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Abstract. Zadeh initiated fuzzy sets, as a generalization, Jun et al. introduced the notion of a cubic set. Pawlak initiated rough set theory to study incomplete and insufficient information. Dubois, Prade first investigated fuzzy rough set and rough fuzzy set. Then many researchers studied the theory of rough sets in variously fuzzy structures. In the paper, we define two rough operators on cubic sets by means of a cubic relation, and investigate some of their properties with respect to two systems operators on cubic sets.

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1. Introduction

In 1965, Zadeh [1] initiated fuzzy sets. As a generalization, Jun et al. [2] introduced the notion of a cubic set. After then, Kang and Kim [3] defined a mapping of cubic sets, Kim et al. [4] investigated a cubic relation between cubic sets. The related contents may be referred [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

Pawlak initiated rough set theory to study incomplete and insufficient information [16]. As a generalization, Yao studied two rough operators induced by an arbitrary binary relation [17]. Dubois, Prade first investigated fuzzy rough set and rough fuzzy set in[18]. After that time, from the point of view of the theory of fuzzy sets, many researchers studied the theory of rough sets in variously fuzzy structures, for example: fuzzy sets, L-sets, Hesitant fuzzy sets, interval fuzzy sets, fuzzy soft sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, etc.. See [19, 20, 21, 22, 23]

In the paper, we define two rough approximations in cubic sets induced by a cubic relation. The contents are arranged into three parts, Section 3: Rough cubic sets, Section 4: A equivalence definition. In Section 2, we give an overview of rough sets and cubic sets, which surveys Preliminaries

2. Preliminaries

In the section, we introduce some main notions for each area, i.e., rough sets [16, 17], fuzzy sets[1], interval-valued fuzzy sets [23, 24], and cubic sets [2, 4].

2.1. **Rough Sets.**

In rough set theory, the approximation of an arbitrary subset of a universe by two definable subsets are called lower and upper approximations, which correspond to two rough operators. The two rough operators were first defined by means of a given indiscernibility relation in [16]. Usually indiscernibility relations are supposed to be equivalences.

Let (X, R) be an approximation space, and $R \subseteq X \times X$ be an equivalence relation, then for $A \subseteq X$, two subsets $\underline{R}(A)$ and $\overline{R}(A)$ of X are defined:

$$\underline{R}(A) = \{x \in X \mid [x]_R \subseteq A\}, \qquad \overline{R}(A) = \{x \in X \mid [x]_R \cap A \neq \varnothing\},$$

where $[x]_R = \{y \in X \mid xRy\}$.

If $\underline{R}(A) = \overline{R}(A)$, A is called a *definable set*; if $\underline{R}(A) \neq \overline{R}(A)$, A is called an *undefinable set*, and $(\underline{R}(A), \overline{R}(A))$ is referred to as a pair of rough set. Therefore, $\underline{R}$ and $\overline{R}$ are called two *rough operators*.

Furthermore, as a generalization, in [17], Yao defined the two rough operators by an arbitrary binary relation. Suppose R is a binary relation on X, for $x \in X$, let $r(x) = \{y \mid xRy\}$, then a pair of lower and upper approximations is defined: for $A \subseteq X$,

$$\underline{appr}A = \{x \mid r(x) \subseteq A\}, \qquad \overline{appr}A = \{x \mid r(x) \cap A \neq \varnothing\}.$$

2.2. **Fuzzy sets.**

In the section, we introduce some basic definitions related to fuzzy sets, fuzzy relations (See [1]).

For a set X, a mapping $\lambda : X \to I$ is called a *fuzzy set* in X, where $I = [0, 1]$. The collection of all fuzzy sets in X is denoted by I^X. In particular, 0 and 1 denote the *fuzzy empty set* and the *fuzzy whole set* in X, respectively.

For any $\lambda,\ \mu \in I^X$, the *join* ($\vee$) and *meet* ($\wedge$) of λ and μ, denoted by $\lambda \vee \mu$ and $\lambda \wedge \mu$, are defined as follows: for each $x \in X$,

$$(\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\}, \qquad (\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\}.$$

For any family $(\lambda_j)_{j\in J}$ of fuzzy sets in X, the *join* ($\vee$) and *meet* ($\wedge$) of $(\lambda_j)_{j\in J}$, denoted by $\bigvee_{j\in J} \lambda_j$ and $\bigwedge_{j\in J} \lambda_j$, are defined as follows: for each $x \in X$,

$$(\bigvee_{j\in J} \lambda_j)(x) = \mathrm{sup}_{j\in J}\lambda_j(x), \qquad (\bigwedge_{j\in J} \lambda_j)(x) = \mathrm{inf}_{j\in J}\lambda_j(x).$$

For two sets X, Y, $r \in I^{X\times Y}$ is called a fuzzy relation from X to Y. In particular, $r \in I^{X\times X}$ is called a fuzzy relation on X.

2.3. **Interval-valued fuzzy sets.** In the section, we list some basic definitions related to interval-valued fuzzy sets and interval-valued fuzzy relations (See [23, 24]).

The set of all closed subintervals of I is denoted by $[I]$, and members of $[I]$ are called *interval numbers* and are denoted by $\widetilde{a}, \widetilde{b}, \widetilde{c}$, etc., where $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$. In particular, if $a^- = a^+$, then we write as $\widetilde{a} = \mathbf{a}$.

