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Abstract. Cubic crisp sets due to classical sets and cubic sets are an important new set theory keeping in mind the underlying dynamics of cubic sets and crisp sets. However, despite its usage importance, not only for theory but also for application, the usage of combine crisp and cubic sets have been overlooked. This dissertation aims to provide insight into a cubic crisp sets and their applications to topology. Firstly, the new concept cubic crisp set is defined and some of its algebraic structures are obtained. In section 4, a cubic topology is defined. In the remainder of this section, we study some properties of cubic crisp topology, with the eventual aim of describing cubic crisp topology directly using cubic crisp closure (interior). These results that are obtained in this section, therefore describe the very close connection between P-cubic crisp topology, P-cubic crisp countable topology, internal P-cubic crisp topology and external P-cubic crisp topology, we are aiming to classify. Examinations with examples of the cubic crisp topology is stated in Section 5. We then considered cubic crisp neighborhood which takes into account the cubic crisp points that are defined in section 6. Moreover, we lay the foundations for a new study of cubic crisp subspaces in Section 7. Throughout this paper, we wish to build up passion in the systematic study of the cubic crisp sets with respect to defined cubic crisp topologies on it.

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Keywords: Cubic crisp set, Internal (external) cubic crisp set, Cubic crisp (vanishing) point, Cubic crisp topological space, Cubic crisp P-base (P-subbase), Cubic crisp neighborhood, Cubic crisp closure (interior), Cubic crisp continuity, Cubic crisp subspace.

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1. Introduction

In order to solve the complexation and uncertainty in real world, Zadeh [1] proposed a fuzzy set as the generalization of classical sets in 1965. After that time, numerous mathematicians have introduced the various notions solving various complex and uncertain problems, for example, Zadeh [2](1975), Pawlak [3](1982), Atanassov [4](1983), Atanassov and Gargov [5](1989), Gau and Buychrer [6](1993), Smarandache [7](1998), Molodtsov [8](1999), Lee [9](2000), Torra [10](2010), Jun et al. [11](2012), and Kim et al. [12, 13](2020) introduced the concept of interval-valued fuzzy sets, rough sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, vague sets, neutrosophic sets, soft sets, bipolar fuzzy sets, hesitant fuzzy sets, cubic sets combined by interval-valued fuzzy sets and fuzzy sets, and octahedron sets combined by interval-valued fuzzy sets, intuitionistic fuzzy sets, and fuzzy sets, IVI-octahedron sets combined by interval-valued intuitionistic fuzzy sets, intuitionistic fuzzy sets and fuzzy sets, in turn.

In 1999, Çoker [14] introduced the concept of an intuitionistic set (called an intuitionistic crisp set by Salama et al. [15]) as the generalzation of an ordinary set and the specialization of an intuitionistic fuzzy set. After then, many researchers applied the notion to topology (See [16, 17, 18, 19, 20, 22, 23, 24]) and category theory (See [25, 26]). In particular, Kim et al. [27] discussed with intuitionistic hyperspaces. Recently, Kim et al [28] dealt with further properties of interval-valued sets (by introduced bu Yao [29]) as the generalization of classical sets and the special case of interval-valued fuzzy set proposed by Zadeh [2] and applied it to topological structures. Jo et al. [30] proposed the notion of interval-valued neutrosophic crisp sets as the generalization of classical sets and the special case of interval-valued neutrosophic sets introduced by Wang et al. [31] and investigated its various topological structures. Moreover, Chae et al. [32] define an interval-valued intuitionistic set as the generalization of ordinary sets and the specialization of interval-valued intuitionistic fuzzy sets by proposed by Atanassov and Gargov [5]. J. Kim et al. [33] introduced the concept of intuitionistic neutrosophic sets as the generalization of classical sets and the special case of intuitionistic neutrosophic sets and discussed with various properties of its topological structures.

The purpose of our study is to introduce a new concept that will provide a tool for modeling and processing partially known concepts by competing interval-valued sets and classical sets as the generalization of ordinary sets and the special case of cubic sets defined by Jun et al. [11], and to deal with its various topological structures. To accomplish such research, this paper is composed of seven sections. In Section 2, we recall some concepts and one result related to interval-valued sets. In Section 3, we introduce the new concept of cubic crisp sets and obtain some of its algebraic structures. In Section 4, we define a cubic topology and find some properties of its topological structures, give some examples. In Section 5, we propose the notion of cubic crisp points of two types and discuss with the characterizations of inclusions, intersections and unions of cubic crisp sets. Also, we introduce the concepts of cubic crisp base and subbase, and find some of their properties. Furthermore, by using cubic crisp points, we define a cubic crisp neighborhood of two types and obtain some of its properties, and give some examples. In Section 6, we define cubic crisp interiors and closures an obtain some of their properties. Also, we define a cubic crisp continuity and find its various properties. In Section 6, we introduce the notion of cubic crisp subspaces and obtain some of its properties.

2. Preliminaries

In this section, we recall some definitions and one result proposed and obtained by [28, 29].

Definition 2.1 ([29])**.** Let X be an non-empty set. Then the form

$$[A^-, A^+] = \{B : A^- \subset B \subset A^+\}$$

is called an interval-valued sets (briefly, IVS) in X, where $A^-,\ A^+ \subset X$ and $A^- \subset A^+$. In particular, $[\varnothing, \varnothing]$ [resp. $[X, X]$] is called the interval-valued empty [resp. whole] set in X and denoted by $\widetilde{\varnothing}$ [resp. $\widetilde{X}$].

We will denote the set of all IVSs in X as $IVS(X)$.

It is obvious that $[A, A] \in IVS(X)$ for classical subset A of X. Then we can consider an IVS in X as the generalization of a classical subset of X. Furthermore, if $A = [A^-, A^+] \in IVS(X)$, then $\chi_A = [\chi_{A^-}, \chi_{A^+}]$ is an interval-valued fuzzy set in X introduced by Zadeh [2]. Thus we can consider an interval-valued fuzzy set as the generalization of an IVS.

Definition 2.2 ([28, 29])**.** Let X be a non-empty set and let $A,\ B \in IVS(X)$. Then

(i) we say that A contained in B, denoted by $A \subset B$, if $A^- \subset B^-$ and $A^+ \subset B^+$,

(ii) we say that A equal to B, denoted by $A = B$, if $A \subset B$ and $B \subset A$,

(iii) the complement of A, denoted A^c, is an interval-valued set in X defined by:

$$A^c = [(A^+)^c, (A^-)^c],$$

(iv) the union of A and B, denoted by $A \cup B$, is an interval-valued set in X defined by:

$$A \cup B = [A^- \cup B^-, A^+ \cup B^+],$$

(v) the intersection of A and B, denoted by $A \cap B$, is an interval-valued set in X defined by:

$$A \cap B = [A^- \cap B^-, A^+ \cap B^+].$$

Definition 2.3 ([28, 29])**.** Let $(A_j)_{j \in J}$ be a family of members of $IVS(X)$. Then

(i) the intersection of $(A_j)_{j \in J}$, denoted by $\bigcap_{j \in J} A_j$, is an IVS in X defined by:

$$\bigcap_{j \in J} A_j = [\bigcap_{j \in J} A_j^-, \bigcap_{j \in J} A_j^+],$$

(ii) the union of $(A_j)_{j \in J}$, denoted by $\bigcup_{j \in J} \widetilde{A}_j$, is an IVS in X defined by:

$$\bigcup_{j \in J} A_j = [\bigcup_{j \in J} A_j^-, \bigcup_{j \in J} A_j^+].$$

Definition 2.4 ([28])**.** Let X be a non-empty set, let $a \in X$ and let $A \in IVS(X)$. Then the form $[\{a\}, \{a\}]$ [resp. $[\varnothing, \{a\}]$] is called an interval-valued [resp. vanishing] point in X and denoted by $a_{_{IVP}}$ [resp. $a_{_{IVVP}}$]. We will denote the set of all interval-valued points in X as $IVP(X)$.

(i) We say that $a_{_{IVP}}$ belongs to A, denoted by $a_{_{IVP}} \in A$, if $a \in A^-$.
(ii) We say that $a_{_{IVVP}}$ belongs to A, denoted by $a_{_{IVVP}} \in A$, if $a \in A^+$.

Result 2.5 ([28], Proposition 3.11)**.** *Let X be a non-empty set and let $A \in IVS(X)$. Then*

$$A = A_{IVP} \cup A_{IVVP},$$

where $A_{IVP} = \bigcup_{a_{_{IVP}} \in A} a_{_{IVP}}$ *and* $A_{IVVP} = \bigcup_{a_{_{IVVP}} \in A} a_{_{IVVP}}$.
In fact, $A_{IVP} = [A^-, A^-]$ *and* $A_{IVVP} = [\varnothing, A^+]$.

Definition 2.6 ([28])**.** Let $X,\ Y$ be two non-empty sets, let $f : X \to Y$ be a mapping and let $A \in IVS(X),\ B \in IVS(Y)$.
(i) The image of A under f, denoted by $f(A)$, is an IVS in Y defined as:

$$f(A) = [f(A^-), f(A^+)].$$

(ii) The preimage of B under f, denoted by $f^{-1}(B)$, is an IVS in X defined as:

$$f^{-1}(B) = [f^{-1}(B^-), f^{-1}(B^+)].$$

It is obvious that $f(a_{_{IVP}}) = f(a)_{_{IVP}}$ and $f(a_{_{IVVP}}) = f(a)_{_{IVVP}}$ for each $a \in X$.

3. Cubic crisp sets

In this section, we introduce the concept of cubic crisp sets and study some of its properties.

Definition 3.1. Let X be a non-empty set. Then the form

$$\mathcal{A} = \langle \mathbf{A}, A \rangle$$

is called a cubic crisp set (briefly, CCS) in X,
where $\mathbf{A} = [A^-, A^+] \in IVS(X)$ and $A \in 2^X$.

A CCS $\mathcal{A} = \langle \mathbf{A}, A \rangle$ with $\mathbf{A} = \widetilde{X},\ A = X$ [resp. $\mathbf{A} = \widetilde{X},\ A = \varnothing$; $\mathbf{A} = \widetilde{\varnothing},\ A = X$ and $\mathbf{A} = \widetilde{\varnothing},\ A = \varnothing$] is denoted by $\hat{X}$ [resp. $\ddot{X}$, $\ddot{\varnothing}$ and $\hat{\varnothing}$].In particular, $\hat{X}$ [resp. $\hat{\varnothing}$] is called a cubic crisp whole [resp. empty] set in X. We will denote the set of all CCSs in X as $CCS(X)$.

It is obvious that $\langle [A, A], A \rangle \in CCS(X)$ for a classical subset A of X. Then we can consider an CCS in X as the generalization of a classical subset of X. If $\mathcal{A} = \langle \mathbf{A}, A \rangle \in CCS(X)$, then $\chi_{\mathcal{A}} = \langle \chi_{A\mathbf{A}}, \chi_A \rangle$ is a cubic set in X proposed by Jun et al. (See [11]). Thus we can consider a CCS as the special case of a cubic set.

Example 3.2. (1) Let $X = \{a, b\}$. Then we can easily obtain all members of $CCS(X)$:

$$\hat{\varnothing},\ \left\langle \widetilde{\varnothing}, \{a\} \right\rangle,\ \left\langle \widetilde{\varnothing}, \{b\} \right\rangle,\ \ddot{\varnothing} = \left\langle \widetilde{\varnothing}, X \right\rangle,\ \langle [\varnothing, \{a\}], \varnothing \rangle,\ \langle [\varnothing, \{a\}], \{a\} \rangle,$$
$$\langle [\varnothing, \{a\}], \{b\} \rangle,\ \langle [\varnothing, \{a\}], X \rangle,\ \langle [\{a\}, \{a\}], \varnothing \rangle,\ \langle [\{a\}, \{a\}], \{a\} \rangle,\ \langle [\{a\}, \{a\}], \{b\} \rangle,$$
$$\langle [\{a\}, \{a\}], X \rangle,\ \langle [\{b\}, \{b\}], \varnothing \rangle,\ \langle [\{b\}, \{b\}], \{a\} \rangle,\ \langle [\{b\}, \{b\}], \{b\} \rangle,\ \langle [\{b\}, \{b\}], X \rangle,$$
$$\langle [\varnothing, X], \varnothing \rangle,\ \langle [\varnothing, X], \{a\} \rangle,\ \langle [\varnothing, X], \{b\} \rangle,\ \langle [\varnothing, X], X \rangle,\ \langle [\{a\}, X], \varnothing \rangle,$$
$$\langle [\{a\}, X], \{a\} \rangle\ \langle [\{a\}, X], \{b\} \rangle,\ \langle [\{a\}, X], X \rangle,\ \langle [\{b\}, X], \varnothing \rangle,\ \langle [\{b\}, X], \{a\} \rangle,$$
$$\langle [\{b\}, X], \{b\} \rangle,\ \langle [\{b\}, X], X \rangle,\ \ddot{X} = \left\langle \widetilde{X}, \varnothing \right\rangle,\ \left\langle \widetilde{X}, \{a\} \right\rangle,\ \left\langle \widetilde{X}, \{b\} \right\rangle, \hat{X}.$$

(2) Let X a non-empty set and let $\mathbf{A} \in IVS(X)$. Then clearly,

$$\langle \mathbf{A}, \varnothing \rangle, \ \langle \mathbf{A}, X \rangle, \ \langle \mathbf{A}, A \rangle, \ \left\langle \mathbf{A}, A^{-c} \right\rangle, \ \left\langle \mathbf{A}, A^{+c} \right\rangle$$

are CCSs in X, where $A^- \subset A \subset A^+$.

For any sets A and B, the family $\{X : A \subsetneqq X \subsetneqq B\}$ is called an open interval of sets and denoted by (A, B).

Definition 3.3. Let X be a non-empty set and let $\mathcal{A} = \langle \mathbf{A}, A \rangle \in CCS(X)$. Then $\mathcal{A}$ is called:

(i) an internal cubic crisp set (briefly, ICCS) in X, if $A^- \subset A \subset A^+$,

(ii) an external cubic crisp set (briefly, ECCS) in X, if $A \notin (A^-, A^+)$, i.e., either $A \subset A^-$ or $A \supset A^+$.

We will denote the set of all ICCSs [resp. SICCSs and ECCSs] in X as

$ICCS(X)$ [resp. $SICCS(X)$ and $ECCS(X)$].

It is obvious that $\hat{\varnothing}, \hat{X} \in ICCS(X) \cap ECCS(X)$ and $\ddot{\varnothing}, \ddot{X} \in ECCS(X)$.

Example 3.4. In Example 3.2 (2), we can easily check that $\langle \mathbf{A}, A \rangle \in ICCS(X)$ and $\left\langle \mathbf{A}, A^{-c} \right\rangle, \ \left\langle \mathbf{A}, A^{+c} \right\rangle \in ECCS(X)$.

Proposition 3.5. *Let $\mathcal{A} = \langle \mathbf{A}, A \rangle \in CCS(X)$ such that $\mathcal{A} \notin ECCS(X)$. Then $A^- \subset A \subset A^+$.*

Proof. Straightforward. □

Definition 3.6. Let X be a non-empty set and let $\mathcal{A} = \langle \mathbf{A}, A \rangle, \ \mathcal{B} = \langle \mathbf{B}, B \rangle \in CCS(X)$. Then we define

(i) (Equality) $\mathcal{A} = \mathcal{B} \Leftrightarrow \mathbf{A} = \mathbf{B}$ and $A = B$,

(ii) (P-order) $\mathcal{A} \subset_P \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B}$ and $A \subset B$,

(iii) (R-order) $\mathcal{A} \subset_P \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B}$ and $A \supset B$.

It is obvious that $\hat{\varnothing} \subset_P \ddot{\varnothing} \subset_P \hat{X}, \ \ddot{X} \subset_P \hat{X}$ and $\ddot{\varnothing} \subset_R \hat{\varnothing} \subset_R \ddot{X}, \ \hat{X} \subset_R \hat{X}$.

Definition 3.7. Let $(\mathcal{A}_j)_{j \in J} = \langle \mathbf{A}_j, A_j \rangle \subset CCS(X)$, where J is an index set. Then the union and intersection of $(\mathcal{A}_j)_{j \in J}$ are defined as follows:

(i) (P-union) $\bigcup_{P, \ j \in J} \mathcal{A}_j = \left\langle \bigcup_{j \in J} \mathbf{A}_j, \bigcup_{j \in J} A_j \right\rangle$,

(ii) (P-intersection) $\bigcap_{P, \ j \in J} \mathcal{A}_j = \left\langle \bigcap_{j \in J} \mathbf{A}_j, \bigcap_{j \in J} A_j \right\rangle$,

(iii) (R-union) $\bigcup_{R, \ j \in J} \mathcal{A}_j = \left\langle \bigcup_{j \in J} \mathbf{A}_j, \bigcap_{j \in J} A_j \right\rangle$,

(iv) (R-intersection) $\bigcap_{R, \ j \in J} \mathcal{A}_j = \left\langle \bigcap_{j \in J} \mathbf{A}_j, \bigcup_{j \in J} A_j \right\rangle$.

Definition 3.8. Let X be a non-empty set and let $\mathcal{A} = \langle \mathbf{A}, A \rangle \in CCS(X)$. Then the complement of $\mathcal{A}$, denoted by $\mathcal{A}^c$, is defined as follows:

$$\mathcal{A}^c = \langle \mathbf{A}^c, A^c \rangle = \left\langle [A^{+c}, A^{-c}], A^c \right\rangle.$$

Example 3.9. Let $X = \{a, b, c\}$. Consider two CCSs $\mathcal{A} = \langle [\{a\}, \{a, b\}], \{c\} \rangle, \ \mathcal{B} = \langle [\{b\}, \{b, c\}], \{a\} \rangle$. Then clearly, we have

$$\mathcal{A}^c = \langle [\{c\}, \{b, c\}], \{a, b\} \rangle, \ \mathcal{B}^c = \langle [\{a\}, \{a, c\}], \{a\} \rangle,$$

$$A \cup_P B = \langle [\{a, b\}, X], \{a, c\} \rangle, \ A \cap_P B = \langle [\varnothing, \{c\}], \{a\} \rangle,$$

$$A \cup_R B = \langle [\{a, b\}, X], \varnothing \rangle, \ A \cap_R B = \langle [\varnothing, \{c\}], \{a, c\} \rangle.$$

The followings are immediate results of Definitions 3.1, 3.6, 3.7 and 3.8.

Proposition 3.10 (See [11], Proposition 3.10)**.** *Let X be a non-empty set and let $\mathcal{A},\ \mathcal{B},\ \mathcal{C} \in CCS(X)$. Then*

(1) $\hat{\varnothing} \subset_P \mathcal{A} \subset \hat{X}$,
(2) *if* $\mathcal{A} \subset_P \mathcal{B}$ *and* $\mathcal{B} \subset_P \mathcal{C}$, *then* $\mathcal{A} \subset_P \mathcal{C}$,
(3) *if* $\mathcal{A} \subset_P \mathcal{B}$, *then* $\mathcal{B}^c \subset_P \mathcal{A}^c$,
(4) *if* $\mathcal{A} \subset_P \mathcal{B}$ *and* $\mathcal{A} \subset_P \mathcal{C}$, *then* $\mathcal{A} \subset_P \mathcal{B} \cap_P \mathcal{C}$,
(5) *if* $\mathcal{A} \subset_P \mathcal{B}$ *and* $\mathcal{C} \subset_P \mathcal{B}$, *then* $\mathcal{A} \cup_P \mathcal{C} \subset_P \mathcal{B}$,
(6) *if* $\mathcal{A} \subset_R \mathcal{B}$ *and* $\mathcal{B} \subset_R \mathcal{C}$, *then* $\mathcal{A} \subset_R \mathcal{C}$,
(7) *if* $\mathcal{A} \subset_R \mathcal{B}$, *then* $\mathcal{B}^c \subset_R \mathcal{A}^c$,
(8) *if* $\mathcal{A} \subset_R \mathcal{B}$ *and* $\mathcal{A} \subset_R \mathcal{C}$, *then* $\mathcal{A} \subset_R \mathcal{B} \cap_R \mathcal{C}$,
(9) *if* $\mathcal{A} \subset_R \mathcal{B}$ *and* $\mathcal{C} \subset_R \mathcal{B}$, *then* $\mathcal{A} \cup_R \mathcal{C} \subset_R \mathcal{B}$.

