Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 22, No. 1, (August 2021) pp. 1–15
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2021.22.1.1

@FMI

Crossing cubic structures as an extension of bipolar fuzzy sets

Young Bae Jun, Kul Hur, Jeong Gon Lee, Junhui Kim

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 22, No. 1, August 2021

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 22, No. 1, (August 2021) pp. 1–15
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2021.22.1.1

@FMI

Crossing cubic structures as an extension of bipolar fuzzy sets

Young Bae Jun, Kul Hur, Jeong Gon Lee, Junhui Kim

Received 28 January 2021; Revised 20 February 2021; Accepted 3 March 2021

Abstract. As an extension of bipolar-valued fuzzy sets, the notion of (inner, outer) crossing cubic structures is introduced by using the notion of $\mathcal{N}$-functions and interval-valued fuzzy sets, and related properties are investigated. The same direction order and the opposite direction order in crossing cubic structures are defined, and several properties are discussed. Also, S-union, S-intersection, O-union and O-intersection of crossing cubic structures are introduced, and their related properties are considered. Crossing cubic subalgebras are studied by applying a crossing cubic structure to BCK/BCI-algebras.

2020 AMS Classification: 03E72, 03G25, 06F35

Keywords: (inner, outer) crossing cubic structure, same direction order, opposite direction order, S-union, S-intersection, O-union, O-intersection, crossing cubic subalgebra

Corresponding Author: Jeong Gon Lee (jukolee@wku.ac.kr)

1. Introduction

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. In [1], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, everyone also feel a need to supply mathematical tool. To attain such object, Jun et al. [2] introduced and used a new function which is called negative-valued function. Using a fuzzy set and an interval-valued fuzzy set, Jun et al. [3] introduced the notion of cubic sets. Fuzzy set theory is established in the paper [4]. In the traditional fuzzy sets, the membership degrees of elements range over the interval $[0,1]$. The traditional fuzzy set representation cannot tell apart contrary elements from irrelevant elements. Only with the membership degrees ranged on the interval $[0,1]$, it is difficult to express the difference of the irrelevant elements from the contrary elements in fuzzy sets. If a set representation could express this kind of difference, it would be more informative than the traditional fuzzy set representation. Based on these observations, Lee [5] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets.

In this paper, using the notion of $\mathcal{N}$-functions and interval-valued fuzzy sets, we introduce the notion of (inner, outer) crossing cubic structures which is an extension of bipolar-valued fuzzy sets, and investigate several properties. We define the same direction order and the opposite direction order in crossing cubic structures. Also, we define S-union, S-intersection, O-union and O-intersection of crossing cubic structures, and discuss their related properties. We study crossing cubic subalgebras by applying crossing cubic structures to BCK/BCI-algebras.

2. Preliminaries

A set X with a binary operation "$\rightsquigarrow$" and a special element 0) is called *BCI-algebra* if it satisfies:

$$(\forall x, y, z \in X)(((x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)) \rightsquigarrow (z \rightsquigarrow y) = 0), \tag{2.1}$$

$$(\forall x, y \in X)((x \rightsquigarrow (x \rightsquigarrow y)) \rightsquigarrow y = 0), \tag{2.2}$$

$$(\forall x \in X)(x \rightsquigarrow x = 0), \tag{2.3}$$

$$(\forall x, y \in X)(x \rightsquigarrow y = 0,\, y \rightsquigarrow x = 0 \;\Rightarrow\; x = y). \tag{2.4}$$

By a *BCK-algebra* we mean a BCI-algebra X satisfying the following condition:

$$(\forall x \in X)(0 \rightsquigarrow x = 0). \tag{2.5}$$

A subset L of a BCK/BCI-algebra X is called a *subalgebra* of X if $x \rightsquigarrow y \in L$ for all $x, y \in L$.

Denote by $\mathcal{F}(X, [-1,0])$ the collection of all functions from a set X to $[-1,0]$. We say that an element of $\mathcal{F}(X, [-1,0])$ is a *negative-valued function* from X to $[-1,0]$ (briefly, *$\mathcal{N}$-function* on X.) Define a relation $\leq$ on $\mathcal{F}(X, [-1,0])$ as follows:

$$\xi \leq \eta \;\Leftrightarrow\; (\forall x \in X)(\xi(x) \leq \eta(x)) \tag{2.6}$$

for all $\xi, \eta \in \mathcal{F}(X, [-1,0])$. The *complement* of $\xi \in \mathcal{F}(X, [-1,0])$, denoted by ξ^c, is defined as follows:

$$(\forall x \in X)(\xi^c(x) = -1 - \xi(x)). \tag{2.7}$$

An *interval number* is defined to be a subinterval $\tilde{a} = [a^-, a^+]$ of $[0,1]$, where $0 \leq a^- \leq a^+ \leq 1$. The interval number $\tilde{a} = [a^-, a^+]$ with $a^- = a^+$ is denoted by $\mathbf{a}$. Denote by $[[0,1]]$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) of two elements in $[[0,1]]$. We also define the symbols "$\succcurlyeq$", "$\preccurlyeq$", "$=$" in case of two elements in $[[0,1]]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\begin{aligned}&\operatorname{rmin}\{\tilde{a}_1, \tilde{a}_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right],\\ &\tilde{a}_1 \succcurlyeq \tilde{a}_2 \;\Leftrightarrow\; a_1^- \geq a_2^-,\; a_1^+ \geq a_2^+,\end{aligned}$$

and similarly we may have $\tilde{a}_1 \preccurlyeq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succcurlyeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preccurlyeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$). Let $\tilde{a}_i \in [[0,1]]$ where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

For any $\tilde{a} \in [[0,1]]$, its *complement*, denoted by $\tilde{a}^c$, is defined to be the interval number

$$\tilde{a}^c = [1 - a^+, 1 - a^-].$$

Let X be a nonempty set. A function $f : X \to [[0,1]]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X. Let $[[0,1]]^X$ stand for the set of all IVF sets in X. For every $f \in [[0,1]]^X$ and $x \in X$, $f(x) = [f^-(x), f^+(x)]$ is called the *degree* of membership of an element x to f, where $f^- : X \to [0,1]$ and $f^+ : X \to [0,1]$ are fuzzy sets in X which are called a *lower fuzzy set* and an *upper fuzzy set* in X, respectively. For simplicity, we denote $f = [f^-, f^+]$. For every $f, g \in [[0,1]]^X$, we define

$$f \subseteq g \;\Leftrightarrow\; f(x) \preccurlyeq g(x) \;\text{ for all } x \in X,$$

and

$$f = g \;\Leftrightarrow\; f(x) = g(x) \;\text{ for all } x \in X.$$

The complement f^c of $f \in [[0,1]]^X$ is defined as follows: $f^c(x) = f(x)^c$ for all $x \in X$, that is,

$$f^c(x) = [1 - f^+(x), 1 - f^-(x)] \;\text{ for all } x \in X.$$

3. Crossing cubic structures

Definition 3.1. By a *crossing cubic structure* on a set X, we mean a pair $\left(X, \mathcal{C}_{(f,\xi)}\right)$ where

$$\mathcal{C}_{(f,\xi)} := \{\langle x, f(x), \xi(x)\rangle \mid x \in X\} \tag{3.1}$$

in which f is an interval-valued fuzzy set in X and ξ is an $\mathcal{N}$-function on X.

Definition 3.2. A crossing cubic structure $\left(X, \mathcal{C}_{(f,\xi)}\right)$ on a set X is said to be

- *inner*, if it satisfies:

$$(\forall x \in X)(-\xi(x) \in [f^-(x), f^+(x)]). \tag{3.2}$$

- *outer*, if it satisfies:

$$(\forall x \in X)(-\xi(x) \leq f^-(x) \text{ or } -\xi(x) \geq f^+(x)). \tag{3.3}$$

Example 3.3. 1. Let f be an interval-valued fuzzy set in X. Then $\mathcal{C}_{(f,\xi_0)} := \{\langle x, f(x), \xi_0(x)\rangle \mid x \in X\}$, $\mathcal{C}_{(f,\xi_{-1})} := \{\langle x, f(x), \xi_{-1}(x)\rangle \mid x \in X\}$ and $\mathcal{C}_{(f,\xi_c)} := \{\langle x, f(x), \xi_c(x)\rangle \mid x \in X\}$ are crossing cubic structures on X, where $\xi_0(x) = 0$, $\xi_{-1}(x) = -1$ and $\xi_c(x) = \frac{1}{2}(f^-(x) + f^+(x))$ for all $x \in X$.