We define relations $\preceq$ and $=$ on $[I]$ as follows:

$$(\forall\ \widetilde{a},\ \widetilde{b} \in [I])(\widetilde{a} \preceq \widetilde{b} \Longleftrightarrow a^- \leq b^- \text{ and } a^+ \leq b^+),$$

$$(\forall\ \widetilde{a},\ \widetilde{b} \in [I])(\widetilde{a} = \widetilde{b} \Longleftrightarrow \widetilde{a} \succeq \widetilde{b} \text{ and } \widetilde{a} \preceq \widetilde{b}), i.e.,$$

$$(\forall\ \widetilde{a},\ \widetilde{b} \in [I])(\widetilde{a} = \widetilde{b} \Longleftrightarrow a^- = b^- \text{ and } a^+ = b^+).$$

To say $\widetilde{a} \prec \widetilde{b}$, we mean $\widetilde{a} \preceq \widetilde{b}$ and $\widetilde{a} \neq \widetilde{b}$.

For any $\widetilde{a},\ \widetilde{b} \in [I]$, their *minimum* and *maximum*, denoted by $\widetilde{a} \wedge \widetilde{b}$ and $\widetilde{a} \vee \widetilde{b}$ are defined as follows:

$$\widetilde{a} \wedge \widetilde{b} = [a^- \wedge b^-, a^+ \wedge b^+], \qquad \widetilde{a} \vee \widetilde{b} = [a^- \vee b^-, a^+ \vee b^+].$$

Let $(\widetilde{a}_j)_{j \in J} \subset [I]$. Then its *inf* and *sup*, denoted by $\bigwedge_{j \in J} \widetilde{a}_j$ and $\bigvee_{j \in J} \widetilde{a}_j$, are defined as follows:

$$\bigwedge_{j \in J} \widetilde{a}_j = [\bigwedge_{j \in J} a_j^-, \bigwedge_{j \in J} a_j^+], \qquad \bigvee_{j \in J} \widetilde{a}_j = [\bigvee_{j \in J} a_j^-, \bigvee_{j \in J} a_j^+].$$

For a nonempty set X, a mapping $A : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an IVF set) in X. Let $[I]^X$ denote the set of all IVF sets in X. For each $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree of membership of an element x to A*, where $A^-, A^+ \in I^X$ are called a *lower fuzzy set* and an *upper fuzzy set* in X, respectively. For each $A \in [I]^X$, we write $A = [A^-, A^+]$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in X, respectively. We define relations $\subset$ and $=$ on $[I]^X$ as follows:

$$(\forall\ A,\ B \in [I]^X)(A \subset B \Longleftrightarrow (x \in X)(A(x) \preceq B(x)),$$

$$(\forall\ A,\ B \in [I]^X)(A = B \Longleftrightarrow (x \in X)(A(x) = B(x)).$$

For each $A \in [I]^X$, the *complement* of A, denoted by A^c, is defined as follows: for each $x \in X$,

$$A^c(x) = [1 - A^+(x), 1 - A^-(x)].$$

For any $(A_j)_{j \in J} \subset [I]^X$, its *intersection* $\bigcap_{j \in J} A_j$ and *union* $\bigcup_{j \in J} A_j$ are defined, respectively as follows: for each $x \in X$,

$$(\bigcap_{j \in J} A_j)(x) = \bigwedge_{j \in J} A_j(x), \qquad (\bigcup_{j \in J} A_j)(x) = \bigvee_{j \in J} A_j(x).$$

For two sets $X,\ Y$, $R \in [I]^{X \times Y}$ is called an *interval-valued fuzzy relation(briefly, IVF relation) from X to Y. In particular, $R \in [I]^{X \times X}$ is called an IVF relation on X.*

2.4. **Cubic sets and cubic relations.** As a generalization of Zadeh's fuzzy set, the notions of a cubic fuzzy set and a cubic fuzzy relation are introduced in [2, 4].

Definition 2.1 ([2])**.** Let X be a nonempty set. Then a complex mapping $\mathcal{A} = \langle A, \lambda\rangle : X \to [I] \times I$ is called a *cubic set* in X.

In special, a cubic set $\mathcal{A} = \langle A, \lambda\rangle$ in which $A(x) = \mathbf{1}$ and $\lambda(x) = 1$ (resp. $A(x) = \mathbf{0}$ and $\lambda(x) = 0$) for each $x \in X$ is denoted by $\hat{0}$ (resp. $\hat{1}$). In this case, $\hat{0}$ (resp. $\hat{1}$) will be called a *cubic empty* (resp. *whole*) *set* in X.

A cubic set $\mathcal{A} = \langle A, \lambda\rangle$ in which $A(x) = \mathbf{0}$ and $\lambda(x) = 1$ (resp. $A(x) = \mathbf{1}$ and $\lambda(x) = 0$) for each $x \in X$ is denoted by $\ddot{0}$ (resp. $\ddot{1}$).

We denote the set of all cubic sets in X as $([I] \times I)^X$.

Example 2.2. Let $X = \{a, b, c\}$ be a set, A be the interval fuzzy set and λ be the fuzzy set on X given, respectively by the following tables:

X	a	b	c
$\mathcal{A} =< A, \lambda >$	$< [0.3, 0.7], 0.5 >$	$< [0.3, 0.4], 0.7 >$	$< [0.1, 0.6], 0.3 >$

Table 2.1

Then $\mathcal{A} =< A, \lambda >$ is a cubic set in X.