Proposition 3.11. *Let X be a non-empty set, let $\mathcal{A},\ \mathcal{B},\ \mathcal{C} \in CCS(X)$ and let $(\mathcal{A}_j)_{j\in J} \subset CCS(X)$. Then*

(1) (Idempotent laws) $\mathcal{A} \cup_P \mathcal{A} = \mathcal{A},\ \mathcal{A} \cap_P \mathcal{A} = \mathcal{A},\ \mathcal{A} \cup_R \mathcal{A} = \mathcal{A},\ \mathcal{A} \cap_R \mathcal{A} = \mathcal{A}$,
(2) (Commutative laws) $\mathcal{A} \cup_P \mathcal{B} = \mathcal{B} \cup_P \mathcal{A},\ \mathcal{A} \cap_P \mathcal{B} = \mathcal{B} \cap_P \mathcal{A}$,
$\mathcal{A} \cup_R \mathcal{B} = \mathcal{B} \cup_R \mathcal{A},\ \mathcal{A} \cap_R \mathcal{B} = \mathcal{B} \cap_R \mathcal{A}$,
(3) (Associative laws) $\mathcal{A} \cup_P (\mathcal{B} \cup_P \mathcal{C}) = (\mathcal{A} \cup_P \mathcal{B}) \cup_P \mathcal{C}$,
$\mathcal{A} \cap_P (\mathcal{B} \cap_P \mathcal{C}) = (\mathcal{A} \cap_P \mathcal{B}) \cap_P \mathcal{A}\mathcal{C}$,
$\mathcal{A} \cup_R (\mathcal{B} \cup_R \mathcal{C}) = (\mathcal{A} \cup_R \mathcal{B}) \cup_R \mathcal{C}$,
$\mathcal{A} \cap_P (\mathcal{B} \cap_R \mathcal{C}) = (\mathcal{A} \cap_R \mathcal{B}) \cap_R \mathcal{A}\mathcal{C}$,
(4) (Distributive laws) $\mathcal{A} \cup_P (\mathcal{B} \cap_P \mathcal{C}) = (\mathcal{A} \cup_P \mathcal{B}) \cap_P (\mathcal{A} \cup_P \mathcal{C})$,
$\mathcal{A} \cap_P (\mathcal{B} \cup_P \mathcal{C}) = (\mathcal{A} \cap_P \mathcal{B}) \cup_P (\mathcal{A} \cap_P \mathcal{C})$,
$\mathcal{A} \cup_R (\mathcal{B} \cap_R \mathcal{C}) = (\mathcal{A} \cup_R \mathcal{B}) \cap_R (\mathcal{A} \cup_R \mathcal{C})$,
$\mathcal{A} \cap_R (\mathcal{B} \cup_R \mathcal{C}) = (\mathcal{A} \cap_R \mathcal{B}) \cup_R (\mathcal{A} \cap_R \mathcal{C})$,
$\mathcal{A} \cup_P (\bigcap_{P,\ j\in J} \mathcal{A}_j) = \bigcap_{P,\ j\in J}(\mathcal{A} \cup_P \mathcal{A}_j)$,
$\mathcal{A} \cap_P (\bigcup_{P,\ j\in J} \mathcal{A}_j) = \bigcup_{P,\ j\in J}(\mathcal{A} \cap_P \mathcal{A}_j)$,
$\mathcal{A} \cup_R (\bigcap_{R,\ j\in J} \mathcal{A}_j) = \bigcap_{R,\ j\in J}(\mathcal{A} \cup_R \mathcal{A}_j)$,
$\mathcal{A} \cap_R (\bigcup_{R,\ j\in J} \mathcal{A}_j) = \bigcup_{R,\ j\in J}(\mathcal{A} \cap_R \mathcal{A}_j)$,
(5) (Absorption laws) $\mathcal{A} \cup_P (\mathcal{A} \cap_P \mathcal{B}) = \mathcal{A},\ \mathcal{A} \cap_P (\mathcal{A} \cup_P \mathcal{B}) = \mathcal{A}$,
$\mathcal{A} \cup_R (\mathcal{A} \cap_R \mathcal{B}) = \mathcal{A},\ \mathcal{A} \cap_R (\mathcal{A} \cup_R \mathcal{B}) = \mathcal{A}$,
(6) (DeMorgan's laws) $(\mathcal{A} \cup_P \mathcal{B})^c = \mathcal{A}^c \cap_P \mathcal{B}^c,\ (\mathcal{A} \cap_P \mathcal{B})^c = \mathcal{A}^c \cup_P \mathcal{B}^c$,
$(\mathcal{A} \cup_R \mathcal{B})^c = \mathcal{A}^c \cap_R \mathcal{B}^c,\ (\mathcal{A} \cap_R \mathcal{B})^c = \mathcal{A}^c \cup_R \mathcal{B}^c$,
$(\bigcup_{P,\ j\in J} \mathcal{A}_j)^c = \bigcap_{P,\ j\in J} \mathcal{A}_j^c,\ (\bigcap_{P,\ j\in J} \mathcal{A}_j)^c = \bigcup_{P,\ j\in J} \mathcal{A}_j^c$,
(7) $(\mathcal{A}^c)^c = \mathcal{A}$,
(8) (8_a) $\mathcal{A} \cup_P \hat{\varnothing} = \mathcal{A},\ \mathcal{A} \cap_P \hat{\varnothing} = \hat{\varnothing}$,
(8_b) $\mathcal{A} \cup_P \hat{X} = \hat{X},\ \mathcal{A} \cap_P \hat{X} = \mathcal{A}$,
(8_c) $\hat{X}^c = \hat{\varnothing},\ \hat{\varnothing}^c = \hat{X},\ \ddot{X}^c = \ddot{\varnothing},\ \ddot{\varnothing}^c = \ddot{X}$,
(8_d) $\mathcal{A} \cup_P \mathcal{A}^c \neq \hat{X},\ \mathcal{A} \cap_P \mathcal{A}^c \neq \hat{\varnothing}$ *in general* (See Example 3.12).

Example 3.12. Let $X = \{a, b, c\}$. Consider a CCS $\mathcal{A} = \langle [\{a\}, \{a, b\}], \{c\}\rangle$ in X. Then clearly, $\mathcal{A}^c = \langle [\{c\}, \{b, c\}], \{a, b\}\rangle$. Thus we have

$$\mathcal{A} \cup_P \mathcal{A}^c = \langle [\{a, c\}, X], X\rangle \neq \hat{X} \text{ and } \mathcal{A} \cap_P \mathcal{A}^c = \langle [\varnothing, \{b\}], \varnothing\rangle \neq \hat{\varnothing}.$$

From Proposition 3.11, we can easily see that $(CCS(X), \cup_P, \cap_P, ^c, \hat{\varnothing}, \hat{X})$ forms a Boolian algebra except the property (8_d).

Now we discuss with some properties for ICCss and ECCSs.

Proposition 3.13. *Let $\mathcal{A} \in CCS(X)$. If $\mathcal{A}$ is an ICCS [resp. ECCS], then so is $\mathcal{A}^c$ is an ICCS [resp. ECCS].*

Proof. Suppose $\mathcal{A}$ is an ICCS. Then clearly, $A^- \subset A \subset A^+$. Thus ${A^+}^c \subset A^c \subset {A^-}^c$. So $\mathcal{A}^c = \langle \mathbf{A}^c, A^c \rangle$ is an ICCS. The proof of the second part is similar. □

Proposition 3.14. *Let $(\mathcal{A}_j)_{j \in J} \subset ICCS(X)$. Then $\bigcup_{P,\ J \in J} \mathcal{A}_j,\ \bigcap_{P,\ J \in J} \mathcal{A}_j \in ICCS(X)$.*

Proof. Since $\mathcal{A}_j \in ICCS(X)$ for each $j \in J$, $A_j^- \subset A_j \subset A_j^+$ for each $j \in J$. Then we have

$$\left(\bigcup_{j \in J} A_j\right)^- \subset \bigcup_{j \in J} A_j \subset \left(\bigcup_{j \in J} A_j\right)^+$$

and

$$\left(\bigcap_{j \in J} A_j\right)^- \subset \bigcap_{j \in J} A_j \subset \left(\bigcap_{j \in J} A_j\right)^+.$$

Thus $\bigcup_{P,\ J \in J} \mathcal{A}_j,\ \bigcap_{P,\ J \in J} \mathcal{A}_j \in ICCS(X)$. □

Remark 3.15. (1) The P-union and P-intersection of ECCSs need not be an ECCS (See Example 3.16 (1)).

(2) The R-union and R-intersection of ICCSs need not be an ICCS (See Example 3.16 (2)).

(3) The R-union and R-intersection of ECCSs need not be an ECCS (See Example 3.16 (3) and (4)).

Example 3.16. Let $X = \{a, b, c, d, e, f, g\}$.

(1) Consider two ECCSs $\mathcal{A}$ and $\mathcal{B}$ given by, respectively:

$$\mathcal{A} = \langle [\{a, b\}, \{a, b, c, d\}], \{a, b, c, d, e\} \rangle,\ \mathcal{B} = \langle [\{a, b, c, d\}, \{a, b, c, d, e, f\}], \{a, b, c\} \rangle.$$

Then we have

$$\mathcal{A} \cup_P \mathcal{B} = \langle [\{a, b, c, d\}, \{a, b, c, d, e, f\}], \{a, b, c, d, e\} \rangle$$

and

$$\mathcal{A} \cap_P \mathcal{B} = \langle [\{a, b\}, \{a, b, c, d\}], \{a, b, c\} \rangle.$$

Thus $\mathcal{A} \cup_P \mathcal{B},\ \mathcal{A} \cap_P \mathcal{B} \in ICCS(X)$. So $\mathcal{A} \cup_P \mathcal{B},\ \mathcal{A} \cap_P \mathcal{B} \notin ECCS(X)$.

(2) Consider two ICCSs $\mathcal{A}$ and $\mathcal{B}$ given by, respectively:

$$\mathcal{A} = \langle [\{a, b\}, \{a, b, c, d\}], \{a, b, c\} \rangle,\ \mathcal{B} = \langle [\{a, b, c, d\}, \{a, b, c, d, e, f\}], \{a, b, c, d, e\} \rangle.$$

Then we have

$$\mathcal{A} \cup_R \mathcal{B} = \langle [\{a, b, c, d\}, \{a, b, c, d, e, f\}], \{a, b, c\} \rangle$$

and

$$\mathcal{A} \cap_R \mathcal{B} = \langle [\{a, b\}, \{a, b, c, d\}], \{a, b, c, d, e\} \rangle.$$

Thus $\mathcal{A} \cup_R \mathcal{B},\ \mathcal{A} \cap_R \mathcal{B} \in ECCS(X)$. So $\mathcal{A} \cup_R \mathcal{B},\ \mathcal{A} \cap_R \mathcal{B} \notin ICCS(X)$.

(3) Consider two ECCSs $\mathcal{A}$ and $\mathcal{B}$ given by, respectively:

$$\mathcal{A} = \langle [\{a, b\}, \{a, b, c\}], \{a, b, c, d\} \rangle \,,\; \mathcal{B} = \langle [\{a, b, c\}, \{a, b, c, d, e\}], \{a, b, c, d, e, f\} \rangle \,.$$

Then we have

$$\mathcal{A} \cup_R \mathcal{B} = \langle [\{a, b, c\}, \{a, b, c, d, e\}], \{a, b, c, d\} \rangle \,.$$

Thus $\mathcal{A} \cup_R \mathcal{B} \in ICCS(X)$. So $\mathcal{A} \cup_R \mathcal{B},\ \mathcal{A} \cap_P \mathcal{B} \notin ECCS(X)$.

(4) Consider two ECCSs $\mathcal{A}$ and $\mathcal{B}$ given by, respectively:

$$\mathcal{A} = \langle [\{a, b\}, \{a, b, c, d\}], \{a\} \rangle \,,\; \mathcal{B} = \langle [\{a, b, c, d\}, \{a, b, c, d, e, f\}], \{a, b, c\} \rangle \,.$$

Then we have

$$\mathcal{A} \cap_R \mathcal{B} = \langle [\{a, b\}, \{a, b, c, d\}], \{a, b, c\} \rangle \,.$$

Thus $\mathcal{A} \cap_R \mathcal{B} \in ICCS(X)$. So $\mathcal{A} \cap_R \mathcal{B},\ \mathcal{A} \cap_P \mathcal{B} \notin ECCS(X)$.

The following provides a condition for the R-union of two ICCSs to be an ICCS.

Proposition 3.17. *Let $\mathcal{A},\ \mathcal{B} \in ICCS(X)$ satisfying the following condition:*

$$max\{A^-, B^-\} \subset A \cap B. \tag{3.1}$$

Then $\mathcal{A} \cup_R \mathcal{B} \in ICCS(X)$.

Proof. Since $\mathcal{A},\ \mathcal{B} \in ICCS(X)$, we have the following inclusions:

$$A^- \subset A \subset A^+,\ B^- \subset B \subset B^+.$$

Then clearly, we have $A \cap B \subset (A \cup B)^+$. Thus by the condition (3.1), we get

$$(A \cup B)^- = max\{A^-, B^-\} \subset A \cap B \subset (A \cup B)^+.$$

So $\mathcal{A} \cup_R \mathcal{B} = \langle \mathbf{A} \cup \mathbf{B}, A \cap B \rangle \in ICCS(X)$. □

The following provides a condition for the R-intersection of two ICCSs to be an ICCS.

Proposition 3.18. *Let $\mathcal{A},\ \mathcal{B} \in ICCS(X)$ satisfying the following condition:*

$$min\{A^+, B^+\} \supset A \cap B. \tag{3.2}$$

Then $\mathcal{A} \cap_R \mathcal{B} \in ICCS(X)$.

Proof. Since $\mathcal{A},\ \mathcal{B} \in ICCS(X)$, we have the following inclusions:

$$A^- \subset A \subset A^+,\ B^- \subset B \subset B^+.$$

Then clearly, we have $(A \cap B)^- \subset A \cup B$. Thus by the condition (3.2), we get

$$(A \cap B)^- \subset A \cup B \subset min\{A^+, B^+\} = (A \cap B)^+.$$

So $\mathcal{A} \cap_R \mathcal{B} = \langle \mathbf{A} \cap \mathbf{B}, A \cup B \rangle \in ICCS(X)$. □

From Example 3.16 (1), we know that the P-intersection of two ECCSs may not be an ECCS. However, we give a condition for the P-intersection of two ECCSs to be an ECCS.

Proposition 3.19. *Let $\mathcal{A},\ \mathcal{B} \in ECCS(X)$ satisfying the following condition:*

(3.3)
$$min\{max\{A^+, B^-\}, max\{A^-, B^+\}\} \supset A \cap B \supsetneqq max\{min\{A^+, B^-\}, min\{A^-, B^+\}\}.$$

Then $\mathcal{A} \cap_P \mathcal{B} \in ECCS(X)$.

Proof. Let us take

$$C := min\{max\{A^+, B^-\}, max\{A^-, B^+\}\}$$

and

$$D := max\{min\{A^+, B^-\}, min\{A^-, B^+\}\}.$$

Then clearly, C is one of A^-, A^+, B^- and B^+.

Case 1: Suppose $C = A^-$. Then clearly, $B^- \subset B^+ \subset A^- \subset A^+$. Thus $D = B^+$. By the condition (3.3), we have

$$B^- = (A \cap B)^- \subset (A \cap B)^+ = B^+ = D \subsetneqq A \cap B.$$

So $\mathcal{A} \cap_P \mathcal{B} = \langle \mathbf{A} \cap \mathbf{B}, A \cap B\rangle \in ECCS(X)$.

Case 2: Suppose $C = A^+$. Then clearly, $B^- \subset A^+ \subset B^+$. Thus $D = max\{A^-, B^-\}$. Assume that $D = A^-$. Then we have

$$B^- \subset A^- \subsetneqq A \cap B \subset A^+ \subset B^+. \tag{3.4}$$

Thus from the inclusion (3.4), we get

$$B^- \subset A^- \subsetneqq A \cap B \subsetneqq A^+ \subset B^+ \tag{3.5}$$

or

$$B^- \subset A^- \subsetneqq A \cap B = A^+ \subset B^+. \tag{3.6}$$

Clearly, the inclusion (3.5) is a contradiction to the fact that $\mathcal{A}$, $\mathcal{B} \in ECCS(X)$. From the inclusion (3.6), we have

$$A \cap B = A^+ = (A \cap B)^+.$$

So $\mathcal{A} \cap_P \mathcal{B} = \langle \mathbf{A} \cap \mathbf{B}, A \cap B\rangle \in ECCS(X)$.

Now assume that $D = B^-$. Then we have

$$A^- \subset B^- \subsetneqq A \cap B \subset A^+ \subset B^+. \tag{3.7}$$

Thus from the inclusion (3.7), we get

$$A^- \subset B^- \subsetneqq A \cap B \subsetneqq A^+ \subset B^+ \tag{3.8}$$

or

$$B^- \subset A^- \subsetneqq A \cap B = A^+ \subset B^+. \tag{3.9}$$

Clearly, the inclusion (3.8) is a contradiction to the fact that $\mathcal{A}$, $\mathcal{B} \in ECCS(X)$. From the inclusion (3.9), we have

$$A \cap B = A^+ = (A \cap B)^+.$$

So $\mathcal{A} \cap_P \mathcal{B} = \langle \mathbf{A} \cap \mathbf{B}, A \cap B\rangle \in ECCS(X)$.

The proof of $C = B^-$ or $C = B^+$ is similar to Cases 1 and 2. This completes the proof. □

Remark 3.20. Let $\mathcal{A}$, $\mathcal{B} \in ECCS(X)$ satisfying the following condition:

$$min\{max\{A^+, B^-\}, max\{A^-, B^+\}\} \supsetneqq A \cap B = max\{min\{A^+, B^-\}, min\{A^-, B^+\}\}. \tag{3.10}$$

Then $\mathcal{A} \cap_P \mathcal{B} \notin ECCS(X)$ (See Example 3.21).

Example 3.21. Let $X = \{a, b, c, d, e, f\}$. Consider two $\mathcal{A},\ \mathcal{B} \in CCS(X)$, respectively given by:

$$\mathcal{A} = \langle [\{a, b, c\}, \{a, b, c, d, e\}], \{a, b\} \rangle\ ,\ \mathcal{B} = \langle [\{a\}, \{a, b\}], \{a, b, c\} \rangle\ .$$

Then clearly, $\mathcal{A},\ \mathcal{B} \in ECCS(X)$ satisfying the condition (3.10). But

$$(A \cap B)^- = \{a\} \subset A \cap B = \{a, b\} = (A \cap B)^+.$$

Thus $A \cap B \in ((A \cap B)^-, (A \cap B)^+)$. So $\mathcal{A} \cap_P \mathcal{B} = \langle [\{a\}, \{a, b\}], \{a, b\} \rangle \notin ECCS(X)$.

The following provide a condition for the P-intersection of two ECCSs to be both an ECCS and an ICCS.

Proposition 3.22. *Let $\mathcal{A},\ \mathcal{B} \in ECCS(X)$ satisfying the following condition:*
(3.11)
$$min\{max\{A^+, B^-\}, max\{A^-, B^+\}\} = A \cap B = max\{min\{A^+, B^-\}, min\{A^-, B^+\}\}.$$
Then $\mathcal{A} \cap_P \mathcal{B} \in ECCS(X) \cap ICCS(X)$.

Proof. Let us take

$$C := min\{max\{A^+, B^-\}, max\{A^-, B^+\}\}$$

and

$$D := max\{min\{A^+, B^-\}, min\{A^-, B^+\}\}.$$

Then clearly, C is one of $A^-,\ A^+,\ B^-$ and B^+. We consider $C = A^-$ or $C = A^+$ only. For the remaining cases, the proofs are similar to theses cases.

Case 1: Suppose $C = A^-$. Then $B^- \subset B^+ \subset A^- \subset A^+$. Thus $D = B^+$. By (3.11), we have

$$A^- = C = A \cap B = D = B^+.$$

This imply that $B^- \subset B^+ = A \cap B = A^- \subset A^+$, i.e., $A \cap B = B^+ = (A \cap B)^+$. So we have $A \cap B \notin ((A \cap B)^-, (A \cap B)^+)$ and $(A \cap B)^- \subset A \cap B \subset (A \cap B)^+$. Hence $\mathcal{A} \cup_P \mathcal{B} \in ECCS(X) \cap ICCS(X)$.

Case 2: Suppose $C = A^+$. Then clearly, $B^- \subset A^+ \subset B^+$. Thus we have

$$A \cap B = A^+ = (A \cap B)^+.$$

So $A \cap B \notin ((A \cap B)^-, (A \cap B)^+)$ and $(A \cap B)^- \subset A \cap B \subset (A \cap B)^+$. Hence $\mathcal{A} \cap_P \mathcal{B} \in ECCS(X) \cap ICCS(X)$. This completes the proof. □

The P-union of two ECCs $\mathcal{A}$ and $\mathcal{B}$ may not be an ECCS, in general (See Example 3.23).

Example 3.23. Let $X = \{a, b, c, d, e, f\}$. Consider two $\mathcal{A},\ \mathcal{B} \in CCS(X)$, respectively given by:

$$\mathcal{A} = \langle [\{a, b, c\}, \{a, b, c, d, e\}], \{a, b\} \rangle\ ,\ \mathcal{B} = \langle [\{a\}, \{a, b\}], \{a, b, c, d\} \rangle\ .$$

Then clearly, $\mathcal{A},\ \mathcal{B} \in ECCS(X)$. But we have

$$(A \cup B)^- = \{a, b, c\} \subset \{a, b, c, d\} = A \cup B \subset \{a, b, c, d, e\} = (A \cup B)^+.$$

Thus $\mathcal{A} \cup_P \mathcal{B} = \langle [\{a, b, c\}, \{a, b, c, d, e\}], \{a, b, c, d\} \rangle \notin ECCS(X)$.

The following provide a condition for the P-union of two ECCSs to be an ECCS.

Proposition 3.24. *Let $\mathcal{A}$, $\mathcal{B} \in ECCS(X)$ satisfying the following condition:*
(3.12)
$min\{max\{A^+, B^-\}, max\{A^-, B^+\}\} \supsetneqq A \cup B \supset max\{min\{A^+, B^-\}, min\{A^-, B^+\}\}$.
Then $\mathcal{A} \cup_P \mathcal{B} \in ECCS(X)$.

Proof. Let us take

$$C := min\{max\{A^+, B^-\}, max\{A^-, B^+\}\}$$

and

$$D := max\{min\{A^+, B^-\}, min\{A^-, B^+\}\}.$$

Then clearly, C is one of A^-, A^+, B^- and B^+. We consider $C = A^-$ or $C = A^+$ only. For the remaining cases, the proofs are similar to theses cases.

