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Abstract. In this paper, the concept of fuzzy Mengerness, fuzzy almost Mengerness and fuzzy nearly Mengerness introduced and studied. Every fuzzy Menger space is fuzzy nearly Menger space and every fuzzy nearly Menger space is fuzzy almost Menger space. We give some characterizations of almost Menger space in terms of fuzzy regular open sets or fuzzy regular closed sets.

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Corresponding Author: A. Haydar Eş (haydares@baskent.edu.tr)

1. Introduction

The concept of fuzzy set was introduced by Zadeh in his classic paper [1]. The theory of fuzzy topological spaces was introduced and developed by Chang [2]. In [3, 4, 5] some weaker forms of fuzzy compactness are considered for the 1st time.

The investigation of covering properties of topological spaces has a long history going back to papers by Hurewicz, Menger and Rothberger [6, 7, 8]. However more recently a new theory called Selection Principles was introduced by Scheepers [9]. The theory of Selection Principles has extraordinary connections with numerous subareas of mathematics, for example set theory and general topology, uniform structures and ditopological texture spaces [10, 11, 12].

In 1999, Kocinac defined and characterized the almost Menger property [11]. Following this concept Parvez and Khan defined and studied nearly Menger and nearly star-Menger spaces [13].

In this paper we are concerned with the weaker forms of the fuzzy Mengerness in fuzzy topological spaces.

2. Preliminaries

Let X be a nonempty set and $F(X) = \{\mu|\mu : X \to [0,1]\}$. The elements of $F(X)$ are called *fuzzy subsets* of X [1]. If $\mu \in F(X)$, then $\mu^{'} = 1 - \mu$. We denote by 0_X and 1_X, the functions on X identically equal 0 and 1 respectively. If $f : X \to Y$ be a function and $\mu_i \in F(Y)$ for all $i \in I$, then $f^{-1}(\bigvee_{i\in I} \mu_i) = \bigvee_{i\in I} f^{-1}(\mu_i)$ and $f^{-1}(\bigwedge_{i\in I} \mu_i) = \bigwedge_{i\in I} f^{-1}(\mu_i)$.

If $\mu \in F(X)$ and $\lambda \in F(Y)$, then $(\mu \times \lambda)(x,y) = min\{\mu(x), \lambda(y)\}$ for every $(x,y) \in X \times Y$.

Now, we recall that a fuzzy topology on X (See [2]) is a subset τ of $F(X)$ such that:

(i) $0_X \in \tau$ and $1_X \in \tau$,

(ii) if $\mu, \lambda \in \tau$, then $\mu \wedge \lambda \in \tau$,

(iii) if $\mu_i \in \tau$ for each $i \in I$, then $\bigvee_{i\in I} \mu_i \in \tau$.

If τ is fuzzy topology on X, then (X, τ) is called a *fuzzy topological space*.

Let (X, τ) be a fuzzy topological space. For a fuzzy set μ of X, the *closure* and the *interior* of μ (See [14]) are defined respectively as:

$$cl(\mu) = inf\left\{\lambda : \lambda \geq \mu, \lambda^{'} \in \tau\right\},$$

$$int(\mu) = sup\{\lambda : \lambda \leq \mu, \lambda \in \tau\}.$$

A fuzzy topological space X is product related to a fuzzy topological space Y iff for every $h \in F(X)$ and $k \in F(Y)$, whenever $f^{'} \not\geq h$ and $g^{'} \not\geq k$ implies $f^{'} \times 1_Y \vee 1_X \times g^{'} \geq h \times k$, where $f \in \tau_X$ and $g \in \tau_Y$, there exists $f_1 \in \tau_X$ and $g_1 \in \tau_Y$ such that $f_1^{'} \geq h$ or $g_1^{'} \geq k$ and $f_1^{'} \times 1_Y \vee 1_X \times g_1^{'} = f^{'} \times 1_Y \vee 1_X \times g^{'}$. If $f \in F(X), g \in F(Y)$ and X is product related to Y, then $cl(f \times g) = cl(f) \times cl(g)$ [14].