Definition 2.3 ([2])**.** Let $\mathcal{A} = \langle A, \lambda\rangle\,,\ \mathcal{B} = \langle B, \mu\rangle \in ([I] \times I)^X$. Then we define the following relations:

(i) (Equality) $\mathcal{A} = \mathcal{B} \Leftrightarrow A = B$ and $\lambda = \mu$,

(ii) (P-order) $\mathcal{A} \sqsubset \mathcal{B} \Leftrightarrow A \subset B$ and $\lambda \leq \mu$,

(iii) (R-order) $\mathcal{A} \Subset \mathcal{B} \Leftrightarrow A \subset B$ and $\lambda \geq \mu$

Definition 2.4 ([2])**.** Let $\mathcal{A} = \langle A, \lambda\rangle\,,\ \mathcal{B} = \langle B, \mu\rangle \in ([I] \times I)^X$ and let $(\mathcal{A}_j)_{j\in J} = (\langle A_j, \lambda_j\rangle)_{j\in J} \subset ([I] \times I)^X$. Then the complement $\mathcal{A}^c$ of $\mathcal{A}$, P-union $\sqcup$, P-intersection $\sqcap$, R-union $\Cup$ and R-intersection $\Cap$ are defined as follows, respectively: for each $x \in X$,

(i) (Complement) $\mathcal{A}^c(x) = \langle A^c(x), \lambda^c(x)\rangle$,

(ii) (P-union) $(\mathcal{A} \sqcup \mathcal{B})(x) = \langle (A \cup B)(x), (\lambda \vee \mu)(x)\rangle$,

$$(\sqcup_{j\in J}\mathcal{A}_j)(x) = \left\langle (\bigcup_{j\in J} A_j)(x), (\bigvee_{j\in J} \lambda_j)(x) \right\rangle,$$

(iii) (P-intersection) $(\mathcal{A} \sqcap \mathcal{B})(x) = \langle (A \cap B)(x), (\lambda \wedge \mu)(x)\rangle$,

$$(\sqcap_{j\in J}\mathcal{A}_j)(x) = \left\langle (\bigcap_{j\in J} A_j)(x), (\bigwedge_{j\in J} \lambda_j)(x) \right\rangle,$$

(iv) (R-union) $(\mathcal{A} \Cup \mathcal{B})(x) = \langle (A \cup B)(x), (\lambda \wedge \mu)(x)\rangle$,

$$(\Cup_{j\in J}\mathcal{A}_j)(x) = \left\langle (\bigcup_{j\in J} A_j)(x), (\bigwedge_{j\in J} \lambda_j)(x) \right\rangle,$$

(v) (R-intersection) $(\mathcal{A} \Cap \mathcal{B})(x) = \langle (A \cap B)(x), (\lambda \vee \mu)(x)\rangle$,

$$(\Cap_{j\in J}\mathcal{A}_j)(x) = \left\langle (\bigcap_{j\in J} A_j)(x), (\bigvee_{j\in J} \lambda_j)(x) \right\rangle.$$

For more references,see [25, 26, 27, 28, 29, 30].

In special, for cubic sets $\langle \widetilde{a}, \lambda\rangle\,,\ \left\langle \widetilde{b}, \mu \right\rangle$, $\langle \widetilde{a}_j, \lambda_j\rangle\,,\ \left\langle \widetilde{b}_j, \mu_j \right\rangle$ $(j \in J)$, we also adopt the above symbols*:*

$$\langle \widetilde{a}, \lambda\rangle \sqsubset \left\langle \widetilde{b}, \mu \right\rangle \Leftrightarrow \widetilde{a} \preceq \widetilde{b} \text{ and } \lambda \leq \mu, \quad < \widetilde{a}, \lambda > = < \widetilde{b}, \mu > \Leftrightarrow \widetilde{a} = \widetilde{b} \text{ and } \lambda = \mu,$$

$$\langle \widetilde{a}, \lambda\rangle \sqcup \left\langle \widetilde{b}, \mu\right\rangle = \left\langle \widetilde{a} \vee \widetilde{b}, \lambda \vee \mu\right\rangle, \qquad \langle \widetilde{a}, \lambda\rangle \sqcap \left\langle \widetilde{b}, \mu\right\rangle = \left\langle \widetilde{a} \wedge \widetilde{b}, \lambda \wedge \mu\right\rangle,$$

$$\bigsqcup_{j\in J} \langle \widetilde{a}_j, \lambda_j\rangle = \left\langle \bigvee_{j\in J} \widetilde{a}_j, \bigvee_{j\in J} \lambda_j \right\rangle, \qquad \sqcap_{j\in J} \langle \widetilde{a}_j, \lambda_j\rangle = \left\langle \bigwedge_{j\in J} \widetilde{a}_j, \bigwedge_{j\in J} \lambda_j \right\rangle,$$

$$\langle \widetilde{a}, \lambda\rangle \Subset \left\langle \widetilde{b}, \mu\right\rangle \Leftrightarrow \widetilde{a} \preceq \widetilde{b} \text{ and } \lambda \geq \mu.$$

Cubic relations are cubic sets, we have the following example [4].

Example 2.5. Let $X = \{a, b, c\}$ be a set, let R be the IVF relation and r be the fuzzy relation on X given, respectively by the following tables:

$\mathcal{R} =< R, r >$	a	b	c
a	$\langle[0.3, 0.7], 06\rangle$	$\langle[0.4, 0.8], 0.4\rangle$	$\langle[0.1, 0.6], 0.\rangle$
b	$\langle[0.1, 0.6], 0.8\rangle$	$\langle[0, 1], 0.5\rangle$	$\langle[0.2, 0.5], 0.9\rangle$
c	$\langle[0.4, 0.9], 0.4\rangle$	$\langle[0.3, 0.8], 0.7\rangle$	$\langle[0, 1], 0.6\rangle$

Table 2.2

Then clearly, $\mathcal{R} =< R, r >$ is a cubic relation on X.

3. Rough cubic sets

In Section 2.4, we know there exist two systems operators P-order, P-union, P-intersection and R-order,R-union, R-intersection on cubic sets. So we define rough operators on cubic sets induced by a cubic relation, and investigate some of their properties respectively.

3.1. **Rough cubic sets I.**

In the section, we investigate rough cubic sets with the operators P-order, P-union, P-intersection.