Case 1: Suppose $C = A^-$. Then clearly, we have

$$B^- \subset B^+ \subset A^- \subset A^+.$$

Thus $D = B^+$. So $(A \cup B)^- = A^- = C \supsetneqq A \cup B$, i.e., $A \cup B \notin ((A \cup B)^-, (A \cup B)^+)$. Hence $\mathcal{A} \cup_P \mathcal{B} \in ECCS(X)$.

Case 2: Suppose $C = A^+$. Then clearly, we have

$$B^- \subset B^- \subset A^+ \subset B^+.$$

Thus $D = max\{A^-, B^-\}$.

Assume that $D = A^-$. Then we have

$$B^- \subset A^- \subset A \cup B \subsetneqq A^+ \subset B^+. \tag{3.13}$$

Thus $B^- \subset A^- \subsetneqq A \cup B \subsetneqq A^+ \subset B^+$ or $B^- \subset A^- = A \cup B \subsetneqq A^+ \subset B^+$. So we have

$$(A \cup B)^- = B^- \subsetneqq A \cup B \subsetneqq B^+ = (A \cup B)^+ \tag{3.14}$$

or

$$(A \cup B)^- = B^- = A \cup B \subsetneqq B^+ = (A \cup B)^+. \tag{3.15}$$

It is clear that (3.14) contradicts to the fact that $\mathcal{A}$, $\mathcal{B} \in ECCS(X)$. Since the inclusion (3.15) holds only, $A \cup B \notin ((A \cup B)^-, (A \cup B)^+)$. Hence $\mathcal{A} \cup_P \mathcal{B} \in ECCS(X)$.

Assume that $D = B^-$. Then we have

$$A^- \subset B^- \subset A \cup B \subset A^+ \subsetneqq B^+. \tag{3.16}$$

Thus $B^- \subset A^- \subsetneqq A \cup B \subsetneqq A^+ \subset B^+$ or $B^- \subset A^- = A \cup B \subsetneqq A^+ \subset B^+$. So we have

$$(A \cup B)^- = A^- \subsetneqq A \cup B \subsetneqq B^+ = (A \cup B)^+ \tag{3.17}$$

or

$$(A \cup B)^- = A^- = A \cup B \subsetneqq B^+ = (A \cup B)^+. \tag{3.18}$$

It is clear that (3.17) contradicts to the fact that $\mathcal{A}$, $\mathcal{B} \in ECCS(X)$. Since the inclusion (3.18) holds only, $A \cup B \notin ((A \cup B)^-, (A \cup B)^+)$. Hence $\mathcal{A} \cup_P \mathcal{B} \in ECCS(X)$. This completes the proof. □

Now we give a condition for the R-union of two ECCSs to be an ECCS.

Proposition 3.25. *Let $\mathcal{A}$, $\mathcal{B} \in ECCS(X)$ satisfying the following condition:*
(3.19)
$$min\{max\{A^+, B^-\}, max\{A^-, B^+\}\} \supsetneqq A \cap B \supset max\{min\{A^+, B^-\}, min\{A^-, B^+\}\}.$$
Then $\mathcal{A} \cup_R \mathcal{B} \in ECCS(X)$.

Proof. Let us take
$$C := min\{max\{A^+, B^-\}, max\{A^-, B^+\}\}$$
and
$$D := max\{min\{A^+, B^-\}, min\{A^-, B^+\}\}.$$
Then clearly, C is one of A^-, A^+, B^- and B^+. We consider $C = B^-$ or $C = B^+$ only. For the remaining cases, the proofs are similar to theses cases.

Case 1: Suppose $C = B^-$. Then clearly, we have
$$A^- \subset A^+ \subset B^- \subset B^+.$$
Thus $D = A^+$. So by (3.19),
$$(A \cup B)^- = A^- = C \supsetneqq A \cap B, \text{ i.e., } A \cap B \notin ((A \cup B)^-, (A \cup B)^+).$$
Hence $\mathcal{A} \cup_R \mathcal{B} \in ECCS(X)$.

Case 2: Suppose $C = A^+$. Then clearly, $A^- \subset B^+ \subset A^+$. Thus $D = max\{A^-, B^-\}$.

Assume that $D = A^-$. Then by (3.19),
$$B^- \subset A^- \subset A \cap B \subsetneqq B^+ \subset A^+. \tag{3.20}$$
Thus $B^- \subset A^- \subsetneqq A \cap B \subsetneqq B^+ \subset A^+$ or $B^- \subset A^- = A \cap B \subsetneqq B^+ \subset A^+$. So we have
$$(A \cup B)^- = A^- \subsetneqq A \cap B \subsetneqq A^+ = (A \cup B)^+ \tag{3.21}$$
or
$$(A \cup B)^- = A^- = A \cap B \subsetneqq A^+ = (A \cup B)^+. \tag{3.22}$$
It is clear that (3.21) contradicts to the fact that $\mathcal{A}$, $\mathcal{B} \in ECCS(X)$. Since the inclusion (3.22) holds only, $A \cap B \notin ((A \cup B)^-, (A \cup B)^+)$. Hence $\mathcal{A} \cup_R \mathcal{B} \in ECCS(X)$.

Suppose $D = B^-$. Then we have
$$A^- \subset B^- \subset A \cap B \subsetneqq B^+ \subset A^+. \tag{3.23}$$
Thus $A^- \subset B^- \subsetneqq A \cap B \subsetneqq B^+ \subset A^+$ or $A^- \subset B^- = A \cap B \subsetneqq B^+ \subset A^+$. So we have
$$(A \cup B)^- = B^- \subsetneqq A \cap B \subsetneqq A^+ = (A \cup B)^+ \tag{3.24}$$
or
$$(A \cup B)^- = B^- = A \cap B \subsetneqq A^+ = (A \cup B)^+. \tag{3.25}$$
It is clear that (3.24) contradicts to the fact that $\mathcal{A}$, $\mathcal{B} \in ECCS(X)$. Since the inclusion (3.25) holds only, $A \cap B \notin ((A \cup B)^-, (A \cup B)^+)$. Hence $\mathcal{A} \cup_R \mathcal{B} \in ECCS(X)$. This completes the proof. $\square$

We give example that for two ECCSs $\mathcal{A}$, $\mathcal{B}$ satisfying the following condition:
(3.26)
$$min\{max\{A^+, B^-\}, max\{A^-, B^+\}\} = A \cap B \subsetneqq max\{min\{A^+, B^-\}, min\{A^-, B^+\}\},$$
$\mathcal{A} \cup_R \mathcal{B}$ may be not an ECCS in X.

Example 3.26. Let $X = \{a, b, c, d, e, f\}$ be a set. Consider two ICCSs $\mathcal{A}$, $\mathcal{B}$, respectively given by:
$$\mathcal{A} = \langle [\{a, b, c, d\}, \{a, b, c, d, e\}], X \rangle, \ \mathcal{B} = \langle [\{a, b, c\}, \{a, b, c\}], \{a, b, c, d\} \rangle.$$
Then we can easily check that the condition (3.26) holds. On the other hand,
$$\mathcal{A} \cup_R \mathcal{B} = \langle [\{a, b, c, d\}, \{a, b, c, d, e\}], \{a, b, c, d\} \rangle.$$
Thus we have
$$(A \cup B)^- = \{a, b, c, d\} = A \cap B \subset \{a, b, c, d, e\} = (A \cup B)^+.$$
So $\mathcal{A} \cup_R \mathcal{B} \notin ECCS(X)$.

We give a condition for the R-intersection of two ECCSs to be an ECCS.

Proposition 3.27. *Let $\mathcal{A}$, $\mathcal{B} \in ECCS(X)$ satisfying the following condition:*
(3.27)
$$min\{max\{A^+, B^-\}, max\{A^-, B^+\}\} \supset A \cup B \supsetneqq max\{min\{A^+, B^-\}, min\{A^-, B^+\}\}.$$
Then $\mathcal{A} \cap_R \mathcal{B} \in ECCS(X)$.

Proof. The proof is similar to Proposition 3.22. □

We give example that for two ECCSs $\mathcal{A}$, $\mathcal{B}$ satisfying the following condition:
(3.28)
$$min\{max\{A^+, B^-\}, max\{A^-, B^+\}\} \supsetneqq A \cup B = max\{min\{A^+, B^-\}, min\{A^-, B^+\}\},$$
$\mathcal{A} \cap_R \mathcal{B}$ may be not an ECCS in X.

Example 3.28. Let $X = \{a, b, c, d, e, f\}$ be a set. Consider two ICCSs $\mathcal{A}$, $\mathcal{B}$, respectively given by:
$$\mathcal{A} = \langle [\{a, b, c\}, \{a, b, c, d\}], \{a, b\} \rangle, \ \mathcal{B} = \langle [\{a, b, c, d, e\}, \{a, b, c, d, e\}], \{a, b, c\} \rangle.$$
Then we can easily check that the condition (3.28) holds. On the other hand,
$$\mathcal{A} \cap_R \mathcal{B} = \langle [\{a, b, c\}, \{a, b, c, d\}], \{a, b, c\} \rangle.$$
Thus we have
$$(A \cap B)^- = \{a, b, c\} = A \cap B \subset \{a, b, c, d\} = (A \cap B)^+.$$
So $\mathcal{A} \cap_R \mathcal{B} \notin ECCS(X)$.

The following provides a condition for the R-intersection of two ECCSs to be both an ECCS and an ICCS.

Proposition 3.29. *Let $\mathcal{A}$, $\mathcal{B} \in ECCS(X)$ satisfying the following condition:*
(3.29)
$$min\{max\{A^+, B^-\}, max\{A^-, B^+\}\} = A \cup B = max\{min\{A^+, B^-\}, min\{A^-, B^+\}\}.$$
Then $\mathcal{A} \cap_R \mathcal{B} \in ECCS(X) \cap ICCS(X)$.

Proof. The proof is similar to 3.22. □

We give a condition for R-union of two ICCSs to be an ECCS.

Proposition 3.30. *Let $\mathcal{A},\ \mathcal{B} \in ICCS(X)$ satisfying the following condition:*

(3.30) $$A \cup B \subset max\{A^-, B^-\}.$$

Then $\mathcal{A} \cup_R \mathcal{B} \in ECCS(X)$.

Proof. Straightforward. □

The following provides a condition for R-intersection of two ICCSs to be an ECCS.

Proposition 3.31. *Let $\mathcal{A},\ \mathcal{B} \in ICCS(X)$ satisfying the following condition:*

(3.31) $$A \cup B \supset min\{A^+, B^+\}.$$

Then $\mathcal{A} \cap_R \mathcal{B} \in ECCS(X)$.

Proof. Straightforward. □

The following provides a condition for R-union of two ECCSs to be an ICCS.

Proposition 3.32. *Let $\mathcal{A},\ \mathcal{B} \in ECCS(X)$ satisfying the following condition:*

(3.32) $$min\{max\{A^+, B^-\}, max\{A^-, B^+\}\} \subset A \cap B \subset max\{A^+, B^+\}.$$

Then $\mathcal{A} \cup_R \mathcal{B} \in ICCS(X)$.

Proof. Straightforward. □

4. Cubic crisp topological spaces

In this section, we introduce the notions of P-cubic crisp topology and R-cubic crisp topology, and study some of their properties, and give some examples.

Definition 4.1 ([34])**.** Let X be a non-empty set and let $\tau_P \subset CS(X)$. Then τ_P is called a P-cubic topology (briefly, PCT) on X, it satisfies the following axioms:

(PCO_1) $\hat{0}, \hat{1} \in \tau_P$,
(PCO_2) $\mathcal{A} \cap_P \mathcal{B} \in \tau_P$, for any $\mathcal{A}, \mathcal{B} \in \tau_P$,
(PCO_3) $\bigcup_{P,\ j \in J} \mathcal{A}_j \in \tau_P$, for each $(\mathcal{A}_j)_{j \in J} \subset \tau_P$.

In this case, the pair (X, τ_P) is called a P-cubic topological space (briefly, PCTS) and each member of τ is called a P-cubic open set (briefly, PCOS) in X. A CS $\mathcal{F}$ of X is called a P-cubic closed set (briefly, PCCS) in X, if $\mathcal{F}^c \in \tau_P$.

In particular, a family of ICCs satisfying the conditions (PCO_1), (PCO_2) and (PCO_3) is called an internal P-cubic topology (briefly, IPCT) on X (See [34]).

Definition 4.2. Let X be a non-empty set and let $\tau_P \subset CCS(X)$. Then τ_P is called a P-cubic crisp topology (briefly, PCCT) on X, if it satisfies the following axioms:

($PCCO_1$) $\hat{\varnothing}, \hat{X} \in \tau_P$,
($PCCO_2$) $\mathcal{A} \cap_P \mathcal{B} \in \tau_P$, for any $\mathcal{A}, \mathcal{B} \in \tau_P$,,
($PCCO_3$) $\bigcup_{P,\ j \in J} \mathcal{A}_j \in \tau_P$, for each $(\mathcal{A}_j)_{j \in J} \subset \tau_P$.

In this case, the pair (X, τ_P) is called a P-cubic crisp topological space (briefly, PCCTS) and each member of τ is called a P-cubic crisp open set (briefly, PCCOS) in X. A CCS $\mathcal{A}$ is called a P-cubic crisp closed set (briefly, PCCCS) in X, if $\mathcal{A}^c \in \tau$.

It is obvious that $\{\hat{\varnothing}, \hat{X}\}$ is a PCCT on X, and called the P-cubic crisp indiscrete topology on X and denoted by $\tau_{_{P,0}}$. Also $CCS(X)$ is a PCCT on X, and called the P-cubic crisp discrete topology on X and denoted by $\tau_{_{P,1}}$. The pair $(X, \tau_{_{P,0}})$ [resp. $(X, \tau_{_{P,1}})$] will be called the P-cubic crisp indiscrete [resp. discrete] space.

We denote the set of all PCCTs on X as $PCCT(X)$. For a PCCTS X, we denote the set of all PCCOs [resp. PCCCSs] in X as $PCCO(X)$ [resp. $PCCC(X)$].

Remark 4.3. (1) For each $\tau_{_P} \in PCCT(X)$, there are an interval-valued topology $\tau_{_{PIV}}$ proposed by Kim et al. [28] and a classical topology $\tau_{_{PC}}$ on X, respectively given as follows:

$$\tau_{_{PIV}} = \{\mathbf{A} \in IVS(X) : \mathcal{A} \in \tau_{_P}\},\ \tau_{_{PC}} = \{A \in 2^X : \mathcal{A} \in \tau_{_P}\}.$$

Also, there are three classical topologies $\tau^-_{_P}$, $\tau^+_{_P}$, $\tau_{_{PC}}$ on X, respectively given by:

$$\tau^-_{_P} = \{A^- \in 2^X : \mathcal{A} \in \tau_{_P}\},\ \tau^+_{_P} = \{A^+ \in 2^X : \mathcal{A} \in \tau_{_P}\},\ \tau_{_{PC}} = \{A \in 2^X : \mathcal{A} \in \tau_{_P}\}.$$

In this case, $\tau_{_{PIV}}$ [resp. $\tau^-_{_P}$, $\tau^+_{_P}$ and $\tau_{_{PC}}$] will be called an interval-valued topology [resp. classical topologies] on X generated by the PCCT $\tau_{_P}$. In particular, we can consider the triple $(\tau^-_{_P}, \tau^+_{_P}, \tau)$ is a tri-topology on X (See [35]).

Furthermore, we can see that

$$\chi_{\tau_{_P}} = \{\chi_{\mathcal{A}} : \chi_{\mathcal{A}} = \langle[\chi_{A^-}, \chi_{A^+}], \chi_A\rangle,\ \mathcal{A} \in \tau_{_P}\}$$

is a P-cubic topology on X proposed by Zeb et al. (See Definition 4.1). Thus a PCCT is the spacial case of a PCCT on X.

(2) Let (X, τ_o) be an ordinary topological space. Then clearly, $\tau = \{\langle [A, A], A^c\rangle \in CCS(X) : A \in \tau_o\} \in RCCT(X)$. Thus a CCT is a generalization of a classical topology.

(3) Let $\tau_{_{IV}}$ be an interval-valued topology on a set X in the sense of Kim et al. [28]. Then we can easily see that the following two families are PCCTs on X:

$$\tau_{_{PIV,1}} = \{\langle \mathcal{A}, A^- \rangle \in CCS(X) : \mathcal{A} \in \tau_{_{IV}}\},$$

$$\tau_{_{PIV,2}} = \{\langle \mathcal{A}, A^+ \rangle \in CCS(X) : \mathcal{A} \in \tau_{_{IV}}\}.$$

Thus a PCCT is a generalization of an interval-valued topology on X.

From the above Remark, we can easily see the relationships among classical topology, IVT, PCCT and PCT:

Classical topology$\Longrightarrow$IVT$\Longrightarrow$PCCT$\Longrightarrow$PCT.

Example 4.4. (1) In Example 3.2, $CCS(X) = \tau_{_{P,1}}$.

(2) Let X be a set and let $\mathcal{A} \in CCS(X)$. Then $\mathcal{A}$ is said to be finite, if A^+ and A are finite. Consider the family $\tau = \{\mathcal{U} \in CCS(X) : \mathcal{U} = \hat{\varnothing}$ or $\mathcal{U}^c$ is finite$\}$. Then we can easily check that $\tau \in PCCT(X)$.

In this case, τ will be called a P-cubic crisp cofinite topology (briefly, PCCCFT) on X.

(3) Let X be a set and let $\mathcal{A} \in CCS(X)$. Then $\mathcal{A}$ is said to be countable, if A^+ and A are countable. Consider the family $\tau = \{\mathcal{U} \in CCS(X) : \mathcal{U} = \hat{\varnothing}$ or $\mathcal{U}^c$ is countable$\}$. Then we can easily prove that $\tau \in PCCT(X)$.

In this case, τ will be called a P-cubic crisp cocountable topology (briefly, PCC-CCT) on X.

The following is the immediate result of Definition 4.2

Proposition 4.5. *Let X be an PCCTS. Then*
(1) $\hat{\varnothing},\ \hat{X} \in PCCC(X)$,
(2) $\mathcal{A} \cup_P \mathcal{B} \in PCCC(X)$ *for any* $\mathcal{A},\ \mathcal{B} \in PCCC(X)$,
(3) $\bigcap_{P,\ j \in J} \mathcal{A}_j \in PCCC(X)$ *for any* $(\mathcal{A}_j)_{j \in J} \subset PCCC(X)$.

Definition 4.6. Let X be a non-empty set and let $\tau_1,\ \tau_2 \in PCCT(X)$. Then we say that τ_1 is contained in τ_2 or τ_1 is coarser than τ_2 or τ_2 is finer than τ_1, if $\tau_1 \subset \tau_2$, i.e., $\mathcal{A} \in \tau_2$ for each $\mathcal{A} \in \tau_1$.

It is obvious that $\tau_{_{P,0}} \subset \tau \subset \tau_{_{P,1}}$ for each $\tau \in PCCT(X)$.

Proposition 4.7. *Let $(\tau_j)_{j \in J} \subset PCCT(X)$. Then $\bigcap_{j \in J} \tau_j \in PCCT(X)$.*
In fact, $\bigcap_{j \in J} \tau_j$ is the finest PCCT on X contained in each τ_j.

Proof. Let $\tau = \bigcap_{j \in J} \tau_j$. Since $(\tau_j)_{j \in J} \subset PCCT(X)$, $\hat{\varnothing}, \hat{X} \in \tau_j$ for each $j \in J$. Then $\hat{\varnothing}, \hat{X} \in \tau$. Thus τ satisfies the condition (PCCO_1).

Now let $\mathcal{A},\ \mathcal{B} \in \tau$. Then clearly, $\mathcal{A},\ \mathcal{B} \in \tau_j$ for each $j \in J$. Since $\tau_j \in PCCT(X)$, $\mathcal{A} \cup_P \mathcal{B} \in \tau_j$ for each $j \in J$. Thus $\mathcal{A} \cup_P \mathcal{B} \in \tau$. So τ satisfies the condition (PCCO_2). Finally, let $(\mathcal{A}_i)_{i \in I}$ be a family of members of τ indexed by a class I. Then clearly, $(\mathcal{A}_i)_{i \in I}$ is a family of members of τ_j for each $j \in J$. Thus $\bigcup_{P, i \in I} \mathcal{A}_i \in \tau_j$ for each $j \in J$. So $\bigcup_{P, i \in I} \mathcal{A}_i \in \tau$. Hence τ satisfies the condition (PCCO_3). This completes the proof. □

The union of Two PCCTs may be not a PCCT on a set X (See Example 4.8).

Example 4.8. Let $X = \{a, b, c\}$ be a set. Consider two PCCTs $\tau_1,\ \tau_2$, respectively given by:

$$\tau_1 = \{\hat{\varnothing}, \hat{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\},\ \tau_2 = \{\hat{\varnothing}, \hat{X}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\},$$

where $\mathcal{A}_1 = \langle [\{a\}, \{a, b\}], \{a, b\} \rangle$, $\mathcal{A}_2 = \langle [\varnothing, \{a, c\}], \{b\} \rangle$, $\mathcal{A}_3 = \langle [\{a\}, X], X \rangle$,
$\mathcal{A}_4 = \langle [\varnothing, \{a\}], \varnothing \rangle$, $\mathcal{B}_1 = \langle [\{b\}, \{a, b\}], \{c\} \rangle$, $\mathcal{B}_2 = \langle [\{c\}, \{b, c\}], \{a, c\} \rangle$,
$\mathcal{B}_3 = \langle [\{b, c\}, X], \{a, c\} \rangle$, $\mathcal{B}_4 = \langle [\varnothing, \{b\}], \{c\} \rangle$.
Then $\tau_1 \cup \tau_2 = \{\hat{\varnothing}, \hat{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\}$. But we have

$$\mathcal{A}_1 \cup_P \mathcal{B}_1 = \langle [\{a, b\}, \{a, b\}], \{a, c\} \rangle \notin \tau_1 \cup \tau_2.$$

Thus $\tau_1 \cup \tau_2 \notin PCCT(X)$.