2. Let $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on $[0,1]$. If $f(x) = [0.4, 0.7]$ and $\xi(x) = -0.5$ for all $x \in [0,1]$, then $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ is an inner crossing cubic structure on $[0,1]$. If $f(x) = [0.4, 0.7]$ and $\xi(x) = -0.75$ for all $x \in [0,1]$, then $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ is an outer crossing cubic structure on $[0,1]$. If $f(x) = [0.4, 0.7]$ and $\xi(x) = -x$ for all $x \in [0,1]$, then $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ is neither an inner crossing cubic structure nor an outer crossing cubic structure .

Example 3.4. Define an interval-valued fuzzy set f and a negative-valued function ξ on the real line $\mathbb{R}$ by

$$f : \mathbb{R} \to [[0,1]],\ x \mapsto \begin{cases} [0, 0.4] & \text{if } x < 0 \\ [0.5, 0.6] & \text{if } x = 0 \\ [0.7, 0.9] & \text{if } x > 0 \end{cases}$$

and

$$\xi : \mathbb{R} \to [-1, 0], x \mapsto -1 + \frac{1}{1 + e^{-x}}.$$

respectively, Then $\left(\mathbb{R}, \mathcal{C}_{(f,\xi)}\right)$ is a crossing cubic structure on the real line $\mathbb{R}$.

Proposition 3.5. *If a crossing cubic structure* $\left(X, \mathcal{C}_{(f,\xi)}\right)$ *on a set* X *is not outer, then* $f^-(a) < -\xi(a) < f^+(a)$ *for some* $a \in X$.

Proof. It is straightforward. □

Proposition 3.6. *If a crossing cubic structure* $\left(X, \mathcal{C}_{(f,\xi)}\right)$ *on a set* X *is inner and outer, then*

$$(\forall x \in X)(-\xi(x) = f^-(x) \text{ or } -\xi(x) = f^+(x)). \tag{3.4}$$

Proof. Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on a set X which is inner and outer. Then $-\xi(x) \in [f^-(x), f^+(x)]$ and $-\xi(x) \le f^-(x)$ or $-\xi(x) \in [f^-(x), f^+(x)]$ and $-\xi(x) \ge f^+(x)$ for all $x \in X$. It follows that $-\xi(x) = f^-(x)$ or $-\xi(x) = f^+(x)$. This completes the proof. □

Definition 3.7. Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on a set X. The *complement* of $\left(X, \mathcal{C}_{(f,\xi)}\right)$ is defined to be the crossing cubic structure

$$\left(X, \mathcal{C}_{(f,\xi)}\right)^c := \left(X, \mathcal{C}_{(f^c,\xi^c)}\right),$$

where $f^c : X \to [[0,1]], x \mapsto [1 - f^+(x), 1 - f^-(x)]$ and $\xi^c : X \to [-1, 0],\ x \mapsto -1 - \xi(x)$.

Example 3.8. Consider the crossing cubic structure $\left(\mathbb{R}, \mathcal{C}_{(f,\xi)}\right)$ on $\mathbb{R}$ which is given in Example 3.4. Then f^c and ξ^c are calculated as follows:

$$f^c : \mathbb{R} \to [[0,1]],\ x \mapsto \begin{cases} [0.6, 1] & \text{if } x < 0 \\ [0.4, 0.5] & \text{if } x = 0 \\ [0.1, 0.3] & \text{if } x > 0 \end{cases}$$

and

$$\xi^c : \mathbb{R} \to [-1,0], x \mapsto \frac{-1}{1+e^{-x}}.$$

Hence the crossing cubic structure $\left(X, \mathcal{C}_{(f,\xi)}\right)^c := \left(X, \mathcal{C}_{(f^c,\xi^c)}\right)$ is the complement of $\left(\mathbb{R}, \mathcal{C}_{(f,\xi)}\right)$.

For a crossing cubic structure $\left(X, \mathcal{C}_{(f,\xi)}\right)$ on a set X, $\tilde{a} = [a^-, a^+] \in [[0,1]]$, $t \in [-1,0]$ and $\varepsilon \in \{\geq, >, \leq, <\}$, we define:

$$\begin{aligned}\mathcal{C}^{(\geq\wedge\geq,\varepsilon)}_{(\tilde{a},t)} &:= f^{\geq\wedge\geq}_{\tilde{a}} \cap \xi^{\varepsilon}_t \\ &:= \{x \in X \mid f^-(x) \geq a^-,\ f^+(x) \geq a^+\} \cap \{x \in X \mid \xi(x)\,\varepsilon\, t\},\end{aligned} \tag{3.5}$$

$$\begin{aligned}\mathcal{C}^{(\geq\wedge>,\varepsilon)}_{(\tilde{a},t)} &:= f^{\geq\wedge>}_{\tilde{a}} \cap \xi^{\varepsilon}_t \\ &:= \{x \in X \mid f^-(x) \geq a^-,\ f^+(x) > a^+\} \cap \{x \in X \mid \xi(x)\,\varepsilon\, t\},\end{aligned} \tag{3.6}$$

$$\begin{aligned}\mathcal{C}^{(>\wedge\geq,\varepsilon)}_{(\tilde{a},t)} &:= f^{>\wedge\geq}_{\tilde{a}} \cap \xi^{\varepsilon}_t \\ &:= \{x \in X \mid f^-(x) > a^-,\ f^+(x) \geq a^+\} \cap \{x \in X \mid \xi(x)\,\varepsilon\, t\},\end{aligned} \tag{3.7}$$

$$\begin{aligned}\mathcal{C}^{(>\wedge>,\varepsilon)}_{(\tilde{a},t)} &:= f^{>\wedge>}_{\tilde{a}} \cap \xi^{\varepsilon}_t \\ &:= \{x \in X \mid f^-(x) > a^-,\ f^+(x) > a^+\} \cap \{x \in X \mid \xi(x)\,\varepsilon\, t\},\end{aligned} \tag{3.8}$$

$$\begin{aligned}\mathcal{C}^{(\leq\vee\leq,\varepsilon)}_{(\tilde{a},t)} &:= f^{\leq\vee\leq}_{\tilde{a}} \cap \xi^{\varepsilon}_t \\ &:= \{x \in X \mid f^-(x) \leq a^- \text{ or } f^+(x) \leq a^+\} \cap \{x \in X \mid \xi(x)\,\varepsilon\, t\},\end{aligned} \tag{3.9}$$

$$\begin{aligned}\mathcal{C}^{(\leq\vee<,\varepsilon)}_{(\tilde{a},t)} &:= f^{\leq\vee<}_{\tilde{a}} \cap \xi^{\varepsilon}_t \\ &:= \{x \in X \mid f^-(x) \leq a^- \text{ or } f^+(x) < a^+\} \cap \{x \in X \mid \xi(x)\,\varepsilon\, t\},\end{aligned} \tag{3.10}$$

$$\begin{aligned}\mathcal{C}^{(<\vee\leq,\varepsilon)}_{(\tilde{a},t)} &:= f^{<\vee\leq}_{\tilde{a}} \cap \xi^{\varepsilon}_t \\ &:= \{x \in X \mid f^-(x) < a^- \text{ or } f^+(x) \leq a^+\} \cap \{x \in X \mid \xi(x)\,\varepsilon\, t\},\end{aligned} \tag{3.11}$$

$$\begin{aligned}\mathcal{C}^{(<\vee<,\varepsilon)}_{(\tilde{a},t)} &:= f^{<\vee<}_{\tilde{a}} \cap \xi^{\varepsilon}_t \\ &:= \{x \in X \mid f^-(x) < a^- \text{ or } f^+(x) < a^+\} \cap \{x \in X \mid \xi(x)\,\varepsilon\, t\}.\end{aligned} \tag{3.12}$$

In a crossing cubic structure $\left(X, \mathcal{C}_{(f,\xi)}\right)$ on a set X, we define:

$$\begin{aligned}
f^{\geq}_{a^-} &:= \{x \in X \mid f^-(x) \geq a^-\},\ f^{\geq}_{a^+} := \{x \in X \mid f^+(x) \geq a^+\},\\
f^{>}_{a^-} &:= \{x \in X \mid f^-(x) > a^-\},\ f^{>}_{a^+} := \{x \in X \mid f^+(x) > a^+\},\\
f^{\leq}_{a^-} &:= \{x \in X \mid f^-(x) \leq a^-\},\ f^{\leq}_{a^+} := \{x \in X \mid f^+(x) \leq a^+\},\\
f^{<}_{a^-} &:= \{x \in X \mid f^-(x) < a^-\},\ f^{<}_{a^+} := \{x \in X \mid f^+(x) < a^+\},
\end{aligned}$$

for all $\tilde{a} = [a^-, a^+] \in [[0,1]]$.