Definition 3.1. Suppose X is a universe set, $\mathcal{R} =< R, r >$ is a cubic relation, and two rough operators N_P, H_P (shorthand for N, H) are defined as follows, for every cubic set $\mathcal{A} =< A, \lambda >$ on X,

$$N(\mathcal{A})(x) = \sqcap_{y\in X} \mathcal{A}(y) \sqcup \mathcal{R}^c(y, x), \qquad H(\mathcal{A})(x) = \bigsqcup_{y\in X} \mathcal{A}(y) \sqcap \mathcal{R}(x, y).$$

If $N(\mathcal{A}) = H(\mathcal{A})$, then $\mathcal{A}$ is called a *definable cubic set*; otherwise, $\mathcal{A}$ is called an *undefinable cubic set*. $(N(\mathcal{A}), H(\mathcal{A}))$ is referred to as a pair of cubic rough set [31].

Next, we introduce some examples.

Example 3.2. Suppose $\mathcal{A}$ and $\mathcal{R}$ are defined as Examples 2.2 and 2.5, then we obtain $N(\mathcal{A})$ and $H(\mathcal{A})$ as follows.

X	a	b	c
$N(\mathcal{A})$	$\langle[0.1, 0.6], 0.5\rangle$	$\langle[0.2, 0.7], 0.3\rangle$	$\langle[0.1, 0.8], 0.4\rangle$

Table 3.1

Where

$$\begin{aligned}
N(\mathcal{A})(a) &= \sqcap_{y\in X}\mathcal{A}(y)\sqcup\mathcal{R}^c(y,a)\\
&= (\mathcal{A}(a)\sqcup\mathcal{R}^c(a,a))\sqcap(\mathcal{A}(b)\sqcup\mathcal{R}^c(b,a))\sqcap(\mathcal{A}(c)\sqcup\mathcal{R}^c(c,a))\\
&= (\langle[0.3,0.7],0.5\rangle\sqcup\langle[0.3,0.7],0.6\rangle^c)\sqcap(\langle[0.3,0.4],0.7\rangle\\
&\quad\sqcup\langle[0.1,0.6],0.8\rangle^c)\sqcap(\langle[0.1,0.6],0.3\rangle\sqcup\langle[0.4,0.9],0.4\rangle^c)\\
&= (\langle[0.3,0.7],0.5\rangle\sqcup\langle[0.3,0.7],0.4\rangle)\sqcap(\langle[0.3,0.4],0.7\rangle\\
&\quad\sqcup\langle[0.4,0.9],0.2\rangle)\sqcap(\langle[0.1,0.6],0.3\rangle\sqcup\langle[0.1,0.6],0.6\rangle)\\
&= \langle[0.3,0.7],0.5\rangle\sqcap\langle[0.4,0.9],0.7\rangle\sqcap\langle[0.1,0.6],0.6\rangle\\
&= \langle[0.1,0.6],0.5\rangle.
\end{aligned}$$

In the similar way, we obtain

$$N(\mathcal{A})(b)=\langle[0.2,0.7],0.3\rangle\,,\ N(\mathcal{A})(c)=\langle[0.1,0.8],0.4\rangle\,.$$

and

X	a	b	c
$H(\mathcal{A})$	$\langle[0.3,0.7],0.5\rangle$	$\langle[0.1,0.6],0.5\rangle$	$\langle[0.3,0.7],0.7\rangle$

Table 3.2

Where

$$\begin{aligned}
H(\mathcal{A})(a) &= \bigsqcup_{y\in X}\mathcal{A}(y)\sqcap\mathcal{R}(a,y)\\
&= (\mathcal{A}(a)\sqcap\mathcal{R}(a,a))\sqcup(\mathcal{A}(b)\sqcap\mathcal{R}(a,b))\sqcup(\mathcal{A}(c)\sqcap\mathcal{R}(a,c))\\
&= (\langle[0.3,0.7],0.5\rangle\sqcap\langle[0.3,0.7],0.6\rangle)\sqcup(\langle[0.3,0.4],0.7\rangle\\
&\quad\sqcap\langle[0.4,0.8],0.4\rangle)\sqcup(\langle[0.1,0.6],0.3\rangle\sqcap\langle[0.1,0.6],0.7\rangle\\
&= \langle[0.3,0.7],0.5\rangle\sqcup\langle[0.3,0.4],0.4\rangle\sqcup\langle[0.1,0.6],0.3\rangle\\
&= \langle[0.3,0.7],0.5\rangle.
\end{aligned}$$

In the similar way, we obtain

$$H(\mathcal{A})(b)=\langle[0.1,0.6],0.5\rangle\,,\ H(\mathcal{A})(c)=\langle[0.3,0.7],0.7\rangle\,.$$

Third, we investigate some properties of the two rough operators.

In the classical case, N and H are monotone increasing, i.e., if $A\subseteq B$, $N(A)\subseteq N(B)$ and $H(A)\subseteq H(B)$ hold. For the P-order $\sqsubset$, we obtain

Proposition 3.3. (1) $N(\hat{1})=\hat{1}$, $H(\hat{0})=\hat{0}$,
(2) $\mathcal{A}\sqsubset\mathcal{B}\Rightarrow N(\mathcal{A})\sqsubset N(\mathcal{B})$,
(3) $\mathcal{A}\sqsubset\mathcal{B}\Rightarrow H(\mathcal{A})\sqsubset H(\mathcal{B})$,

For every $\mathcal{A}=<A,\lambda>$, the *complement* of $\mathcal{A}$ is defined by $\mathcal{A}^c=\langle A^c,\lambda^c\rangle$, and $\mathcal{A}^{cc}=\mathcal{A}$, combine with the two rough operator $N,\ H$, we have

Proposition 3.4. *Suppose $\mathcal{A}=\langle A,\lambda\rangle$ is a cubic set, $\mathcal{R}=\langle R,r\rangle$ is symmetric, we have*
(1) $H(\mathcal{A}^c)=(N(\mathcal{A}))^c$,
(2) $(H(\mathcal{A}))^c=N(\mathcal{A}^c)$