From Proposition 4.7 and Example 4.8, we can see that $(PCCT(X), \subset)$ forms a meet-complete lattice.

Definition 4.9. Let $\tau \subset ICCS(X)$. Then τ is called an internal P-cubic crisp topology (briefly, IPCCT) on X, if it satisfies the conditions (PCCO_1), (PCCO_2) and (PCCO_3). The pair (X, τ) is called an internal P-cubic crisp topological space (briefly, IPCCTS). Each member of τ is called an internal cubic crisp open set (briefly, ICCOS) in X and $ICCO(X)$ denotes the set of all ICCOSs in X. $\mathcal{A} \in ICCS(X)$ is called an internal cubic crisp closed set (briefly, ICCCS) in X, if $\mathcal{A}^c \in \tau$. The set of all ICCCSs in X is denoted by $ICCC(X)$.

Example 4.10. Let X be a set. Then clearly, $\hat{\varnothing},\ \hat{X} \in ICCC(X)$. Moreover, by Proposition 3.14, we can easily see that the following:

$$\mathcal{A} \cap_P \mathcal{B} \in ICCC(X) \text{ for any } \mathcal{A},\ \mathcal{B} \in ICCC(X)$$

and

$$\bigcup_{P,\ j \in J} \mathcal{A}_j \in ICCC(X) \text{ for any } (\mathcal{A}_j)_{j \in J} \subset ICCC(X).$$

Thus $ICCC(X)$ is an ICCT on X.

From Propositions 3.11, 3.13 and 3.14, we get the following.

Proposition 4.11. *Let X be an ICCTS. Then*
(1) $\hat{\varnothing},\ \hat{X} \in ICCC(X)$,
(2) $\mathcal{A} \cup_P \mathcal{B} \in ICCC(X)$ *for any* $\mathcal{A},\ \mathcal{B} \in ICCC(X)$,
(3) $\bigcap_{P,\ j \in J} \mathcal{A}_j \in ICCC(X)$ *for each* $(\mathcal{A}_j)_{j \in J} \subset ICCC(X)$.

A collection of ECCSs need not to be a PCCT (See Example 4.12).

Example 4.12. Let $X = \{a, b, c, d, e, f\}$ be a set. Consider $\tau \subset ECCS(X)$ containing two ECCSs $\mathcal{A},\ \mathcal{B}$, respectively given by:

$$\mathcal{A} = \langle [\{a, b\}, \{a, b, c\}], \{a, b, c, d\} \rangle\,,\ \mathcal{B} = \langle [\{a, b, c\}, \{a, b, c, d, e\}], \{a, b\} \rangle\,.$$

Then $\mathcal{A} \cup_P \mathcal{B} = \langle [\{a, b, c\}, \{a, b, c, d, e\}], \{a, b, c, d\} \rangle$. Thus we have

$$(A \cup B)^- = \{a, b, c\} \subset \{a, b, c, d\} = A \cup B \subset \{a, b, c, d, e\} = (A \cup B)^+.$$

So $\mathcal{A} \cup_P \mathcal{B} \notin \tau$. Hence $\tau \notin PCCT(X)$.

Definition 4.13. Let $\tau \subset ECCS(X)$. Then τ is called an external P-cubic crisp topology (briefly, EPCCT) on X, if it satisfies the conditions ($PCCO_1$), ($PCCO_2$) and ($PCCO_3$). The pair (X, τ) is called an external P-cubic crisp topological space (briefly, EPCCTS). Each member of τ is called an external cubic crisp open set (briefly, ECCOS) in X and $ECCO(X)$ denotes the set of all ECCOSs in X. $\mathcal{A} \in ECCS(X)$ is called an external cubic crisp closed set (briefly, ECCCS) in X, if $\mathcal{A}^c \in \tau$. The set of all ECCCSs in X is denoted by $ECCC(X)$.

Example 4.14. Let X be a set. Then we can easily check that $ECCC(X)$ is an ECCT on X.

From Propositions 3.11, 3.13 and 3.14, we get the following.

Proposition 4.15. *Let X be an ECCTS. Then*
(1) $\hat{\varnothing},\ \hat{X} \in ECCC(X)$,
(2) $\mathcal{A} \cup_P \mathcal{B} \in ECCC(X)$ *for any* $\mathcal{A},\ \mathcal{B} \in ECCC(X)$,
(3) $\bigcap_{P,\ j \in J} \mathcal{A}_j \in ECCC(X)$ *for each* $(\mathcal{A}_j)_{j \in J} \subset ECCC(X)$.

Definition 4.16. Let X be a non-empty set and let $\tau_{_R} \subset CCS(X)$. Then $\tau_{_P}$ is called a P-cubic crisp topology (briefly, PCCT) on X, if it satisfies the following axioms:

(RCCO$_1$) $\ddot{\varnothing}, \ddot{X}, \hat{\varnothing}, \hat{X} \in \tau_{_R}$,

(RCCO$_2$) $\mathcal{A} \cap_P \mathcal{B} \in \tau_{_R}$, for any $\mathcal{A}, \mathcal{B} \in \tau_P$,

(RCCO$_3$) $\bigcup_{P,\ j \in J} \mathcal{A}_j \in \tau_{_R}$, for each $(\mathcal{A}_j)_{j \in J} \subset \tau_P$.

In this case, the pair $(X, \tau_{_R})$ is called an R-cubic crisp topological space (briefly, RCCTS) and each member of τ is called a R-cubic crisp open set (briefly, RCCOS) in X. A CCS $\mathcal{A}$ is called an R-cubic crisp closed set (briefly, RCCCS) in X, if $\mathcal{A}^c \in \tau$.

It is obvious that $\{\ddot{\varnothing}, \ddot{X}, \hat{\varnothing}, \hat{X}\}$ is an RCCT on X, and called the R-cubic crisp indiscrete topology on X and denoted by $\tau_{_{R,0}}$. Also $CCS(X)$ is an RCCT on X, and called the R-cubic crisp discrete topology on X and denoted by $\tau_{_{R,1}}$. The pair $(X, \tau_{_{R,0}})$ [resp. $(X, \tau_{_{R,1}})$] will be called the R-cubic crisp indiscrete [resp. discrete] space. In fact, $\tau_{_{P,1}} = \tau_{_{R,1}}$.

We denote the set of all RCCTs on X as $RCCT(X)$. For an RCCTS X, we denote the set of all RCCOs [resp. RCCCSs] in X as $RCCO(X)$ [resp. $RCCC(X)$].

Remark 4.17. (1) For each $\tau_{_P} \in RCCT(X)$, there are an interval-valued topology $\tau_{_{RIV}}$ and a classical topology $\tau_{_{RC}}$ on X, respectively given as follows:

$$\tau_{_{RIV}} = \{\mathbf{A} \in IVS(X) : \mathcal{A} \in \tau_{_R}\},\ \tau_{_{RC}} = \{A^c \in 2^X : \mathcal{A} \in \tau_{_R}\}.$$

Also, there are three classical topologies $\tau_{_R}^-,\ \tau_{_R}^+,\ \tau_{_{RC}}$ on X, respectively given by:

$$\tau_{_R}^- = \{A^- \in 2^X : \mathcal{A} \in \tau_{_R}\},\ \tau_{_R}^+ = \{A^+ \in 2^X : \mathcal{A} \in \tau_{_R}\},\ \tau_{_{RC}} = \{A^c \in 2^X : \mathcal{A} \in \tau_{_R}\}.$$

We can consider the triple $(\tau_{_R}^-, \tau_{_R}^+, \tau_{_{RC}})$ is a tri-topology on X .

Furthermore, we can see that

$$\chi_{\tau_{_R}} = \{\chi_{\mathcal{A}} : \chi_{\mathcal{A}} = \langle [\chi_{A^-}, \chi_{A^+}], \chi_A \rangle,\ \mathcal{A} \in \tau_{_R}\}$$

is an R-cubic topology on X proposed by Zeb et al. (See [34]). Thus an RCCT is the spacial case of an RCCT on X.

(2) Let (X, τ_o) be an ordinary topological space. Then clearly,

$$\tau = \{\ddot{\varnothing}, \ddot{X}\} \cup \{\langle [A, A], A^c \rangle \in CCS(X) : A \in \tau_o\} \in RCCT(X).$$

Thus an RCCT is a generalization of a classical topology.

(3) Let $\tau_{_{IV}}$ be an interval-valued topology on a set X. Then we can easily see that the following two families are PCCTs on X:

$$\tau_{_{RIV,1}} = \{\ddot{\varnothing}, \ddot{X}\} \cup \{\langle \mathcal{A}, (A^-)^c \rangle \in CCS(X) : \mathcal{A} \in \tau_{_{IV}}\},$$

$$\tau_{_{RIV,2}} = \{\ddot{\varnothing}, \ddot{X}\} \cup \{\langle \mathcal{A}, (A^+)^c \rangle \in CCS(X) : \mathcal{A} \in \tau_{_{IV}}\}.$$

Thus an RCCT is a generalization of an interval-valued topology on X.

From the above Remark, we can easily see the relationships among classical topology, IVT, RCCT and RCT:

Classical topology$\Longrightarrow$IVT$\Longrightarrow$RCCT$\Longrightarrow$RCT.

Example 4.18. (1) In Example 3.2, $CCS(X) = \tau_{_{R,1}}$.

(2) Let $X = \{a, b\}$ be a set. Consider the family τ of CCSs in X given by:

$$\tau = \{\ddot{\varnothing}, \ddot{X}, \hat{\varnothing}, \hat{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_1\},$$

where $\mathcal{A}_1 = \langle [\varnothing, \{a\}], \{b\} \rangle\,,\ \mathcal{A}_2 = \langle [\{a\}, X], \{a\} \rangle\,,$
$\mathcal{A}_3 = \langle [\varnothing, \{a\}], X \rangle\,,\ \mathcal{A}_4 = \langle [\{a\ ], X], \varnothing \rangle\,.$
Then we can easily check that $\tau \in RCCT(X)$.

The following is the immediate result of Definition 4.16

Proposition 4.19. *Let X be an RCCTS. Then*
(1) $\ddot{\varnothing},\ \ddot{X},\ \hat{\varnothing},\ \hat{X} \in RCCC(X)$,
(2) $\mathcal{A} \cap_R \mathcal{B} \in RCCC(X)$ *for any* $\mathcal{A},\ \mathcal{B} \in RCCC(X)$,
(3) $\bigcup_{P,\ j \in J} \mathcal{A}_j \in RCCC(X)$ *for any* $(\mathcal{A}_j)_{j \in J} \subset RCCC(X)$.

Proposition 4.20. *Let $(\tau_j)_{j \in J} \subset RCCT(X)$. Then $\bigcap_{j \in J} \tau_j \in RCCT(X)$.*
In fact, $\bigcap_{j \in J} \tau_j$ is the finest RCCT on X contained in each τ_j.

Proof. The proof is similar to Proposition 4.7. □

The union of Two RCCTs may be not an RCCT on a set X (See Example 4.21).

Example 4.21. Let $X = \{a, b, c\}$ be a set. Consider two PCCTs $\tau_1,\ \tau_2$, respectively given by:

$$\tau_1 = \{\ddot{\varnothing}, \ddot{X}, \hat{\varnothing}, \hat{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\},\ \tau_2 = \{\ddot{\varnothing}, \ddot{X}, \hat{\varnothing}, \hat{X}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\},$$

where $\mathcal{A}_1 = \langle [\{a\}, \{a, b\}], \{a, b\} \rangle\,,\ \mathcal{A}_2 = \langle [\varnothing, \{a, c\}], \{b\} \rangle\,,\ \mathcal{A}_3 = \langle [\{a\}, X], \{b\} \rangle\,,$
$\mathcal{A}_4 = \langle [\varnothing, \{a\}], \{a, b\} \rangle\,,\ \mathcal{B}_1 = \langle [\{b\}, \{a, b\}], \{c\} \rangle\,,\ \mathcal{B}_2 = \langle [\{c\}, \{b, c\}], \{a, c\} \rangle\,,$
$\mathcal{B}_3 = \langle [\{b, c\}, X], \{c\} \rangle\,,\ \mathcal{B}_4 = \langle [\varnothing, \{b\}], \{a, c\} \rangle\,.$
Then $\tau_1 \cup_P \tau = \{\ddot{\varnothing}, \ddot{X}, \hat{\varnothing}, \hat{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\}$. But we have

$$\mathcal{A}_1 \cup_R \mathcal{B}_1 = \langle [\{a, b\}, \{a, b\}], \varnothing \rangle \notin \tau_1 \cup \tau_2.$$

Thus $\tau_1 \cup \tau_2 \notin RCCT(X)$.

From Proposition 4.20 and Example 4.21, we can see that $(RCCT(X), \subset)$ forms a meet-complete lattice with the least element $\tau_{_{R,0}}$ and the largest element $\tau_{_{P,1}}$.

The collection of ECCSs need not to be an RCCT (See Example 4.22).

Example 4.22. Let $X = \{a, b, c, d, e, f\}$ and let τ be a family of ECCSs containing $\mathcal{A},\ \mathcal{B}$ given in Example 3.23:

$$\mathcal{A} = \langle [\{a, b, c\}, \{a, b, c, d, e\}], \{a, b\} \rangle\,,\ \mathcal{B} = \langle [\{a\}, \{a, b\}], \{a, b, c, d\} \rangle\,.$$

Then we can easily check that $\mathcal{A} \cup_R \mathcal{B} \notin ECCS(X)$. Thus $\tau \in TCCT(X)$.

Also, the collection of ICCSs need not to be an RCCT (See Example 4.23).

Example 4.23. Let $X = \{a, b, c, d, e, f\}$ and let τ be a family of ICCSs containing $\mathcal{A},\ \mathcal{B}$ given :

$$\mathcal{A} = \langle [\{a, b, c\}, \{a, b, c, d, e\}], \{a, b, c, d\} \rangle\,,\ \mathcal{B} = \langle [\{a\}, \{a, b, c\}], \{a, b\} \rangle\,.$$

Then $\mathcal{A} \cup_R \mathcal{B} = \langle [\{a, b, c\}, \{a, b, c, d, e\}], \{a, b\} \rangle$. Thus $\mathcal{A} \cup_R \mathcal{B} \notin ICCS(X)$. So $\tau \in TCCT(X)$.

5. Cubic crisp bases and neighborhoods

In this section, we define a cubic crisp point and some of its properties. Also, we introduce the concepts of cubic crisp base, subbase and neighborhood, and obtain their some properties.

Definition 5.1. Let $a,\ b \in X$. Then the form $\langle [\{a\}, \{a\}], \{b\} \rangle$ [resp. $\langle [\varnothing, \{a\}], \{b\} \rangle$] is called a cubic crisp point (briefly, CCP) [resp. vanishing point (briefly, CCVP)] with the support a and b, and denoted by $(a,b)_{CCP}$ [resp. $(a,b)_{CCVP}$]. In particular, $(a,a)_{CCP}$ [resp. $(a,a)_{CCVP}$] will be denoted by a_{CCP} [resp. a_{CCVP}]. The set of all CCPs and CCVPs in X is denoted by $CC_P(X)$.

Definition 5.2. Let $\mathcal{A} \in CCS(X)$ and $a,\ b \in X$. Then
(i) $(a,b)_{CCP}$ is said to belong to $\mathcal{A}$, denoted by $(a,b)_{CCP} \in \mathcal{A}$, if $a \in A^-,\ b \in A$,
(ii) $(a,b)_{CCVP}$ is said to belong to $\mathcal{A}$, denoted by $(a,b)_{CCVP} \in \mathcal{A}$, if $a \in A^+,\ b \in A$.

From Definition 2.4, we can easily see that $(a,b)_{CCP} \in \mathcal{A} \Leftrightarrow a_{IVP} \in \mathbf{A}$, $b \in A$ and $(a,b)_{CCVP} \in \mathcal{A} \Leftrightarrow a_{IVVP} \in \mathbf{A}$, $b \in A$.

Proposition 5.3. *Let X be a non-empty set and let $\mathcal{A} \in CCS(X)$. Then*

$$\mathcal{A} = \mathcal{A}_{CCP} \cup_P \mathcal{A}_{CCVP},$$

where $A_{CCP} = \bigcup_{P,\ (a,b)_{CCP} \in \mathcal{A}} (a,b)_{CCP}$ *and* $A_{CCVP} = \bigcup_{(a,b)_{P,\ CCVP} \in \mathcal{A}} (a,b)_{IVVP}$.
In fact, $\mathcal{A}_{CCP} = \langle [A^-, A^-], A \rangle$ *and* $\mathcal{A}_{CCVP} = \langle [\varnothing, A^+], A \rangle$.

Proof. $\mathcal{A}_{CCP} = \bigcup_{P,\ (a,b)_{CCP} \in \mathcal{A}} (a,b)_{CCP} = \left\langle \bigcup_{a_{IVP} \in \mathbf{A}} a_{IVP}, \bigcup_{b \in A} \{b\} \right\rangle$
$= \langle [A^-, A^-], A \rangle$ by Result 2.5.
Similarly, we have $\mathcal{A}_{CCVP} = \langle [\varnothing, A^+], A \rangle$. Then the result holds. □

We give the characterization of the P-union and P-intersection by CCPs and CCVPs.

Theorem 5.4 (See Theorem 3.14)**.** *Let $(\mathcal{A}_j)_{j \in J} \subset CCS(X)$ and let $a,\ b \in X$.*
(1) $(a,b)_{CCP} \in \bigcap_{P,\ j \in J} \mathcal{A}_j$ [resp. $(a,b)_{CCVP} \in \bigcap_P \mathcal{A}_j$] if and only if $(a,b)_{CCP} \in \mathcal{A}_j$ [resp. $(a,b)_{CCVP} \in \mathcal{A}_j$], for each $j \in J$.
(2) $(a,b)_{CCP} \in \bigcup_P \mathcal{A}_j$ [resp. $(a,b)_{CCVP} \in \bigcup_P \mathcal{A}_j$] if and only if there exists $j \in J$ such that $(a,b)_{CCP} \in \mathcal{A}_j$ [resp. $(a,b)_{CCVP} \in \mathcal{A}_j$.

Proof. Straightforward. □

The following provides the characterization of the P-inclusion by CCPs and CCVPs.

Theorem 5.5 (See Theorem 3.15)**.** *Let $\mathcal{A},\ \mathcal{V} \in CCS(X)$. Then*
(1) $\mathcal{A} \subset_P \mathcal{B}$ if and only if $(a,b)_{CCP} \in \mathcal{A} \Rightarrow (a,b)_{CCP} \in \mathcal{B}$ [resp. $(a,b)_{CCVP} \in \mathcal{A} \Rightarrow (a,b)_{CCVP} \in \mathcal{B}$] for any $a,\ b \in X$,
(2) $\mathcal{A} = \mathcal{B}$ if and only if $(a,b)_{CCP} \in \mathcal{A} \Leftrightarrow (a,b)_{CCP} \in \mathcal{B}$ [resp. $(a,b)_{CCVP} \in \mathcal{A} \Leftrightarrow (a,b)_{CCVP} \in \mathcal{B}$] for any $a,\ b \in X$.

Proof. Straightforward. □

Definition 5.6. Let (X, τ) be a PCCTS.

(i) A subfamily β of τ is called a cubic crisp P-base (briefly, CCPB) for τ, if for each $\mathcal{A} \in \tau$, $\mathcal{A} = \hat{\varnothing}$ or there is $\beta^{'} \subset \beta$ such that $\mathcal{A} = \bigcup_P \beta^{'}$.

(ii) A subfamily σ of τ is called a cubic crisp P-subbase (briefly, CCPSB) for τ, if the family $\beta = \{\bigcap_P \sigma^{'} : \sigma^{'}$ is a finite subset of $\sigma\}$ is a CCPB for τ.

Remark 5.7. (1) Let β be a CCPB for a PCCT τ on a non-empty set X. Consider two families $\beta_{_{PIV}}$ and $\beta_{_{PC}}$ given by:

$$\beta_{_{PIV}} = \{\mathbf{A} \in IVS(X) : \mathcal{A} \in \beta\},\ \beta_{_{PC}} = \{A \in 2^X : \mathcal{A} \in \beta\}.$$

Then we can easily see that $\beta_{_{PIV}}$ is an IVB for an IVT $\tau_{_{PIV}}$ and $\beta_{_{PC}}$ is an ordinary base for an ordinary topology $\tau_{_{PC}}$ (See Remark 4.3 (1)).

Now let us consider the following families of subsets of X given by:

$$\beta^-_{_P} = \{A^- \in 2^X : \mathcal{A} \in \beta\},\ \beta^+_{_P} = \{A^+ \in 2^X : \mathcal{A} \in \beta\}.$$

Then clearly, $\beta^-_{_P}$ [resp. $\beta^+_{_P}$] is an ordinary base for the ordinary topology $\tau^-_{_P}$ [resp. $\tau^+_{_P}$] (See Remark 4.3 (1)). In this case, $\beta_{_{PIV}}$ [resp. $\beta^-_{_P}$, $\beta^+_{_P}$ and $\beta_{_{PC}}$] will be called an Interval-valued base [resp. ordinary bases] for τ generated by the CCPB β.