Proposition 3.9. *Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on a set X. For any $\tilde{a} = [a^-, a^+] \in [[0,1]]$, $t \in [-1,0]$ and $\varepsilon \in \{\geq, >, \leq, <\}$, we have:*

$$\begin{aligned}
\mathcal{C}^{(\geq\wedge\geq,\varepsilon)}_{(\tilde{a},t)} &= f^{\geq}_{a^-} \cap f^{\geq}_{a^+} \cap \xi^{\varepsilon}_t,\ \mathcal{C}^{(\geq\wedge>,\varepsilon)}_{(\tilde{a},t)} = f^{\geq}_{a^-} \cap f^{>}_{a^+} \cap \xi^{\varepsilon}_t,\\
\mathcal{C}^{(>\wedge\geq,\varepsilon)}_{(\tilde{a},t)} &= f^{>}_{a^-} \cap f^{\geq}_{a^+} \cap \xi^{\varepsilon}_t,\ \mathcal{C}^{(>\wedge>,\varepsilon)}_{(\tilde{a},t)} = f^{>}_{a^-} \cap f^{>}_{a^+} \cap \xi^{\varepsilon}_t.
\end{aligned}$$

Proof. Straightforward. □

Proposition 3.10. *Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on a set X. For any $\tilde{a} = [a^-, a^+] \in [[0,1]]$, $t \in [-1,0]$ and $\varepsilon \in \{\geq, >, \leq, <\}$, we have:*

$$\mathcal{C}_{(\tilde{a},t)}^{(\leq\vee\leq,\varepsilon)} = (f_{a^-}^{\leq} \cap \xi_t^{\varepsilon}) \cup (f_{a^+}^{\leq} \cap \xi_t^{\varepsilon}),$$
$$\mathcal{C}_{(\tilde{a},t)}^{(\leq\vee<,\varepsilon)} = (f_{a^-}^{\leq} \cap \xi_t^{\varepsilon}) \cup (f_{a^+}^{<} \cap \xi_t^{\varepsilon}),$$
$$\mathcal{C}_{(\tilde{a},t)}^{(<\vee\leq,\varepsilon)} = (f_{a^-}^{<} \cap \xi_t^{\varepsilon}) \cup (f_{a^+}^{\leq} \cap \xi_t^{\varepsilon}),$$
$$\mathcal{C}_{(\tilde{a},t)}^{(<\vee<,\varepsilon)} = (f_{a^-}^{<} \cap \xi_t^{\varepsilon}) \cup (f_{a^+}^{<} \cap \xi_t^{\varepsilon}).$$

Proof. Straightforward. □

Proposition 3.11. *Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on a set X. For any $\tilde{a} = [a^-, a^+] \in [[0,1]]$, $t \in [-1,0]$ and $\varepsilon \in \{\geq, >, \leq, <\}$, we have:*

$$\mathcal{C}_{(\tilde{a},t)}^{(>\wedge>,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(\geq\wedge>,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(\geq\wedge\geq,\varepsilon)},$$
$$\mathcal{C}_{(\tilde{a},t)}^{(>\wedge>,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(>\wedge\geq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(\geq\wedge\geq,\varepsilon)},$$
$$\mathcal{C}_{(\tilde{a},t)}^{(<\vee<,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(\leq\vee<,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(\leq\vee\leq,\varepsilon)},$$
$$\mathcal{C}_{(\tilde{a},t)}^{(<\vee<,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(<\vee\leq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(\leq\vee\leq,\varepsilon)}.$$

Proof. Straightforward. □

Proposition 3.12. *Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on a set X and let $\varepsilon \in \{\geq, >, \leq, <\}$. Let $\tilde{a} = [a^-, a^+], \tilde{b} = [b^-, b^+] \in [[0,1]]$ and $t, s \in [-1,0]$ be such that $\tilde{a} \prec \tilde{b}$ and $t < s$. If $\varepsilon \in \{\geq, >\}$, then $\mathcal{C}_{(\tilde{b},s)}^{(\geq\wedge\geq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(>\wedge>,>)}$ and $\mathcal{C}_{(\tilde{a},s)}^{(\leq\vee\leq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{b},t)}^{(<\vee<,>)}$. If $\varepsilon \in \{\leq, <\}$, then $\mathcal{C}_{(\tilde{b},t)}^{(\geq\wedge\geq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},s)}^{(>\wedge>,<)}$ and $\mathcal{C}_{(\tilde{a},t)}^{(\leq\vee\leq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{b},s)}^{(<\vee<,>)}$.*

Proof. Assume that $\tilde{a} \prec \tilde{b}$ and $t < s$. Then $a^- < b^-$ and $a^+ < b^+$. If $x \in \mathcal{C}_{(\tilde{b},s)}^{(\geq\wedge\geq,\varepsilon)}$ for $\varepsilon \in \{\geq, >\}$, then $x \in f_{b^-}^{\geq} \cap f_{b^+}^{\geq} \cap \xi_s^{\varepsilon}$. Thus $f^-(x) \geq b^- > a^-$, $f^+(x) \geq b^+ > a^+$ and $\xi(x)\, \varepsilon\, s > t$. This shows that $x \in \mathcal{C}_{(\tilde{a},t)}^{(>\wedge>,>)}$. So $\mathcal{C}_{(\tilde{b},s)}^{(\geq\wedge\geq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(>\wedge>,>)}$. Let $x \in \mathcal{C}_{(\tilde{a},s)}^{(\leq\vee\leq,\varepsilon)}$. Then $x \in f_{a^-}^{\leq} \cap \xi_s^{\varepsilon}$ or $x \in f_{a^+}^{\leq} \cap \xi_s^{\varepsilon}$. If $x \in f_{a^-}^{\leq} \cap \xi_s^{\varepsilon}$, then $f^-(x) \leq a^- < b^-$ and $\xi(x)\, \varepsilon\, s > t$, that is, $x \in f_{b^-}^{<} \cap \xi_t^{>}$. If $x \in f_{a^+}^{\leq} \cap \xi_s^{\varepsilon}$, then $f^+(x) \leq a^+ < b^+$ and $\xi(x)\, \varepsilon\, s > t$, that is, $x \in f_{b^+}^{<} \cap \xi_t^{>}$. Hence $x \in (f_{a^+}^{\leq} \cap \xi_t^{>}) \cup (f_{b^+}^{<} \cap \xi_t^{>}) = \mathcal{C}_{(\tilde{b},t)}^{(<\vee<,>)}$. Similarly, we can verify that $\mathcal{C}_{(\tilde{b},t)}^{(\geq\wedge\geq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},s)}^{(>\wedge>,<)}$ and $\mathcal{C}_{(\tilde{a},t)}^{(\leq\vee\leq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{b},s)}^{(<\vee<,>)}$ for $\varepsilon \in \{\leq, <\}$. □

Applying the De Morgan's laws to Propositions 3.9 and 3.10 induces the following results.

Proposition 3.13. *Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on a set X, $\tilde{a} = [a^-, a^+] \in [[0,1]]$ and $t \in [-1,0]$. For $\alpha =\geq$ (resp., $>$, $\leq$ and $<$), let $\alpha^c =<$ (resp., $\leq$, $>$ and $\geq$). Then $\left(\mathcal{C}_{(\tilde{a},t)}^{(\alpha\wedge\beta,\gamma)}\right)^c = f_{a^-}^{\alpha^c} \cup f_{a^+}^{\beta^c} \cup \xi_t^{\gamma^c}$ and $\left(\mathcal{C}_{(\tilde{a},t)}^{(\alpha\vee\beta,\gamma)}\right)^c = (f_{a^-}^{\alpha^c} \cap f_{a^+}^{\beta^c}) \cup \xi_t^{\gamma^c}$ for $\alpha, \beta, \gamma \in \{\geq, >, \leq, <\}$.*

Denote by $CCS(X)$ the set of all crossing cubic structures on a set X. We define a binary relation "$\lessdot$", called the *same direction order* (briefly, S-order), on $CCS(X)$ as follows:

$$(X, \mathcal{C}_{(f,\xi)}) \lessdot (X, \mathcal{C}_{(g,\eta)}) \ \Leftrightarrow \ f \subseteq g, \ \xi \leq \eta \tag{3.13}$$

for all $(X, \mathcal{C}_{(f,\xi)}), (X, \mathcal{C}_{(g,\eta)}) \in CCS(X)$. It is clear that $(CCS(X), \lessdot)$ is a poset.