Proof. (1) For every $x\in X$, we have

$$\begin{aligned}
H(\mathcal{A}^c)(x) &= \bigsqcup_{y\in X}\mathcal{A}^c(y)\sqcap\mathcal{R}(x,y)\\
&= \bigsqcup_{y\in X}(\mathcal{A}(y)\sqcup\mathcal{R}^c(x,y))^c
\end{aligned}$$

$$= (\sqcap_{y\in X}\mathcal{A}(y) \sqcup \mathcal{R}^c(x,y))^c$$
$$= (\sqcap_{y\in X}\mathcal{A}(y) \sqcup \mathcal{R}^c(y,x))^c$$
$$= (N(\mathcal{A})(x))^c.$$

Then $H(\mathcal{A}^c) = (N(\mathcal{A}))^c$.

(2) In the similar way. □

Note that $\mathcal{R} = \langle R, r\rangle$ is said to be *symmetric*, if for every $x,\ y \in X$, $R(x,y) = R(y,x)$, $r(x,y) = r(y,x)$[4].

Combine with the operators P-order, P-union, P-intersection and the two rough operators N, H, we obtain

Proposition 3.5. *Suppose $\mathcal{A} = \langle A, \lambda\rangle, \mathcal{B} = \langle B, \mu\rangle$ are two cubic sets, we have*
(1) $N(\mathcal{A}) \sqcup N(\mathcal{B}) \sqsubset N(\mathcal{A} \sqcup \mathcal{B})$,
(2) $N(\mathcal{A} \sqcap \mathcal{B}) = N(\mathcal{A}) \sqcap N(\mathcal{B})$,
(3) $H(\mathcal{A} \sqcup \mathcal{B}) = H(\mathcal{A}) \sqcup H(\mathcal{B})$,
(4) $H(\mathcal{A} \sqcap \mathcal{B}) \sqsubset H(\mathcal{A}) \sqcap H(\mathcal{B})$.

Proof. (1) Since $A \subset A \cup B, B \subset A \cup B$, $N(\mathcal{A}) \sqcup N(\mathcal{B}) \sqsubset N(\mathcal{A} \sqcup \mathcal{B})$. The converse does not hold (See Example 3.6).

(2) For every $x \in X$,

$$\begin{aligned} N(\mathcal{A} \sqcap \mathcal{B})(x) &= \sqcap_{y\in X}(\mathcal{A} \sqcap \mathcal{B})(y) \sqcup \mathcal{R}^c(y,x) \\ &= \sqcap_{y\in X}(\mathcal{A}(y) \sqcap \mathcal{B}(y)) \sqcup \mathcal{R}^c(y,x) \\ &= \sqcap_{y\in X}(\mathcal{A}(y) \sqcup \mathcal{R}^c(y,x)) \sqcap (\mathcal{B}(y) \sqcup \mathcal{R}^c(y,x)) \\ &= (\sqcap_{y\in X}\mathcal{A}(y) \sqcup \mathcal{R}^c(y,x)) \sqcap (\sqcap_{y\in X}\mathcal{B}(y) \sqcup \mathcal{R}^c(y,x)) \\ &= N(\mathcal{A})(x) \sqcap N(\mathcal{B})(x). \end{aligned}$$

Then $N(\mathcal{A}) \sqcap N(\mathcal{B}) = N(\mathcal{A} \sqcap \mathcal{B})$.

(3) For every $x \in X$,

$$\begin{aligned} H(\mathcal{A} \sqcup \mathcal{B})(x) &= \bigsqcup_{y\in X}(\mathcal{A} \cup \mathcal{B})(y) \sqcap \mathcal{R}(x,y) \\ &= \bigsqcup_{y\in X}(\mathcal{A}(y) \sqcup \mathcal{B}(y)) \sqcap \mathcal{R}(x,y) \\ &= \bigsqcup_{y\in X}(\mathcal{A}(y) \sqcap \mathcal{R}(x,y)) \sqcup (\mathcal{B}(y) \sqcap \mathcal{R}(x,y)) \\ &= (\bigsqcup_{y\in X}\mathcal{A}(y) \sqcap \mathcal{R}(x,y)) \sqcup (\bigsqcup_{y\in X}\mathcal{B}(y) \sqcap \mathcal{R}(x,y)) \\ &= H(\mathcal{A})(x) \sqcup H(\mathcal{B})(x). \end{aligned}$$

Then $H(\mathcal{A}) \sqcup H(\mathcal{B}) = H(\mathcal{A} \sqcup \mathcal{B})$.

(4) Since $\mathcal{A} \sqcap \mathcal{B} \sqsubset \mathcal{A}$, $\mathcal{A} \sqcap \mathcal{B} \sqsubset \mathcal{B}$, $H(\mathcal{A} \sqcap \mathcal{B}) \sqsubset H(\mathcal{A}) \sqcap H(\mathcal{B})$. □

The converse of Proposition 3.5 (1) does not hold (See the following example).