(2) Let σ be a CCPSB for a PCCT τ on a non-empty set X. Then there are an IVSB $\sigma_{_{PIV}}$ for an IVT $\tau_{_{PIV}}$ and an ordinary subbase $\sigma_{_{PC}}$ for an ordinary topology $\tau_{_{PC}}$, respectively given by:

$$\sigma_{_{PIV}} = \{\mathbf{A} \in IVS(X) : \mathcal{A} \in \sigma\},\ \sigma_{_{PC}} = \{A \in 2^X : \mathcal{A} \in \sigma\}.$$

Moreover, we have an ordinary subbase $\sigma^-_{_P}$ [resp. $\sigma^+_{_P}$] for the ordinary topology $\tau^-_{_P}$ [resp. $\tau^+_{_P}$] given by:

$$\sigma^-_{_P} = \{A^- \in 2^X : \mathcal{A} \in \sigma\} \text{ [resp. } \sigma^+_{_P} = \{A^+ \in 2^X : \mathcal{A} \in \sigma\}].$$

In this case, $\sigma_{_{PIV}}$ [resp. $\sigma^-_{_P}$, $\sigma^+_{_P}$ and $\sigma_{_{PC}}$] will be called an Interval-valued subbase [resp. ordinary subbases] for τ generated by the CCPSB σ.

Example 5.8. Let X be a non-empty set and let $\sigma = \{(a, b)_{_{CCP}} : a,\ b \in X\} \cup \{(a, b)_{_{CCVP}} : a,\ b \in X\}$. Then σ is a CCPSB for the P-cubic discrete topology $\tau_{_{P,\ 1}}$ on X. In fact, in Example 3.2 (1), we can easily see that $CCS(X)$ has the below set as its CCPSB

$$\{a_{_{CCP}}, a_{_{CCVP}}, b_{_{CCP}}, b_{_{CCVP}}, (a, b)_{_{CCP}}, (a, b)_{_{CCVP}}, (b, a)_{_{CCP}}, (b, a)_{_{CCVP}}\}.$$

The following provides a necessary and sufficient condition for a family of CCSs to be a CCPB for a PCCT on a set X.

Theorem 5.9. *Let X be a non-empty set and let $\beta \subset CCS(X)$. Then β is a CCPB for a PCCT τ on X if and only if it satisfies the followings:*

(1) $\hat{X} = \bigcup_P \beta$,

(2) if $\mathcal{B}_1, \mathcal{B}_2 \in \beta$ and $(a, b)_{_{CCP}} \in \mathcal{B}_1 \cap_P \mathcal{B}_2$ [resp. $(a, b)_{_{CCVP}} \in \mathcal{B}_1 \cap_P \mathcal{B}_2$], then there exists $\mathcal{B} \in \beta$ such that $(a, b)_{_{CCP}} \in \mathcal{B} \subset_P \mathcal{B}_1 \cap_P \mathcal{B}_2$ [resp. $(a, b)_{_{CCVP}} \in \mathcal{B} \subset_P \mathcal{B}_1 \cap_P \mathcal{B}_2$].

Proof. The proof is the same as one in classical topological spaces. □

A family β of CCSs in X such that $\hat{X} = \bigcup_P \beta$ may not to be a PCCT on X (See Example 5.10 (1)).

Example 5.10. Let $X = \{a, b, c\}$.

(1) Let $\beta = \{[\{a, b\}, \{a, b\}], \{c\}\rangle, \langle[\{b, c\}, X], X\rangle, \hat{X}\}$. Assume that β is a CCPB for a PCCT τ on X. Then by the definition of base, $\beta \subset \tau$. Thus we have

$$[\{a, b\}, \{a, b\}], \{c\}\rangle,\ \langle[\{b, c\}, X], X\rangle \in \tau.$$

So we get

$$[\{a, b\}, \{a, b\}], \{c\}\rangle \cap_P \langle[\{b, c\}, X], X\rangle = [\{b\}, \{a, b\}], \{c\}\rangle \in \tau.$$

But for any $\beta^{'} \subset \beta$, $[\{b\}, \{a, b\}], \{c\}\rangle \neq \bigcup_P \beta^{'}$. Hence β is not a CCPB for a PCCT on X.

(2) Consider the family β of IVISs in X given by:

$$\beta = \{\langle[\{a\}, \{a\}], \{b, c\}\rangle, \langle[\{a, b\}, \{a, b\}], \{a, c\}\rangle, \langle[\{a, c\}, \{a, c\}], \{a, b\}\rangle\}.$$

Then clearly, β satisfies two conditions of Theorem 5.9. Thus β is a CCPB for a PCCT τ on X. Furthermore, we can easily check that τ is the family of CCSs in X given by:

$$\tau = \{\hat{\varnothing}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \hat{X}\},$$

where $\mathcal{A}_1 = \langle[\{a\}, \{a\}], \{b, c\}\rangle,\ \mathcal{A}_2 = \langle[\{a, b\}, \{a, b\}], \{a, c\}\rangle,$
$\mathcal{A}_3 = \langle[\{a, c\}, \{a, c\}], \{a, b\}\rangle,\ \mathcal{A}_4 = \langle[\{a, b\}, \{a, b\}], X\rangle,\ \mathcal{A}_5 = \langle[\{a, c\}, \{a, c\}], X\rangle.$

Proposition 5.11. *Let X be a non-empty set and let $\sigma \subset CCS(X)$ such that $\hat{X} = \bigcup_P \sigma$. Then there exists a unique PCCT τ on X such that σ is a CCPSB for τ.*

Proof. Let $\beta = \{B \in CCS(X) : B = \bigcup_{P,\ i=1}^{n} S_i$ and $S_i \in \sigma\}$. Let $\tau = \{U \in CCS(X) : U = \hat{\varnothing}$ or there is a subcollection $\beta^{'}$ of β such that $U = \bigcup_P \beta^{'}\}$. Then we can prove that τ is the unique PCCT on X such that σ is a CCPSB for τ. □

In Proposition 5.11, τ is called the PCCT on X generated by σ.

Example 5.12. Let $X = \{a, b, c\}$ and let us consider the family of IVISs in X given by:

$$\sigma == \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\},$$

where $\mathcal{A}_1 = \langle[\{a\}, \{a, b\}], \{b, c\}\rangle,\ \mathcal{A}_2 = \langle[\{b, c\}, \{b, c\}], \{a\}\rangle,\ \mathcal{A}_3 = \langle[\{c\}, \{a, c\}], \{b\}\rangle.$
Then clearly, $\bigcup_P \sigma = \hat{X}$. Let β be the collection of all finite P-intersections of members of σ. Then we have

$$\beta = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6\},$$

where $\mathcal{A}_4 = \langle[\varnothing, \{b\}], \varnothing\rangle,\ \mathcal{A}_5 = \langle[\varnothing, \{a\}], \{b\}\rangle,\ \mathcal{A}_6 = \langle[\{c\}, \{c\}], \varnothing\rangle.$
Thus we obtain the generated IVIT τ by σ:

$$\tau = \{\hat{\varnothing}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \mathcal{A}_8, \mathcal{A}_9, \mathcal{A}_{10}, \mathcal{A}_{11}, \mathcal{A}_{12}, \hat{X}\},$$

where $\mathcal{A}_7 = \langle[\{a, c\}, X], \{b, c\}\rangle,\ \mathcal{A}_8 = \langle[\{b, c\}, X], \{a, b\}\rangle,\ \mathcal{A}_9 = \langle[\{c\}, X], \{b\}\rangle,$
$\mathcal{A}_{10} = \langle[\varnothing, \{a, b\}], \{b\}\rangle,\ \mathcal{A}_{11} = \langle[\{c\}, \{b, c\}], \varnothing\rangle,\ \mathcal{A}_{12} = \langle[\{c\}, X], \{b\}\rangle.$

Now, we define neighborhoods of cubic crisp points of two types, and discuss with their various properties and give some examples.

Definition 5.13 ([28])**.** Let X be an IVTS, $a \in X$ and let $N \in IVS(X)$. Then

(i) N is called an interval-valued neighborhood (briefly, IVN) of a_{IV}, if there exists a $U \in IVO(X)$ such that

$$a_{IVP} \in U \subset N, \text{ i.e., } a \in U^- \subset N^-,$$

(ii) N is called an interval-valued vanishing neighborhood (briefly, IVVN) of a_{IVV}, if there exists a $U \in IVO(X)$ such that

$$a_{IVVP} \in U \subset N, \text{ i.e., } a \in U^+ \subset N^+.$$

We will denote the set of all IVNs [resp. IVVNs] of a_{IVP} [resp. a_{IVVP}] by $N(a_{IVP})$ [resp. $N(a_{IVVP})$].

Definition 5.14. Let X be a PCCTS, $a, b \in X$ and let $\mathcal{N} \in CCS(X)$. Then

(i) $\mathcal{N}$ is called a cubic crisp neighborhood (briefly, CCN) of $(a,b)_{CCP}$, if there exists a $\mathcal{U} \in PCCO(X)$ such that

$$(a,b)_{CCP} \in \mathcal{U} \subset_P \mathcal{N}, \text{ i.e., } a_{IVP} \in \mathbf{U} \subset \mathbf{N} \text{ and } b \in U \subset N,$$

(ii) $\mathcal{N}$ is called a cubic crisp vanishing neighborhood (briefly, CCVN) of $(a,b)_{CCVP}$, if there exists a $\mathcal{U} \in PCCO(X)$ such that

$$(a,b)_{CCVP} \in \mathcal{U} \subset_P \mathcal{N}, \text{ i.e., } a_{IVVP} \in \mathbf{U} \subset \mathbf{N} \text{ and } b \in U \subset N.$$

We will denote the set of all CCNs [resp. CCVNs] of $(a,b)_{CCP}$ [resp. $(a,b)_{CCVP}$] by $N((a,b)_{CCP})$ [resp. $N((a,b)_{CCVP})$].

Remark 5.15. Let (X,τ) be a CCTS and let $N((a,b)_{CCP})$ [resp. $N((a,b)_{CCVP})$]. Then we can consider a IVN $\mathbf{N} \in N(a_{IVP})$ [resp. $N(a_{IVVP})$] and a classical neighborhood $N \in N(b)$.

Example 5.16. Let $X = \{a,b,c,d\}$ and let τ be the IVIT on X given by:

$$\tau = \{\hat{\varnothing}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \mathcal{A}_8, \mathcal{A}_9, \hat{X}\},$$

where $\mathcal{A}_1 = \langle [\varnothing, \{a\}], \{c,d\} \rangle$, $\mathcal{A}_2 = \langle [\{a\}, \{a\}], \{c\} \rangle$,
$\mathcal{A}_3 = \langle [\{b\}, \{b\}], \{a,c,d\} \rangle$, $\mathcal{A}_4 = \langle [\{a\}, \{d\}], \{c,d\} \rangle$,
$\mathcal{A}_5 = \langle [\{b\}, \{a,b\}, \{a,c,d\} \rangle$, $\mathcal{A}_6 = \langle [\{a,b\}, \{a,b\}], \{a,c,d\} \rangle$,
$\mathcal{A}_7 = \langle [\varnothing, \{a\}], \varnothing \rangle$, $\mathcal{A}_8 = \left\langle \widetilde{\varnothing}, \{c,d\} \right\rangle$, $\mathcal{A}_9 = \left\langle \widetilde{\varnothing}, \{c\} \right\rangle$.

Let $\mathcal{N} = \langle [\{a,b\}, \{a,b,d\}], \{a,c,d\} \rangle$. Then we can easily see that

$$\mathbf{N} \in N(a_{IVP}) \cap N(a_{IVVP}),\ \mathbf{N} \in N(b_{IVP}) \cap N(b_{IVVP}),\ \mathbf{N} \in N(d_{IVVP})$$

and

$$N \in N(a),\ N \in N(c),\ N \in N(d).$$

Thus we have

$$\mathcal{N} \in N((a,c)_{CCP}) \cap N((a,c)_{CCVP}),\ \mathcal{N} \in N((b,c)_{CCP}) \cap N((b,c)_{CCVP}),$$
$$\mathcal{N} \in N((d,c)_{CCVP}).$$

Proposition 5.17. *Let X be a PCCTS and let $a, b, c, d \in X$.*

[CCN1] *If $\mathcal{N} \in N((a,b)_{CCP})$, then $(a,b)_{CCP} \in \mathcal{N}$.*

[CCN2] *If $\mathcal{N} \in N((a,b)_{CCP})$ and $\mathcal{N} \subset_P \mathcal{M}$, then $\mathcal{M} \in N((a,b)_{CCP})$.*

[CCN3] *If $\mathcal{N}, \mathcal{M} \in N((a,b)_{CCP})$, then $\mathcal{N} \cap_P \mathcal{M} \in N((a,b)_{CCP})$.*

[CCN4] *If* $\mathcal{N} \in N((a,b)_{CCP})$, *then there exists* $\mathcal{M} \in N((a,b)_{CCP})$ *such that* $\mathcal{N} \in N((c,d)_{CCP})$ *for each* $(c,d)_{CCP} \in \mathcal{M}$.

Proof. [CCN1] Suppose $\mathcal{N} \in N((a,b)_{CCP})$. Then by Definition 5.14 (i), there is a $\mathcal{U} = \langle \mathbf{U}, U \rangle \in PCCO(X)$ such that $(a,b)_{CCP} \in \mathcal{U} \subset_P \mathcal{N}$. Thus by Remark 4.3 (1) and Definition 5.13, $\mathbf{N} \in N(a_{IVP})$ and $N \in N(b)$. So $a_{IVP} \in \mathbf{N}$ and $b \in N$. Hence $(a,b)_{CCP} \in \mathcal{N}$.

[CCN2] The proof is easy.

[CCN3] Suppose $\mathcal{N},\ \mathcal{M} \in N((a,b)_{CCP})$. Then there are $\mathcal{U},\ \mathcal{V} \in PCCO(X)$ such that

$$(a,b)_{CCP} \in \mathcal{U} \subset_P \mathcal{N} \text{ and } (a,b)_{CCP} \in \mathcal{V} \subset_P \mathcal{M}.$$

Let $\mathcal{W} = \mathcal{U} \cap_P \mathcal{V}$. Then clearly, $\mathcal{W} \in PCCO(X)$ and $(a,b)_{CCP} \in \mathcal{W} \subset_P \mathcal{N} \cap_P \mathcal{M}$. Thus $\mathcal{N} \cap_P \mathcal{M} \in N((a,b)_{CCP})$.

[CCN4] The proof is easy. □

Proposition 5.18. *Let* X *be a PCCTS and let* $a,\ b,\ c,\ d \in X$.

[CCVN1] *If* $\mathcal{N} \in N((a,b)_{CCVP})$, *then* $(a,b)_{CCVP} \in \mathcal{N}$.

[CCVN2] *If* $\mathcal{N} \in N((a,b)_{CCVP})$ *and* $\mathcal{N} \subset_P \mathcal{M}$, *then* $\mathcal{M} \in N((a,b)_{CCVP})$.

[CCVN3] *If* $\mathcal{N},\ \mathcal{M} \in N((a,b)_{CCVP})$, *then* $\mathcal{N} \cap_P \mathcal{M} \in N((a,b)_{CCVP})$.

[CCVN4] *If* $\mathcal{N} \in N((a,b)_{CCVP})$, *then there exists* $\mathcal{M} \in N((a,b)_{CCVP})$ *such that* $\mathcal{N} \in N((c,d)_{CCVP})$ *for each* $(c,d)_{CCVP} \in \mathcal{M}$.

Proof. The proof is similar to one of Proposition 5.17. □

In a classical topological space X, it is well-known that if $A \in N(a)$ for each subset A of X and each $a \in A$, then A is an open set of X. But such property does not hold in PCCTS, in general. Then first of all, we introduce the result related to such property in an interval-valued topological space.

Result 5.19 ([28], Proposition 5.5)**.** *Let* (X,τ) *be an IVTS and let us define two families:*

$$\tau_{IVP} = \{U \in IVS(X) : U \in N(a_{IVP}) \text{ for each } a_{IVP} \in U\}$$

and

$$\tau_{IVVP} = \{U \in IVS(X) : U \in N(a_{IVVP}) \text{ for each } a_{IVVP} \in U\}.$$

Then we have

(1) $\tau_{IVP},\ \tau_{IVVP} \in IVT(X)$,

(2) $\tau \subset \tau_{IVP}$ *and* $\tau \subset \tau_{IVVP}$.

Now we get the similar property to the above Result in an PCCTS.

Proposition 5.20. *Let* (X,τ) *be a PCCTS and let us define two families:*

$$\tau_{CCP} = \{\mathcal{U} \in CCS(X) : \mathcal{U} \in N((a,b)_{CCP}) \text{ for each } (a,b)_{CCP} \in \mathcal{U}\}$$

and

$$\tau_{CCVP} = \{\mathcal{U} \in CCS(X) : \mathcal{U} \in N((a,b)_{CCVP}) \text{ for each } (a,b)_{CCVP} \in \mathcal{U}\}.$$

Then we have

(1) $\tau_{CCP},\ \tau_{CCVP} \in PCCT(X)$,

(2) $\tau \subset \tau_{CCP}$ *and* $\tau \subset \tau_{CCVP}$.

Proof. (1) We only prove that $\tau_{CCP} \in PCCT(X)$.

(PCCO$_1$) From the definition of τ_{CCP}, we have $\hat{\varnothing},\ \hat{\tilde{X}} \in \tau_{CCP}$.

(IVIO$_2$) Let $\mathcal{U}\ ,\mathcal{V} \in CCS(X)$ such that $\mathcal{U}\ ,\mathcal{V} \in \tau_{CCP}$ and let $(a,b)_{CCP} \in \mathcal{U} \cap_P \mathcal{V}$. Then clearly, $\mathcal{U},\ \mathcal{V} \in N((a,b)_{CCP})$. Thus by [CCN3], $\mathcal{U} \cap_P \mathcal{V} \in N((a,b)_{CCP})$. So $\mathcal{U} \cap_P \mathcal{V} \in \tau_{CCP}$.

(IVIO$_3$) Let $(\mathcal{U}_j)_{j\in J}$ be any family of members of τ_{CCP}, let $\mathcal{U} = \bigcup_{P,\ j\in J} \mathcal{U}_j$ and let $(a,b)_{CCP} \in \mathcal{U}$. Then by Theorem 5.4 (2), there is $j_0 \in J$ such that $(a,b)_{CCP} \in \mathcal{U}_{j_0}$. Since $\mathcal{U}_{j_0} \in \tau_{CCP}$, $\mathcal{U}_{j_0} \in N((a,b)_{CCP})$ by the definition of τ_{CCP}. Since $\mathcal{U}_{j_0} \subset_P \mathcal{U}$, $\mathcal{U} \in N((a,b)_{CCP})$ by [CCN2]. So $\mathcal{U} \in \tau_{CCP}$.

(2) Let $\mathcal{U} \in \tau$. Then clearly, $\mathcal{U} \in N((a,b)_{CCP})$ and $\mathcal{U} \in N((a,b)_{CCVP})$ for each $(a,b)_{CCP} \in \mathcal{U}$ and $(a,b)_{CCVP} \in \mathcal{U}$, respectively. Thus $\mathcal{U} \in \tau_{CCP}$ and $\mathcal{U} \in \tau_{CCVP}$. So the results hold. □

Remark 5.21. (1) From the definitions of τ_{CCP} and τ_{CCVP}, and Remark 5.6 (1) in [28], we can easily obtain the followings:

$$\tau_{CCP} = \tau \cup \{\langle [U^-, S], U\rangle : U^+ \subset S,\ \mathcal{U} \in \tau\},$$

$$\tau_{CCVP} = \tau \cup \{\langle \mathbf{S}, U\rangle : \varnothing \neq S^- \subset X \setminus U^+,\ S^+ = S^- \cup U^+,\ \mathcal{U} = \langle [\varnothing, U^+], U\rangle\}.$$

In fact, it is clear that if $U^- = \varnothing$ for each $U \in \tau$, then $\tau_{CCVP} = \tau$.