For any $(X, \mathcal{C}_{(f,\xi)}) \in CCS(X)$, $\tilde{a} = [a^-, a^+] \in [[0,1]]$ and $t \in [-1, 0]$, we define a scalar $\odot$-product and a scalar $*$-product of $\mathcal{C}_{(f,\xi)}$ by $(\tilde{a}, t) \odot \mathcal{C}_{(f,\xi)} := \mathcal{C}_{(\tilde{a}\odot f, t\odot\xi)}$ and $(\tilde{a}, t) * \mathcal{C}_{(f,\xi)} := \mathcal{C}_{(\tilde{a}*f, t*\xi)}$ where

$$\tilde{a} \odot f : X \to [[0,1]], \ x \mapsto [\min\{a^-, f^-(x)\}, \min\{a^+, f^+(x)\}],$$

$$t \odot \xi : X \to [-1, 0], \ x \mapsto \min\{t, \xi(x)\},$$

$$\tilde{a} * f : X \to [[0,1]], \ x \mapsto [\max\{a^-, f^-(x)\}, \max\{a^+, f^+(x)\}],$$

$$t * \xi : X \to [-1, 0], \ x \mapsto \max\{t, \xi(x)\}.$$

Proposition 3.14. *Let $(X, \mathcal{C}_{(f,\xi)})$ and $(X, \mathcal{C}_{(g,\eta)})$ be crossing cubic structures on a set X, $\tilde{a} = [a^-, a^+], \tilde{b} = [b^-, b^+] \in [[0,1]]$ and $t, s \in [-1, 0]$. If $\tilde{a} \preccurlyeq \tilde{b}$ and $t \leq s$, then $(X, (\tilde{a}, t) \odot \mathcal{C}_{(f,\xi)}) \lessdot (X, (\tilde{b}, s) \odot \mathcal{C}_{(f,\xi)})$ and $(X, (\tilde{a}, t) * \mathcal{C}_{(f,\xi)}) \lessdot (X, (\tilde{b}, s) * \mathcal{C}_{(f,\xi)})$. If $(X, \mathcal{C}_{(f,\xi)}) \lessdot (X, \mathcal{C}_{(g,\eta)})$, then $(X, (\tilde{a}, t) \odot \mathcal{C}_{(f,\xi)}) \lessdot (X, (\tilde{a}, t) \odot \mathcal{C}_{(g,\eta)})$ and $(X, (\tilde{a}, t) * \mathcal{C}_{(f,\xi)}) \lessdot (X, (\tilde{a}, t) * \mathcal{C}_{(g,\eta)})$.*

Proof. For any $x \in X$, we have

$$\begin{aligned}(\tilde{a} \odot f)(x) &= [\min\{a^-, f^-(x)\}, \min\{a^+, f^+(x)\}] \\ &\preccurlyeq [\min\{b^-, f^-(x)\}, \min\{b^+, f^+(x)\}] \\ &= (\tilde{b} \odot f)(x),\end{aligned}$$

$$(t \odot \xi)(x) = \min\{t, \xi(x)\} \leq \min\{s, \xi(x)\} = (s \odot \xi)(x),$$

$$\begin{aligned}(\tilde{a} * f)(x) &= [\max\{a^-, f^-(x)\}, \max\{a^+, f^+(x)\}] \\ &\preccurlyeq [\max\{b^-, f^-(x)\}, \max\{b^+, f^+(x)\}] \\ &= (\tilde{b} * f)(x),\end{aligned}$$

and $(t * \xi)(x) = \max\{t, \xi(x)\} \leq \max\{s, \xi(x)\} = (s * \xi)(x)$. Then $(X, (\tilde{a}, t) \odot \mathcal{C}_{(f,\xi)}) \lessdot (X, (\tilde{b}, s) \odot \mathcal{C}_{(f,\xi)})$ and $(X, (\tilde{a}, t) * \mathcal{C}_{(f,\xi)}) \lessdot (X, (\tilde{b}, s) * \mathcal{C}_{(f,\xi)})$. Assume that $(X, \mathcal{C}_{(f,\xi)}) \lessdot (X, \mathcal{C}_{(g,\eta)})$. Then $f \subseteq g$ and $\xi \leq \eta$, that is, $[f^-(x), f^+(x)] \preccurlyeq [g^-(x), g^+(x)]$ and $\xi(x) \leq \eta(x)$ for all $x \in X$. Thus

$$\begin{aligned}(\tilde{a} \odot f)(x) &= [\min\{a^-, f^-(x)\}, \min\{a^+, f^+(x)\}] \\ &\preccurlyeq [\min\{a^-, g^-(x)\}, \min\{a^+, g^+(x)\}] \\ &= (\tilde{a} \odot g)(x)\end{aligned}$$

and $(t \odot \xi)(x) = \min\{t, \xi(x)\} \leq \min\{s, \xi(x)\} = (s \odot \xi)(x)$ for all $x \in X$, that is, $\tilde{a} \odot f \subseteq \tilde{a} \odot g$ and $t \odot \xi \leq t \odot \eta$. So $(X, (\tilde{a}, t) \odot \mathcal{C}_{(f,\xi)}) \lessdot (X, (\tilde{a}, t) \odot \mathcal{C}_{(g,\eta)})$. Also we have

$$\begin{aligned}(\tilde{a} * f)(x) &= [\max\{a^-, f^-(x)\}, \max\{a^+, f^+(x)\}] \\ &\preccurlyeq [\max\{a^-, g^-(x)\}, \max\{a^+, g^+(x)\}] \\ &= (\tilde{a} * g)(x),\end{aligned}$$

and $(t * \xi)(x) = \max\{t, \xi(x)\} \leq \max\{t, \eta(x)\} = (t * \eta)(x)$ for all $x \in X$. i.e., $\tilde{a} * f \subseteq \tilde{a} * g$ and $t * \xi \leq t * \eta$. Hence $(X, (\tilde{a}, t) * \mathcal{C}_{(f,\xi)}) \lessdot (X, (\tilde{a}, t) * \mathcal{C}_{(g,\eta)})$. □

Theorem 3.15. *If we define a binary operation "$\cdot$" on $CCS(X)$ as follows:*

$$\left(X, \mathcal{C}_{(f,\xi)}\right) \cdot \left(X, \mathcal{C}_{(g,\eta)}\right) = \left(X, \mathcal{C}_{(f \wedge_r g, \xi \wedge \eta)}\right), \tag{3.14}$$

where $(f \wedge_r g)(x) = \text{rmin}\{f(x), g(x)\}$ *and* $(\xi \wedge \eta)(x) = \min\{\xi(x), \eta(x)\}$ *for all* $x \in X$, *then* $(CCS(X), \cdot)$ *is a semigroup.*

Proof. Straightworwad. □

Definition 3.16. Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ and $\left(X, \mathcal{C}_{(g,\eta)}\right)$ be crossing cubic structures on a set X. We define the *equality* "=" and the *opposite direction order* (briefly, O-order) "$\ll$" in $CCS(X)$ as follows:

$$\begin{aligned}\left(X, \mathcal{C}_{(f,\xi)}\right) = \left(X, \mathcal{C}_{(g,\eta)}\right) &\Leftrightarrow f = g,\ \xi = \eta, \\ \left(X, \mathcal{C}_{(f,\xi)}\right) \ll \left(X, \mathcal{C}_{(g,\eta)}\right) &\Leftrightarrow f \subseteq g,\ \xi \geq \eta.\end{aligned}$$

Theorem 3.17. $(CCS(X), \ll)$ *is a poset.*

Proof. Straightforward. □

Definition 3.18. Let $\left\{\left(X, \mathcal{C}^i_{(f,\xi)}\right) \mid i \in \Lambda\right\}$ be a family of crossing cubic structures on a set X, where Λ is any index set and $\mathcal{C}^i_{(f,\xi)} = \{\langle x, f_i(x), \xi_i(x)\rangle \mid x \in X\}$. Then

(i) the S-*union*, denoted by $\Cup_{S\, i\in\Lambda} \left(X, \mathcal{C}^i_{(f,\xi)}\right)$, of $\left\{\left(X, \mathcal{C}^i_{(f,\xi)}\right) \mid i \in \Lambda\right\}$ is defined to be the crossing cubic structure $\left(X, \Cup_{S\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)}\right)$ in which

$$\Cup_{S\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)} := \left\{\left\langle x, \left(\bigcup_{i\in\Lambda} f_i\right)(x), \left(\bigvee_{i\in\Lambda} \xi_i\right)(x)\right\rangle \mid x \in X\right\},$$