Example 3.6. Let $X = \{a, b\}$ be a set, and $\mathcal{R}$ a cubic relation, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A} \sqcup \mathcal{B}$ cubic sets inn X given, respectively by the following tables:

X	a	b
$\mathcal{A}$	$\langle[0.1, 0.2], 1\rangle$	$\langle[0.6, 0.7], 1\rangle$
$\mathcal{B}$	$\langle[0.3, 0.4], 1\rangle$	$\langle[0.2, 0.3], 1\rangle$
$\mathcal{A} \sqcup \mathcal{B}$	$\langle[0.3, 0.4], 1\rangle$	$\langle[0.6, 0.7], 1\rangle$

Table 3.3

$\mathcal{R}$	a	b
a	$\langle \mathbf{1}, \mathbf{1} \rangle$	$\langle [0, 0.5], 1 \rangle$
b	$\langle [0.5, 1], 1 \rangle$	$\langle \mathbf{1}, \mathbf{1} \rangle$

Table 3.4

Then we have

$$\begin{aligned} N(\mathcal{A})(a) &= (\mathcal{A}(a) \sqcup \mathcal{R}^c(a,a)) \sqcap (\mathcal{A}(b) \sqcup \mathcal{R}^c(a,b)) \\ &= (\langle [0.1, 0.2], 1 > \sqcup \langle \mathbf{0}, \mathbf{0} \rangle) \sqcap (\langle [0.6, 0.7], 1 \rangle \sqcup < [0.5, 1], 0 \rangle) \\ &= \langle [0.1, 0.2], 1 \rangle \sqcap \langle [0.6, 1], 1 \rangle = \langle [0.1, 0.2], 1 \rangle, \\ N(\mathcal{B})(a) &= (\mathcal{B}(a) \sqcup \mathcal{R}^c(b,a)) \sqcap (\mathcal{B}(b) \sqcup \mathcal{R}^c(b,b)) \\ &= (\langle [0.3, 0.4], 1 \rangle \sqcup \langle [0, 0.5], 0 \rangle) \sqcap (\langle [0.2, 0.3], 1 \rangle \sqcup \langle \mathbf{0}, \mathbf{0} \rangle \\ &= \langle [0.3, 0.5], 1 \rangle \sqcap \langle [0.2, 0.3], 1 \rangle = \langle [0.2, 0.3], 1 \rangle, \\ N(\mathcal{A} \sqcup \mathcal{B})(a) &= ((\mathcal{A} \sqcup \mathcal{B})(a) \sqcup \mathcal{R}^c(a,a)) \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \sqcup \mathcal{R}^c(a,b)) \\ &= (\langle [0.3, 0.4], 1 \rangle \sqcup \langle \mathbf{0}, \mathbf{0} \rangle) \sqcap (\langle [0.6, 0.7], 1 \rangle \sqcup \langle [0.5, 1], 0 \rangle) \\ &= \langle [0.3, 0.4], 1 \rangle \sqcap \langle [0.6, 1], 1 \rangle = \langle [0.3, 0.4], 1 \rangle. \end{aligned}$$

Thus we get

$$N(\mathcal{A})(a) \sqcup N(\mathcal{B})(a) = \langle [0.2, 0.3], 1 \rangle \neq \langle [0.3, 0.4], 1 \rangle = N(\mathcal{A} \sqcup \mathcal{B})(a).$$

So $N(\mathcal{A}) \sqcup N(\mathcal{B}) \neq N(\mathcal{A} \sqcup \mathcal{B})$.

Suppose $\{\mathcal{A}_j, j \in J\}$ is a family of cubic sets, about P-order, P-union and P-intersection, we hav$e$

Proposition 3.7. *Suppose $\mathcal{A}_j, j \in J$ is a family of cubic sets, we have*
(1) $\sqcup_{j \in J} N(\mathcal{A}_{j \in J}) \sqsubset N(\sqcup_{j \in J} \mathcal{A}_j)$,
(2) $N(\sqcap_{j \in J} \mathcal{A}_j) = \sqcap_{j \in J} N(\mathcal{A}_{j \in J})$,
(3) $\sqcup_{j \in J} H(\mathcal{A}_{j \in J}) = H(\sqcup_{j \in J} \mathcal{A}_j)$,
(4) $H(\sqcap_{j \in J} \mathcal{A}_j) \sqsubset \sqcap_{j \in J} H(\mathcal{A}_j)$.

So the set $\{N(\mathcal{A}), \sqsubset\}$ forms a $\sqcap$-semilattice with the maximal element $\hat{1}$. The set $\{H(\mathcal{A}), \sqsubset\}$ also forms a $\sqcup$-semilattice with the minimal element $\hat{0}$.

We consider a special case. $\mathcal{R}$ is reflexive.

A cubic relation $\mathcal{R} = \langle R, r \rangle$ is said to be $reflexive$, if for every $x \in X$, $R(x,x) = \mathbf{1}$, $r(x,x) = 1$ [4], i.e., $\mathcal{R}(x,x) = \hat{1}$.

Proposition 3.8. *Suppose $\mathcal{A} =< A, \lambda >$ is a cubic set, $\mathcal{R} =< R, r >$ is reflexive, we have*
(1) $N(\mathcal{A}) \sqsubset \mathcal{A}$,
(2) $\mathcal{A} \sqsubset H(\mathcal{A})$.

Proof. (1) For each $x \in X$,

$$\begin{aligned} N(\mathcal{A})(x) &= \sqcap_{y \in X} \mathcal{A}(y) \sqcup \mathcal{R}^c(y,x) \\ &\sqsubset \mathcal{A}(x) \sqcup \mathcal{R}^c(x,x) \\ &= \mathcal{A}(x) \sqcup \langle \mathbf{1}, \mathbf{1} \rangle^c \\ &= \mathcal{A}(x) \sqcup \langle \mathbf{0}, \mathbf{0} \rangle \\ &= \mathcal{A}(x). \end{aligned}$$

Then $N(\mathcal{A}) \sqsubset \mathcal{A}$.

(2) For each $x \in X$,

$$\begin{aligned} H(\mathcal{A})(x) &= \bigsqcup_{y\in X} \mathcal{A}(y) \sqcap \mathcal{R}(x,y) \\ &\sqsupset \mathcal{A}(x) \sqcap \mathcal{R}(x,x) \\ &= \mathcal{A}(x) \sqcap \langle \mathbf{1}, \mathbf{1}\rangle \\ &= \mathcal{A}(x). \end{aligned}$$

Then $\mathcal{A} \sqsubset H(\mathcal{A})$. □

3.2. **Rough cubic sets II.** In the section, we define rough cubic sets with these operators R-order, R-union and R-intersection.

Definition 3.9. Suppose X is a universe set, $\mathcal{R} =< R, r >$ is a cubic relation, and two rough operators N_R, H_R are defined as follows, for every cubic set $\mathcal{A} = \langle A, \lambda\rangle$ in X,

$$N_R(\mathcal{A})(x) = \Cap_{y\in X} \mathcal{A}(y) \Cup \mathcal{R}^c(y,x), \quad H_R(\mathcal{A})(x) = \Cup_{y\in X} \mathcal{A}(y) \Cap \mathcal{R}(x,y).$$

If $N_R(\mathcal{A}) = H_R(\mathcal{A})$, then $\mathcal{A}$ is called a *definable cubic set*; otherwise, $\mathcal{A}$ is called an *undefinable cubic set*. $(N_R(\mathcal{A}), H_R(\mathcal{A}))$ is referred to as a pair of cubic rough set [31].

Next, we give an example to computer $N_R(\mathcal{A}), H_R(\mathcal{A})$.

Example 3.10. Suppose X, $\mathcal{A}, \mathcal{R}$ defined as Example 3.2. Then

$$\begin{aligned} N_R(\mathcal{A})(a) &= \Cap_{y\in X} \mathcal{A}(y) \Cup \mathcal{R}^c(y,a) \\ &= (\mathcal{A}(a) \Cup \mathcal{R}^c(a,a)) \Cap (\mathcal{A}(b) \Cup \mathcal{R}^c(b,a)) \Cap (\mathcal{A}(c) \Cup \mathcal{R}^c(c,a)) \\ &= (\langle[0.3,0.7],0.5\rangle \Cup \langle[0.3,0.7],0.6\rangle^c) \Cap (\langle[0.3,0.4],0.7\rangle \\ &\quad \Cup \langle[0.1,0.6],0.8\rangle^c) \Cap (\langle[0.1,0.6],0.3\rangle \Cup \langle[0.4,0.9],0.4\rangle^c) \\ &= (\langle[0.3,0.7],0.5\rangle \Cup \langle[0.3,0.7],0.4\rangle) \Cap (\langle[0.3,0.4],0.7\rangle \\ &\quad \Cup \langle[0.4,0.9],0.2\rangle) \Cap (\langle[0.1,0.6],0.3\rangle \Cup \langle[0.1,0.6],0.6\rangle) \\ &= \langle[0.3,0.7],0.4\rangle \Cap \langle[0.4,0.9],0.2\rangle \Cap \langle[0.1,0.6],0.3\rangle \\ &= \langle[0.1,0.6],0.4\rangle, \\ H_R(\mathcal{A})(a) &= \Cup_{y\in X} \mathcal{A}(y) \Cap \mathcal{R}(a,y) \\ &= (\mathcal{A}(a) \Cap \mathcal{R}(a,a)) \Cup (\mathcal{A}(b) \Cap \mathcal{R}(a,b)) \Cup (\mathcal{A}(c) \Cap \mathcal{R}(a,c)) \\ &= (\langle[0.3,0.7],0.5\rangle \Cap \langle[0.3,0.7],0.6\rangle) \Cup (\langle[0.3,0.4],0.7\rangle \\ &\quad \Cap \langle[0.4,0.8],0.4\rangle) \Cup (\langle[0.1,0.6],0.3\rangle \Cap \langle[0.1,0.6],0.7\rangle \\ &= \langle[0.3,0.7],0.6\rangle \Cup \langle[0.3,0.4],0.7\rangle \Cup \langle[0.1,0.6],0.7\rangle \\ &= \langle[0.3,0.7],0.6\rangle. \end{aligned}$$

Similarly to Proposition 3.3, for R-order, we have

Proposition 3.11. (1) $N_R(\ddot{1}) = \ddot{1}$*,* $H_R(\ddot{0}) = \ddot{0}$*,*
(2) $\mathcal{A} \Subset \mathcal{B} \Rightarrow N_R(\mathcal{A}) \Subset N_R(\mathcal{B})$*,*
(3) $\mathcal{A} \Subset \mathcal{B} \Rightarrow H_R(\mathcal{A}) \Subset H_R(\mathcal{B})$*.*

Propositions 3.4 also holds for the R-order.

Proposition 3.12. *Suppose* $\mathcal{A} = \langle A, \lambda\rangle$ *is a cubic set and* $\mathcal{R} = \langle R, r\rangle$ *is symmetric. Then we have*
(1) $H_R(\mathcal{A}^c) = (N_R(\mathcal{A}))^c$*,*
(2) $(H_R(\mathcal{A}))^c = N_R(\mathcal{A}^c)$*.*

About R-order, R-union, R-intersection, we have the following proposition.

Proposition 3.13. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are two cubic sets. Then we have*
(1) $N_R(\mathcal{A}) \Cup N_R(\mathcal{B}) \Subset N_R(\mathcal{A} \Cup \mathcal{B})$,
(2) $H_R(\mathcal{A}) \Cup H_R(\mathcal{B}) = H_R(\mathcal{A} \Cup \mathcal{B})$,
(3) $N_R(\mathcal{A} \Cap \mathcal{B}) = N_R(\mathcal{A}) \Cap N_R(\mathcal{B})$,
(4) $H_R(\mathcal{A} \Cap \mathcal{B}) \Subset H_R(\mathcal{A}) \Cap H_R(\mathcal{B})$.