(2) Let τ be any PCCT on a set X. Then from 4.3 (1) and Proposition 5.20, we get seven ordinary topologies on X given by:

$$\tau_P^- = \{U^- \in 2^X : \mathcal{U} \in \tau\},\ \tau_P^+ = \{U^+ \in 2^X : \mathcal{U} \in \tau\},\ \tau_{PC} = \{U \in 2^X : \mathcal{U} \in \tau\},$$

$$\tau_{CCP}^- = \{U^- \in 2^X : \mathcal{U} \in \tau_{CCP}\},\ \tau_{CCP}^+ = \{U^+ \in 2^X : \mathcal{U} \in \tau_{CCP}\},$$

$$\tau_{CCVP}^- = \{U^- \in 2^X : \mathcal{U} \in \tau_{CCVP}\},\ \tau_{CCVP}^+ = \{U^+ \in 2^X : \mathcal{U} \in \tau_{CCVP}\}.$$

Example 5.22. Let $X = \{a,b,c,d\}$ and consider the family τ of CCSs in X given by:

$$\tau = \{\hat{\varnothing}, \hat{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7\},$$

where $\mathcal{A}_1 = \langle [\{a,b\},\{a,b,c\}],\{b,d\}\rangle,\ \mathcal{A}_2 = \langle [\{c\},\{b,c\}],\{a,b,d\}\rangle,$
$\mathcal{A}_3 = \langle [\varnothing,\{a,c\}],\{d\}\rangle,\ \mathcal{A}_4 = \langle [\varnothing,\{b,c\}],\{b,d\}\rangle,\ \mathcal{A}_5 = \langle [\varnothing,\{c\}],\{d\}\rangle$
$\mathcal{A}_6 = \langle [\{a,b,c\},\{a,b,c\}],\{a,b,d\}\rangle,\ \mathcal{A}_7 = \langle [\{c\},\{a,b,c\}],\{a,b,d\}\rangle.$

Then we can easily see that (X,τ) is a PCCTS. Thus we have:

$$\tau_{CCP} = \tau \cup \{\mathcal{A}_8, \mathcal{A}_9, \mathcal{A}_{10}, \mathcal{A}_{11}, \mathcal{A}_{12}, \mathcal{A}_{13}, \mathcal{A}_{14}, \mathcal{A}_{15}, \mathcal{A}_{16}, \mathcal{A}_{17}, \mathcal{A}_{18}, \mathcal{A}_{19}, \mathcal{A}_{20}, \mathcal{A}_{21}, \mathcal{A}_{22}, \mathcal{A}_{23}\mathcal{A}_{24}, \mathcal{A}_{25}, \mathcal{A}_{26}\}$$

and

$$\tau_{CCVP} = \tau \cup \{\mathcal{A}_{27}, \mathcal{A}_{28}, \mathcal{A}_{29}, \mathcal{A}_{30}, \mathcal{A}_{31}, \mathcal{A}_{32}, \mathcal{A}_{33}, \mathcal{A}_{34}, \mathcal{A}_{35}, \mathcal{A}_{36}, \mathcal{A}_{37}, \mathcal{A}_{38}, \mathcal{A}_{39}\},$$

where $\mathcal{A}_8 = \langle [\{a,b\},X],\{b,d\}\rangle,\ \mathcal{A}_9 = \langle [\{c\},\{a,b,c\}],\{a,b,d\}\rangle,$
$\mathcal{A}_{10} = \langle [\{c\},\{b,c,d\}],\{a,b,d\}\rangle,\ \mathcal{A}_{11} = \langle [\{c\},X],\{a,b,d\}\rangle,$
$\mathcal{A}_{12} = \langle [\varnothing,\{a,b,c\}],\{d\}\rangle,\ \mathcal{A}_{13} = \langle [\varnothing,\{b,c,d\}],\{d\}\rangle,$
$\mathcal{A}_{14} = \langle [\varnothing,X],\{d\}\rangle,\ \mathcal{A}_{15} = \langle [\varnothing,\{a,b,c\}],\{b,d\}\rangle,$
$\mathcal{A}_{16} = \langle [\varnothing,\{b,c,d\}],\{b,d\}\rangle,\ \mathcal{A}_{17} = \langle [\varnothing,X],\{b,d\}\rangle,$
$\mathcal{A}_{18} = \langle [\varnothing,\{a,c\}],\{d\}\rangle,\ \mathcal{A}_{19} = \langle [\varnothing,\{b,c\}],\{d\}\rangle,$
$\mathcal{A}_{20} = \langle [\varnothing,\{c,d\}],\{d\}\rangle,\ \mathcal{A}_{21} = \langle [\varnothing,\{a,b,c\}],\{d\}\rangle,$
$\mathcal{A}_{22} = \langle [\varnothing,\{a,c,d\}],\{d\}\rangle,\ \mathcal{A}_{23} = \langle [\varnothing,\{b,c,d\}],\{d\}\rangle,$
$\mathcal{A}_{24} = \langle [\varnothing,X],\{d\}\rangle,\ \mathcal{A}_{25} = \langle [\{a,b,c\},X],\{a,b,d\}\rangle,$

$\mathcal{A}_{26} = \langle [\{c\}, X], \{a, b, d\} \rangle$, $\mathcal{A}_{27} = \langle [\{b\}, \{a, b, c\}], \{d\} \rangle$,
$\mathcal{A}_{28} = \langle [\{d\}, \{a, c, d\}], \{d\} \rangle$, $\mathcal{A}_{29} = \langle [\{a, d\}, X], \{d\} \rangle$,
$\mathcal{A}_{30} = \langle [\{a\}, \{a, b, c\}], \{b, d\} \rangle$, $\mathcal{A}_{31} = \langle [\{d\}, \{a, b, d\}], \{b, d\} \rangle$,
$\mathcal{A}_{32} = \langle [\{a, d\}, X], \{b, d\} \rangle$, $\mathcal{A}_{33} = \langle [\{a\}, \{a, c\}], \{d\} \rangle$,
$\mathcal{A}_{34} = \langle [\{b\}, \{b, c\}], \{d\} \rangle$, $\mathcal{A}_{35} = \langle [\{d\}, \{c, d\}], \{d\} \rangle$,
$\mathcal{A}_{36} = \langle [\{a, b\}, \{a, b, c\}], \{d\} \rangle$, $\mathcal{A}_{37} = \langle [\{a, d\}, \{a, c, d\}], \{d\} \rangle$,
$\mathcal{A}_{38} = \langle [\{b, d\}, \{b, c, d\}], \{d\} \rangle$, $\mathcal{A}_{39} = \langle [\{a, b, d\}, X], \{d\} \rangle$.

So we can confirm that Proposition 5.20 holds.

Furthermore, we obtain seven ordinary topologies on X for the PCCT τ:

$$\tau^-_P = \{\varnothing, X, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\},\ \tau^+_P = \{\varnothing, X, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$

$$\tau_{PC} = \{\varnothing, X, \{d\}, \{b, d\}, \{a, b, d\}\},\ \tau^-_{CCP} = \{\varnothing, X, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\},$$

$$\tau^+_{CCP} = \{\varnothing, X, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\},$$

$$\tau^-_{CCVP} = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}\},$$

$$\tau^+_{CCVP} = \{\varnothing, X, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}.$$

The following is the immediate result of Proposition 5.20 (2).

Corollary 5.23. *Let (X, τ) be a PCCTS and let $PCCC_\tau$ [resp. $PCCC_{\tau_{CCP}}$ and $PCCC_{\tau_{CCVP}}$] be the set of all CCSs w.r.t. τ [resp. τ_{CCP} and τ_{CCVP}]. Then*

$$PCCC_\tau \subset PCCC_{\tau_{CCP}}, \text{ and } PCCC_\tau \subset PCCC_{\tau_{CCVP}}.$$

Example 5.24. Let (X, τ) be the PCCTS given in Example 5.22. Then we have:

$$PCCC_\tau = \{\hat{\varnothing}, \hat{X}, \mathcal{A}^c_1, \mathcal{A}^c_2, \mathcal{A}^c_3, \mathcal{A}^c_4, \mathcal{A}^c_5, \mathcal{A}^c_6, \mathcal{A}^c_7\},$$
$$PCCC_{\tau_{CCP}} = PCCC_\tau \cup \{\mathcal{A}^c_8, \mathcal{A}^c_9, \mathcal{A}^c_{10}, \mathcal{A}^c_{11}, \mathcal{A}^c_{12}, \mathcal{A}^c_{13}, \mathcal{A}^c_{14}, \mathcal{A}^c_{15}, \mathcal{A}^c_{16}, \mathcal{A}^c_{17}, \mathcal{A}^c_{18}, \mathcal{A}^c_{19}, \mathcal{A}^c_{20}, \mathcal{A}^c_{21}, \mathcal{A}^c_{22}, \mathcal{A}^c_{23}, \mathcal{A}^c_{24}, \mathcal{A}^c_{25}, \mathcal{A}^c_{26}\},$$
$$PCCC_{\tau_{CCVP}} = PCCC_\tau \cup \{\mathcal{A}^c_{27}, \mathcal{A}^c_{28}, \mathcal{A}^c_{29}, \mathcal{A}^c_{30}, \mathcal{A}^c_{31}, \mathcal{A}^c_{32}, \mathcal{A}^c_{33}, \mathcal{A}^c_{34}, \mathcal{A}^c_{35}, \mathcal{A}^c_{36}, \mathcal{A}^c_{37}, \mathcal{A}^c_{38}, \mathcal{A}^c_{39}\},$$

where $\mathcal{A}^c_1 = \langle [\{d\}, \{c, d\}], \{a, b\} \rangle$, $\mathcal{A}^c_2 = \langle [\{a, d\}, \{a, b, d\}], \{c\} \rangle$,
$\mathcal{A}^c_3 = \langle [\{b, d\}, X], \{a, b, c\} \rangle$, $\mathcal{A}^c_4 = \langle [\{a, d\}, X], \{a, c\} \rangle$
$\mathcal{A}^c_5 = \langle [\{a, b, d\}, X], \{a, b, c\} \rangle$, $\mathcal{A}^c_6 = \langle [\{d\}, \{d\}], \{c\} \rangle$,
$\mathcal{A}^c_7 = \langle [\{d\}, \{a, b, d\}], \{c\} \rangle$,
$\mathcal{A}^c_8 = \langle [\varnothing, \{c, d\}], \{a, c\} \rangle$, $\mathcal{A}^c_9 = \langle [\{d\}, \{a, b, d\}], \{c\} \rangle$,
$\mathcal{A}^c_{10} = \langle [\{a\}, \{a, b, d\}], \{c\} \rangle$, $\mathcal{A}^c_{11} = \langle [\varnothing, \{a, b, d\}], \{c\} \rangle$,
$\mathcal{A}^c_{12} = \langle [\varnothing, \{a, b, c\}], \{d\} \rangle$, $\mathcal{A}^c_{13} = \langle [\varnothing, \{b, c, d\}], \{d\} \rangle$,
$\mathcal{A}^c_{14} = \langle [\varnothing, X], \{a, b, c\} \rangle$, $\mathcal{A}^c_{15} = \langle [\{d\}, X], \{a, c\} \rangle$,
$\mathcal{A}^c_{16} = \langle [\{a\}, X], \{a, c\} \rangle$, $\mathcal{A}^c_{17} = \langle [\varnothing, X], \{a, c\} \rangle$,
$\mathcal{A}^c_{18} = \langle [\{b, d\}, X], \{a, b, c\} \rangle$, $\mathcal{A}^c_{19} = \langle [\{a, d\}, X], \{a, b, c\} \rangle$,
$\mathcal{A}^c_{20} = \langle [\{a, b\}, X], \{a, b, c\} \rangle$, $\mathcal{A}^c_{21} = \langle [\{d\}, X], \{a, b, c\} \rangle$,
$\mathcal{A}^c_{22} = \langle [\{b\}, X], \{a, b, c\} \rangle$, $\mathcal{A}^c_{23} = \langle [\{a\}, X], \{a, b, c\} \rangle$,
$\mathcal{A}^c_{24} = \langle [\varnothing, X], \{a, b, c\} \rangle$, $\mathcal{A}^c_{25} = \langle [\varnothing, \{d\}], \{c\} \rangle$,
$\mathcal{A}^c_{26} = \langle [\varnothing, \{a, b, d\}], \{c\} \rangle$, $\mathcal{A}^c_{27} = \langle [\{d\}, \{a, b, d\}], \{a, b, c\} \rangle$,
$\mathcal{A}^c_{28} = \langle [\{d\}, \{a, c, d\}], \{d\} \rangle$, $\mathcal{A}^c_{29} = \langle [\{a, d\}, X], \{d\} \rangle$,
$\mathcal{A}^c_{30} = \langle [\{d\}, \{b, c, d\}], \{a, c\} \rangle$, $\mathcal{A}^c_{31} = \langle [\{c\}, \{a, b, c\}], \{a, c\} \rangle$,
$\mathcal{A}^c_{32} = \langle [\varnothing, \{b, c\}], \{a, c\} \rangle$, $\mathcal{A}^c_{33} = \langle [\{b, d\}, \{b, c, d\}], \{a, b, c\} \rangle$,
$\mathcal{A}^c_{34} = \langle [\{a, d\}, \{a, c, d\}], \{a, b, c\} \rangle$, $\mathcal{A}^c_{35} = \langle [\{a, b\}, \{a, b, c\}], \{a, b, c\} \rangle$,

$\mathcal{A}^c_{36} = \langle [\{d\}, \{c,d\}], \{a,b,c\} \rangle$, $\mathcal{A}^c_{37} = \langle [\{b\}, \{b,c\}], \{d\} \rangle$,
$\mathcal{A}^c_{38} = \langle [\{b,d\}, \{b,c,d\}], \{a,b,c\} \rangle$, $\mathcal{A}^c_{39} = \langle [\varnothing, \{c\}], \{a,b,c\} \rangle$.
Thus we can confirm that Corollary 5.23 holds.

Now let us the converses of Propositions 5.17 and 5.18.

Proposition 5.25. *Let X be a non-empty set. Suppose to each $a,\ b \in X$, there corresponds a set $N_*((a,b)_{CCVP})$ of CCSs in X satisfying the conditions [CCVN1], [CCVN2], [CCVN3] and [CCVN4] in Proposition 5.18. Then there is a PCCT on X such that $N_*((a,b)_{CCVP})$ is the set of all CCNs of $(a,b)_{CCVP}$ in this PCCT for any $a,\ b \in X$.*

Proof. Let

$$\tau_{CCVP} = \{\mathcal{U} \in CCS(X) : \mathcal{U} \in N((a,b)_{CCVP}) \text{ for each } (a,b)_{CCVP} \in \mathcal{U}\},$$

where $N((a,b)_{CCVP})$ denotes the set of all CCVNs of $(a,b)_{CCVP}$ in τ.
Then clearly, $\tau_{CCVP} \in PCCT(X)$ by Proposition 5.18. we will prove that $N_*((a,b)_{CCVP})$ is the set of all CCVNs of $(a,b)_{CCVP}$ in τ_{CCVP} for any $a,\ b \in X$.

Let $\mathcal{V} \in CCS(X)$ such that $\mathcal{V} \in N_*((a,b)_{CCVP})$ and let U be the union of all the CCVPs $(a_1,b_1)_{CCVP}$ in X such that $\mathcal{U} \in N_*((a,b)_{CCVP})$. If we can prove that

$$\mathcal{V} \in \mathcal{U} \subset_P \mathcal{V} \text{ and } \mathcal{U} \in \tau_{CCVP},$$

then the proof will be complete.

Since $\mathcal{V} \in N_*((a_1,b_1)_{CCVP})$, $(a,b)_{CCVP} \in \mathcal{U}$. Moreover, $\mathcal{U} \subset_P \mathcal{V}$. Suppose $(c,d)_{CCVP} \in \mathcal{U}$. Then by [CCVN4], there is a CCS $\mathcal{W} \in N_*((a_1,b_1)_{CCVP})$ such that $\mathcal{V} \in N_*((a_2,b_2)_{CCVP})$ for each $(a_2,b_2)_{CCVP} \in \mathcal{W}$. Thus $(a_2,b_2)_{CCVP} \in \mathcal{U}$. By Theorem 5.5, $\mathcal{W} \subset_P \mathcal{U}$. So by [CCVN2], $\mathcal{U} \in N_*((a_1,b_1)_{CCVP})$ for each $(a_1,b_1)_{CCVP} \in \mathcal{U}$. Hence by the definition of τ_{CCVP}, $\mathcal{U} \in \tau_{CCVP}$. This completes the proof. □

Proposition 5.26. *Let X be a non-empty set. Suppose to each $a,\ b \in X$, there corresponds a set $N_*((a,b)_{CCP})$ of CCSs in X satisfying the conditions [CCN1], [CCN2], [CCN3] and [CCN4] in Proposition 5.17. Then there is a PCCT on X such that $N_*((a,b)_{CCP})$ is the set of all CCNs of $(a,b)_{CCP}$ in this PCCT for any $a,\ b \in X$.*

Proof. The proof is similar to Proposition 5.25. □

The following provides a necessary and sufficient condition for a CCS to be a PCCOS.

Theorem 5.27. *Let (X,τ) be a PCCTS and let $\mathcal{A} \in CCS(X)$. Then $\mathcal{A} \in \tau$ if and only if $\mathcal{A} \in N((a,b)_{CCP})$ and $\mathcal{A} \in N((a,b)_{CCVP})$ for any $(a,b)_{CCP},\ (a,b)_{CCVP} \in \mathcal{A}$.*

Proof. Suppose $\mathcal{A} \in N((a,b)_{CCP})$ and $\mathcal{A} \in N((a,b)_{CCVP})$ for any $(a,b)_{CCP},\ (a,b)_{CCVP} \in \mathcal{A}$. Then there are $\mathcal{U}_{(a,b)_{CCP}},\ \mathcal{V}_{(a,b)_{CCVP}} \in \tau$ such that $(a,b)_{CCP} \in \mathcal{U}_{(a,b)_{CCP}} \subset_P \mathcal{A}$ and $(a,b)_{CCVP} \in \mathcal{V}_{(a,b)_{CCVP}} \subset_P \mathcal{A}$. Thus we have

$$\begin{aligned}\mathcal{A} &= \left(\bigcup_{P,\ (a,b)_{CCP} \in \mathcal{A}} (a,b)_{CCP}\right) \cup_P \left(\bigcup_{P,\ (a,b)_{CCVP} \in \mathcal{A}} (a,b)_{CCVP}\right) \\ &\subset_P \left(\bigcup_{P,\ (a,b)_{CCP} \in \mathcal{A}} \mathcal{U}_{(a,b)_{CCP}}\right) \cup_P \left(\bigcup_{P,\ (a,b)_{CCVP} \in \mathcal{A}} \mathcal{V}_{(a,b)_{CCVP}}\right)\end{aligned}$$

$\subset_P \mathcal{A}$.

So we get

$$\mathcal{A} = \left(\bigcup_{P,\ (a,b)_{CCP} \in \mathcal{A}} \mathcal{U}_{(a,b)_{CCP}} \right) \cup_P \left(\bigcup_{P,\ (a,b)_{CCVP} \in \mathcal{A}} \mathcal{V}_{(a,b)_{CCVP}} \right).$$

Since $\mathcal{U}_{(a,b)_{CCP}},\ \mathcal{V}_{(a,b)_{CCVP}} \in \tau$, $\mathcal{A} \in \tau$.

The proof of the necessary condition is easy. □

Now, we give the relation among three PCCTs, τ, τ_{CCP} and τ_{CCVP}.

Proposition 5.28. $\tau = \tau_{CCP} \cap \tau_{CCVP}$.

Proof. From Proposition 5.20 (2), it is clear that $\tau \subset \tau_{CCP} \cap \tau_{CCVP}$.

Conversely, let $\mathcal{U} \in \tau_{CCP} \cap \tau_{CCVP}$. Then clearly, $\mathcal{U} \in \tau_{CCP}$ and $\mathcal{U} \in \tau_{CCVP}$. Thus $\mathcal{U}$ is a CCN of each of its CCPs $(a,b)_{CCP}$ and a CCVN of each of its CCVPs $(a,b)_{CCVP}$. So there are $\mathcal{U}(a,b)_{CCP},\ \mathcal{U}_{(a,b)_{CCVP}} \in \tau$ such that $(a,b)_{CCP} \in \mathcal{U}_{(a,b)_{CCP}} \subset_P \mathcal{U}$ and $(a,b)_{CCVP} \in \mathcal{U}_{(a,b)_{CCVP}} \subset_P \mathcal{U}$. Hence we have

$$\mathcal{U}_{CCP} = \bigcup_{P,\ (a,b)_{CCP} \in \mathcal{U}} (a,b)_{CCP} \subset_P \mathcal{U} \text{ and } \mathcal{U}_{CCVP} = \bigcup_{P,\ (a,b)_{CCVP} \in \mathcal{U}} (a,b)_{CCVP} \subset_P \mathcal{U}.$$

By Proposition 5.3, we get

$$\mathcal{U} = \mathcal{U}_{CCP} \cup_P \mathcal{U}_{CCVP} \subset_P \left(\bigcup_{P,\ (a,b)_{CCP} \in \mathcal{U}} \mathcal{U}_{(a,b)_{CCP}} \right) \cup_P \left(\bigcup_{(a,b)_{CCVP} \in \mathcal{U}} \mathcal{U}_{(a,b)_{CCVP}} \right) \subset_P \mathcal{U},$$

$$\text{i.e., } \mathcal{U} = \left(\bigcup_{P,\ (a,b)_{CCP} \in \mathcal{U}} \mathcal{U}_{(a,b)_{CCP}} \right) \cup_P \left(\bigcup_{P,\ (a,b)_{CCVP} \in \mathcal{U}} \mathcal{U}_{(a,b)_{CCVP}} \right).$$

It is obvious that $\mathcal{U} \in \tau$. Therefore $\tau_{CCP} \cap_P \tau_{CCVP} \subset_P \tau$. This completes the proof. □

The following is the immediate result of Proposition 5.28.

Corollary 5.29. *Let (X,τ) be a PCCTS. Then*

$$PCCC_\tau = PCCC_{\tau_{CCP}} \cap_P PCCC_{\tau_{CCVP}}.$$

6. Interiors and closures of CCSs and cubic crisp continuities

In this section, we introduce the concepts of cubic crisp interiors and closures, and study some of their properties and give some examples. In particular, we show that there is a unique PCCT on a set X from the cubic crisp closure [resp. interior] operator. Furthermore, we define a cubic crisp continuity and discuss with some of its properties.

Definition 6.1. Let (X,τ) be a CCTS and let $\mathcal{A} \in CCS(X)$.

(i) The cubic crisp closure of $\mathcal{A}$ w.r.t. τ, denoted by $CCcl(\mathcal{A})$, is a CCS in X defined as:

$$CCcl(\mathcal{A}) = \bigcap_P \{\mathcal{K} : \mathcal{K}^c \in \tau \text{ and } \mathcal{A} \subset_P \mathcal{K}\}.$$

(ii) The cubic crisp interior of $\mathcal{A}$ w.r.t. τ, denoted by $CCint(\mathcal{A})$, is a CCS in X defined as:

$$CCint(\mathcal{A}) = \bigcup_P \{\mathcal{G} : \mathcal{G} \in \tau \text{ and } \mathcal{G} \subset_P \mathcal{A}\}.$$

(iii) The cubic crisp closure of $\mathcal{A}$ w.r.t. τ_{CCP}, denoted by $cl_{CCP}(\mathcal{A})$, is a CCS in X defined as:

$$cl_{CCP}(\mathcal{A}) = \bigcap_P \{\mathcal{K} : \mathcal{K}^c \in \tau_{CCP} \text{ and } \mathcal{A} \subset_P \mathcal{K}\}.$$

(iv) The cubic crisp interior of A w.r.t. τ_{CCP}, denoted by $int_{CCP}(\mathcal{A})$, is a CCS in X defined as:

$$int_{CCP}(\mathcal{A}) = \bigcup_P \{\mathcal{G} : \mathcal{G} \in \tau_{CCP} \text{ and } \mathcal{G} \subset_P \mathcal{A}\}.$$

(v) The cubic crisp closure of A w.r.t. τ_{CCVP}, denoted by $cl_{CCVP}(\mathcal{A})$, is a CCS in X defined as:

$$cl_{CCVP}(\mathcal{A}) = \bigcap_P \{\mathcal{K} : \mathcal{K}^c \in \tau_{CCVP} \text{ and } \mathcal{A} \subset_P \mathcal{K}\}.$$

(vi) The cubic crisp interior of A w.r.t. τ_{CCVP}, denoted by $int_{CCVP}(\mathcal{A})$, is a CCS in X defined as:

$$int_{CCVP}(\mathcal{A}) = \bigcup_P \{\mathcal{G} : \mathcal{G} \in \tau_{CCVP} \text{ and } \mathcal{G} \subset_P \mathcal{A}\}.$$

It is obvious that $CCcl(\mathcal{A})$ [resp. $CCint(\mathcal{A})$] is the smallest PCCCS in X containing $\mathcal{A}$ [resp. the largest PCCOS in X contained in $\mathcal{A}$.

Remark 6.2. From the above definition, it is obvious that the followings hold:

$$CCint(\mathcal{A}) \subset int_{CCP}(\mathcal{A}),\ CCint(\mathcal{A}) \subset int_{CCVP}(\mathcal{A})$$

and

$$cl_{CCP}(\mathcal{A}) \subset CCcl(\mathcal{A}),\ cl_{CCVP}(\mathcal{A}) \subset CCcl(\mathcal{A}).$$

Example 6.3. Let (X, τ) be the PCCTS given in Example 5.22. Consider a CCS $\mathcal{A} = \langle [\{b\}, \{b, c\}], \{d\} \rangle$ in X. Then

$CCint(\mathcal{A}) = \mathcal{A}_1 \cup_P \mathcal{A}_6 = \mathcal{A}_6$,
$int_{CCP}(\mathcal{A}) = \mathcal{A}_1 \cup_P \mathcal{A}_6 \cup_P \mathcal{A}_8 \cup_P \mathcal{A}_{25} = \mathcal{A}_{25}$,
$int_{CCVP}(\mathcal{A}) = \mathcal{A}_1 \cup_P \mathcal{A}_6 \cup_P \mathcal{A}_{34} \cup_P \mathcal{A}_{36} \cup_P \mathcal{A}_{38} \cup_P \mathcal{A}_{39} = \left\langle \widetilde{X}, \{a, b, d\} \right\rangle$
$CCcl(\mathcal{A}) = cl_{CCP}(\mathcal{A}) = \hat{X}$,
$cl_{CCCP}(\mathcal{A}) = \hat{X} \cap_P \mathcal{A}^c_{37} = \mathcal{A}^c_{37}$.

Thus we can confirm that Remark 6.2 holds.

Proposition 6.4. *Let (X, τ) be a PCCTS and let $\mathcal{A} \in CCS(X)$. Then*

$$CCint(\mathcal{A}^c) = (CCcl(\mathcal{A}))^c \text{ and } CCcl(\mathcal{A}^c) = (CCint(\mathcal{A})^c.$$

Proof. $CCint(\mathcal{A}^c)$
$= \bigcup_P \{\mathcal{G} \in \tau : \text{and } \mathcal{G} \subset_P \mathcal{A}^c\}$
$= \bigcup_P \{\mathcal{G} \in \tau : \mathbf{G} = [G^-, G^+] \subset [A^{+c}, A^{-c}] = \mathbf{A}^c,\ G \subset A^c\}$
$= \bigcup_P \{\mathcal{G} \in \tau : \mathbf{A} \subset \mathbf{G}^c,\ A \subset G^c\}$
$= (\bigcap_P \{\mathcal{G}^c : \mathcal{G} \in \tau, \mathcal{A} \subset_P \mathcal{G}^c\})^c$

$= (CCl(\mathcal{A}))^c$.
Similarly, we can show that $CCl(\mathcal{A}^c) = (CCint(\mathcal{A})^c$. □

Proposition 6.5. *Let (X, τ) be a PCCTS and let $\mathcal{A} \in CCS(X)$. Then*

$$CCint(\mathcal{A}) = int_{CCP}(\mathcal{A}) \cap_P int_{CCVP}(\mathcal{A}).$$

Proof. The proof is straightforward from Proposition 5.28 and Definition 6.1. □

The following is the immediate result of Definition 6.1, and Propositions 6.4 and 6.5.

Corollary 6.6. *Let (X, τ) be a PCCTS and let $\mathcal{A} \in CCS(X)$. Then*

$$CCl(\mathcal{A}) = cl_{CCP}(\mathcal{A}) \cup_P cl_{CCVP}(\mathcal{A}).$$

Theorem 6.7. *Let X be a PCCTS and let $\mathcal{A} \in CCS(X)$. Then*
(1) *$\mathcal{A} \in PCCC(X)$ if and only if $\mathcal{A} = CCl(\mathcal{A})$,*
(2) *$\mathcal{A} \in PCCO(X)$ if and only if $\mathcal{A} = CCint(\mathcal{A})$.*

Proof. The proofs are easy from Definition 6.1. □

Proposition 6.8 (Kuratowski Closure Axioms)**.** *Let X be a PCCTS and let $\mathcal{A}$, $\mathcal{B} \in CCS(X)$. Then*
[CCK0] *if $\mathcal{A} \subset_P \mathcal{B}$, then $CCl(\mathcal{A}) \subset_P CCl(\mathcal{B})$,*
[CCK1] *$CCl(\hat{\varnothing}) = \hat{\varnothing}$,*
[CCK2] *$\mathcal{A} \subset_P CCl(A)$,*
[CCK3] *$CCl(CCl(A)) = CCl(\mathcal{A})$,*
[CCK4] *$CCl(\mathcal{A} \cup_P \mathcal{B}) = CCl(\mathcal{A}) \cup_P CCl(\mathcal{B})$.*

Proof. Straightforward. □

Let $CCl^* : CCS(X) \to CCS(X)$ be the mapping satisfying the properties [CCK1], [CCK2],[CCK3] and [CCK4]. Then the mapping CCl^* will be called the cubic crisp closure operator(briefly, CCCO) on X.

Proposition 6.9. *Let CCl^* be the CCCO on X. Then there exists a unique PCCT τ on X such that $CCl^*(\mathcal{A}) = CCl(\mathcal{A})$, for each $\mathcal{A} \in CCS(X)$, where $CCl(\mathcal{A})$ denotes the cubic crisp closure of $\mathcal{A}$ in the PCCTS (X, τ). In fact,*

$$\tau = \{\mathcal{A}^c \in CCS(X) : CCl^*(\mathcal{A}) = \mathcal{A}\}.$$

Proof. The proof is almost similar to the case of classical topological spaces. □

Proposition 6.10. *Let X be a PCCTS and let $\mathcal{A}$, $\mathcal{B} \in CCS(X)$. Then*
[CCI0] *if $\mathcal{A} \subset_P \mathcal{B}$, then $CCint(\mathcal{A}) \subset_P CCint(\mathcal{B})$,*
[CCI1] *$CCint(\hat{X}) = \hat{X}$,*
[CCI2] *$CCint(\mathcal{A}) \subset_P \mathcal{A}$,*
[CCI3] *$CCint(CCint(\mathcal{A})) = CCint(\mathcal{A})$,*
[CCI4] *$CCint(\mathcal{A} \cap_P \mathcal{B}) = CCint(\mathcal{A}) \cap_P CCint(\mathcal{B})$.*

Proof. Straightforward. □

Let $CCint^* : CCS(X) \to CCS(X)$ be the mapping satisfying the properties [CCI1], [CCI2],[CCI3] and [CCI4]. Then the mapping $CCint^*$ will be called the cubic crisp interior operator (briefly, CCIO) on X.

Proposition 6.11. *Let $CCint^*$ be the CCIO on X. Then there exists a unique PCCT τ on X such that $CCint^*(\mathcal{A}) = CCint(\mathcal{A})$, for each $\mathcal{A} \in CCS(X)$, where $CCint(\mathcal{A})$ denotes the cubic crisp interior of $\mathcal{A}$ in the PCCTS (X, τ). In fact,*

$$\tau = \{\mathcal{A} \in CCS(X) : CCint^*(\mathcal{A}) = \mathcal{A}\}.$$

Proof. The proof is similar to one of Proposition 6.9. □

Before we introduce the concept of cubic crisp continuous mappings, we define the image and preimage of a cubic crisp set under a mapping and find their properties.

Definition 6.12. Let $X,\ Y$ be two non-empty sets, let $f : X \to Y$ be a mapping and let $\mathcal{A} \in CCS(X),\ \mathcal{B} \in CCS(Y)$.

(i) The image of $\mathcal{A}$ under f, denoted by $f(\mathcal{A})$, is a CCS in Y defined by:

$$f(\mathcal{A}) = \langle f(\mathbf{A}, f(A)\rangle = \left\langle [f(A^-), f(A^{-+})], f(A)\right\rangle.$$

(ii) The preimage of $\mathcal{B}$ under f, denoted by $f^{-1}(\mathcal{B})$, is a CCS in X defined by:

$$f^{-1}(\mathcal{B}) = \left\langle f^{-1}(\mathbf{B}), f^{-1}(B)\right\rangle = \left\langle [f^{-1}(B^-), f^{-1}(B^+)], f^{-1}(B)\right\rangle.$$

It is obvious that $f((a,b)_{CCP}) = (f(a), f(b))_{CCP}$ and $f(a_{CCVP}) = (f(a), f(b))_{CCVP}$ for any $a,\ b \in X$.

Proposition 6.13. *Let $X,\ Y$ be two non-empty sets, let $f : X \to Y$ be a mapping, let $\mathcal{A},\ \mathcal{A}_1,\ \mathcal{A}_2 \in CCS(X)$, $(\mathcal{A}_j)_{j\in J} \subset CCS(X)$ and let $\mathcal{B},\ \mathcal{B}_1,\ \mathcal{B}_2 \in CCS(Y)$, $(\mathcal{B}_j)_{j\in J} \subset CCS(Y)$. Then*

(1) *if $\mathcal{A}_1 \subset_P \mathcal{A}_2$, then $f(\mathcal{A}_1) \subset_P f(\mathcal{A}_2)$,*
(2) *if $\mathcal{B}_1 \subset_P \mathcal{B}_2$, then $f^{-1}(\mathcal{B}_1) \subset_P f^{-1}(\mathcal{B}_2)$,*
(3) *$\mathcal{A} \subset_P f^{-1}(f(\mathcal{A}))$ and if f is injective, then $\mathcal{A} = f^{-1}(f(\mathcal{A}))$,*
(4) *$f(f^{-1}(\mathcal{B})) \subset \mathcal{B}$ and if f is surjective, $f(f^{-1}(\mathcal{B})) = \mathcal{B}$,*
(5) *$f^{-1}(\bigcup_{P,\ j\in J} \mathcal{B}_j) = \bigcup_{P,\ j\in J} f^{-1}(\mathcal{B}_j)$,*
(6) *$f^{-1}(\bigcap_{P,\ j\in J} \mathcal{B}_j) = \bigcap_{P,\ j\in J} f^{-1}(\mathcal{B}_j)$,*
(7) *$f(\bigcup_{P,\ j\in J} \mathcal{A}_j) = \bigcup_{P,\ j\in J} f(\mathcal{A}_j)$,*
(8) *$f(\bigcap_{P,\ j\in J} \mathcal{A}_j) \subset_P \bigcap_{P,\ j\in J} f(\mathcal{A}_j)$ and if f is injective, then $f(\bigcap_{p,\ j\in J} \mathcal{A}_j) = \bigcap_{P,\ j\in J} f(\mathcal{A}_j)$,*
(9) *if f is surjective, then $f(\mathcal{A})^c \subset_P f(\mathcal{A}^c)$.*
(10) *$f^{-1}(\mathcal{B}^c) = f^{-1}(\mathcal{B})^c$.*
(11) *$f^{-1}(\hat{\varnothing}) = \hat{\varnothing}$, $f^{-1}(\hat{X}) = \hat{X}$,*
(12) *$f(\hat{\varnothing}) = \hat{\varnothing}$ and if f is surjective, then $f(\hat{X}) = \hat{X}$,*
(13) *if $g : Y \to Z$ is a mapping, then $(g \circ f)^{-1}(\mathcal{C}) = f^{-1}(g^{-1}(\mathcal{C}))$, for each $\mathcal{C} \in CCS(Z)$.*

Proof. The proofs are straightforward. □

Definition 6.14. Let $X,\ Y$ be PCCTSs and let $f : X \to Y$ be a mapping. Then f is said to cubic crisp continuous, if $f^{-1}(\mathcal{V}) \in PCCO(X)$ for each $\mathcal{V} \in PCCO(Y)$.

Proposition 6.15. *Let $X,\ Y,\ Z$ be PCCTSs and let $f : X \to Y$ and $g : Y \to Z$ be mappings.*

(1) *The identity mapping $id : X \to X$ is cubic crisp continuous.*
(2) *If $f,\ g$ are cubic crisp continuous, then $g \circ f$ is cubic crisp continuous.*

Proof. From Definition 6.14 and Proposition 6.13 (13), the proofs are easy. □

Remark 6.16. Let $\mathbf{CC_{Top}}$ be the collection of all PCCTSs and all cubic crisp mappings between them. Then we can easily see that $\mathbf{CC_{Top}}$ forms a concrete category from Proposition 6.15.

Definition 6.17. Let $X,\ Y$ be PCCTSs, let $a,\ b \in X$ and let $f : X \to Y$ be a mapping. Then f is said to be:

(i) cubic crisp point-wise continuous (briefly, CCPC) at $(a,b)_{CCP}$, if $f^{-1}(\mathcal{V}) \in N((a,b)_{CCP})$ for each $\mathcal{V} \in N((f(a), f(b))_{CCP})$,

(ii) cubic crisp vanishing point-wise continuous (briefly, CCVPC) at $(a,b)_{CCVP}$, if $f^{-1}(\mathcal{V}) \in N((a,b)_{CCVP})$ for each $\mathcal{V} \in N((f(a), f(b))_{CCVP})$.

Proposition 6.18. *Let $X,\ Y$ be two PCCTSs. Then a mapping $f : X \to Y$ is cubic crisp continuous if and only if it is CCPC at each $(a,b)_{CCVP}$ and CCVPC at each $(a,b)_{CCVP}$.*

Proof. Suppose f is cubic crisp continuous and let $\mathcal{V} \in N((f(a), f(b))_{CCP})$ for any $(a,b)_{CCP}$. Then there is $\mathcal{U} \in PCCO(Y)$ such that $(f(a), f(b))_{CCP} \in \mathcal{U} \subset_P \mathcal{V}$. Thus Proposition 6.13 (2), we have

$$(a,b)_{CCP} \in f^{-1}(\mathcal{U}) \subset_P f^{-1}(\mathcal{V}) \text{ and } f^{-1}(\mathcal{U}) \in PCCO(X).$$

So f is CCPC at $(a,b)_{CCVP}$. Similarly, the second part is proved.

Conversely, suppose the necessary condition hold and let $\mathcal{V} \in PCCO(Y)$ such that $(f(a), f(b))_{CCP} \in \mathcal{V}$ and $(f(a), f(b))_{CCVP} \in \mathcal{V}$ for any $(a,b)_{CCP},\ (a,b)_{CCVP}$. Then by the hypotheses and Proposition 5.3, there are $\mathcal{U}_{CCP},\ \mathcal{U}_{CCVP} \in PCCO(X)$ such that $(f(a), f(b))_{CCP} \in \mathcal{U}_{CCP} \subset_P \mathcal{V}_{CCP}$, $(f(a), f(b))_{CCVP} \in \mathcal{U}_{CCVP} \subset_P \mathcal{V}_{CCVP}$ and $\mathcal{U} = \mathcal{U}_{CCP} \cup_P \mathcal{U}_{CCVP}$, $\mathcal{V} = \mathcal{V}_{CCP} \cup_P \mathcal{V}_{CCVP}$. Thus Proposition 6.13 (2), we get

$$(a,b)_{CCP} \in f^{-1}(\mathcal{U}_{CCP}) \subset_P f^{-1}(\mathcal{V}_{CCP}) \text{ and } (a,b)_{CCVP} \in f^{-1}(\mathcal{U}_{CCVP}) \subset_P f^{-1}(\mathcal{V}_{CCVP}).$$

So by Proposition 6.13 (5), we have

$$\begin{aligned} f^{-1}(\mathcal{V}) &= f^{-1}(\mathcal{V}_{CCP}) \cup_P f^{-1}(\mathcal{V}_{CCVP}) \\ &= \left(\bigcup\nolimits_{P,\ (a,b)_{CCP} \in f^{-1}(\mathcal{V}_{CCP})} f^{-1}(\mathcal{U}_{CCP})\right) \\ &\quad \cup_P \left(\bigcup\nolimits_{P,\ (a,b)_{CCVP} \in f^{-1}(\mathcal{V}_{CCVP})} f^{-1}(\mathcal{U}_{CCVP})\right). \end{aligned}$$

Hence $f^{-1}(\mathcal{V}) \in PCCO(X)$. Therefore f is cubic crisp continuous. □

There are other equivalent formulations of cubic crisp continuity that are useful at various times, and its proof is almost similar to classical case.

Theorem 6.19. *Let $(X,\tau),\ (Y,\delta)$ be PCCTSs, $f : X \to Y$ be a mapping and let $\beta,\ \sigma$ be a base and subbase for τ, respectively. Then the followings are equivalent:*

(1) *f is cubic crisp continuous,*

(2) *$f^{-1}(\mathcal{C}) \in PCCC(X)$ for each $\mathcal{C} \in PCCC(Y)$,*

(3) *$f(CCcl(\mathcal{A})) \subset_P CCcl(f(\mathcal{A}))$ for each $\mathcal{A} \in CCS(X)$,*

(4) *$CCcl(f^{-1}(\mathcal{B}) \subset_P f^{-1}(CCcl(\mathcal{B}))$ for each $\mathcal{B} \in CCS(Y)$,*

(5) *$f^{-1}(\mathcal{B}) \in \tau$ for each $\mathcal{B} \in \beta$,*

(6) *$f^{-1}(\mathcal{S}) \in \tau$ for each $\mathcal{S} \in \sigma$.*

Definition 6.20. Let X, Y be PCCTSs. Then a mapping $f : X \to Y$ is said to be cubic crisp open [resp. closed], if $f^{-1}(\mathcal{B}) \in PCCO(X)$ for each $\mathcal{B} \in PCCO(Y)$ [resp. $f^{-1}(\mathcal{C}) \in PCCC(X)$ for each $\mathcal{C} \in PCCC(Y)$].

From Proposition 6.13 (13) and Definition 6.20, we have the following.

Proposition 6.21. *Let X, Y, Z be PCCTSs and let $f : X \to Y$ and $g : Y \to Z$ be mappings. If f, g are cubic crisp open [resp. closed], then so is $g \circ f$.*

We give a necessary and sufficient condition for a mapping to be cubic crisp open.

Theorem 6.22. *Let X, Y be PCCTSs and let $f : X \to Y$. Then f is cubic crisp open if and only if $f(CCint(\mathcal{A})) \subset_P CCint(f(\mathcal{A}))$ for each $\mathcal{A} \in CCS(X)$.*

Proof. Suppose f is cubic crisp open and let $\mathcal{A} \in CCS(X)$. Since $CCint(\mathcal{A}) \in PCCO(X)$, $f(CCint(\mathcal{A})) \in PCCO(Y)$ by the hypothesis. Since $CCint(\mathcal{A}) \subset_P \mathcal{A}$, $f(CCint(\mathcal{A})) \subset_P f(\mathcal{A})$ by Proposition 6.13 (1). On the other hand, $CCint(f(\mathcal{A}))$ is the largest PCCOS in X contained in $f(\mathcal{A})$. Then we have $f(CCint(\mathcal{A})) \subset_P CCint(f(\mathcal{A}))$.

Conversely, suppose the necessary condition holds and let $\mathcal{U} \in PCCO(X)$. Then by Theorem 6.7 (2), $\mathcal{U} = CCint(\mathcal{U})$. Thus by the hypothesis, $f(\mathcal{U}) = f(CCint(\mathcal{U})) \subset CCint(f(\mathcal{U}))$. On the other hand, it is obvious that $CCint(f(\mathcal{U})) \subset_P f(\mathcal{U})$. So $f(\mathcal{U}) = CCint(f(\mathcal{U}))$. Hence $f(\mathcal{U}) \in PCCO(Y)$. Therefore f is cubic crisp open. □

Proposition 6.23. *Let X, Y be PCCTSs and let $f : X \to Y$. If f is cubic crisp continuous, then $CCint(f(\mathcal{A}))) \subset_P f(CCint(\mathcal{A}))$ for each $\mathcal{A} \in CCS(X)$.*

Proof. Suppose f is cubic crisp continuous and let $\mathcal{A} \in CCS(X)$. Since $f(CCint(\mathcal{A})) \in PCCO(Y)$, $f^{-1}(f(CCint(\mathcal{A}))) \in PCCO(X)$ by the hypothesis. Since f is injective, from Proposition 6.13 (3), we have

$$f^{-1}(f(CCint(\mathcal{A}))) \subset_P f^{-1}(f(\mathcal{A})) = \mathcal{A}.$$

On the other hand, $CCint(\mathcal{A})$ is the largest PCCOS in X contained in $\mathcal{A}$. Then $f^{-1}(f(CCint(\mathcal{A}))) \subset_P CCint(\mathcal{A})$. Thus $CCintf(\mathcal{A})) \subset_P f(CCint(\mathcal{A}))$. □

The following is the immediate result of Theorem 6.22 and Proposition 6.23.

Corollary 6.24. *Let X, Y be PCCTSs and let $f : X \to Y$. If f is cubic crisp continuous, open and injective, then $f(CCint(\mathcal{A})) = CCintf((\mathcal{A})))$ for each $\mathcal{A} \in CCS(X)$.*

The following gives a necessary and sufficient condition for a mapping to be cubic crisp closed.

Theorem 6.25. *Let X, Y be PCCTSs and let $f : X \to Y$. Then f is cubic crisp closed if and only if $CCcl(f(\mathcal{A})) \subset_P f(CCcl(\mathcal{A}))$ for each $\mathcal{A} \in CCS(X)$.*

Proof. Suppose f is cubic crisp closed and let $\mathcal{A} \in CCS(X)$. Then clearly, $\mathcal{A} \subset_P CCcl(\mathcal{A})$. Since $CCcl(\mathcal{A}) \in PCCC(X)$, $f(CCcl(\mathcal{A})) \in PCCC(Y)$ by the hypothesis. Thus $CCcl(f(\mathcal{A})) \subset_P f(CCcl(\mathcal{A}))$.

Conversely, suppose the necessary condition holds and let $\mathcal{C} \in PCCC(X)$. Since $\mathcal{C} = CCcl(\mathcal{C})$, we have

$$CCcl(f(\mathcal{C})) \subset_P f(CCcl(\mathcal{C})) = f(\mathcal{C}) \subset CCcl(f(\mathcal{C})).$$

Then $f(\mathcal{C}) = CCcl(f(\mathcal{C}))$. Thus $f(\mathcal{C}) \in PCCC(Y)$. So f is cubic crisp closed. □

Theorem 6.26. *X, Y be PCCTSs and let $f : X \to Y$. Then f is cubic crisp continuous and closed if and only if $f(CCcl(\mathcal{A})) = CCcl(f(\mathcal{A}))$ for each $\mathcal{A} \in CCS(X)$.*

Proof. Let $\mathcal{A} \in CCS(X)$. Then from Theorem 6.19 (3), we have

f is cubic crisp continuous if and only $f(CCcl(\mathcal{A})) \subset_P CCcl(f(\mathcal{A}))$.

Also, by Theorem 6.25, $CCcl(f(\mathcal{A})) \subset_P f(CCcl(\mathcal{A}))$. Thus the result holds. □

Definition 6.27. Let X, Y be PCCTSs and let $f : X \to Y$. Then f is called a cubic crisp homeomorphism, if it is bijective, cubic crisp continuous and open.

We want very often to know if there is a PCCT on a set X such that a mapping or a family of mappings of X into a PCCTS Y is cubic crisp continuous. The following Propositions answer this question.

Proposition 6.28. *Let X be a set, let (Y, δ) be a PCCTS and let $f : X \to Y$. Then there is a coarsest PCCT τ on X such that f is cubic crisp continuous.*

Proof. Let $\tau = \{f^{-1}(\mathcal{V}) \in CCS(X) : \mathcal{V} \in \delta\}$. Then we can easily check that τ satisfies the conditions ($PCCO_1$), ($PCCO_2$) and ($PCCO_3$). Thus τ is a PCCT on X. By the definition of τ, it is clear that $f : (X, \tau) \to (Y, \delta)$ is cubic crisp continuous. It is easy to prove that τ is the coarsest PCCT on X such that $f : (X, \tau) \to (Y, \delta)$ is cubic crisp continuous. □

Proposition 6.29. *Let X be a set, let (Y, δ) be a PCCTS and let $(f_j : X \to Y)_{j \in J}$ be be a family of mappings, where J is an index set. Then there is a coarsest PCCT τ on X such that f_j is cubic crisp continuous for each $j \in J$.*

Proof. Let $\sigma = \{f_j^{-1}(\mathcal{V}) \in CCS(X) : \mathcal{V} \in \delta,\ j \in J\}$. Then we can easily check that τ is the PCCT on X having σ as its CCPSB. Thus τ is the coarsest PCCT on X such that $f_j : (X, \tau) \to (Y, \delta)$ is cubic crisp continuous for each $j \in J$. □

Now let us think the dual of Proposition 6.28.

Proposition 6.30. *Let (X, τ) be a PCCTS, let Y be a set and let $f : X \to Y$ be be a mapping. Then there is a finest PCCT δ on Y such that f is cubic crisp continuous.*

Proof. Let $\delta = \{\mathcal{V} \in CCS(X) : f^{-1}(\mathcal{V}) \in \tau\}$. Then we can easily check that δ is the is the finest PCCT on Y such that $f : (X, \tau) \to (Y, \delta)$ is cubic crisp continuous. □

Definition 6.31. Let (X, τ) be a PCCTS, let Y be a set and let $f : X \to Y$ be be a sujective mapping. Then $\delta = \{\mathcal{V} \in CCS(X) : f^{-1}(\mathcal{V}) \in \tau\}$ is called the P-cubic crisp quotient topology (briefly, PCCQT) on Y induced by f. The pair (Y, δ) is called a P-cubic crisp quotient space (briefly, PCCQS) and f is called a P-cubic crisp quotient mapping (briefly, PCCQM).

From Proposition 6.30, it is obvious that $\delta \in PCCT(Y)$. Moreover, it is easy to see that if (Y, δ) is a PCCQS of (X, τ) with PCCQM f, then for a CCS $\mathcal{C}$ in Y $\mathcal{C} \in PCCC(Y)$ if and only if $f^{-1}(\mathcal{C}) \in PCCC(X)$.

Let (X, τ), (Y, η) be PCCTSs and let $f : X \to Y$ be be a sujective mapping. Then the following gives conditions on f such that $\eta = \delta$, where δ is the PCCQT on Y induced by f.

Proposition 6.32. *Let (X, τ), (Y, η) be PCCTSs, let $f : (X, \tau)X \to (Y, \eta)$ be be a cubic crisp continuous surjective mapping and let δ is the PCCQT on Y induced by f. If f is cubic crisp open or closed, then $\eta = \delta$.*

Proof. Suppose f is cubic crisp open and let let δ is the PCCQT on Y induced by f. Then clearly by Proposition 6.30, δ is the finest PCCT on Y for which f is cubic crisp continuous. Thus $\eta \subset \delta$. Let $\mathcal{U} \in \delta$. Then clearly $f^{-1}(\mathcal{U}) \in \delta$ by the definition of δ. Since f is cubic crisp open and surjective, $\mathcal{U} = f(f^{-1}(\mathcal{U})) \in \eta$. Thus $\delta \subset \eta$. So $\eta = \delta$.

The proof that if f is cubic crisp closed, then $\eta = \delta$ is similar. □

Proposition 6.33. *The composition of two PCCQMs is a PCCQM.*

Proof. Let $f : (X, \tau) \to (Y, \delta)$ and $g : (Y, \delta) \to (Z, \gamma)$ be two PCCQMs. Let η be the PCCQM on Z induced by $g \circ f$. We prove that $\eta = \gamma$. Let $\mathcal{V} \in \gamma$. Since $g : (Y, \delta) \to (Z, \gamma)$ is a PCCQM, $g^{-1}(\mathcal{V}) \in \delta$. Since $f : (X, \tau) \to (Y, \delta)$ is a PCCQM, $(g \circ f)^{-1}(\mathcal{V}) = f^{-1}(g^{-1}(\mathcal{V})) \in \tau$. Then $\mathcal{V} \in \eta$. Thus $\gamma \subset \eta$. Moreover, we can easily show that $\eta \subset \gamma$. Thus $\eta = \gamma$. So $g \circ f$ is a PCCQM. □

The following is a basic result about PCCQS.

Theorem 6.34. *Let (X, τ), (Z, η) be two PCCTSs, let Y be a set, let $f : X \to Y$ be a surjective mapping and let δ be the PCCQT on Y induced by f. Then $g : (X, \tau) \to (Z, \eta)$ is cubic crisp continuous if and only if $g \circ f : (X, \tau) \to (Z, \eta)$ is cubic crisp continuous*

Proof. Suppose g is cubic crisp continuous. Since $f : (X, \tau) \to (Y, \delta)$ is cubic crisp continuous, $g \circ f : (X, \tau) \to (Z, \eta)$ is cubic crisp continuous by Proposition 6.15 (2).

Suppose $g \circ f$ is cubic crisp continuous and let $\mathcal{V} \in \eta$. Then clearly, $(g \circ f)^{-1}(\mathcal{V}) \in \tau$ and $(g \circ f)^{-1}(\mathcal{V}) = f^{-1}(g^{-1}(\mathcal{V}))$. Thus by the definition of δ, $g^{-1}(\mathcal{V}) \in \delta$. So g is cubic crisp continuous. □

7. Cubic crisp subspaces

In this section, we introduce the notion of cubic crisp subspaces and some of its properties.

Proposition 7.1. *Let (X, τ) be a PCCTS and let $\mathcal{H} \in CCS(X)$. Then the family of CCSs in X, denoted by $\tau_{_\mathcal{H}}$, given by:*

$$\tau_{_\mathcal{H}} = \{\mathcal{U} \cap_P \mathcal{H} : \mathcal{U} \in \tau\}$$

is a PCCT on $\mathcal{H}$.

In this case, $\tau_{_\mathcal{H}}$ is called the P-cubic crisp relative topology on $\mathcal{H}$ determined by τ. The pair $(\mathcal{H}, \tau_{_\mathcal{H}})$ is called a P-cubic crisp subspace of (X, τ). The members of $\tau_{_\mathcal{H}}$ are called cubic crisp open sets in $\mathcal{H}$.

Proof. The proof is easy. □

For convenience, we sometimes omit mention of the PCCT τ and $\tau_{_\mathcal{H}}$ refer to $\mathcal{H}$ as a P-cubic crisp subspace of X. throughout the remainder of this section, unless otherwise stated, we assume that if (X, τ) is a PCCTS and $\mathcal{H} \in CCS(X)$, then $\mathcal{H}$ has the P-cubic crisp relative topology.

Example 7.2. (1) Let τ be the P-cubic crisp discrete topology on a set X and let $\mathcal{H} \in CCS(X)$. Then $\tau_{\mathcal{H}}$ is the P-cubic crisp discrete topology on $\mathcal{H}$.

(2) Let τ be the P-cubic crisp indiscrete topology on a set X and let $\mathcal{H} \in CCS(X)$. Then $\tau_{\mathcal{H}}$ is the P-cubic crisp indiscrete topology on $\mathcal{H}$.

Proposition 7.3. *Let (X, τ) be a PCCTS and let $\mathcal{A}$, $\mathcal{B} \in CCS(X)$ such that $\mathcal{A} \subset_P \mathcal{B}$. Then $\tau_{\mathcal{A}} = \tau_{\tau_{\mathcal{B}}}$.*

Proof. The proof is easy. □

Proposition 7.4. *Let (X, τ) be a PCCTS, let $\mathcal{A} \in CCS(X)$ and let β be a base for τ. Then $\beta_{\mathcal{A}} = \{\mathcal{B} \cap_P \mathcal{A} : \mathcal{B} \in \beta\}$ is a base for $\tau_{\mathcal{A}}$.*

Proof. It is obvious that $\mathcal{A} = \bigcup_P \beta_{\mathcal{A}}$. Let $\mathcal{U} \in \tau$ and suppose $(a, b)_{CCP} \in \mathcal{U} \cap_P \mathcal{A}$ and $(a, b)_{CCVP} \in \mathcal{U} \cap_P \mathcal{A}$. Then by Definition 5.6 and Theorem 5.4, there are $\mathcal{B}_1$, $\mathcal{B}_2 \in \beta$ such that $(a, b)_{CCP} \in \mathcal{B}_1 \subset_P \mathcal{U}$ and $(a, b)_{CCVP} \in \mathcal{B}_2 \subset_P \mathcal{U}$. Thus we have

$$(a, b)_{CCP} \in \mathcal{B}_1 \cap_P \mathcal{A} \subset_P \mathcal{U} \cap_P \mathcal{A} \text{ and } (a, b)_{CCVP} \in \mathcal{B}_2 \cap_P \mathcal{A} \subset_P \mathcal{U} \cap_P \mathcal{A}.$$

So by Theorem 5.9, $\beta_{\mathcal{A}}$ is a base for $\tau_{\mathcal{A}}$. □

We give a special situation in which every member of the P-cubic crisp relative topology is also a member of the PCCT on X.

Proposition 7.5. *Let (X, τ) be a PCCTS and let $\mathcal{A} \in \tau$. If $\mathcal{U} \in \tau_{\mathcal{A}}$, then $\mathcal{U} \in \tau$.*

Proof. Suppose $\mathcal{U} \in \tau_{\mathcal{A}}$. Then by Proposition 7.1, there is $\mathcal{V} \in \tau$ such that $\mathcal{U} = \mathcal{A} \cap_P \mathcal{V}$. Since $\mathcal{A}$, $\mathcal{V} \in \tau$, $\mathcal{A} \cap_P \mathcal{V} \in \tau$. Thus $\mathcal{U} \in \tau$. □

Definition 7.6. Let X be a set and let $\mathcal{A}$, $\mathcal{B} \in CCS(X)$ such that $\mathcal{B} \subset_P \mathcal{A}$. Then the difference of $\mathcal{B}$ and $\mathcal{A}$, denoted by $\mathcal{A} - \mathcal{B}$, is a CCS in $\mathcal{A}$ defined as follows:

$$\mathcal{A} - \mathcal{B} = \mathcal{A} \cap_P \mathcal{B}^c.$$

From Proposition 3.11 (8_d), we can see that $\mathcal{C} \cap_P \mathcal{C}^c \neq \hat{\varnothing}$ in general. Then we define the new set of all CCSs in X, denoted by $CCS^*(X)$:

$$CCS^*(X) = \{\mathcal{C} \in CCS(X) : \mathcal{C} \cap_P \mathcal{C}^c = \hat{\varnothing}\}.$$

Let $\mathcal{A}$, $\mathcal{B} \in CCS^*(X)$ such that $\mathcal{B} \subset_P \mathcal{A}$. Then we can easily see that the following holds:

$$\mathcal{B} = \mathcal{A} - (\mathcal{A} - \mathcal{B}). \tag{7.1}$$

Theorem 7.7. *Let (X, τ) be a PCCTS such that $\tau \subset CCS^*(X)$, let $(\mathcal{A}, \tau_{\mathcal{A}})$ is called a P-cubic crisp subspace of (X, τ) and let $\mathcal{C} \in CCS^*(X)$ such that $\mathcal{C} \subset \mathcal{A}$. Then $\mathcal{C}$ is cubic crisp closed in $(\mathcal{A}, \tau_{\mathcal{A}})$ if and only if there is a $\mathcal{D} \in PCCC(X)$ such that $\mathcal{C} = \mathcal{A} \cap_P \mathcal{D}$.*

Proof. Suppose $\mathcal{C}$ is cubic crisp closed in $(\mathcal{A}, \tau_{\mathcal{A}})$. Then clearly, $\mathcal{A} - \mathcal{C} \in \tau_{\mathcal{A}}$. Thus there is a $\mathcal{U} \in \tau$ such that $\mathcal{A} - \mathcal{C} = \mathcal{A} \cap_P \mathcal{U}$. So by (7.1), Definition 7.6 and Proposition P3.11 (6), $\mathcal{C} = \mathcal{A} - (\mathcal{A} - \mathcal{C}) = \mathcal{A} - (\mathcal{A} \cap_P \mathcal{U}) = \mathcal{A} \cap_P \mathcal{U}^c$. Since $\mathcal{U}^c \in PCCC(X)$, the proof is complete.

Let $\mathcal{C} \in CCS^*(X)$ such that $\mathcal{C} \subset_P \mathcal{A}$ and suppose there is a $\mathcal{D} \in PCCC(X)$ such that $\mathcal{C} = \mathcal{A} \cap_P \mathcal{D}$. Then clearly, $\mathcal{D}^c \in \tau$. Moreover, we have

$$\mathcal{A} - \mathcal{C} = \mathcal{A} - (\mathcal{A} \cap_P \mathcal{D}) = \mathcal{A} \cap_P \mathcal{D}^c.$$

Since $\mathcal{D}^c \in \tau$, $\mathcal{A} - \mathcal{C} \in \tau_{\mathcal{A}}$. Thus $\mathcal{C}$ is cubic crisp closed in $(\mathcal{A}, \tau_{\mathcal{A}})$. □

There is a criterion for a PCCCS in a P-cubic crisp subspace to be PCCC in the PCCTS.

Corollary 7.8. *Let (X, τ) be a PCCTS such that $\tau \subset CCS^*(X)$ and let $\mathcal{A} \in PCCC(X)$. If $\mathcal{C}$ is PCCC in $(\mathcal{A}, \tau_{\mathcal{A}})$, then $\mathcal{C} \in PCCC(X)$.*

Proof. The proof is easy from Theorem 7.7. □

When we deal with P-cubic crisp subspaces of a PCCTS, we need to exercise care in taking cubic crisp closures of a CCS because the cubic crisp closure in the P-cubic crisp subspace may be wuite different from the cubic crisp closure in the PCCTS. The following establishes a criterion for dealing with this situation.

Proposition 7.9. *Let (X, τ) be a PCCTS such that $\tau \subset CCS^*(X)$ and let $\mathcal{A}$, $\mathcal{B} \in CCS^*(X)$ such that $\mathcal{B} \subset \mathcal{A}$. Then $CCcl_{\tau_{\mathcal{A}}}(\mathcal{B}) = \mathcal{A} \cap_P CCcl_{\tau}(\mathcal{B})$.*

Proof. It is obvious that $CCcl_{\tau}(\mathcal{B}) \in PCCC(X)$. Then by Theorem 7.7, $\mathcal{A} \cap_P CCcl_{\tau}(\mathcal{B})$ is PCCC in $(\mathcal{A}, \tau_{\mathcal{A}})$. By Definition 6.1, we have

$$CCcl_{\tau_{\mathcal{A}}}(\mathcal{B}) = \bigcap_P \{\mathcal{K} : \mathcal{K}^c \in \tau_{\mathcal{A}} \text{ and } \mathcal{B} \subset_P \mathcal{K}\}.$$

Moreover, $\mathcal{B} \subset_P \mathcal{A} \cap_P CCcl_{\tau}(\mathcal{B})$. Thus $CCcl_{\tau_{\mathcal{A}}}(\mathcal{B}) \subset_P \mathcal{A} \cap_P CCcl_{\tau}(\mathcal{B})$.

Since $CCcl_{\tau_{\mathcal{A}}}(\mathcal{B}) \in PCCC(\mathcal{A})$, by Theorem 7.7, there is a $\mathcal{C} \in PCCC(X)$ such that $CCcl_{\tau_{\mathcal{A}}}(\mathcal{B}) = \mathcal{A} \cap_P \mathcal{C}$. So $\mathcal{C} \in PCCC(X)$ such that $\mathcal{B} \subset_P \mathcal{C}$. From the definition of $CCcl_{\tau}(\mathcal{B})$, it is clear that $CCcl_{\tau}(\mathcal{B}) \subset_P \mathcal{C}$. Hence we have

$$\mathcal{A} \cap_P CCcl_{\tau}(\mathcal{B}) \subset_P \mathcal{A} \cap_P \mathcal{C} = CCcl_{\tau_{\mathcal{A}}}(\mathcal{B}).$$

Therefore $CCcl_{\tau_{\mathcal{A}}}(\mathcal{B}) = \mathcal{A} \cap_P CCcl_{\tau}(\mathcal{B})$. □

8. Summary and concluding remarks

We introduced the new concept of cubic crisp sets which are the generalization of classical sets and the special case of cubic sets, and obtained its various properties. Also, we defined an internal (external) cubic crisp sets, P-(R-) orders, P-(R-) intersections and found some properties of each concept. Next, we defined a cubic crisp topology and obtained its various properties, and gave some examples. Moreover, we defined cubic crisp points of two types. By using them, we proposed cubic crisp neighborhoods of two types and investigated some of their properties. Also, we defined cubic crisp P-base and P-subbase, and obtained some of properties. Finally, we discussed with cubic crisp closures (interiors), cubic crisp continuities and cubic crisp subspaces. In the future, we expect that one can apply the concept of cubic crisp sets to group and ring theory, BCK-algebra and category theory, etc. Furthermore, we expect that one can propose the new notions of cubic crisp soft sets, octahedron crisp sets and octahedron crisp soft sets, etc.

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