(ii) the S-*intersection*, denoted by $\Cap_{S\, i\in\Lambda} \left(X, \mathcal{C}^i_{(f,\xi)}\right)$, of $\left\{\left(X, \mathcal{C}^i_{(f,\xi)}\right) \mid i \in \Lambda\right\}$ is defined to be the crossing cubic structure $\left(X, \Cap_{S\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)}\right)$ in which

$$\Cap_{S\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)} := \left\{\left\langle x, \left(\bigcap_{i\in\Lambda} f_i\right)(x), \left(\bigwedge_{i\in\Lambda} \xi_i\right)(x)\right\rangle \mid x \in X\right\},$$

(iii) the O-*union*, denoted by $\Cup_{O\, i\in\Lambda} \left(X, \mathcal{C}^i_{(f,\xi)}\right)$, of $\left\{\left(X, \mathcal{C}^i_{(f,\xi)}\right) \mid i \in \Lambda\right\}$ is defined to be the crossing cubic structure $\left(X, \Cup_{O\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)}\right)$ in which

$$\Cup_{O\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)} := \left\{\left\langle x, \left(\bigcup_{i\in\Lambda} f_i\right)(x), \left(\bigwedge_{i\in\Lambda} \xi_i\right)(x)\right\rangle \mid x \in X\right\},$$

(iv) the O-*intersection*, denoted by $\Cap_{O\, i\in\Lambda}\left(X, \mathcal{C}^i_{(f,\xi)}\right)$, of $\left\{\left(X, \mathcal{C}^i_{(f,\xi)}\right) \mid i\in\Lambda\right\}$ is defined to be the crossing cubic structure $\left(X, \Cap_{O\, i\in\Lambda}\mathcal{C}^i_{(f,\xi)}\right)$ in which

$$\Cap_{O\, i\in\Lambda}\mathcal{C}^i_{(f,\xi)} := \left\{\left\langle x, \left(\bigcap_{i\in\Lambda} f_i\right)(x), \left(\bigvee_{i\in\Lambda}\xi_i\right)(x)\right\rangle \mid x\in X\right\},$$

where $\left(\bigcup_{i\in\Lambda} f_i\right)(x) = \operatorname{rsup}_{i\in\Lambda} f_i(x)$, $\left(\bigvee_{i\in\Lambda}\xi_i\right)(x) = \sup\{\xi_i(x) \mid i\in\Lambda\}$,
$\left(\bigcap_{i\in\Lambda} f_i\right)(x) = \operatorname{rinf}_{i\in\Lambda} f_i(x)$ and $\left(\bigwedge_{i\in\Lambda}\xi_i\right)(x) = \inf\{\xi_i(x) \mid i\in\Lambda\}$.

Note that

$$\left(X, \Cup_{S\, i\in\Lambda}\mathcal{C}^i_{(f,\xi)}\right) = \left(X, \mathcal{C}_{(\bigcup_{i\in\Lambda} f_i, \bigvee_{i\in\Lambda}\xi_i)}\right),\ \left(X, \Cap_{S\, i\in\Lambda}\mathcal{C}^i_{(f,\xi)}\right) = \left(X, \mathcal{C}_{(\bigcap_{i\in\Lambda} f_i, \bigwedge_{i\in\Lambda}\xi_i)}\right),$$
$$\left(X, \Cup_{O\, i\in\Lambda}\mathcal{C}^i_{(f,\xi)}\right) = \left(X, \mathcal{C}_{(\bigcup_{i\in\Lambda} f_i, \bigwedge_{i\in\Lambda}\xi_i)}\right),\ \left(X, \Cap_{O\, i\in\Lambda}\mathcal{C}^i_{(f,\xi)}\right) = \left(X, \mathcal{C}_{(\bigcap_{i\in\Lambda} f_i, \bigvee_{i\in\Lambda}\xi_i)}\right).$$

Proposition 3.19. *Given crossing cubic structures*

$$\left(X, \mathcal{C}_{(f,\xi)}\right), \left(X, \mathcal{C}_{(g,\eta)}\right), \left(X, \mathcal{C}_{(h,\zeta)}\right) \text{ and } \left(X, \mathcal{C}_{(k,\varrho)}\right)$$

on a set X, we have

(1) *if* $\left(X,\mathcal{C}_{(f,\xi)}\right) \lessdot \left(X,\mathcal{C}_{(g,\eta)}\right)$, *then* $\left(X,\mathcal{C}_{(g,\eta)}\right)^c \lessdot \left(X,\mathcal{C}_{(f,\xi)}\right)^c$,

(2) *if* $\left(X,\mathcal{C}_{(f,\xi)}\right) \lessdot \left(X,\mathcal{C}_{(g,\eta)}\right)$ *and* $\left(X,\mathcal{C}_{(f,\xi)}\right) \lessdot \left(X,\mathcal{C}_{(h,\zeta)}\right)$, *then*
$$\left(X,\mathcal{C}_{(f,\xi)}\right) \lessdot \left(X,\mathcal{C}_{(g,\eta)}\right) \Cap_S \left(X,\mathcal{C}_{(h,\zeta)}\right),$$

(3) *if* $\left(X,\mathcal{C}_{(f,\xi)}\right) \lessdot \left(X,\mathcal{C}_{(h,\zeta)}\right)$ *and* $\left(X,\mathcal{C}_{(g,\eta)}\right) \lessdot \left(X,\mathcal{C}_{(h,\zeta)}\right)$, *then*
$$\left(X,\mathcal{C}_{(f,\xi)}\right) \Cup_S \left(X,\mathcal{C}_{(g,\eta)}\right) \lessdot \left(X,\mathcal{C}_{(h,\zeta)}\right),$$

(4) *if* $\left(X,\mathcal{C}_{(f,\xi)}\right) \lessdot \left(X,\mathcal{C}_{(h,\zeta)}\right)$ *and* $\left(X,\mathcal{C}_{(g,\eta)}\right) \lessdot \left(X,\mathcal{C}_{(k,\varrho)}\right)$, *then*
$$\left(X,\mathcal{C}_{(f,\xi)}\right) \Cup_S \left(X,\mathcal{C}_{(g,\eta)}\right) \lessdot \left(X,\mathcal{C}_{(h,\zeta)}\right) \Cup_S \left(X,\mathcal{C}_{(k,\varrho)}\right) \text{ and}$$
$$\left(X,\mathcal{C}_{(f,\xi)}\right) \Cap_S \left(X,\mathcal{C}_{(g,\eta)}\right) \lessdot \left(X,\mathcal{C}_{(h,\zeta)}\right) \Cap_S \left(X,\mathcal{C}_{(k,\varrho)}\right),$$

(5) *if* $\left(X,\mathcal{C}_{(f,\xi)}\right) \ll \left(X,\mathcal{C}_{(g,\eta)}\right)$, *then* $\left(X,\mathcal{C}_{(g,\eta)}\right)^c \ll \left(X,\mathcal{C}_{(f,\xi)}\right)^c$,

(6) *if* $\left(X,\mathcal{C}_{(f,\xi)}\right) \ll \left(X,\mathcal{C}_{(g,\eta)}\right)$ *and* $\left(X,\mathcal{C}_{(f,\xi)}\right) \ll \left(X,\mathcal{C}_{(h,\zeta)}\right)$, *then*
$$\left(X,\mathcal{C}_{(f,\xi)}\right) \ll \left(X,\mathcal{C}_{(g,\eta)}\right) \Cap_O \left(X,\mathcal{C}_{(h,\zeta)}\right),$$

(7) *if* $\left(X,\mathcal{C}_{(f,\xi)}\right) \ll \left(X,\mathcal{C}_{(h,\zeta)}\right)$ *and* $\left(X,\mathcal{C}_{(g,\eta)}\right) \ll \left(X,\mathcal{C}_{(h,\zeta)}\right)$, *then*
$$\left(X,\mathcal{C}_{(f,\xi)}\right) \Cup_O \left(X,\mathcal{C}_{(g,\eta)}\right) \ll \left(X,\mathcal{C}_{(h,\zeta)}\right),$$