Proof. Similarly to Proposition 3.5. □

Furthermore, suppose $\{\mathcal{A}_j, j \in J\}$ is a family of cubic sets, about R-order, R-union and R-intersection, we also have

Proposition 3.14. *Suppose $\mathcal{A}_j, j \in J$ is a family of cubic sets. Then we have*
(1) $\Cup_{j\in J} N_R(\mathcal{A}_{j\in J}) \Subset N_R(\Cup_{j\in J}\mathcal{A}_j)$,
(2) $\Cup_{j\in J} H_R(\mathcal{A}_{j\in J}) = H_R(\Cup_{j\in J}\mathcal{A}_j)$,
(3) $N_R(\Cap_{j\in J}\mathcal{A}_j) = \Cap_{j\in J} N_R(\mathcal{A}_{j\in J})$,
(4) $H_R(\Cap_{j\in J}\mathcal{A}_j) \Subset \Cap_{j\in J} H_R(\mathcal{A}_{j\in J})$.

Clearly, $\{N_R(\mathcal{A}), \Subset\}$ forms a $\Cap-$semilattice with the maximal element $\ddot{1}$, and $\{H_R(\mathcal{A}), \Subset\}$ forms a $\Cup-$semilattice with minimal element $\ddot{0}$.

Similarly to Proposition 3.8, we also have

Proposition 3.15. *Suppose $\mathcal{A} = \langle A, \lambda\rangle$ is a cubic set and $\mathcal{R} = \langle R, r\rangle$ is a cubic relation satisfying the condition:*

$$\mathcal{R}(x, x) = \langle \mathbf{1}, \mathbf{0}\rangle = \ddot{1} \ for\ every\ x \in X.$$

Then we have
(1) $N_R(\mathcal{A}) \Subset \mathcal{A}$,
(2) $\mathcal{A} \Subset H_R(\mathcal{A})$.

Proof. (1) For each $x \in X$,

$$\begin{aligned} N_R(\mathcal{A})(x) &= \Cap_{y\in X}\mathcal{A}(y) \Cup \mathcal{R}^c(y, x) \\ &\Subset \mathcal{A}(x) \Cup \mathcal{R}^c(x, x) \\ &= \mathcal{A}(x) \Cup \langle \mathbf{1}, \mathbf{0}\rangle^{\ c} \\ &= \mathcal{A}(x) \Cup \langle \mathbf{0}, \mathbf{1}\rangle \\ &= \mathcal{A}(x). \end{aligned}$$

Then $N_R(\mathcal{A}) \Subset \mathcal{A}$.

(2) For each $x \in X$,

$$\begin{aligned} H_R(\mathcal{A})(x) &= \Cup_{y\in X}\mathcal{A}(y) \Cap \mathcal{R}(x, y) \\ &\Supset \mathcal{A}(x) \Cap \mathcal{R}(x, x) \\ &= \mathcal{A}(x) \Cap \langle \mathbf{1}, \mathbf{0}\rangle \\ &= \mathcal{A}(x). \end{aligned}$$

Then $\mathcal{A} \Subset H_R(\mathcal{A})$. □

4. Note

In Definition 3.1, for a cubic set $\mathcal{A}$, we define two rough operators $N(\mathcal{A})$ and $H(\mathcal{A})$. In the follows, we give a equivalence definition.

Definition 4.1. Suppose X is a universe set, $\mathcal{R} =< R, r >$ is a cubic relation and two rough operators N, H are defined as follows: for every cubic set $\mathcal{A} = \langle A, \lambda\rangle$ in X,

$$N(\mathcal{A})(x) = \langle N(A)(x), N(\lambda)(x) \rangle,$$

$$H(\mathcal{A})(x) = \langle H(A)(x), H(\lambda)(x) \rangle,$$

where

$$N(A)(x) = \bigwedge_{y \in X} A(y) \vee R^c(y,x), \quad N(\lambda)(x) = \bigwedge_{y \in X} \lambda(y) \vee r^c(y,x),$$

$$H(A)(x) = \bigvee_{y \in X} A(y) \wedge R(x,y), \quad H(\lambda)(x) = \bigvee_{y \in X} \lambda(y) \wedge r(x,y).$$

We investigate the relation between $N(A)$, $H(A)$ and $N(A^-), N(A^+)$, $H(A^-)$, $H(A^+)$.

Proposition 4.2. *Suppose $\mathcal{A} = \langle A, \lambda \rangle$ is a cubic set, $A = [A^-, A^+]$. Then we have*
(1) $(N(A))^- = N(A^-)$, $(N(A))^+ = N(A^+)$,
(2) $(H(A))^- = H(A^-)$, $(H(A))^+ = H(A^+)$.

Proof. (1) For every $x \in X$.

$$\begin{aligned}
&[N(A))^-(x), N(A))^+(x)] \\
&= N(A(x)) \\
&= \bigwedge_{y \in Y} A(y) \vee R^c(y,x) \\
&= \bigwedge_{y \in Y} [A^-(y), A^+(y)] \vee [(R^c)^-(y,x), (R^c)^+(y,x)] \\
&= \bigwedge_{y \in Y} [A^-(y) \vee [(R^c)^-(y,x), A^+(y) \vee (R^c)^+(y,x)] \\
&= [\bigwedge_{y \in Y} A^-(y) \vee (R^c)^-(y,x), \bigwedge_{y \in Y} A^+(y) \vee (R^c)^+(y,x)] \\
&= [N(A^-)(x), N(A^+)(x)].
\end{aligned}$$

Then $(N(A))^- = N(A^-)$, $(N(A))^+ = N(A^+)$.
(2) In the similar way. □

5. Conclusion

In the paper, we define two rough operators on a cubic set induced by a cubic relation, give some examples, and investigate some of their properties with respect to the P-order $\sqsubset$, P-union $\sqcup$, P-intersection $\sqcap$, and R-order $\Subset$, R-union $\Cup$, R-intersection $\Cap$ on cubic sets, provide a new platform to further study.

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