A collection $\{\mu_i\}_{i\in I}$ of fuzzy (open) sets of X is called a (*open*) *cover* of X, if $\bigvee_{i\in I} \mu_i = 1_X$. A fuzzy topological space is said to be *compact*, if every open cover has a finite subcover (See [2]).

A fuzzy set μ is called *regularly open*, if $\mu = int(cl(\mu))$. A fuzzy set λ is called *regularly closed*, if $\lambda = cl(int(\lambda))$. A function $f : X \to Y$ is called *almost continuous*, if the preimage of each regularly open fuzzy set of Y is a fuzzy open set of X. A fuzzy topological space X is called *regular*, if each fuzzy open set μ of X is a join of fuzzy open sets λ_i of X such that $cl(\lambda_i) \leq \mu$ for each i [14]. The definitions of fuzzy weak continuity, fuzzy strong continuity, almost compactness, near compactness and other concepts can be found in [3, 4, 14].

Recall that a topological space (X, τ) is called *Menger* (resp. *almost Menger*),if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exists a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of finite families such that for each $n, \mathcal{V}_n \subset \mathcal{U}_n$ and $\cup_{n\in\mathbb{N}}\mathcal{V}_n$ (resp. $\cup_{n\in\mathbb{N}}\mathcal{V}_{\mathrm{n}}^{'}$ where $\mathcal{V}_{\mathrm{n}}^{'} = \{cl(V) : V \in \mathcal{V}_n\}$) covers X (See [6, 7, 8]). A topological space (X, τ) is called *nearly Menger*, if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exists a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of finite families such that for each $n, \mathcal{V}_n \subset \mathcal{U}_n$ and $\cup_{n\in\mathbb{N}}\mathcal{V}^{'}_{\mathrm{n}}$ covers X, where $\mathcal{V}^{'}_n = \{int(cl(V)) : V \in \mathcal{V}_n\}$.

3. Almost Menger fuzzy topological spaces

We introduce and study fuzzy almost Mengerness for fuzzy topological spaces.

Definition 3.1. A fuzzy topological space (X, τ) is called:

(i) *fuzzy Menger*, if for any sequence $\{\mu_n : n \in \mathbb{N}\}$ of fuzzy open covers of X, there exists a sequence $\{\lambda_n : n \in \mathbb{N}\}$ of finite families of fuzzy sets in X such that for every $n \in \mathbb{N}, \lambda_n$ is a finite fuzzy subset of μ_n and $\bigvee_{n\in\mathbb{N}} \lambda_n = 1_X$.

(ii) *fuzzy almost Menger*, if for any sequence $\{\mu_n : n \in \mathbb{N}\}$ of fuzzy open covers of X, there exists a sequence $\{\lambda_n : n \in \mathbb{N}\}$ of finite families of fuzzy sets in X such that for every $n \in \mathbb{N}, \lambda_n$ is a finite fuzzy subset of μ_n and $\bigvee_{n\in\mathbb{N}} \lambda^{'}_n = 1_X$, where $\lambda^{'}_n = \{cl(\lambda) : \lambda \leq \lambda_n\}$.

We immediately note that every fuzzy Menger space is fuzzy almost Menger. But the reverse implication does not hold in general as the following example shows.

Example 3.2. Let $X = \{2, 3, 4, ...\}$ and τ be the fuzzy topology with subbase

$$\{\mu_n, \lambda_n : n = 2, 3, 4, ...\},$$

where

$$\lambda_n(m) = \begin{cases} 1, \text{if } m = n \\ \frac{1}{2} - \frac{1}{n}, \text{if } m \neq n \end{cases} \quad \text{for } n = 2, 3, 4, ...$$

$$\lambda_2(m) = \begin{cases} 0, \text{if } m = 2 \\ \frac{1}{2}, \text{if } m > n \end{cases}$$

$$\lambda_n(m) = \begin{cases} 0, \text{if } m \geq n \\ \frac{1}{2} + \frac{1}{n}, \text{if } m < n \end{cases} \quad \text{for } n = 3, 4, 5, ... \text{ [4]}.$$

Then we have

(1) $cl(\mu_n = 1_X - \lambda_n$, for $n = 2, 3, 4, ...,$

(2) $int(cl(\mu_2)) = \mu_2 \vee \lambda_2$,

(3) $int(cl(\mu_3)) = \mu_3 \vee \lambda_2$ and $int(cl(\mu_n)) = \mu_n$ for $n \geq 4$.

Thus (X, τ) is fuzzy almost Menger. But for the fuzzy open cover $\{\mu_n : n = 2, 3, ...\}$, we have $\{int(cl(\mu_n)) : n = 2, 3, 4, ...\} = \{\mu_2 \vee \lambda_2, \mu_3 \vee \lambda_2, \mu_n : n \geq 4\}$ which has no finite fuzzy subset of μ_n and $\bigvee_{n\in\mathbb{N}} \lambda_n \neq 1_X$. So (X, τ) is not fuzzy Menger.

Theorem 3.3. *A fuzzy topological space (X, τ) is fuzzy almost Menger if and only if for every sequence $\{\mu_n : n \in \mathbb{N}\}$ of fuzzy open subsets of X has the finite intersection property, we have $\bigwedge_{n\in\mathbb{N}} cl(\mu_n) \neq 0_X$.*

Proof. Let $\{\mu_n : n \in \mathbb{N}\}$ be a sequence of fuzzy open subsets of X having the finite intersection property. Assume that $\bigwedge_{n\in\mathbb{N}} cl(\mu_n) = 0_X$. Then we have $\bigvee_{n\in\mathbb{N}} int(\mu^{'}_n) = 1_X$. Since X is fuzzy almost Menger, for every $n \in \mathbb{N}$, there exists a sequence $\{\lambda_n : n \in \mathbb{N}\}$ such that λ_n is a finite subset of $int(\mu^{'}_n)$ and $\bigvee_{n\in\mathbb{N}} \lambda^{'}_n = 1_X$, where $\lambda^{'}_n = \{cl(\lambda) : \lambda \leq \lambda_n\}$. But from $\lambda_n \leq int(\mu^{'}_n)$ and $\mu_n \leq int(cl(\mu_n))$, we see that $\bigwedge_{n\in\mathbb{N}} \mu_n = 0_X$, which is a contradiction with the finite intersection property of $\{\mu_n : n \in \mathbb{N}\}$.

Conversely, suppose the necessary condition holds. We have to show that X is fuzzy almost Menger. Let $\{\mu_n : n \in \mathbb{N}\}$ be a fuzzy open cover of X. If $\bigvee_{n\in\mathbb{N}} \lambda_n^{'} \neq 1_X$, where $\lambda_n^{'} = \{cl(\lambda) : \lambda \leq \lambda_n\}$ and λ_n is a finite subset of μ_n, then $\left\{1 - \lambda_n^{'} : n \in \mathbb{N}\right\}$ is a fuzzy open sequence with the finite intersection property. Thus from the hypothesis, it follows that $\bigwedge_{n\in\mathbb{N}} cl(1-\lambda_n^{'}) \neq 0_X$ and then $\bigvee_{n\in\mathbb{N}}(1-cl(1-\lambda_n^{'})) \neq 1_X$. Since $\bigvee_{n\in\mathbb{N}} \mu_n \leq \bigvee_{n\in\mathbb{N}}(1-cl(1-\lambda_n^{'})) \neq 1_X$, $\bigvee_{n\in\mathbb{N}} \mu_n \neq 1_X$, which is a contradiction. □

Theorem 3.4. *A fuzzy almost Menger regular fuzzy topological space (X, τ) is fuzzy Menger.*

Proof. Let $\{\mu_n : n \in \mathbb{N}\}$ be a sequence of fuzzy open covers of X. From the regularity of X, it follows that $\mu_n = \bigvee_{n\in\mathbb{N}} \lambda_j^n$, where λ_j^n is a fuzzy open set such that $cl(\lambda_j^n) \leq \mu_n$. Since $\bigvee_{n\in\mathbb{N}} \mu_n = \bigvee_{n\in\mathbb{N}} \lambda_j^n = 1_X$, there exists a finite subset F of $\mathbb{N}$ such that $\bigvee_{n\in F} cl(\lambda_j^n) = 1_X$. But $\bigvee_{n\in F} cl(\lambda_j^n) = \bigvee_{n\in F} \mu_n$. Then $\bigvee_{n\in F} \mu_n = 1_X$. Thus (X, τ) is fuzzy Menger. □

Theorem 3.5. *If (X_1, τ_{X_1}), (X_2, τ_{X_2}) are fuzzy almost Menger fuzzy topological space and (X_1, τ_{X_1}) is product related to (X_2, τ_{X_2}), then their fuzzy topological product (X, τ) is fuzzy almost Menger.*

Proof. From the hypothesis, it follows that $cl(\lambda \times \mu) = cl(\lambda) \times cl(\mu)$ for every $\lambda \in F(X_1)$ and $\mu \in F(X_2)$. To show that the product space $X_1 \times X_2$ is fuzzy almost Menger, by Theorem 3.3, it is sufficient to prove that if

$$\{\lambda_n \times \mu_n : \lambda_n \in \tau_{X_1},\ \mu_n \in \tau_{X_2},\ n \in \mathbb{N}\}$$

is a sequence with the finite intersection property, then $\bigwedge_{n\in\mathbb{N}} cl(\lambda_n \times \mu_n) \neq 0_X$ holds. We observe that $\{\lambda_n : n \in \mathbb{N}\}$ and $\{\mu_n : n \in \mathbb{N}\}$ are fuzzy open sets family of X_1 and X_2 respectively with finite intersection property. Then $\bigwedge_{n\in\mathbb{N}} cl(\lambda_n) \neq 0_{X_1}$ and $\bigwedge_{n\in\mathbb{N}} cl(\mu_n) \neq 0_{X_2}$. Thus there exists $(x_1, x_2) \in X_1 \times X_2$ such that $\bigwedge_{n\in\mathbb{N}} cl(\lambda_n(x_1)) \neq 0_{X_1}$ and $\bigwedge_{n\in\mathbb{N}} cl(\mu_n(x_1)) \neq 0_{X_2}$. So

$$\begin{aligned}\bigwedge_{n\in\mathbb{N}} cl(\lambda_n \times \mu_n) &= \bigwedge_{n\in\mathbb{N}} (cl(\lambda_n(x_1))) \wedge (cl(\mu_n(x_2))) \\ &= \left(\bigwedge_{n\in\mathbb{N}} (cl(\lambda_n(x_1))\right) \wedge \left(\bigwedge_{n\in\mathbb{N}} (cl(\mu_n(x_2))\right) \\ &\neq 0_{X_1\times X_2}.\end{aligned}$$

□

Theorem 3.6. *In a fuzzy topological space (X, τ) the following conditions are equivalent:*

(1) *(X, τ) is fuzzy almost Menger,*

(2) *For every sequence $\{\mu_n : n \in \mathbb{N}\}$ of fuzzy regular closed sets such that $\wedge_{n\in\mathbb{N}}\mu_n = 0_X$, there exists a sequence $\{\lambda_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}, \lambda_n$ is finite subset of μ_n and $\wedge_{n\in\mathbb{N}}\lambda_n^{'} = 0_X$, where $\lambda_n^{'} = \{int(\lambda) : \lambda \leq \lambda_n\}$,*

(3) *$\bigwedge_{n\in\mathbb{N}} cl(\mu_n) \neq 0_X$ holds for every sequence of fuzzy regular open sets $\{\mu_n : n \in \mathbb{N}\}$ with the finite intersection property,*

(4) *For each sequence* $\{\mu_n : n \in \mathbb{N}\}$ *of fuzzy regular open covers of* X*, there exists a sequence* $\{\lambda_n : n \in \mathbb{N}\}$ *such that for each* $n \in \mathbb{N}, \lambda_n$ *is a finite fuzzy subset of* μ_n *and* $\bigvee_{n\in\mathbb{N}} \lambda_n^{'} = 1_X$*, where* $\lambda_n^{'} = \{cl(\lambda) : \lambda \leq \lambda_n\}$.

Proof. The proof of this theorem follows a similar pattern to Theorem 3.3. □

Theorem 3.7. *Let* (X, τ_1) *be a fuzzy almost Menger topological space and* (Y, τ_2) *be a fuzzy topological space. If* $f : X \to Y$ *is a fuzzy almost continuous surjection, then* (Y, τ_2) *is fuzzy almost Menger.*

Proof. Let $\{\mu_n : n \in \mathbb{N}\}$ be a sequence of covers of Y by fuzzy open sets. Then $\{int(cl(\mu_n)) : n \in \mathbb{N}\}$ is also a fuzzy open covers of Y. For every $n \in \mathbb{N}$ and $\lambda \leq \lambda_n$, we can choose a member $\mu_\lambda \leq \mu_n$ such that $\lambda = \mathrm{f}^{-1}(\mu_\lambda)$. From the almost continuity of f, it follows that $\left\{\mathrm{f}^{-1}(int(cl(\mu_n))) : n \in \mathbb{N}\right\}$ is a fuzzy open covers of X. Thus there is a sequence $\{\lambda_n : n \in \mathbb{N}\}$ such that λ_n is a finite subset of $\mathrm{f}^{-1}(int(cl(\mu_n)))$ and $1_X = \bigvee_{n\in\mathbb{N}} \lambda_n^{'}$, where $\lambda_n^{'} = \{cl(\lambda) : \lambda \leq \lambda_n\}$. From the surjectivity of f, we have

$$f\left(\bigvee_{n\in\mathbb{N}} cl(\mathrm{f}^{-1}(int(cl(\mu_\lambda))))\right) = \bigvee_{n\in\mathbb{N}} f(cl(\mathrm{f}^{-1}(int(cl(\mu_\lambda))))) = 1_Y.$$

But from $int(cl(\mu_\lambda)) \leq cl(\mu_\lambda)$ and almost continuity of f, $\mathrm{f}^{-1}(cl(\mu_\lambda))$ must be a fuzzy closed set containing $\mathrm{f}^{-1}(int(cl(\mu_\lambda)))$ and then $cl(\mathrm{f}^{-1}(int(cl(\mu_\lambda))))$. Thus

$$f(cl(\mathrm{f}^{-1}(int(cl(\mu_\lambda))))) \leq cl(\mu_\lambda).$$

So $\bigvee_{n\in\mathbb{N}} \{cl(\mu_\lambda) : \lambda \leq \lambda_n\} = 1_Y$. Hence Y is almost Menger. □

Theorem 3.8. *A fuzzy weakly continuous image of a fuzzy Menger space is fuzzy almost Menger.*

Proof. Let X be a fuzzy Menger space and let $f : X \to Y$ be a fuzzy weakly continuous and surjective function. If $\{\mu_n : n \in \mathbb{N}\}$ is a sequence fuzzy open covers of Y, then $\bigvee_{n\in\mathbb{N}} \mathrm{f}^{-1}(\mu_n) = 1_X$. Since f is fuzzy weakly continuous, for each $n \in \mathbb{N}$, we have $\mathrm{f}^{-1}(\mu_n) \leq int(\mathrm{f}^{-1}(cl(\mu_n)))$. Thus $\left\{int(\mathrm{f}^{-1}(cl((\mu_n))) : n \in \mathbb{N}\right\}$ is a fuzzy open cover of X. Since X is fuzzy Menger, there is a sequence $\{\lambda_n : n \in \mathbb{N}\}$ such that λ_n is a finite subset of $int(\mathrm{f}^{-1}(cl(\mu_n)))$ and $1_X = \bigvee_{n\in\mathbb{N}} \lambda_n \leq \bigvee_{n\in\mathbb{N}} int(\mathrm{f}^{-1}(cl(\mu_n)))$. For every $n \in \mathbb{N}$ and $\lambda \leq \lambda_n$, we choose a member $\mu_\lambda \leq \mu_n$ such that $\lambda = \mathrm{f}^{-1}(\mu_\lambda)$. From the surfectivity of f, we have

$$1_Y = f\left(\bigvee_{n\in\mathbb{N}} int(\mathrm{f}^{-1}(cl(\mu_\lambda)))\right) = \bigvee_{n\in\mathbb{N}} f(int(\mathrm{f}^{-1}(cl(\mu_\lambda)))).$$

From the weak continuity of f, we deduce $1_Y = \bigvee_{n\in\mathbb{N}} \{cl(\mu_\lambda) : \lambda \leq \lambda_n\}$. So Y is almost Menger. □

Theorem 3.9. *A fuzzy strongly continuous image of a fuzzy almost Menger space is fuzzy Menger.*

Proof. Let X be a fuzzy almost Menger space and let $f : X \to Y$ be a fuzzy strongly continuous and surjective function. Let $\{\mu_n : n \in \mathbb{N}\}$ be a sequence of fuzzy open covers of Y. Then from the strongly continuity of f, it follows that $\bigvee_{n\in\mathbb{N}} \mathrm{f}^{-1}(\mu_n) = 1_X$. Since X is almost Menger, there exists a family $\{\lambda_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}, \lambda_n$ is a finite fuzzy subset of $\mathrm{f}^{-1}(\mu_n)$ and $\bigvee_{n\in\mathbb{N}} \lambda_n^{'} = 1_X$, where $\lambda_n^{'} = \{cl(\lambda) : \lambda \leq \lambda_n\}$. For every $n \in \mathbb{N}$ and $\lambda \leq \lambda_n$, we choose a member $\mu_\lambda \leq \mu_n$ such that $\lambda = \mathrm{f}^{-1}(\mu_\lambda)$. Thus from the surjectivity and the strongly continuity of f, we deduce

$$\begin{aligned} 1_Y = f(1_X) = f\left(\bigvee_{n\in\mathbb{N}} cl(\mathrm{f}^{-1}(\mu_\lambda))\right) \\ &= \bigvee_{\lambda\leq\lambda_n} f(cl(\mathrm{f}^{-1}(\mu_\lambda))) \\ &\leq \bigvee_{\lambda\leq\lambda_n} f(\mathrm{f}^{-1}(\mu_\lambda)) \\ &= \bigvee_{\lambda\leq\lambda_n} \mu_\lambda. \end{aligned}$$

So Y is fuzzy Menger. □

4. Nearly Menger fuzzy topological spaces

Parvez and Khan [13] have introduced near Mengerness in topological spaces. In this section, we introduce and study fuzzy near Mengerness for fuzzy topological spaces.

Definition 4.1. A fuzzy topological space (X, τ) is called *fuzzy nearly Menger*, if for every sequence $\{\mu_n : n \in \mathbb{N}\}$ of fuzzy open covers of X there exists a sequence $\{\lambda_n : n \in \mathbb{N}\}$ of finite families such that for every $n \in \mathbb{N}, \lambda_n$ is a finite fuzzy subset of μ_n and $\bigvee_{n\in\mathbb{N}} \lambda_n^{'} = 1_X$, where $\lambda_n^{'} = \{int(cl(\lambda)) : \lambda \leq \lambda_n\}$.

It is clear that in fuzzy topological spaces, we have the following implications:

fuzzy Mengerness ⇒ fuzzy near Mengerness ⇒ fuzzy almost Mengerness.

The reverse implications do not hold.

Example 4.2. Consider the fuzzy topological space (X, τ) given in the Example 3.2. Then clearly, (X, τ) is fuzzy almost Menger. On the other hand, for the fuzzy open cover $\{\mu_n : n = 2, 3, 4, ...\}$ we have $\{int(cl(\mu_n)) : n \geq 4\}$ which has no finite fuzzy subset of μ_n and $\bigvee_{n\in\mathbb{N}} \lambda_n^{'} \neq 1_X$, where $\lambda_n^{'} = \{int(cl(\lambda)) : \lambda \leq \lambda_n\}$. Thus (X, τ) is not nearly Menger.

Corollary 4.3. *A fuzzy nearly Menger regular fuzzy topological space (X, τ) is Menger.*

Theorem 4.4. *A fuzzy topological space (X, τ) is fuzzy nearly Menger if and only if for every sequence $\{\mu_n : n \in \mathbb{N}\}$ of covers of X by regular open sets, there is a sequence $\{\lambda_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}, \lambda_n$ is a finite subset of μ_n and $\bigvee_{n\in\mathbb{N}} \lambda_n = 1_X$.*

Proof. The direct part of is obvious from the definition of nearly Menger space. Conversely, let $\{\mu_n : n \in \mathbb{N}\}$ be a sequence of fuzzy open covers of X. Also, consider a sequence $\{int(cl(\mu_n)) : n \in \mathbb{N}\}$. Then for every $n \in \mathbb{N}, \{int(cl(\mu_n)) : n \in \mathbb{N}\}$ is a cover of X by regular open sets. Thus from the hypothesis, there exists a sequence $\{\lambda_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}, \lambda_n$ is a finite subset of $int(cl(\mu_n))$ and $\bigvee_{n\in\mathbb{N}} \lambda_n = 1_X$. For every $n \in \mathbb{N}$ and $\lambda \leq \lambda_n$, there is an $\mu_\lambda \leq \mu_n$ such that $\lambda = int(cl(\mu_\lambda))$. So $\bigvee_{n\in\mathbb{N}} \{int(cl(\mu_\lambda)) : \lambda \leq \lambda_n\} = 1_X$. Hence (X, τ) is nearly Menger. □

Theorem 4.5. *A fuzzy topological space (X, τ) is fuzzy nearly Menger if and only if for every sequence $\{\mu_n : n \in \mathbb{N}\}$ of fuzzy regularly closed sets of X having the finite intersection property we have $\bigwedge_{n\in\mathbb{N}} \mu_n \neq 0_X$.*

Proof. It follows from Theorem 3.3. □

Theorem 4.6. *The image of fuzzy nearly Menger space under a fuzzy almost continuous and fuzzy almost open function is nearly Menger.*

Proof. Similar to the proof of Theorem 3.8. □

Corollary 4.7. *A fuzzy almost continuous image of a fuzzy Menger topological space is nearly Menger.*

Corollary 4.8. *A fuzzy continuous open surjective image of a fuzzy Menger topological space is nearly Menger.*

5. Conclusion

In this paper, we introduced and characterized the Menger property in fuzzy topological spaces. Some interesting properties are also established. The results in this work can be extended to the fuzzy selection properties.

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A. Haydar Eş (haydares@baskent.edu.tr)
Department of Mathematics Education, Faculty of Education, Başkent University, postal code 06790, Ankara, Turkey