(8) *if* $\left(X,\mathcal{C}_{(f,\xi)}\right) \ll \left(X,\mathcal{C}_{(h,\zeta)}\right)$ *and* $\left(X,\mathcal{C}_{(g,\eta)}\right) \ll \left(X,\mathcal{C}_{(k,\varrho)}\right)$, *then*
$$\left(X,\mathcal{C}_{(f,\xi)}\right) \Cup_O \left(X,\mathcal{C}_{(g,\eta)}\right) \ll \left(X,\mathcal{C}_{(h,\zeta)}\right) \Cup_O \left(X,\mathcal{C}_{(k,\varrho)}\right) \text{ and}$$
$$\left(X,\mathcal{C}_{(f,\xi)}\right) \Cap_O \left(X,\mathcal{C}_{(g,\eta)}\right) \ll \left(X,\mathcal{C}_{(h,\zeta)}\right) \Cap_O \left(X,\mathcal{C}_{(k,\varrho)}\right).$$

Proof. Straightforward. □

Theorem 3.20. *If a crossing cubic structure $\left(X,\mathcal{C}_{(f,\xi)}\right)$ on a set X is inner (resp., outer), then its complement is also inner (resp., outer).*

Proof. Assume that $\left(X, \mathcal{C}_{(f,\xi)}\right)$ is an inner crossing cubic structure on a set X. Then $-\xi(x) \in [f^-(x), f^+(x)] = f(x)$, that is, $f^-(x) \leq -\xi(x) \leq f^+(x)$ for all $x \in X$. It follows that $1 - f^+(x) \leq -\xi^c(x) \leq 1 - f^-(x)$, i.e., $-\xi^c(x) \in [1 - f^+(x), 1 - f^-(x)] = f^c(x)$ for all $x \in X$. Thus $\left(X, \mathcal{C}_{(f,\xi)}\right)^c$ is an inner crossing cubic structure on X. Now if $\left(X, \mathcal{C}_{(f,\xi)}\right)$ is an outer crossing cubic structure on a set X, then $-\xi(x) \leq f^-(x)$ or $-\xi(x) \geq f^+(x)$ for all $x \in X$. So $-\xi^c(x) = -(-1 - \xi(x)) = 1 + \xi(x) \geq 1 - f^-(x)$ or $-\xi^c(x) = -(-1 - \xi(x)) = 1 + \xi(x) \leq 1 - f^+(x)$ for all $x \in X$. Hence $\left(X, \mathcal{C}_{(f,\xi)}\right)^c$ is an outer crossing cubic structure on X. □

Theorem 3.21. *If $\left(X, \mathcal{C}_{(f,\xi)}\right)$ and $\left(X, \mathcal{C}_{(g,\eta)}\right)$ are inner crossing cubic structures on a set X, then so is their O-union.*

Proof. Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ and $\left(X, \mathcal{C}_{(g,\eta)}\right)$ be inner crossing cubic structures on a set X. Then $f^-(x) \leq -\xi(x) \leq f^+(x)$ and $g^-(x) \leq -\eta(x) \leq g^+(x)$ for all $x \in X$. It follows that

$$\begin{aligned}(f \cup g)^-(x) &= \max\{f^-(x), g^-(x)\} \leq \max\{-\xi(x), -\eta(x)\} \\ &= -\min\{\xi(x), \eta(x)\} = -(\xi \wedge \eta)(x)\end{aligned}$$

and

$$\begin{aligned}-(\xi \wedge \eta)(x) &= -\min\{\xi(x), \eta(x)\} = \max\{-\xi(x), -\eta(x)\} \\ &\leq \max\{f^+(x), g^+(x)\} = (f \cup g)^+(x)\end{aligned}$$

for all $x \in X$. Thus $\left(X, \mathcal{C}_{(f,\xi)}\right) \Cup_O \left(X, \mathcal{C}_{(g,\eta)}\right)$ is an inner crossing cubic structure on X. □

Theorem 3.22. *If $\left(X, \mathcal{C}_{(f,\xi)}\right)$ and $\left(X, \mathcal{C}_{(g,\eta)}\right)$ are inner crossing cubic structures on a set X, then so is their O-intersection.*

Proof. Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ and $\left(X, \mathcal{C}_{(g,\eta)}\right)$ be inner crossing cubic structures on a set X. Then $f^-(x) \leq -\xi(x) \leq f^+(x)$ and $g^-(x) \leq -\eta(x) \leq g^+(x)$ for all $x \in X$. Thus

$$\begin{aligned}(f \cap g)^-(x) &= \min\{f^-(x), g^-(x)\} \leq \min\{-\xi(x), -\eta(x)\} \\ &= -\max\{\xi(x), \eta(x)\} = -(\xi \vee \eta)(x)\end{aligned}$$

and

$$\begin{aligned}-(\xi \vee \eta)(x) &= -\max\{\xi(x), \eta(x)\} = \min\{-\xi(x), -\eta(x)\} \\ &\leq \min\{f^+(x), g^+(x)\} = (f \cap g)^+(x)\end{aligned}$$

for all $x \in X$. So $\left(X, \mathcal{C}_{(f,\xi)}\right) \Cap_O \left(X, \mathcal{C}_{(g,\eta)}\right)$ is an inner crossing cubic structure on X. □

In the following example, we know that the S-union and the S-intersection of inner crossing cubic structures may not be an inner crossing cubic structure.

Example 3.23. 1. Let $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ be crossing cubic structures on $[0,1]$ in which $f(x) = [0.1, 0.8]$, $\xi(x) = -0.2$, $g(x) = [0.4, 0.9]$ and $\eta(x) = -0.5$ for all $x \in [0,1]$. Then $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ are inner crossing cubic structures on $[0,1]$. The S-union of $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ is

$$\left([0,1], \mathcal{C}_{(f,\xi)}\right) \Cup_S \left([0,1], \mathcal{C}_{(g,\eta)}\right) = \left([0,1], \mathcal{C}_{(f \cup g, \xi \vee \eta)}\right) = \left([0,1], \mathcal{C}_{(g,\xi)}\right)$$

We can check that $-\xi(x) = 0.2 \notin [0.4, 0.9] = g(x)$ which shows that $\left([0,1], \mathcal{C}_{(f,\xi)}\right) \Cup_S \left([0,1], \mathcal{C}_{(g,\eta)}\right)$ is not an inner crossing cubic structure on $[0,1]$.

2. Let $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ be crossing cubic structures on $[0,1]$ in which $f(x) = [0.2, 0.4]$, $\xi(x) = -0.35$, $g(x) = [0.2, 0.3]$ and $\eta(x) = -0.25$ for all $x \in [0,1]$. Then $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ are inner crossing cubic structures on $[0,1]$. The S-intersection of $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ is

$$\left([0,1], \mathcal{C}_{(f,\xi)}\right) \Cap_S \left([0,1], \mathcal{C}_{(g,\eta)}\right) = \left([0,1], \mathcal{C}_{(f\cap g, \xi\wedge\eta)}\right) = \left([0,1], \mathcal{C}_{(g,\xi)}\right)$$

and it is not an inner crossing cubic structure on $[0,1]$ since $-\xi(x) = 0.35 \notin [0.2, 0.3] = g(x)$.

The following example shows that the S-union and the S-intersection of outer crossing cubic structures may not be an outer crossing cubic structure.

Example 3.24. (1) Let $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ be crossing cubic structures on $[0,1]$ in which $f(x) = [0.31, 0.53]$, $\xi(x) = -0.76$, $g(x) = [0.72, 0.83]$ and $\eta(x) = -0.87$ for all $x \in [0,1]$. Then $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ are outer crossing cubic structures on $[0,1]$. The S-union of $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ is

$$\left([0,1], \mathcal{C}_{(f,\xi)}\right) \Cup_S \left([0,1], \mathcal{C}_{(g,\eta)}\right) = \left([0,1], \mathcal{C}_{(f\cup g, \xi\vee\eta)}\right) = \left([0,1], \mathcal{C}_{(g,\xi)}\right)$$

and it is not an outer crossing cubic structure on $[0,1]$ since $-\xi(x) = 0.76 \in [0.72, 0.83] = [g^-(x), g^+(x)]$.

(2) Let $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ be crossing cubic structures on $[0,1]$ in which $f(x) = [0.4, 0.6]$, $\xi(x) = -0.28$, $g(x) = [0.5, 0.7]$ and $\eta(x) = -0.47$ for all $x \in [0,1]$. Then $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ are outer crossing cubic structures on $[0,1]$. The S-intersection of $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ is

$$\left([0,1], \mathcal{C}_{(f,\xi)}\right) \Cap_S \left([0,1], \mathcal{C}_{(g,\eta)}\right) = \left([0,1], \mathcal{C}_{(f\cap g, \xi\wedge\eta)}\right) = \left([0,1], \mathcal{C}_{(f,\eta)}\right)$$

and it is not an outer crossing cubic structure on $[0,1]$ since $-\eta(x) = 0.47 \in [0.4, 0.6] = [f^-(x), f^+(x)]$.

