then (1) $(\tau_{e_X}, \vee, \wedge, \otimes, \Rightarrow, 0_X, 1_X)$ is a c-quantale, where

$$\begin{aligned} (A \otimes B)(z) &= \bigvee_{y \in X} A(y \nearrow z) \odot B(y) = \bigvee_{x \in X} A(x) \odot B(x \nearrow z), \\ (A \Rightarrow B)(x) &= \bigwedge_{z \in X} (A(x \nearrow z) \rightarrow B(z)) = \bigwedge_{y \in X} (A(y) \rightarrow B(x * y)). \end{aligned}$$

(2) If it is an sc-quantale and $e_X(x * y, z) \leq e_X(x, z) \wedge e_X(y, z)$ for each $x, y, z \in X$, then $(\tau_{e_X}, \vee, \wedge, \otimes, \Rightarrow, 0_X, 1_X)$ is an sc-quantale.

Proof. (1) Put $P(x, y, z) = e_X(x * y, z)$. By Theorem 3.3, (X, e_X, P) is a cq-frame. Since $\bigvee_{x \in X} (e_X(x * y, z) \odot A(x)) = \bigvee_{x \in X} (e_X(x, y \nearrow z) \odot A(x)) = A(y \nearrow z)$ for $A \in \tau_{e_X}$ and $\bigvee_{y \in X} (e_X(x * y, z) \odot B(y)) = \bigvee_{y \in X} (e_X(y, x \nearrow z) \odot B(y)) = B(x \nearrow z)$ for $A \in \tau_{e_X}$,

$$\begin{aligned} (A \otimes B)(z) &= \bigvee_{x, y \in X} e_X(x * y, z) \odot A(x) \odot B(y) \\ &= \bigvee_{y \in X} A(y \nearrow z) \odot B(y) \\ &= \bigvee_{x \in X} A(x) \odot B(x \nearrow z). \end{aligned}$$

Since $\bigvee_{x \in X} (e_X(x * y, z) \odot A(x)) = \bigvee_{x \in X} (e_X(x, y \nearrow z) \odot A(x)) = A(y \nearrow z)$ for $A \in \tau_{e_X}$ and $\bigvee_{y \in X} (e_X(x * y, z) \odot B(y)) = \bigvee_{y \in X} (e_X(y, x \nearrow z) \odot B(y)) = B(x \nearrow z)$ for $A \in \tau_{e_X}$,

$$\begin{aligned} (A \Rightarrow B)(x) &= \bigwedge_{y, z \in X} (e_X(x * y, z) \odot A(y) \rightarrow B(z)) \\ &= \bigwedge_{z \in X} (\bigvee_{y \in X} (e_X(y, x \nearrow z) \odot A(y)) \rightarrow B(z)) \\ &= \bigwedge_{z \in X} (A(x \nearrow z) \rightarrow B(z)), \end{aligned}$$

$$\begin{aligned} (A \Rightarrow B)(x) &= \bigwedge_{y \in X} (A(y) \rightarrow \bigwedge_{z \in X} (e_X(x * y, z) \rightarrow B(z))) \\ &= \bigwedge_{y \in X} (A(y) \rightarrow B(x * y)). \end{aligned}$$

(2) It follows from Theorem 3.8 (2). $\square$

Example 3.10. Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a c-quantale with an r-fuzzy poset (L, e_L) and $e_L(x, y) = x \rightarrow y$. For $A, B \in \tau_{e_L}$, we define

$$\begin{aligned} (A \otimes B)(z) &= \bigvee_{x, y \in L} ((x \odot y) \rightarrow z) \odot A(x) \odot B(y) \\ &= \bigvee_{y \in L} A(y \rightarrow z) \odot B(y) \\ &= \bigvee_{x \in L} A(x) \odot B(x \rightarrow z), \end{aligned}$$

$$\begin{aligned} (A \Rightarrow B)(x) &= \bigwedge_{y, z \in L} ((x \odot y) \rightarrow z) \odot A(y) \rightarrow B(z) \\ &= \bigwedge_{z \in L} (A(x \rightarrow z) \rightarrow B(z)) \\ &= \bigwedge_{y \in L} (A(y) \rightarrow B(x \odot y)). \end{aligned}$$

Then $(\tau_{e_L}, \vee, \wedge, \otimes, \Rightarrow, 0_L, 1_L)$ is a c-quantale.

Example 3.11. Let $(P(X), e, P)$ be a cq-frame from Example 3.4 (3). Let $\tau_e = \{\alpha \in L^{P(X)} \mid \alpha(A) \odot e(A, B) \leq \alpha(B)\} = \{\alpha \in L^{P(X)} \mid \alpha(A) \leq \alpha(B), \text{ for } A \subset B\}$. For $\alpha, \beta \in \tau_e$, we define

$$\begin{aligned} (\alpha \otimes \beta)(C) &= \bigvee_{A, B \in P(X)} (P(A, B, C) \odot \alpha(A) \odot \beta(B)) \\ &= \bigvee_{A \cap B \subset C} (\alpha(A) \odot \beta(B)) \\ &= \bigvee_{B \in P(X)} (\alpha(B^c \cup C) \odot \beta(B)) \\ &= \bigvee_{A \in P(X)} (\alpha(A) \odot \beta(A^c \cup C)), \end{aligned}$$

$$\begin{aligned} (\alpha \Rightarrow \beta)(A) &= \bigwedge_{B, C \in P(X)} (P(A, B, C) \odot \alpha(B) \rightarrow \beta(C)) \\ &= \bigwedge_{A \subset B^c \cup C} (\alpha(B) \rightarrow \beta(C)) \\ &= \bigwedge_{C \in P(X)} (\alpha(A^c \cup C) \rightarrow \beta(C)) \end{aligned}$$

$$= \bigwedge_{B \in P(X)} (\alpha(B) \rightarrow \beta(A \cap B)).$$

Then by Theorem 3.8, $(\tau_e, \vee, \wedge, \otimes, \Rightarrow, 0_{P(X)}, 1_{P(X)})$ is a c-quantale, where $0_{P(X)}(A) = 0, 1_{P(X)}(A) = 1$ for each $A \in P(X)$.

Example 3.12. (1) Define $(A \rightarrow B)(x) = A(x) \rightarrow B(x)$ and $(A \odot B)(x) = A(x) \odot B(x)$ for each $x \in X$. Then $(L^X, \vee, \wedge, \odot, \rightarrow, 0_X, 1_X)$ is a sc-quantale.

(2) Let (L^X, e_{L^X}, P) be a cq-frame where $P(A, B, C) = \bigwedge_{x \in X} (A(x) \odot B(x) \rightarrow C(x))$ from Example 3.4 (4). Let $\tau_{e_{L^X}} = \{\alpha \in L^{L^X} \mid \alpha(A) \odot e_{L^X}(A, B) \leq \alpha(B)\}$. For $\alpha, \beta \in \tau_{e_{L^X}}$, we define

$$\begin{aligned} (\alpha \otimes \beta)(C) &= \bigvee_{A, B \in L^X} (P(A, B, C) \odot \alpha(A) \odot \beta(B)) \\ &= \bigvee_{B \in L(X)} (\alpha(B \rightarrow C) \odot \beta(B)) \\ &= \bigvee_{A \in L(X)} (\alpha(A) \odot \beta(A \rightarrow C)), \end{aligned}$$

$$\begin{aligned} (\alpha \Rightarrow \beta)(A) &= \bigwedge_{B, C \in L^X} (P(A, B, C) \odot \alpha(B) \rightarrow \beta(C)) \\ &= \bigwedge_{A \leq B \rightarrow C} (\alpha(B) \rightarrow \beta(C)) \\ &= \bigwedge_{C \in L^X} (\alpha(A \rightarrow C) \rightarrow \beta(C)) \\ &= \bigwedge_{B \in L^X} (\alpha(B) \rightarrow \beta(A \odot B)). \end{aligned}$$

Then by Theorem 3.8, $(\tau_{e_{L^X}}, \vee, \wedge, \otimes, \Rightarrow, 0_{L^X}, 1_{L^X})$ is a c-quantale, where $0_{L^X}(A) = 0, 1_{L^X}(A) = 1$ for each $A \in L^X$.

Theorem 3.13. Let $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be a sc-quantale and e_X is an r -fuzzy poset. For $A, B \in L^X$, we define $A \otimes B \in L^X$ as

$$(A \otimes B)(x) = \bigvee_{y, z \in X} (A(y) \odot B(z) \odot e_X(y * z, x)).$$

Then the following properties hold.

(1) If $A, B \in L^X$ with $B(1) = 1$, then $A \otimes B \geq A$ and $A \otimes E = A$ for $E(1) = 1, E(x) = 0$, otherwise. Moreover, $(L^X, \vee, \wedge, \otimes, 0_X, 1_X)$ is a c-quantale but not an sc-quantale.

(2) If $A, B \in M(L^X)$ and $e_X(a, b) \odot e_X(c, d) \leq e_X(a * c, b * d)$, where

$$M(L^X) = \{A \in L^X \mid A(a * b) \geq A(a) \odot A(b)\},$$

then $A \otimes B \in M(L^X)$.

(3) If $A \in M(L^X) \cap \tau_{e_X}$ and $e_X(a, b) \odot e_X(c, d) \leq e_X(a * c, b * d)$, then $A \otimes A \leq A$.

Proof. (1) For $A, B \in L^X$ with $B(1) = 1$,

$$\begin{aligned} (A \otimes B)(a) &= \bigvee_{x, y \in X} (A(x) \odot B(y) \odot e_X(x * y, a)) \\ &\geq A(a) \odot B(1) \odot e_X(a * 1, a) \\ &= A(a). \end{aligned}$$

Moreover, $(A \otimes E)(a) = A(a) \odot E(1) \odot e_X(a * 1, a) = A(a)$ for $E(1) = 1, E(x) = 0$, otherwise. But $E \neq 1_X$, that is, E is not a top element. Then $(L^X, \vee, \wedge, \otimes, 0_X, 1_X)$ is not an sc-quantale.

(2) For $A, B \in M(L^X)$,

$$\begin{aligned} &(A \otimes B)(a) \odot (A \otimes B)(b) \\ &= \left(\bigvee_{x, y \in X} A(x) \odot B(y) \odot e_X(x * y, a) \right) \odot \left(\bigvee_{z, w \in X} A(z) \odot B(w) \odot e_X(z * w, b) \right) \\ &= \bigvee_{x, y, z, w \in X} \left(A(x) \odot A(z) \odot B(y) \odot B(w) \odot e_X(x * y, a) \odot e_X(z * w, b) \right) \end{aligned}$$

$$\begin{aligned}
 &\leq \bigvee_{x,y,z,w \in X} \left(A(x * z) \odot B(y * w) \odot e_X(x * y * z * w, a * b) \right) \\
 &\leq (A \otimes B)(a * b). \\
 (3) \text{ For } A \in M(L^X) \cap \tau_{e_X}, \\
 &\quad (A \otimes A)(a) = \bigvee_{x,y \in X} (A(x) \odot A(y) \odot e_X(x * y, a)) \\
 &\quad \leq \bigvee_{x,y \in X} (A(x * y) \odot e_X(x * y, a)) \\
 &\quad \leq A(a). \quad \square
 \end{aligned}$$

Theorem 3.14. Let $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be a c -quantale and e_X is an r -fuzzy poset on X . Define $P : L^X \times L^X \times L^X \rightarrow L$ as follows:

$$\begin{aligned}
 P(A, B, C) &= \bigwedge_{x \in X} ((A \otimes B)(x) \rightarrow C(x)), \\
 (A \otimes B)(x) &= \bigvee_{y,z \in X} (A(y) \odot B(z) \odot e_X(y * z, x)).
 \end{aligned}$$

Then

- (1) (L^X, e_{L^X}, P) is a cq -frame,
- (2) if $(X, \wedge, \vee, *, \nearrow, 0, 1)$ is a sc -quantale and $P : W(L^X) \times W(L^X) \times W(L^X) \rightarrow L$, where $W(L^X) = \{A \in L^X \mid A(1) = 1\}$, then $(W(L^X), e_{W(L^X)}, P)$ is an scq -frame,
- (3) if $P : \tau_{e_X} \times \tau_{e_X} \times \tau_{e_X} \rightarrow L$, then $(\tau_{e_X}, e_{\tau_{e_X}}, P)$ is a cq -frame.

Proof. (1) (P1) For $A, B, C, A', B', C' \in L^X$,

$$\begin{aligned}
 &A'(y) \odot B'(z) \odot e_X(y * z, x) \odot (A'(y) \rightarrow A(y)) \odot (B'(z) \rightarrow B(z)) \\
 &\quad \odot (A(y) \odot B(z) \odot e_X(y * z, x) \rightarrow C(x)) \odot (C(x) \rightarrow C'(x)) \\
 &\leq A(y) \odot B(z) \odot e_X(y * z, x) \odot (A(y) \odot B(z) \\
 &\quad \odot e_X(y * z, x) \rightarrow C(x)) \odot (C(x) \rightarrow C'(x)) \leq C'(x).
 \end{aligned}$$

Then we have

$$\begin{aligned}
 &(A'(y) \rightarrow A(y)) \odot (B'(z) \rightarrow B(z)) \odot (A(y) \odot B(z) \\
 &\quad \odot e_X(y * z, x) \rightarrow C(x)) \odot (C(x) \rightarrow C'(x)) \\
 &\leq (A'(y) \odot B'(z) \odot e_X(y * z, x)) \rightarrow C'(x).
 \end{aligned}$$

Thus we get

$$e_{L^X}(A', A) \odot e_{L^X}(B', B) \odot P(A, B, C) \odot e_{L^X}(C, C') \leq P(A', B', C').$$

(P2) For $A, B, C \in L^X$,

$$\begin{aligned}
 P(A, B, C) &= \bigwedge_{x \in X} ((A \otimes B)(x) \rightarrow C(x)) \\
 &= \bigwedge_{x \in X} ((B \otimes A)(x) \rightarrow C(x)) \\
 &= P(B, A, C).
 \end{aligned}$$

(P3) For $A, B, C, B', C' \in L^X$,

$$\begin{aligned}
 &P(A, B, C) \odot P(C, B', C') \\
 &\leq \left(A(a) \odot B(b) \odot e_X(a * b, c) \rightarrow C(c) \right) \odot \left(C(c) \odot B'(b') \odot e_X(c * b', c') \rightarrow C'(c') \right) \\
 &= \left(A(a) \odot B(b) \odot e_X(a * b, c) \rightarrow C(c) \right) \odot \left(C(c) \rightarrow (B'(b') \odot e_X(c * b', c') \rightarrow C'(c')) \right) \\
 &= \left(A(a) \odot B(b) \odot e_X(a * b, c) \right) \odot \left((B'(b') \odot e_X(c * b', c') \rightarrow C'(c')) \right) \\
 &\leq A(a) \odot B(b) \odot e_X(a * b, c) \odot B'(b') \odot e_X(c * b', c') \rightarrow C'(c').
 \end{aligned}$$

Since

$$\begin{aligned}
 e_X(a * w, c') \odot e_X(b * b', w) &= e_X(w, a \nearrow c') \odot e_X(b * b', w) \\
 &\leq e_X(b * b', a \nearrow c') \\
 &= e_X(a * b, b' \nearrow c') \odot e_X(b' * (b' \nearrow c'), c')
 \end{aligned}$$

$$\leq \bigvee_{c \in X} (e_X(a * b, c) \odot e_X(c * b', c')),$$

$$\begin{aligned} & P(A, B, C) \odot P(C, B', C') \\ & \leq \bigvee_{c \in X} \left(A(a) \odot B(b) \odot e_X(a * b, c) \odot B'(b') \odot e_X(c * b', c') \right) \rightarrow C'(c') \\ & \leq A(a) \odot B(b) \odot e_X(a * w, c') \odot B'(b') \odot e_X(b * b', w) \rightarrow C'(c'). \end{aligned}$$

Then we get

$$\begin{aligned} & P(A, B, C) \odot P(C, B', C') \\ & \leq \bigwedge_{b, b'} \left(A(a) \odot B(b) \odot e_X(a * w, c') \odot B'(b') \odot e_X(b * b', w) \rightarrow C'(c') \right) \\ & = A(a) \odot e_X(a * w, c') \odot \left(\bigvee_{b, b'} B(b) \odot B'(b') \odot e_X(b * b', w) \right) \rightarrow C'(c'). \end{aligned}$$

Put $U(w) = (B \otimes B')(w) = \bigvee_{b, b'} B(b) \odot B'(b') \odot e_X(b * b', w)$. Then

$$\begin{aligned} & P(A, B, C) \odot P(C, B', C') \\ & \leq \left(\bigwedge_w (U(w) \rightarrow U(w)) \odot \bigwedge_{c'} \left(\bigvee_{a, w} A(a) \odot e_X(a * w, c') \odot U(w) \rightarrow C'(c') \right) \right) \\ & \leq \bigvee_U P(B, B', U) \odot P(A, U, C'). \\ \text{(P4) For } A, B, C, A', C' \in L^X, \\ & P(A, B, C) \odot P(A', C, C') \\ & \leq \left(A(a) \odot B(b) \odot e_X(a * b, c) \rightarrow C(c) \right) \odot \left(A'(a') \odot C(c) \odot e_X(a' * c, c') \rightarrow C'(c') \right) \\ & = \left(A(a) \odot B(b) \odot e_X(a * b, c) \rightarrow C(c) \right) \odot \left(C(c) \rightarrow (A'(a') \odot e_X(a' * c, c') \rightarrow C'(c')) \right) \\ & \leq A(a) \odot B(b) \odot e_X(a * b, c) \odot A'(a') \odot e_X(a' * c, c') \rightarrow C'(c'). \end{aligned}$$

Since

$$\begin{aligned} e_X(a * a', w) \odot e_X(w * b, c') &= e_X(a * a', w) \odot e_X(w, b \nearrow c') \\ &\leq e_X(a * a', b \nearrow c') \\ &= e_X(a * b, a' \nearrow c') \odot e_X(a' * (a' \nearrow c'), c') \\ &\leq \bigvee_{c \in X} (e_X(a * b, c) \odot e_X(a' * c, c')), \end{aligned}$$

$$\begin{aligned} & P(A, B, C) \odot P(A', C, C') \\ & \leq \bigvee_{c \in X} \left(A(a) \odot B(b) \odot e_X(a * b, c) \odot A'(a') \odot e_X(a' * c, c') \right) \rightarrow C'(c') \\ & \leq A(a) \odot A'(a') \odot e_X(a * a', w) \odot B(b) \odot e_X(w * b, c') \rightarrow C'(c'). \end{aligned}$$

Then we get

$$\begin{aligned} & P(A, B, C) \odot P(A', C, C') \\ & \leq \bigwedge_{a, a'} \left(A(a) \odot A'(a') \odot e_X(a * a', w) \odot B(b) \odot e_X(w * b, c') \rightarrow C'(c') \right) \\ & = \left(\bigvee_{a, a'} \left(A(a) \odot A'(a') \odot e_X(a * a', w) \right) \odot B(b) \odot e_X(w * b, c') \right) \rightarrow C'(c'). \end{aligned}$$

Put $D(w) = (A \otimes A')(w) = \bigvee_{a, a'} (A(a) \odot A'(a') \odot e_X(a * a', w))$. Then

$$\begin{aligned} & P(A, B, C) \odot P(A', C, C') \\ & \leq \bigwedge_w (D(w) \rightarrow D(w)) \odot \bigwedge_{c'} \left(\bigvee_{b, w} (D(w) \odot B(b) \odot e_X(w * b, c')) \rightarrow C'(c') \right) \\ & \leq \bigvee_D P(A', A, D) \odot P(D, B, C'). \end{aligned}$$

(2) (P5) Since $E \otimes B = B$ for $B \in W(L^X)$ and $E(1) = 1, E(x) = 0$, otherwise,

$$\begin{aligned} \bigvee_A P(A, B, B) &= \bigvee_A \left(\bigwedge_{x \in X} ((A \otimes B)(x) \rightarrow B(x)) \right) \\ &\geq \bigwedge_{x \in X} ((E \otimes B)(x) \rightarrow B(x)) = 1. \end{aligned}$$

(P6) Since $A \otimes B \geq A$ and $A \otimes B \geq B$ for $A, B \in W(L^X)$ from Theorem 3.13(1), we have

$$P(A, B, C) \leq e_{W(L^X)}(A, C) \text{ and } P(A, B, C) \leq e_{W(L^X)}(B, C).$$

Then $P(A, B, C) \leq e_{W(L^X)}(A, C) \wedge e_{W(L^X)}(B, C)$. $\square$

Example 3.15. Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a c-quantale and $(X, \vee, \wedge, *, \nearrow, 0, 1)$ be an sc-quantale with a fuzzy partially order e_X and $P(x, y, z) = e_X(x * y, z)$ for each $x, y, z \in X$ in Example 3.4 (2).

(1) By Example 3.4 (2), (X, e_X, P) is a cq-frame.

$$\begin{aligned} (2) \quad \tau_{e_X} &= \{A \in L^X \mid A(x) \odot e_X(x, y) \leq A(y)\} \\ &= \{A \in L^X \mid \bigvee_{x \leq y} A(x) \leq A(y)\} \\ &= \{A \in L^X \mid A(x) \leq A(y), x \leq y\}. \end{aligned}$$

(3) By Theorem 3.9 (1), for $A, B \in \tau_{e_X}$,

$$\begin{aligned} (A \otimes B)(z) &= \bigvee_{x, y \in X} (P(x, y, z) \odot A(x) \odot B(y)) \\ &= \bigvee_{x * y \leq z} (A(x) \odot B(y)) \\ &= \bigvee_{y \in X} (A(y \nearrow z) \odot B(y)) \\ &= \bigvee_{x \in X} (A(x) \odot B(x \nearrow z)), \end{aligned}$$

$$\begin{aligned} (A \Rightarrow B)(x) &= \bigwedge_{y, z \in X} (P(x, y, z) \odot A(y) \rightarrow B(z)) \\ &= \bigwedge_{x \leq (y \nearrow z)} (A(y) \rightarrow B(z)) \\ &= \bigwedge_{z \in X} (A(x \nearrow z) \rightarrow B(z)) \\ &= \bigwedge_{y \in X} (A(y) \rightarrow B(x * y)). \end{aligned}$$

Then $(\tau_{e_X}, \vee, \wedge, \otimes, \Rightarrow, 0_X, 1_X)$ is a c-quantale.

(4) If $A, B \in \tau_{e_X}$ with $A(1) = B(1) = 1$, then

$$(A \otimes B)(z) = \bigvee_{x * y \leq z} (A(x) \odot B(y)) \geq A(z) \odot A(1) = A(z),$$

that is, $A \otimes B \geq A$, $A \otimes B \geq B$ and, for $E(1) = 1, E(x) = 0$, otherwise, $A \otimes E = A$ with $E \in \tau_{e_X}$.

(5) Let $A, B \in M(L^X)$, where $M(L^X) = \{A \in M(L^X) \mid A(a * b) \geq A(a) \odot A(b)\}$. Since $a \leq b$ and $c \leq d$, $a * c \leq b * d$. Then $e_X(a, b) \odot e_X(c, d) \leq e_X(a * c, b * d)$. Thus $A \otimes B \in M(L^X)$.

(6) If $A \in M(L^X) \cap \tau_{e_X}$, $e_X(a, b) \odot e_X(c, d) \leq e_X(a * c, b * d)$, then $A \otimes A \leq A$ from:

$$\begin{aligned} (A \otimes A)(a) &= \bigvee_{x * y \leq a} (A(x) \odot A(y)) \geq A(a) \odot A(1) = A(a), \\ (A \otimes A)(a) &= \bigvee_{x, y \in X} (A(x) \odot A(y) \odot e_X(x * y, a)) \\ &\leq \bigvee_{x, y \in X} (A(x * y) \odot e_X(x * y, a)) \leq A(a). \end{aligned}$$

If $A(1) = 1$, then $A \otimes A \geq A$ from:

$$(A \otimes A)(z) = \bigvee_{x * y \leq z} (A(x) \odot A(y)) \geq A(z) \odot A(1) = A(z).$$

(7) Put $W(L^X) = \{A \in L^X \mid A(1) = 1\}$. Define $P : W(L^X) \times W(L^X) \times W(L^X) \rightarrow L$ as follows:

$$P(A, B, C) = \bigwedge_{x \in X} ((A \otimes B)(x) \rightarrow C(x)).$$

(P5) Since $E \otimes B = B$ from (4),

$$\begin{aligned} \bigvee_A P(A, B, B) &= \bigvee_A \left(\bigwedge_{x \in X} ((A \otimes B)(x) \rightarrow B(x)) \right) \\ &\geq \bigwedge_{x \in X} ((E \otimes B)(x) \rightarrow B(x)) = 1. \end{aligned}$$

(P6) Since $A \otimes B \geq A$ and $A \otimes B \geq B$, we have

$$P(A, B, C) \leq e_{W(L^X)}(A, C) \text{ and } P(A, B, C) \leq e_{W(L^X)}(B, C).$$

Then $P(A, B, C) \leq e_{W(L^X)}(A, C) \wedge e_{W(L^X)}(B, C)$. Thus $(W(L^X), e_{W(L^X)}, P)$ is an scq-frame.

Example 3.16. Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ on $L = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ and $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be two sc-quantales with an r-fuzzy poset (X, e_X) as in Example 3.4 (5).

For $A = (0, \frac{1}{4}, \frac{1}{2}, 0)$, since $\frac{1}{2} = A(y) \odot e_X(y, 1) \not\leq A(1) = 0$, $A \notin \tau_{e_X}$. Put $\bar{A} = \bigvee_{x \in X} (A(x) \odot e_X(x, y)) = (0, \frac{1}{4}, \frac{1}{2}, \frac{1}{2})$. Then $\bar{A} \in \tau_{e_X}$.

For $B = (\frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{1}{4})$, Since $\frac{3}{4} = B(y) \odot e_X(y, 1) \not\leq B(1) = \frac{1}{4}$, $B \notin \tau_{e_X}$. Put $\bar{B} = \bigvee_{x \in X} (B(x) \odot e_X(x, y)) = (\frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4})$. Then $\bar{B} \in \tau_{e_X}$.

For $\bar{A}, \bar{B} \in \tau_{e_X}$,

$$\begin{aligned} (\bar{A} \otimes \bar{B})(z) &= \bigvee_{x \in X} (\bar{A}(x) \odot \bar{B}(x \nearrow z)) \\ &= \bigvee_{y \in X} (\bar{A}(y \nearrow z) \odot \bar{B}(y)) \\ &= (0, 0, \frac{1}{4}, \frac{1}{4}), \end{aligned}$$

$$\begin{aligned} (\bar{A} \Rightarrow \bar{B})(x) &= \bigwedge_{y \in X} (\bar{A}(y) \rightarrow \bar{B}(x \odot y)) \\ &= \bigwedge_{z \in X} (\bar{A}(x \nearrow z) \rightarrow \bar{B}(z)) \\ &= (1, 1, 1, 1), \end{aligned}$$

$$\begin{aligned} (\bar{B} \Rightarrow \bar{A})(x) &= \bigwedge_{y \in X} (\bar{B}(y) \rightarrow \bar{A}(x \odot y)) \\ &= \bigwedge_{z \in X} (\bar{B}(x \nearrow z) \rightarrow \bar{A}(z)) \\ &= (\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}). \end{aligned}$$

In Theorem 3.9 (2), $1 = e_X(x * x, 0) \not\leq e_X(x, 0) \wedge e_X(x, 0) = \frac{1}{2}$. For $1_X \in \tau_{e_X}$,

$$\bar{A} \otimes 1_X = \bigvee_{x \in X} (\bar{A}(x) \odot 1_X(x \nearrow z)) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \geq \bar{A},$$

$$\bar{B} \otimes 1_X = \bigvee_{x \in X} (\bar{B}(x) \odot 1_X(x \nearrow z)) = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}) \geq \bar{B}.$$

For $E = (1, 0, 0, 0)$, $\bar{E} = \bigvee_{x \in X} (E(x) \odot e_X(x, y)) = 1_X$, $\bar{B} \otimes E = \bar{B}$.

Then $(\tau_{e_X}, \vee, \wedge, \otimes, \Rightarrow, 0_X, 1_X)$ is a c-quantale but not a sc-quantale.

Example 3.17. Let $(X = [0, \infty], \leq_{op}, \vee_{op}, +, \wedge_{op}, \infty, 0)$ be an sc-quantale where $\leq_{op} = \geq, \vee_{op} = \wedge, \wedge_{op} = \vee$ and

$$\begin{aligned} x \nearrow y &= \bigvee_{op} \{z \in [0, \infty] \mid x + z \leq_{op} y\} \\ &= \bigwedge \{z \in [0, \infty] \mid z \geq -x + y\} \\ &= (y - x) \vee 0, \end{aligned}$$

$$e_X^0(x, y) = \begin{cases} 0 & \text{if } x \geq y \\ \infty & \text{if } x < y. \end{cases}$$

Define $P(x, y, z) = e_X^0(x + y, z)$ for each $x, y, z \in [0, \infty]$.

(1) By Theorem 3.3, $([0, \infty], e_X^0, P)$ is a cq-frame with (P5)

$$\bigvee_{x \in X} P(x, y, y) = \left(\bigvee_{op} \right)_{x \in X} e_X^0(x + y, y) \geq_{op} e_X^\infty(0 + y, y) = 0.$$

But $0 = e_X^0(2 + 3, 4) \not\leq_{op} e_X^0(2, 4) \wedge e_X^0(3, 4) = \infty$. Then $([0, \infty], e_X^0, P)$ is not an scq-frame.

$$\begin{aligned} (2) \quad \tau_{e_X^0} &= \{A \in [0, \infty]^{[0, \infty]} \mid A(x) + e_X^0(x, y) \leq_{op} A(y)\} \\ &= \{A \in [0, \infty]^{[0, \infty]} \mid \bigwedge_{x \geq y} A(x) \leq_{op} A(y)\} \\ &= \{A \in [0, \infty]^{[0, \infty]} \mid A(x) \geq A(y), x \geq y\}, \end{aligned}$$

because $A(x) \geq \bigwedge_{x \geq y} A(x) \geq A(y)$ for $y \leq x$.

(3) For $A, B \in \tau_{e_X^0}$, we define

$$\begin{aligned} (A \otimes B)(z) &= \left(\bigvee_{op} \right)_{x, y \in X} (P(x, y, z) + A(x) + B(y)) \\ &= \bigwedge_{x+y \geq z} (A(x) + B(y)) \\ &= \bigwedge_{y \in X} (A((z - y) \vee 0) + B(y)) \\ &= \bigwedge_{x \in X} (A(x) + B((z - x) \vee 0)), \end{aligned}$$

$$\begin{aligned} (A \Rightarrow B)(x) &= \left(\bigwedge_{op} \right)_{y, z \in X} (P(x, y, z) + A(y)) \nearrow B(z) \\ &= \bigvee_{x \geq (-y+z) \vee 0} ((B(z) - A(y)) \vee 0) \\ &= \bigvee_{z \in X} ((B(z) - A((z - x) \vee 0)) \vee 0) \\ &= \bigvee_{y \in X} ((B(x + y) - A(y)) \vee 0). \end{aligned}$$

Then $(\tau_{e_X^0}, \leq_{op} = \geq, \vee_{op}, \wedge_{op}, \otimes, \Rightarrow, \infty_X, 0_X)$ is a c-quantale where $0_X(x) = 0$ and $\infty_X(x) = \infty$ for each $x \in [0, \infty]$.

(4) If $A, B \in \tau_{e_X^0}$ with $A(0) = B(0) = 0$, then

$$(A \otimes B)(z) = \bigwedge_{x+y \geq z} (A(x) + B(y)) \leq A(z) + A(0) = A(z).$$

Thus $A \otimes B \geq_{op} A$, $A \otimes B \geq_{op} B$ and, for $E(0) = 0, E(x) = \infty$, otherwise, $A \otimes E = A$ with $E \in \tau_{e_X^0}$.

(5) Let $A, B \in M([0, \infty]^{[0, \infty]})$, where

$$M([0, \infty]^{[0, \infty]}) = \{A \in [0, \infty]^{[0, \infty]} \mid A(a + b) \geq_{op} A(a) + A(b)\}.$$

Since $a \geq b$ and $c \geq d$, $a + c \geq b + d$. Then $e_X^0(a, b) + e_X^0(c, d) \leq_{op} e_X^0(a + c, b + d)$. Thus $A \otimes B \in M([0, \infty]^{[0, \infty]})$.

(6) Let $A \in M([0, \infty]^{[0, \infty]}) \cap \tau_{e_X^0}$. Since $e_X^0(a, b) + e_X^0(c, d) \leq_{op} e_X^0(a + c, b + d)$, $A \otimes A \geq A$ from:

$$\begin{aligned} (A \otimes A)(a) &= \left(\bigvee_{op} \right)_{x, y \in X} (A(x) + A(y) + e_X^0(x + y, a)) \\ &\leq_{op} \left(\bigvee_{op} \right)_{x, y \in X} (A(x + y) + e_X^0(x + y, a)) \\ &\leq_{op} A(a). \end{aligned}$$

If $A(0) = 0$, then $A \otimes A \leq A$ from:

$$(A \otimes A)(a) = \bigwedge_{x+y \geq z} (A(x) + A(y)) \leq A(a) + A(0) = A(a).$$

(7) Put $W([0, \infty]^{[0, \infty]}) = \{A \in [0, \infty]^{[0, \infty]} \mid A(0) = 0\}$. Define

$$P : W([0, \infty]^{[0, \infty]}) \times W([0, \infty]^{[0, \infty]}) \times W([0, \infty]^{[0, \infty]}) \rightarrow [0, \infty]$$

as follows:

$$P(A, B, C) = (\bigwedge_{op, x \in X} ((A \otimes B)(x) \nearrow C(x))).$$

(P5) Since $E \otimes B = B$, from (4),

$$\begin{aligned} (\bigvee_{op})_A P(A, B, B) &= (\bigvee_{op})_A ((\bigwedge_{op})_{x \in X} ((A \otimes B)(x) \nearrow B(x))) \\ &\geq_{op} (\bigwedge_{op})_{x \in X} ((E \otimes B)(x) \nearrow B(x)) = 0. \end{aligned}$$

(P6) Since $A \otimes B \geq_{op} A$ and $A \otimes B \geq_{op} B$, we have

$$P(A, B, C) \leq_{op} e_{W([0, \infty]^{[0, \infty]})}(A, C) \text{ and } P(A, B, C) \leq_{op} e_{W([0, \infty]^{[0, \infty]})}(B, C).$$

Then we get

$$P(A, B, C) \leq_{op} e_{W([0, \infty]^{[0, \infty]})}(A, C) \wedge_{op} e_{W([0, \infty]^{[0, \infty]})}(B, C).$$

Thus $(W([0, \infty]^{[0, \infty]}), e_{W([0, \infty]^{[0, \infty]})}, P)$ is an scq-frame.

Example 3.18. Let $(P(X), \cap, \cup, \odot = \cap, \nearrow, \emptyset, X)$ be an sc-quantale, where

$$A \nearrow B = \bigcup \{C \in P(X) \mid A \cap C \subset B\} = A^c \cup B.$$

(1) Define a map $e^1 : P(X) \times P(X) \rightarrow P(X)$ as

$$e^1(A, B) = A \nearrow B = A^c \cup B.$$

Then $(P(X), e^1)$ is a fuzzy poset.

(2) $(P(X), e^1, P^1)$ is a cq-frame, where $P^1(A, B, C) = e^1(A \cap B, C)$ for each $A, B, C \in P(X)$ from Theorem 3.3. But it is not an sc-quantale frame, because $X = \{a, b, c\}, A = \{a, b\}, B = \{b, c\}$ and $P^1(A, B, A) = e^1(A \cap B, A) = X \not\subset e^1(A, A) \cap e^1(B, A) = \{a, b\}$.

(3) Let $\tau_{e^1} = \{\alpha \in P(X)^{P(X)} \mid \alpha(A) \cap e^1(A, B) \subset \alpha(B)\}$. For $\alpha, \beta \in \tau_{e^1}$, we define

$$\begin{aligned} (\alpha \otimes \beta)(C) &= \bigvee_{A, B \in P(X)} (e^1(A \cap B, C) \cap \alpha(A) \cap \beta(B)) \\ &= \bigvee_{B \in P(X)} (\alpha(B^c \cup C) \cap \beta(B)) \\ &= \bigvee_{A \in P(X)} (\alpha(A) \cap \beta(A^c \cup C)), \end{aligned}$$

$$\begin{aligned} (\alpha \Rightarrow \beta)(A) &= \bigwedge_{B, C \in P(X)} (e^1(A \cap B, C) \cap \alpha(B) \nearrow \beta(C)) \\ &= \bigwedge_{B \in P(X)} (\alpha(A^c \cup B) \nearrow \beta(B)) \\ &= \bigwedge_{C \in P(X)} (\alpha(C) \nearrow \beta(A \cap C)). \end{aligned}$$

Then $(\tau_{e^1}, \vee, \wedge, \otimes, \Rightarrow, 0_{P(X)}, 1_{P(X)})$ is a commutative co-quantale, where $0_{P(X)}(A) = \emptyset, 1_{P(X)}(A) = X$ for each $A \in P(X)$.

(4) If $\alpha, \beta \in \tau_{e^1}$ with $\alpha(X) = \beta(X) = X$, then

$$(\alpha \otimes \beta)(C) \supset e^1(C \cap X, C) \cap \alpha(C) \cap \beta(X) = \alpha(C).$$

For $\gamma(X) = X, \gamma(A) = \emptyset$, otherwise, we have $\gamma \in \tau_{e^1}$,

$$(\alpha \otimes \gamma)(C) = e^1(C \cap X, C) \cap \alpha(C) \cap \gamma(X) = \alpha(C).$$

Thus $\alpha \otimes \beta \supset \alpha, \alpha \otimes \beta \supset \beta$ and $\alpha \otimes \gamma = \alpha$.

(5) Let $\alpha, \beta \in M(P(X)^{P(X)})$, where

$$M(P(X)^{P(X)}) = \{\alpha \in P(X)^{P(X)} \mid \alpha(A \cap B) \supset \alpha(A) \cap \alpha(B)\}.$$

Then by Theorem 3.13 (2), $\alpha \otimes \beta \in M(P(X)^{P(X)})$.

- (6) If $\alpha \in M(P(X)^{P(X)}) \cap \tau_{e^1}$ and $\alpha(X) = X$, then $\alpha \otimes \alpha = \alpha$ from:
 $(\alpha \otimes \alpha)(C) \supset e^1(C \cap X, C) \cap \alpha(C) \cap \alpha(X) = \alpha(C),$

$$\begin{aligned} (\alpha \otimes \alpha)(C) &= \bigvee_{A, B \in P(X)} (e^1(A \cap B, C) \cap \alpha(A) \cap \alpha(B)) \\ &\subset \bigvee_{A, B \in P(X)} (e^1(A \cap B, C) \cap \alpha(A \cap B)) \\ &\subset \alpha(C). \end{aligned}$$

- (7) Put $W(\tau_{e^1}) = \{\alpha \in \tau_{e^1} \mid \alpha(X) = X\}$. Define $e_W : W(\tau_{e^1}) \times W(\tau_{e^1}) \rightarrow P(X)$ and $P : W(\tau_{e^1}) \times W(\tau_{e^1}) \times W(\tau_{e^1}) \rightarrow P(X)$ as follows:

$$\begin{aligned} e_W(\alpha, \beta) &= \bigwedge_{A \in P(X)} ((\alpha(A) \nearrow \beta(A))), \\ P(\alpha, \beta, \eta) &= \bigwedge_{A \in P(X)} ((\alpha \otimes \beta)(A) \nearrow \eta(A)) = e_W(\alpha \otimes \beta, \eta). \end{aligned}$$

- (P5) Since $\beta \otimes \gamma = \beta$, from (4),

$$\begin{aligned} \bigvee_{\alpha} P(\alpha, \beta, \beta) &= \bigvee_{\alpha} \left(\bigwedge_{A \in P(X)} ((\alpha \otimes \beta)(A) \nearrow \beta(A)) \right) \\ &\geq \bigwedge_{A \in P(X)} ((\gamma \otimes \beta)(A) \nearrow \beta(A)) = X. \end{aligned}$$

- (P6) Since $\alpha \otimes \beta \supset \alpha$ and $\alpha \otimes \beta \supset \beta$, we have

$$P(\alpha, \beta, \eta) \subset e_W(\alpha, \eta) \text{ and } P(\alpha, \beta, \eta) \subset e_W(\alpha, \eta).$$

Then $P(\alpha, \beta, \eta) \subset e_W(\alpha, \eta) \cap e_W(\beta, \eta)$. Thus $(W(P(X)^{P(X)}), e_W, P)$ is an scq-frame.

Funding: This work was supported by the Research Institute of Natural Science of Gangneung-Wonju National University.

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JU-MOK OH (jumokoh@gwnu.ac.kr)

Department of Mathematics, Faculty of Natural Science, University of Gangneung-Wonju, Gangneung, Gangwondo, 25457, Korea

YONG CHAN KIM (yck@gwnu.ac.kr)

Department of Mathematics, Faculty of Natural Science, University of Gangneung-Wonju, Gangneung, Gangwondo, 25457, Korea