The O-union of two outer crossing cubic structures is not an outer crossing cubic structure as seen in the following example.

Example 3.25. Let $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ be crossing cubic structures on $[0,1]$ in which $f(x) = [0.4, 0.7]$, $\xi(x) = -0.8$, $g(x) = [0.6, 0.9]$ and $\eta(x) = -0.5$ for all $x \in [0,1]$. Then $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ are outer crossing cubic structures on $[0,1]$. The O-union of $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ is

$$\left([0,1], \mathcal{C}_{(f,\xi)}\right) \Cup_O \left([0,1], \mathcal{C}_{(g,\eta)}\right) = \left([0,1], \mathcal{C}_{(f\cup g, \xi\wedge\eta)}\right) = \left([0,1], \mathcal{C}_{(g,\xi)}\right),$$

and it is not an outer crossing cubic structure on $[0,1]$.

The O-intersection of two outer crossing cubic structures is not an outer crossing cubic structure as seen in the following example.

Example 3.26. Let $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ be crossing cubic structures on $[0,1]$ in which $f(x) = [0.47, 0.75]$, $\xi(x) = -0.83$, $g(x) = [0.68, 0.87]$ and $\eta(x) = -0.45$

for all $x \in [0,1]$. Then $\big([0,1], \mathcal{C}_{(f,\xi)}\big)$ and $\big([0,1], \mathcal{C}_{(g,\eta)}\big)$ are outer crossing cubic structures on $[0,1]$. The O-intersection of $\big([0,1], \mathcal{C}_{(f,\xi)}\big)$ and $\big([0,1], \mathcal{C}_{(g,\eta)}\big)$ is

$$\big([0,1], \mathcal{C}_{(f,\xi)}\big) \Cap_O \big([0,1], \mathcal{C}_{(g,\eta)}\big) = \big([0,1], \mathcal{C}_{(f\cap g, \xi\vee\eta)}\big) = \big([0,1], \mathcal{C}_{(f,\eta)}\big),$$

and it is not an outer crossing cubic structure on $[0,1]$.

4. Application to BCK/BCI-algebras

In this section, let X denote a BCK/BCI-algebra unless otherwise specified.

Definition 4.1. A crossing cubic structure $\big(X, \mathcal{C}_{(f,\xi)}\big)$ on X is called a *crossing cubic subalgebra* of X, if it satisfies:

$$(\forall x, y \in X)\left(\begin{array}{l} f(x \rightsquigarrow y) \succcurlyeq \mathrm{rmin}\{f(x), f(y)\} \\ \xi(x \rightsquigarrow y) \leq \max\{\xi(x), \xi(y)\} \end{array}\right). \tag{4.1}$$

Example 4.2. Consider a BCK-algebra $X = \{0,1,2,3\}$ with the binary operation $\rightsquigarrow$ given by Table 1.

TABLE 1. Cayley table for the binary operation "$\rightsquigarrow$"

$\rightsquigarrow$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Let $\big(X, \mathcal{C}_{(f,\xi)}\big)$ be a crossing cubic structure on X which is given by Table 2. It is

TABLE 2. Tabular representation for $\big(X, \mathcal{C}_{(f,\xi)}\big)$

X	$f(x)$	$\xi(x)$
0	$[0.33, 0.83]$	-0.8
1	$[0.15, 0.56]$	-0.5
2	$[0.33, 0.83]$	-0.7
3	$[0.15, 0.56]$	-0.3

routine to verify that $\big(X, \mathcal{C}_{(f,\xi)}\big)$ is a crossing cubic subalgebra of X.

Proposition 4.3. *If $\big(X, \mathcal{C}_{(f,\xi)}\big)$ is a crossing cubic subalgebra of X, then $f(0) \succcurlyeq f(x)$ and $\xi(0) \leq \xi(x)$ for all $x \in X$.*

Proof. Let $\big(X, \mathcal{C}_{(f,\xi)}\big)$ be a crossing cubic subalgebra of X. Using (2.3) and (4.1), we get

$$\begin{aligned} f(0) &= f(x \rightsquigarrow x) \succcurlyeq \mathrm{rmin}\{f(x), f(y)\} \\ &= \mathrm{rmin}\{[f^-(x), f^-(x)], [f^+(x), f^+(x)]\} \\ &= [f^-(x), f^-(x)] = f(x) \end{aligned}$$

and $\xi(0) = \xi(x \rightsquigarrow x) \leq \max\{\xi(x), \xi(x)\} = \xi(x)$ for all $x \in X$. □

Theorem 4.4. *Let* $\left(X, \mathcal{C}_{(f,\xi)}\right)$ *be a crossing cubic structure on* X*. Then it is a crossing cubic subalgebra of* X *if and only if* f^- *and* f^+ *are fuzzy subalgebras of* X*, and* ξ *is an* $\mathcal{N}$*-subalgebra of* X*.*

Proof. It is easy to verify that if f^- and f^+ are fuzzy subalgebras of X, and ξ is an $\mathcal{N}$-subalgebra of X, then $\left(X, \mathcal{C}_{(f,\xi)}\right)$ is a crossing cubic subalgebra of X.

Conversely, assume that $\left(X, \mathcal{C}_{(f,\xi)}\right)$ is a crossing cubic subalgebra of X. It is clear that ξ is an $\mathcal{N}$-subalgebra of X. For any $x, y \in X$, we have

$$\begin{aligned}[f^-(x \rightsquigarrow y), f^+(x \rightsquigarrow y)] &= f(x \rightsquigarrow y) \succcurlyeq \mathrm{rmin}\{f(x), f(y)\} \\ &= \mathrm{rmin}\{[f^-(x), f^+(x)], [f^-(y), f^+(y)]\} \\ &= [\min\{f^-(x), f^-(y)\}, \min\{f^+(x), f^+(y)\}].\end{aligned}$$

It follows that $f^-(x \rightsquigarrow y) \geq \min\{f^-(x), f^-(y)\}$ and $f^+(x \rightsquigarrow y) \geq \min\{f^+(x), f^+(y)\}$. Therefore f^- and f^+ are fuzzy subalgebras of X. □

Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on X. We define a level set of $\left(X, \mathcal{C}_{(f,\xi)}\right)$, written as $\ell\left(X, \mathcal{C}_{(f,\xi)}\right)$, as follows:

$$\ell\left(X, \mathcal{C}_{(f,\xi)}, [\alpha, \beta], t\right) = \ell(X, f, [\alpha, \beta]) \cap \ell(X, \xi, t) \tag{4.2}$$

where $\ell(X, f, [\alpha, \beta]) = \{x \in X \mid f(x) \succcurlyeq [\alpha, \beta]\}$ and $\ell(X, \xi, t) = \{x \in X \mid \xi(x) \leq t\}$ for $[\alpha, \beta] \in [[0, 1]]$ and $t \in [-1, 0]$. We say that $\ell(X, f, [\alpha, \beta])$ and $\ell(X, \xi, t)$ are *f-level set* and *ξ-level set* of $\left(X, \mathcal{C}_{(f,\xi)}\right)$ with level indices $[\alpha, \beta]$ and t, respectively.

Theorem 4.5. *If* $\left(X, \mathcal{C}_{(f,\xi)}\right)$ *is a crossing cubic subalgebra of* X*, then its nonempty* f*-level set and* ξ*-level set are subalgebras of* X *for all level indices.*

Proof. Let $[\alpha, \beta] \in [[0, 1]]$ and $t \in [-1, 0]$ be level indices of $\left(X, \mathcal{C}_{(f,\xi)}\right)$ such that $\ell(X, f, [\alpha, \beta])$ and $\ell(X, \xi, t)$ are nonempty. Let $x, y \in \ell(X, f, [\alpha, \beta])$ and $a, b \in \ell(X, \xi, t)$. Then $f(x) \succcurlyeq [\alpha, \beta]$, $f(y) \succcurlyeq [\alpha, \beta]$, $\xi(a) \leq t$ and $\xi(b) \leq t$. It follows from (4.1) that $f(x \rightsquigarrow y) \succcurlyeq \mathrm{rmin}\{f(x), f(y)\} \succcurlyeq \mathrm{rmin}\{[\alpha, \beta], [\alpha, \beta]\} = [\alpha, \beta]$ and $\xi(a \rightsquigarrow b) \leq \max\{\xi(a), \xi(b)\} \leq \max\{t, t\} = t$. Thus $x \rightsquigarrow y \in \ell(X, f, [\alpha, \beta])$ and $a \rightsquigarrow b \in \ell(X, \xi, t)$. So $\ell(X, f, [\alpha, \beta])$ and $\ell(X, \xi, t)$ are subalgebras of X. □

Corollary 4.6. *If* $\left(X, \mathcal{C}_{(f,\xi)}\right)$ *is a crossing cubic subalgebra of* X*, then its nonempty level set* $\ell\left(X, \mathcal{C}_{(f,\xi)}, [\alpha, \beta], t\right)$ *is a subalgebra of* X *for all* $[\alpha, \beta] \in [[0, 1]]$ *and* $t \in [-1, 0]$*.*

Theorem 4.7. *Let* $\left(X, \mathcal{C}_{(f,\xi)}\right)$ *be a crossing cubic structure on* X *in which its nonempty* f*-level set and* ξ*-level set are subalgebras of* X *for all level indices. Then* $\left(X, \mathcal{C}_{(f,\xi)}\right)$ *is a crossing cubic subalgebra of* X*.*

Proof. Assume that $\ell(X, f, [\alpha, \beta])$ and $\ell(X, \xi, t)$ are nonempty subalgebras of X for all level indices $[\alpha, \beta] \in [[0, 1]]$ and $t \in [-1, 0]$. Suppose that there exist $x, y, a, b \in X$ such that $f(x \rightsquigarrow y) \prec \mathrm{rmin}\{f(x), f(y)\}$ and $\xi(a \rightsquigarrow b) > \max\{\xi(a), \xi(b)\}$. Taking $[\alpha_x, \beta_y] := \mathrm{rmin}\{f(x), f(y)\}$ and $t_{a \rightsquigarrow b} := \max\{\xi(a), \xi(b)\}$ induces $x, y \in \ell(X, f, [\alpha_x, \beta_y])$ and $a, b \in \ell(X, \xi, t_{a \rightsquigarrow b})$. But $x \rightsquigarrow y \notin \ell(X, f, [\alpha_x, \beta_y])$ and $a \rightsquigarrow b \notin \ell(X, \xi, t_{a \rightsquigarrow b})$. This is a contradiction, and then $f(x \rightsquigarrow y) \succcurlyeq \mathrm{rmin}\{f(x), f(y)\}$ and $\xi(x \rightsquigarrow y) \leq \max\{\xi(x), \xi(y)\}$ for all $x, y \in X$. Thus $\left(X, \mathcal{C}_{(f,\xi)}\right)$ is a crossing cubic subalgebra of X. □

Theorem 4.8. *Given a subset* L *of* X*, we define a crossing cubic structure* $\big(X, \mathcal{C}_{(f,\xi)}\big)$ *as follows:*

$$f : X \to [[0,1]],\ x \mapsto \begin{cases} [\alpha, \beta] & \text{if } x \in L, \\ [0,0] & \text{otherwise,} \end{cases}$$

$$\xi : X \to [-1,0],\ x \mapsto \begin{cases} t & \text{if } x \in L, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha, \beta \in (0,1]$ *with* $\alpha < \beta$ *and* $t \in [-1,0)$*. Then* L *is a subalgebra of* X *if and only if* $\big(X, \mathcal{C}_{(f,\xi)}\big)$ *is a crossing cubic subalgebra of* X*.*

Proof. We know that $\ell(X, f, [\alpha, \beta]) = L$, $\ell(X, f, [0,0]) = X$, $\ell(X, \xi, t) = L$ and $\ell(X, \xi, 0) = X$. Using Theorems 4.5 and 4.7, we have the desired result. □

Theorem 4.9. *If* $\big(X, \mathcal{C}_{(f,\xi)}\big)$ *is a crossing cubic subalgebra of* X*, then the set*

$$X_{\left(X, \mathcal{C}_{(f,\xi)}\right)} := \{x \in X \mid f(x) = f(0),\ \xi(x) = \xi(0)\}$$

is a subalgebra of X*.*

Proof. Let $x, y \in X_{\left(X, \mathcal{C}_{(f,\xi)}\right)}$. Then $f(x) = f(0) = f(y)$ and $\xi(x) = \xi(0) = \xi(y)$. Thus

$$\begin{aligned} f(x \rightsquigarrow y) &\succcurlyeq \text{rmin}\{f(x), f(y)\} = \text{rmin}\{f(0), f(0)\} = f(0), \\ \xi(x \rightsquigarrow y) &\leq \max\{\xi(x), \xi(y)\} = \max\{\xi(0), \xi(0)\} = \xi(0). \end{aligned} \tag{4.3}$$

We get $f(x \rightsquigarrow y) = f(0)$ and $\xi(x \rightsquigarrow y) = \xi(0)$ by combining Proposition 4.3 and (4.3). Thus $x \rightsquigarrow y \in X_{\left(X, \mathcal{C}_{(f,\xi)}\right)}$. So $X_{\left(X, \mathcal{C}_{(f,\xi)}\right)}$ is a subalgebra of X. □

The following theorem describes how to create a new crossing cubic subalgebra from a given crossing cubic subalgebra in BCI-algebras.

Theorem 4.10. *Let* $\big(X, \mathcal{C}_{(f,\xi)}\big)$ *be a crossing cubic subalgebra on a BCI-algebra* X *and let* $\big(X, \mathcal{C}_{(f^{\rightsquigarrow},\xi^{\rightsquigarrow})}\big)$ *be a crossing cubic structure on* X *in which*

$$f^{\rightsquigarrow} : X \to [[0,1]],\ x \mapsto f(0 \rightsquigarrow x) \text{ and } \xi^{\rightsquigarrow} : X \to [-1,0],\ x \mapsto \xi(0 \rightsquigarrow x). \tag{4.4}$$

Then $\big(X, \mathcal{C}_{(f^{\rightsquigarrow},\xi^{\rightsquigarrow})}\big)$ *is a crossing cubic subalgebra of* X*.*

Proof. Note that every BCI-algebra X satisfies:

$$(\forall x, y \in X)(0 \rightsquigarrow (x \rightsquigarrow y) = (0 \rightsquigarrow x) \rightsquigarrow (0 \rightsquigarrow y)).$$

It follows from (4.1) and (4.4) that

$$\begin{aligned} f^{\rightsquigarrow}(x \rightsquigarrow y) &= f(0 \rightsquigarrow (x \rightsquigarrow y)) = f((0 \rightsquigarrow x) \rightsquigarrow (0 \rightsquigarrow y)) \\ &\succcurlyeq \text{rmin}\{f(0 \rightsquigarrow x), f(0 \rightsquigarrow y)\} = \text{rmin}\{f^{\rightsquigarrow}(x), f^{\rightsquigarrow}(y)\} \end{aligned}$$

and

$$\begin{aligned} \xi^{\rightsquigarrow}(x \rightsquigarrow y) &= \xi(0 \rightsquigarrow (x \rightsquigarrow y)) = \xi(0 \rightsquigarrow x) \rightsquigarrow (0 \rightsquigarrow y)) \\ &\leq \max\{\xi(0 \rightsquigarrow x), \xi(0 \rightsquigarrow y)\} = \max\{\xi^{\rightsquigarrow}(x), \xi^{\rightsquigarrow}(y)\}. \end{aligned}$$

Therefore $\big(X, \mathcal{C}_{(f^{\rightsquigarrow},\xi^{\rightsquigarrow})}\big)$ is a crossing cubic subalgebra of X. □

Acknowledgements. The authors wish to thank the anonymous reviewers for their valuable suggestions.

References

[1] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, Inform. Sci. 8 (1975) 199–249.

[2] Y. B. Jun, K. J. Lee and S. Z. Song, $\mathcal{N}$-ideals of BCK/BCI-algebras, J. Chungcheong Math. Soc. 22 (2009), 417–437.

[3] Y. B. Jun, C. S. Kim and K. O. Yang, Cubic sets, Annal. Fuzzy Math. Inform. 4 (1) (2012) 83–98.

[4] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

[5] K. M. Lee, Bipolar-valued fuzzy sets and their operations, Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand (2000) 307–312.

Young Bae Jun (skywine@gmail.com)
Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea

Kul Hur (kulhur@wku.ac.kr)
Division of Applied Mathematics, Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea

Jeong Gon Lee (jukolee@wku.ac.kr)
Division of Applied Mathematics,
Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea

Junhui Kim (junhikim@wku.ac.kr)
Department of Mathematics Education,
Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea