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On fuzzy sets having the fuzzy Baire property in fuzzy topological spaces

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Abstract. In this paper, fuzzy sets having the property of fuzzy Baire in the fuzzy topological spaces are introduced by means of fuzzy first category sets. The conditions for the existence of fuzzy Baireness and fuzzy resolvability of fuzzy topological spaces are established by means of fuzzy sets having the property of fuzzy Baire. It is established that there are no fuzzy sets having the property of fuzzy Baire in fuzzy hyperconnected spaces and fuzzy sets having the property of fuzzy Baire in fuzzy Baire spaces are fuzzy second category sets.

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Keywords: Fuzzy second category set, Fuzzy residual set, Fuzzy simply open set, Fuzzy Baire space, Fuzzy hyperconnected space, Fuzzy P-space.

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1. Introduction

In 1965, the concept of fuzzy sets as an new approach for modeling uncertainities was introduced by Zadeh [1]. The concept of fuzzy topological spaces was introduced by Chang [2] in 1968 and his works paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. The concepts of fuzzy first category and fuzzy residual sets were introduced in [3, 4]. The concept of Baire spaces in fuzzy setting was introduced and studied by Thangaraj and Anjalmose in [4].

The purpose of this paper is to introduce the notion of fuzzy sets having the fuzzy Baire property in fuzzy topological spaces in terms of fuzzy first category sets. Several characterizations of fuzzy sets having the property of fuzzy Baire, are established in this paper. The conditions for the existence of fuzzy Baire and fuzzy resolvable spaces are established by means of fuzzy sets having the property of fuzzy Baire.

2. Preliminaries

Some basic notions and results used in the sequel, are given in order to make the exposition self-contained. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang. Let X be a non-empty set and I be the unit interval $[0, 1]$. The fuzzy set λ in X is a mapping from X into I. The fuzzy set 0_X is defined as $0_X(x) = 0$, for all $x \in X$ and the fuzzy set 1_X is defined as $1_X(x) = 1$, for all $x \in X$.

Definition 2.1 ([2]). Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). The interior, the closure and the complement of λ are defined respectively as follows:

(i) $int(\lambda) = \vee\{\mu \mid \mu \leq \lambda \text{ and } \mu \in T\}$,
(ii) $cl(\lambda) = \wedge\{\mu \mid \lambda \leq \mu \text{ and } 1 - \mu \in T\}$,
(iii) $\lambda'(x) = 1 - \lambda(x),\ \forall\, x \in X$.

Lemma 2.2 ([5]). *For a fuzzy set λ of an fuzzy topological space X,*

(1) $1 - int(\lambda) = cl(1 - \lambda)$,
(2) $1 - cl(\lambda) = int(1 - \lambda)$.

Definition 2.3 ([6]). Let λ be a fuzzy set in an fuzzy topological space X. Then the fuzzy boundary of λ is defined as $Bd(\lambda) = cl(\lambda) \wedge cl(\lambda^c)$.

Definition 2.4. The fuzzy set λ in the fuzzy topological space (X, T) is called a:

(i) fuzzy regular-open set in (X, T), if $\lambda = intcl(\lambda)$, and fuzzy regular-closed set in (X, T), if $\lambda = clint(\lambda)$ [5].
(ii) fuzzy G_δ-set in (X, T), if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$ for $i \in I$, and fuzzy F_σ-set in (X, T), if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $1 - \lambda_i \in T$ for $i \in I$ [7].
(iii) fuzzy dense set in (X, T), if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$, i.e., $cl(\lambda) = 1$, in (X, T) [3],
(iv) fuzzy nowhere dense set, if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$, i.e., $intcl(\lambda) = 0$, in (X, T) [3].
(v) fuzzy first category set in (X, T), if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy second category [3].
(vi) fuzzy residual set in (X, T), if $1 - \lambda$ is an fuzzy first category set in (X, T) [4].
(vii) fuzzy Baire set in (X, T), if $\lambda = \mu \wedge \delta$, where μ is an fuzzy open set and δ is an fuzzy residual set in (X,T) [8].
(viii) fuzzy simply open set in (X, T), if $bd(\lambda)$ is an fuzzy nowhere dense set in (X, T) [9].

Definition 2.5. The fuzzy topological space (X, T) is called a:

(i) fuzzy resolvable space, if there exists an fuzzy dense set λ in (X, T) such that $cl(1 - \lambda) = 1$. Otherwise (X, T) is called an fuzzy irresolvable space [10].
(ii) fuzzy Baire space, if $int\left(\vee_{i=1}^{\infty}(\lambda_i)\right) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T) [4].

(iii) fuzzy hyperconnected space, if every non-null fuzzy open subset of (X, T) is a fuzzy dense set in (X, T) [11].
(iv) fuzzy P-space, if each fuzzy G_δ-set in (X, T) is an fuzzy open set in (X, T) [12].
(v) fuzzy submaximal space, if for each fuzzy set λ in (X, T) such that $cl(\lambda) = 1$, $\lambda \in T$ [7].
(vi. fuzzy ∂-space, if each fuzzy G_δ-set in (X, T) is an fuzzy simply open set in (X, T) [13].
(vii) fuzzy first category space, if $1_X = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). A fuzzy topological space which is not of fuzzy first category is said to be of fuzzy second category [3].

Theorem 2.6 ([5]). *In the fuzzy topological space,*
(1) *the closure of a fuzzy open set is a fuzzy regular closed set,*
(2) *the interior of a fuzzy closed set is a fuzzy regular open set.*

Theorem 2.7 ([8]). *If $\lambda \leq \mu$ and μ is a fuzzy first category set in an fuzzy topological space (X, T), then λ is also a fuzzy first category set in (X, T).*

Theorem 2.8 ([4]). *Let (X, T) be a fuzzy topological space. Then the following are equivalent:*
(1) *(X, T) is a fuzzy Baire space,*
(2) *$int(\lambda) = 0$, for every fuzzy first category set λ in (X, T),*
(3) *$cl(\mu) = 1$, for every fuzzy residual set μ in (X, T).*

Theorem 2.9 ([8]). *If λ is a fuzzy residual set in an fuzzy topological space (X, T), then there exists a fuzzy G_δ-set μ in (X, T) such that $\mu \leq \lambda$.*

Theorem 2.10 ([12]). *If λ is a fuzzy residual set in an fuzzy submaximal and fuzzy P-space (X, T), then λ is a fuzzy open set in (X, T).*

Theorem 2.11 ([8]). *Let (X, T) be a fuzzy topological space. Then the following are equivalent:*
(1) *(X, T) is a fuzzy Baire space,*
(2) *each non-zero fuzzy open set is a fuzzy second category set in (X, T).*

Theorem 2.12 ([12]). *If λ is a fuzzy G_δ-set in a fuzzy Baire and fuzzy P-space (X, T), then λ is a fuzzy second category set in (X, T).*

Theorem 2.13 ([8]). *If λ is a fuzzy Baire set in a fuzzy submaximal and fuzzy P-space (X, T), then λ is a fuzzy open set in (X, T).*

Theorem 2.14 ([13]). *If λ is a fuzzy residual set in a fuzzy submaximal and fuzzy ∂-space (X, T), then λ is a fuzzy simply open set in (X, T).*

Theorem 2.15 ([12]). *If λ is a fuzzy residual set in a fuzzy P-space (X, T), then $int(\lambda) \neq 0$.*

Theorem 2.16 ([8]). *If λ is a fuzzy residual set in a fuzzy submaximal space (X, T), then λ is a fuzzy G_δ-set in (X, T).*

Theorem 2.17 ([4]). *If the fuzzy topological space (X, T) is a fuzzy Baire space, then (X, T) is a fuzzy second category space.*

3. Fuzzy sets having the fuzzy Baire property

Definition 3.1. Let (X, T) be a fuzzy topological space. The fuzzy set λ defined on X is said to have the property of fuzzy Baire , if there exists a fuzzy open set μ in (X, T) such that $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets in (X, T).

Example 3.2. Let $X = \{a, b, c\}$. Consider the fuzzy sets α, β, γ, λ, δ and θ defined on X as follows:

$\alpha : X \to [0, 1]$ is defined as $\alpha(a) = 0.8$; $\alpha(b) = 0.6$; $\alpha(c) = 0.7$,
$\beta : X \to [0, 1]$ is defined as $\beta(a) = 0.6$; $\beta(b) = 0.9$; $\beta(c) = 0.8$,
$\gamma : X \to [0, 1]$ is defined as $\gamma(a) = 0.7$; $\gamma(b) = 0.5$; $\gamma(c) = 0.9$,
$\lambda : X \to [0, 1]$ is defined as $\lambda(a) = 0.7$; $\lambda(b) = 0.9$; $\lambda(c) = 0.9$,
$\delta : X \to [0, 1]$ is defined as $\delta(a) = 0.4$; $\delta(b) = 0.1$; $\delta(c) = 0.2$,
$\theta : X \to [0, 1]$ is defined as $\theta(a) = 0.3$; $\theta(b) = 0.1$; $\theta(c) = 0.1$.

Then $T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, \alpha \vee (\beta \wedge \gamma), \beta \vee (\alpha \wedge \gamma), \gamma \vee (\alpha \wedge \beta), \alpha \wedge (\beta \vee \gamma), \beta \wedge (\alpha \vee \gamma), \gamma \wedge (\alpha \vee \beta), \alpha \vee \beta \vee \gamma, \alpha \wedge \beta \wedge \gamma, 1\}$ is a fuzzy topology on X. On computation, one can find that $1 - \alpha$, $1 - \beta$, $1 - \gamma$, $1 - (\alpha \vee \beta)$, $1 - (\alpha \vee \gamma)$, $1 - (\beta \vee \gamma)$, $1 - (\alpha \wedge \beta)$, $1 - (\alpha \wedge \gamma)$, $1 - (\beta \wedge \gamma)$, $1 - [\alpha \vee (\beta \wedge \gamma)]$, $1 - [\beta \vee (\alpha \wedge \gamma)]$, $1 - [\gamma \vee (\alpha \wedge \beta)]$, $1 - [\alpha \wedge (\beta \vee \gamma)]$, $1 - [\beta \wedge (\alpha \vee \gamma)]$, $1 - [\gamma \wedge (\alpha \vee \beta)]$, $1 - (\alpha \vee \beta \vee \gamma)$ and $1 - (\alpha \wedge \beta \wedge \gamma)$ are fuzzy nowhere dense sets in (X, T). Also on computation, $\delta = (1 - \beta) \vee [1 - (\alpha \wedge \beta)] \vee [1 - (\beta \wedge \gamma)] \vee \{1 - [\beta \wedge (\alpha \vee \gamma)]\} \vee [1 - (\alpha \wedge \beta \wedge \gamma)]$ and $\theta = (1 - \gamma) \vee [1 - (\alpha \wedge \gamma)] \vee [1 - (\beta \vee \gamma)] \vee \{1 - [\beta \vee (\alpha \wedge \gamma)]\} \vee \{1 - [\gamma \vee (\alpha \wedge \beta)]\}$ are fuzzy first category sets in (X, T) and for the fuzzy set λ, there exists an fuzzy open set β in (X, T) such that $\delta = \lambda \wedge (1 - \beta)$ and $\theta = \beta \wedge (1 - \lambda)$. Thus the fuzzy set λ is having the property of fuzzy Baire in (X, T).

Proposition 3.3. *If λ is a fuzzy set having the property of fuzzy Baire in a fuzzy topological space (X, T), then*

(1) *there exists a fuzzy open set μ in (X, T) such that $\mu \leq \wedge_{i=1}^{\infty}(\delta_i)$, where (δ_i)'s are fuzzy dense sets in (X, T),*
(2) *$\lambda \leq \wedge_{i=1}^{\infty}(\gamma_i)$, where (γ_i)'s are fuzzy dense sets in (X, T).*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then there exists a fuzzy open set μ in (X, T) such that $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets in (X, T). This implies that $\lambda \wedge (1 - \mu) = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T) and $\mu \wedge (1 - \lambda) = \vee_{i=1}^{\infty}(\mu_i)$, where (μ_i)'s are fuzzy nowhere dense sets in (X, T).

(1) Since $\lambda \wedge (1 - \mu) \leq (1 - \mu)$, $\mu \leq \{1 - [\lambda \wedge (1 - \mu)]\} = 1 - \vee_{i=1}^{\infty}(\lambda_i) = \wedge_{i=1}^{\infty}(1 - \lambda_i)$. Since (λ_i)'s are fuzzy nowhere dense sets in (X, T), $intcl(\lambda_i) = 0$ and $int(\lambda_i) \leq intcl(\lambda_i)$. Then $int(\lambda_i) = 0$. Thus $cl(1 - \lambda_i) = 1 - int(\lambda_i) = 1 - 0 = 1$. So $(1 - \lambda_i)$'s are fuzzy dense sets in (X, T). Let $\delta_i = 1 - \lambda_i$. Then $\mu \leq \wedge_{i=1}^{\infty}(\delta_i)$, where (δ_i)'s are fuzzy dense sets in (X, T).

(2) Since $\mu \wedge (1 - \lambda) \leq (1 - \lambda)$, $\lambda \leq \{1 - [\mu \wedge (1 - \lambda)]\} = 1 - \vee_{i=1}^{\infty}(\mu_i) = \wedge_{i=1}^{\infty}(1 - \mu_i)$. Since (μ_i)'s are fuzzy nowhere dense sets, $intcl(\mu_i) = 0$, in (X, T) and $int(\mu_i) \leq intcl(\mu_i)$. Then $int(\mu_i) = 0$. Thus $cl(1 - \mu_i) = 1 - int(\mu_i) = 1 - 0 = 1$. So $(1 - \mu_i)$'s are fuzzy dense sets in (X, T). Let $\gamma_i = 1 - \mu_i$. Then $\lambda \leq \wedge_{i=1}^{\infty}(\gamma_i)$, where (γ_i)'s are fuzzy dense sets in (X, T). □

Proposition 3.4. *If λ is a fuzzy set having the property of fuzzy Baire in a fuzzy topological space (X, T), then*

(1) *there exists a fuzzy first category set δ in (X, T) such that $\delta \leq \lambda$,*
(2) *there exists a fuzzy first category set η in (X, T) such that $\eta \leq (1 - \lambda)$.*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then there exists a fuzzy open set μ in (X, T) such that $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets in (X, T).

(1) Clearly, $\lambda \wedge (1 - \mu) \leq \lambda$ in (X, T). Let $\delta = \lambda \wedge (1 - \mu)$. Then δ is a fuzzy first category set in (X, T). Thus for the fuzzy set λ having the property of fuzzy Baire in (X, T), there exists a fuzzy first category set δ in (X, T) such that $\delta \leq \lambda$.

(2) Clearly, $\mu \wedge (1 - \lambda) \leq (1 - \lambda)$ in (X, T). Let $\eta = \mu \wedge (1 - \lambda)$. Then $\eta \leq (1 - \lambda)$. Thus there exists a fuzzy first category set η in (X, T) such that $\eta \leq (1 - \lambda)$. □

Proposition 3.5. *If the fuzzy set λ is having the property of fuzzy Baire in an fuzzy topological space (X, T), then there exists a fuzzy closed set δ in (X, T) such that $\lambda \vee \delta$ and $(1 - \lambda) \vee (1 - \delta)$ are fuzzy residual sets in (X, T).*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then there exists a fuzzy open set μ in (X, T) such that $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets in (X, T). Thus $1 - [\lambda \wedge (1 - \mu)]$ and $1 - [\mu \wedge (1 - \lambda)]$ are fuzzy residual sets in (X, T). This implies that $(1 - \lambda) \vee (1 - (1 - \mu))$ and $(1 - \mu) \vee [1 - (1 - \lambda)] = (1 - \mu) \vee \lambda$ are fuzzy residual sets in (X, T). Let $\delta = 1 - \mu$. Then δ is a fuzzy closed set in (X, T). Thus for the fuzzy set λ having the property of fuzzy Baire in (X, T), there exists a fuzzy closed set δ in (X, T) such that $\lambda \vee \delta$ and $(1 - \lambda) \vee (1 - \delta)$ are fuzzy residual sets in (X, T). □

The following propositions show that fuzzy open and fuzzy first category sets are having the property of fuzzy Baire in fuzzy topological space.

Proposition 3.6. *If λ is a fuzzy open and fuzzy first category set in a fuzzy topological space (X, T), then the fuzzy set λ is having the property of fuzzy Baire in (X, T).*

Proof. Let λ be a fuzzy open and fuzzy first category set in (X, T). Now for the fuzzy set, $\lambda \wedge (1 - \lambda) \leq \lambda$ in (X, T). Since λ is the fuzzy first category set in (X, T), by Theorem 2.7, $\lambda \wedge (1 - \lambda)$ is a fuzzy first category set in (X, T). Let $\mu = \lambda$. Then μ is a fuzzy open set in (X, T). Also $\lambda \wedge (1 - \mu) = \lambda \wedge (1 - \lambda)$ and $\mu \wedge (1 - \lambda) = \lambda \wedge (1 - \lambda)$. Thus there exists a fuzzy open set μ in (X, T) such that $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets in (X, T). So the fuzzy open and fuzzy first category set λ is having the property of fuzzy Baire in (X, T). □

Proposition 3.7. *If $\lambda \leq (1 - \mu)$, where λ is a fuzzy first category set and μ is an fuzzy open and fuzzy first category set in a fuzzy topological space (X, T), then the fuzzy set λ is having the property of fuzzy Baire in (X, T).*

Proof. Let λ be a fuzzy first category set and μ be a fuzzy open and fuzzy first category set in (X, T). By hypothesis, $\lambda \leq (1 - \mu)$ in (X, T). This implies that $\mu \leq (1 - \lambda)$ in (X, T). Then $\lambda \wedge (1 - \mu) = \lambda$ and $\mu \wedge (1 - \lambda) = \mu$. Thus there exists a fuzzy open set μ in (X, T) such that $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets in (X, T). So the fuzzy first category set λ is having the property of fuzzy Baire in (X, T). □

Proposition 3.8. *If $\lambda \leq (1 - \mu)$, where λ is a fuzzy first category set and μ is a fuzzy open and fuzzy first category set in a fuzzy topological space (X, T), then λ and μ are the fuzzy sets having the property of fuzzy Baire in (X, T).*

Proof. Suppose that $\lambda \leq (1 - \mu)$, where λ is a fuzzy first category set and μ is a fuzzy open and fuzzy first category set in (X, T). Then by Proposition 3.3, the fuzzy set μ is having the property of fuzzy Baire in (X, T). Thus by Proposition 3.4, the fuzzy first category set λ is having the property of fuzzy Baire in (X, T). So λ and μ are the fuzzy sets having the property of fuzzy Baire in (X, T). □

Proposition 3.9. *If $\lambda \leq \mu$, where λ is a fuzzy first category set and μ is a fuzzy closed and fuzzy residual set in an fuzzy topological space (X, T), then the fuzzy first category set λ is having the property of fuzzy Baire in (X, T).*

Proof. Let λ be a fuzzy first category set and μ be a fuzzy closed and fuzzy residual set in (X, T). By hypothesis, $\lambda \leq \mu$ in (X, T). This implies that $\lambda \leq (1 - (1 - \mu))$ in (X, T). Since μ is an fuzzy closed and fuzzy residual set in (X, T), $1 - \mu$ is a fuzzy open and fuzzy first category set in (X, T). Then by Proposition 3.7, the fuzzy first category set λ is having the property of fuzzy Baire in (X, T). □

Proposition 3.10. *If $\lambda \leq \mu$, where λ is a fuzzy open and fuzzy first category set and μ is an fuzzy residual set in a fuzzy topological space (X, T), then the fuzzy first category set $1 - \mu$ is having the property of fuzzy Baire in (X, T).*

Proof. Let λ be a fuzzy open and fuzzy first category set and μ be a fuzzy residual set in (X, T). By hypothesis, $\lambda \leq \mu$, in (X, T). This implies that $(1 - \mu) \leq (1 - \lambda)$, in (X, T). Since μ is an fuzzy residual set, $1 - \mu$ is a fuzzy first category set in (X, T). Since λ is an fuzzy open and fuzzy first category set, $1 - \lambda$ is a fuzzy closed and fuzzy residual set in (X, T). Then by Proposition 3.9, the fuzzy first category set $1 - \mu$ is having the property of fuzzy Baire in (X, T). □

Proposition 3.11. *If the fuzzy set λ is having the property of fuzzy Baire in an fuzzy topological space (X, T), then there exists a fuzzy open set μ in (X, T) such that $[\lambda \wedge (1 - \mu)] \vee [\mu \wedge (1 - \lambda)]$ is a fuzzy first category set in (X, T).*

Proof. Let λ be the fuzzy set defined on X having the property of fuzzy Baire in (X, T). Then there exists a fuzzy open set μ in (X, T) such that $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are both fuzzy first category sets in (X, T). This implies that $[\lambda \wedge (1 - \mu)] \vee [\mu \wedge (1 - \lambda)]$ is a fuzzy first category set in (X, T). □

Proposition 3.12. *If λ is a fuzzy open and fuzzy first category set in a fuzzy topological space (X, T), then $\lambda = (\mu \wedge \delta) \vee \eta$, where μ is a fuzzy open set, δ is a fuzzy residual set and η is a fuzzy first category set in (X, T).*

Proof. Let λ be a fuzzy open and fuzzy first category set in (X, T). Now, for the fuzzy set λ, $\lambda \wedge (1 - \lambda) \leq \lambda$ in (X, T). Also, $[\lambda \wedge (1 - \lambda)] \vee \lambda = \lambda$. Since λ is the fuzzy first category set in (X, T), $(1 - \lambda)$ is a fuzzy residual set in (X, T). Let $\delta = 1 - \lambda$ and $\mu = \lambda$. Then δ is a fuzzy residual set and μ is an fuzzy open set in (X, T). Thus

$\lambda = (\mu \wedge \delta) \vee \eta$, where μ is a fuzzy open set, δ is a fuzzy residual set and η $(= \lambda)$ is a fuzzy first category set in (X, T). □

Proposition 3.13. *If the fuzzy open set λ is having the property of fuzzy Baire in a fuzzy topological space (X, T), then $\lambda = (\alpha \wedge \delta) \vee \eta$, where α is a fuzzy open set, δ is a fuzzy residual set and η is a fuzzy first category set in (X, T).*

Proof. Let λ be a fuzzy open set having the property of fuzzy Baire in (X, T). Then there exists a fuzzy open set μ in (X, T) such that $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets in (X, T). Now, for the fuzzy set λ, $\lambda \wedge (1 - \mu) \leq \lambda$ in (X, T). Also, we have

$$[\lambda \wedge (1 - \mu)] \vee \lambda = \lambda. \tag{3.1}$$

Since $\mu \wedge (1 - \lambda) \leq (1 - \lambda)$ in (X, T), $[\mu \wedge (1 - \lambda)] \vee (1 - \lambda) = 1 - \lambda$. Thus we have

$$\lambda = 1 - \{[\mu \wedge (1 - \lambda)] \vee (1 - \lambda)\} = \{1 - [\mu \wedge (1 - \lambda)]\} \wedge [1 - (1 - \lambda)] = \{1 - [\mu \wedge (1 - \lambda)]\} \wedge \lambda. \tag{3.2}$$

Since $\mu \wedge (1 - \lambda)$ is a fuzzy first category set, $1 - [\mu \wedge (1 - \lambda)]$ is a fuzzy residual set in (X, T). Let $\delta = 1 - [\mu \wedge (1 - \lambda)]$, $\eta = \lambda \wedge (1 - \mu)$ and $\alpha = \lambda$. Then δ is a fuzzy residual set and η is the fuzzy first category set in (X, T). So from (3.1) and (3.2), $\lambda = \delta \wedge \lambda$ and $\lambda = \eta \vee \lambda$ in (X, T). This implies that $\lambda = \eta \vee (\delta \wedge \lambda)$. Let $\alpha = \lambda$. Then $\lambda = (\alpha \wedge \delta) \vee \eta$, where α is a fuzzy open set, δ is a fuzzy residual set and η is a fuzzy first category set in (X, T). □

Proposition 3.14. *If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy Baire space (X, T), then there exists a fuzzy open set μ in (X, T) such that $int(\lambda) \leq cl(\mu) \leq cl(\lambda)$ in (X, T).*

Proof. Let λ be a fuzzy set λ having the property of fuzzy Baire in (X, T). Then there exists a fuzzy open set μ in (X, T) such that $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets in (X, T). Since (X, T) is a fuzzy Baire space, by Theorem 2.8, $int[\lambda \wedge (1 - \mu)] = 0$ and $int[\mu \wedge (1 - \lambda)] = 0$ in (X, T). Thus $int(\lambda) \wedge int(1 - \mu) = 0$ and $int(\mu) \wedge int(1 - \lambda) = 0$. This implies that $int(\lambda) \leq [1 - int(1 - \mu)]$ and $int(\mu) \leq [1 - int(1 - \lambda)]$, in (X, T). So $int(\lambda) \leq \{1 - [1 - cl(\mu)]\}$ and $int(\mu) \leq \{1 - [1 - cl(\lambda)]\}$. That is., $int(\lambda) \leq cl(\mu)$ and $int(\mu) \leq cl(\lambda)$. Since μ is a fuzzy open set in (X, T), $\mu = int(\mu) \leq cl(\lambda)$. Hence $cl(\mu) \leq cl(cl(\lambda) = cl(\lambda)$. Therefore $int(\lambda) \leq cl(\mu) \leq cl(\lambda)$ in (X, T). □

Proposition 3.15. *If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy Baire space (X, T), then there exists a fuzzy regular closed set δ in (X, T) such that $int(\lambda) \leq \delta \leq cl(\lambda)$ in (X, T).*

Proof. Let λ be a fuzzy set λ having the property of fuzzy Baire in (X, T). Since (X, T) is a fuzzy Baire space, by Proposition 3.14, there exists an fuzzy open set μ in (X, T) such that $int(\lambda) \leq cl(\mu) \leq cl(\lambda)$. Let $\delta = cl(\mu)$. Since μ is a fuzzy open set in (X, T), $cl(\mu)$ is a fuzzy regular closed set δ in (X, T). Then for the fuzzy set λ having the property of fuzzy Baire in (X, T), there exists a fuzzy regular closed set δ in (X, T) such that $int(\lambda) \leq \delta \leq cl(\lambda)$, in (X, T). □

Proposition 3.16. *If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy topological space (X, T), then there exists a fuzzy open set μ in (X, T) such that $[\mu \wedge (1-\lambda)] \leq \mu \leq \{1-[\lambda \wedge (1-\mu)]\}$, where $\lambda \wedge (1-\mu)$ and $\mu \wedge (1-\lambda)$ are fuzzy first category sets in (X, T).*

Proof. Let λ be an fuzzy set having the property of fuzzy Baire in (X, T). Then there exists a fuzzy open set μ in (X, T) such that $\lambda \wedge (1-\mu)$ and $\mu \wedge (1-\lambda)$ are fuzzy first category sets in (X, T). Now $\lambda \wedge (1-\mu) \leq (1-\mu)$ and this implies that $\mu \leq 1-[\lambda \wedge (1-\mu)]$. Also $\mu \wedge (1-\lambda) \leq \mu$ in (X, T). Then $[\mu \wedge (1-\lambda)] \leq \mu \leq \{1-[\lambda \wedge (1-\mu)]\}$ in (X, T). □

Proposition 3.17. *If λ is a fuzzy set having the property of fuzzy Baire in a fuzzy topological space (X, T), then there exist fuzzy first category sets $\lambda \wedge (1-\mu)$ and $\mu \wedge (1-\lambda)$ such that $[\lambda \wedge (1-\mu)] \wedge [\mu \wedge (1-\lambda)] \leq bd(\lambda)$, where $\mu \in T$ in (X, T).*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then there exists a fuzzy open set μ in (X, T) such that $\lambda \wedge (1-\mu)$ and $\mu \wedge (1-\lambda)$ are fuzzy first category sets in (X, T). Now, $\lambda \wedge (1-\mu) \leq \lambda$ and $\mu \wedge (1-\lambda) \leq (1-\lambda)$. Thus $\lambda \wedge (1-\mu) \leq \lambda \leq cl(\lambda)$ and $\mu \wedge (1-\lambda) \leq (1-\lambda) \leq cl(1-\lambda)$. So $[\lambda \wedge (1-\mu)] \wedge [\mu \wedge (1-\lambda)] \leq cl(\lambda) \wedge cl(1-\lambda)$ in (X, T). Since $bd(\lambda) = cl(\lambda) \wedge cl(1-\lambda)$, for the fuzzy set λ having the property of fuzzy Baire in (X, T), there exist fuzzy first category sets $\lambda \wedge (1-\mu)$ and $\mu \wedge (1-\lambda)$ such that $[\lambda \wedge (1-\mu)] \wedge [\mu \wedge (1-\lambda)] \leq bd(\lambda)$ in (X, T). □

Proposition 3.18. *If the fuzzy set λ having the property of fuzzy Baire is a fuzzy simply open set in a fuzzy topological space (X, T), then $int\{[\lambda \wedge (1-\mu)] \wedge [\mu \wedge (1-\lambda)]\} = 0$, where $\mu \in T$.*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then by Proposition 3.17, there exist fuzzy first category sets $\lambda \wedge (1-\mu)$ and $\mu \wedge (1-\lambda)$ such that $[\lambda \wedge (1-\mu)] \wedge [\mu \wedge (1-\lambda)] \leq bd(\lambda)$, where $\mu \in T$. This implies that $int\{[\lambda \wedge (1-\mu)] \wedge [\mu \wedge (1-\lambda)]\} \leq int[bd(\lambda)]$. By the hypothesis, λ is a fuzzy simply open set in (X, T). Thus $bd(\lambda)$ is a fuzzy nowhere dense set in (X, T). So $intcl[bd(\lambda)] = 0$ in (X, T). Now $int[bd(\lambda)] \leq intcl[bd(\lambda)]$ implies that $int[bd(\lambda)] = 0$. Hence $int\{[\lambda \wedge (1-\mu)] \wedge [\mu \wedge (1-\lambda)]\} = 0$, where $\mu \in T$. □

Proposition 3.19. *If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy topological space (X, T), then $\lambda \wedge (1-\mu)$ is a fuzzy first category set such that $cl[\lambda \wedge (1-\mu)] \neq 1$, where $\mu \in T$, in (X, T).*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then there exists an fuzzy open set μ in (X, T) such that $\lambda \wedge (1-\mu)$ and $\mu \wedge (1-\lambda)$ are fuzzy first category sets in (X, T). By Proposition 3.16, $[\mu \wedge (1-\lambda)] \leq \mu \leq \{1-[\lambda \wedge (1-\mu)]\}$. Then $int[\mu \wedge (1-\lambda)] \leq int(\mu) \leq int\{1-[\lambda \wedge (1-\mu)]\}$ in (X, T). If $cl[\lambda \wedge (1-\mu)] = 1$, then $int\{1-[\lambda \wedge (1-\mu)]\} = 1 - cl[\lambda \wedge (1-\mu)] = 1-1 = 0$ and this implies that $int(\mu) = 0$, a contradiction since $\mu \in T$. Thus $\lambda \wedge (1-\mu)$ is a fuzzy first category set such that $cl[\lambda \wedge (1-\mu)] \neq 1$, where $\mu \in T$ in (X, T). □

Proposition 3.20. *If λ is a fuzzy set having the property of fuzzy Baire in an fuzzy topological space (X, T), then there exist fuzzy first category sets δ and η in (X, T) such that $\delta \leq \lambda \leq (1-\eta)$.*

Proof. Let λ be an fuzzy set having the property of fuzzy Baire in (X, T). Then by Proposition 3.4, there exist fuzzy first category sets δ and η in (X, T) such that $\delta \leq \lambda$ and $\eta \leq (1 - \lambda)$ in (X, T). This implies that $\delta \leq \lambda \leq (1 - \eta)$ in (X, T). □

Proposition 3.21. *If the fuzzy set λ is having the property of fuzzy Baire in an fuzzy topological space (X, T), then $\lambda \geq (\alpha \wedge \beta) \vee \gamma$, where α is a fuzzy open set, β is an fuzzy residual set and γ is a fuzzy first category set in (X, T).*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then by Proposition 3.4, there exist fuzzy first category sets δ and η in (X, T) such that $\delta \leq \lambda$ and $\eta \leq (1 - \lambda)$ in (X, T). Since δ is an fuzzy first category set in (X, T) and $int(\delta) \leq \delta$, by Theorem 2.7, $int(\delta)$ is an fuzzy first category set in (X, T). Now, $int(\delta) \leq int(\lambda) \leq \lambda$ implies that

$$int(\delta) \vee int(\lambda) = int(\lambda) \leq \lambda. \tag{3.3}$$

Now $int(\lambda) \leq \lambda \leq (1 - \eta)$ implies that

$$int(\lambda) \wedge (1 - \eta) = int(\lambda). \tag{3.4}$$

From (3.3) and (3.4), $int(\delta) \vee [int(\lambda) \wedge (1 - \eta)] = int(\lambda) \leq \lambda$. Let $\alpha = int(\lambda)$, $\beta = 1 - \eta$ and $\gamma = int(\delta)$. Then for the fuzzy set λ having the property of fuzzy Baire, $\lambda \geq (\alpha \wedge \beta) \vee \gamma$, where α is a fuzzy open set, β is an fuzzy residual set and γ is a fuzzy first category set in (X, T) □

Proposition 3.22. *If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy topological space (X, T), then $\lambda \geq (\delta \vee \gamma)$, where δ is a fuzzy Baire set and γ is a fuzzy first category set in (X, T).*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then by Proposition 3.21, $\lambda \geq (\alpha \wedge \eta) \vee \gamma$, where α is a fuzzy open set, β is an fuzzy residual set and γ is a fuzzy first category set in (X, T). Let $\delta = \alpha \wedge \beta$. Thus δ is a fuzzy Baire set in (X, T). So $\lambda \geq (\delta \vee \gamma)$, where δ is a fuzzy Baire set and γ is a fuzzy first category set in (X, T). □

Remark 3.23. From the above proposition one will have the following result: "If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy topological space (X, T), then there exists a fuzzy Baire set δ in (X, T) such that $\delta \leq \lambda$".

Proposition 3.24. *If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy topological space (X, T), then there exist fuzzy G_δ-sets β_1 and β_2 in (X, T) such that $\beta_1 \leq 1 - [\lambda \wedge (1 - \mu)]$ and $\beta_2 \leq 1 - [\mu \wedge (1 - \lambda)]$, where $\mu \in T$.*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then there exists a fuzzy open set μ in (X, T) such that $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets in (X, T). Then $1 - [\lambda \wedge (1 - \mu)]$ and $1 - [\mu \wedge (1 - \lambda)]$ are fuzzy residual sets in (X, T). Thus by Theorem 2.9, there exist fuzzy G_δ-sets β_1 and β_2 in (X, T) such that $\beta_1 \leq 1 - [\lambda \wedge (1 - \mu)]$ and $\beta_2 \leq 1 - [\mu \wedge (1 - \lambda)]$, where $\mu \in T$. □

Proposition 3.25. *If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy topological space (X, T), then there exist fuzzy F_σ-sets δ_1 and δ_2 such that $\lambda \wedge (1 - \mu) \leq \delta_1$ and $\mu \wedge (1 - \lambda) \leq \delta_2$, where $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets and $\mu \in T$.*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then there exists a fuzzy open set μ in (X, T) such that $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets in (X, T). Thus by Proposition 3.24, there exist fuzzy G_δ-sets β_1 and β_2 in (X, T) such that $\beta_1 \leq 1 - [\lambda \wedge (1 - \mu)]$ and $\beta_2 \leq 1 - [\mu \wedge (1 - \lambda)]$. So $[\lambda \wedge (1-\mu)] \leq 1 - \beta_1$ and $[\mu \wedge (1-\lambda)] \leq 1 - \beta_2$. Let $\delta_1 = 1 - \beta_1$ and $\delta_2 = 1 - \beta_2$. Then δ_1 and δ_2 are fuzzy F_σ-sets in (X, T). Thus for the fuzzy set λ having the property of fuzzy Baire in (X, T), there exist fuzzy F_σ-sets δ_1 and δ_2 such that $\lambda \wedge (1-\mu) \leq \delta_1$ and $\mu \wedge (1 - \lambda) \leq \delta_2$, where $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets and $\mu \in T$. □

Proposition 3.26. *If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy topological space (X, T), then there exists an fuzzy closed set η in (X, T) such that $(1 - \lambda) \wedge (1 - \eta)$ and $\lambda \wedge \eta$ are fuzzy first category sets in (X, T).*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then there exists an fuzzy open set μ in (X, T) such that $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets in (X, T). Now $\mu \wedge (1 - \lambda) = (1 - (1 - \mu)) \wedge (1 - \lambda)$. Let $\eta = 1 - \mu$. Then $\mu \wedge (1 - \lambda) = (1 - \eta)) \wedge (1 - \lambda)$ and $\lambda \wedge (1 - \mu) = \lambda \wedge \eta$. Since $\mu \in T$, η is a fuzzy closed set in (X, T). Thus $(1 - \lambda) \wedge (1 - \eta)$ and $\lambda \wedge \eta$ are fuzzy first category sets in (X, T). □

4. Fuzzy sets having the property of fuzzy Baire in fuzzy Baire spaces, fuzzy hyperconnected spaces and fuzzy P-spaces

The following propositions give conditions for fuzzy topological spaces to become fuzzy Baire and fuzzy resolvable spaces.

Proposition 4.1. *If there exists a fuzzy set λ having the property of fuzzy Baire in a fuzzy topological space (X, T) such that $int(\lambda) = 0$ and $cl(\lambda) = 1$, then (X, T) is a fuzzy Baire and fuzzy resolvable space.*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then by Proposition 3.4 (1), there exists a fuzzy first category set δ in (X, T) such that $\delta \leq \lambda$. If η is any fuzzy set defined on X such that $\eta \leq \delta$, then by Theorem 2.7, η is also a fuzzy first category set in (X, T). Now $\eta \leq \delta \leq \lambda$ implies that $int(\eta) \leq int(\delta) \leq int(\lambda)$ and $int(\lambda) = 0$ implies that $int(\eta) = 0$ and $int(\delta) = 0$ in (X, T). Thus for the fuzzy first category set $\eta \leq \lambda$ in (X, T), $int(\eta) = 0$.

Also, for the fuzzy set λ having the property of fuzzy Baire in (X, T), by Proposition 3.20, there exist fuzzy first category sets δ and η in (X, T) such that $\delta \leq \lambda \leq (1 - \eta)$. Now $\lambda \leq (1 - \eta)$ implies that $\eta \leq (1 - \lambda)$. Then $int(\eta) \leq int(1 - \lambda) = 1 - cl(\lambda) = 1 - 1 = 0$. Thus for the fuzzy first category set η in (X, T), $int(\eta) = 0$. By Theorem 2.8,

$$(X, T) \text{ is a fuzzy Baire space.} \tag{3.5}$$

From the hypothesis, $int(\lambda) = 0$ and thus $cl(1 - \lambda) = 1 - int(\lambda) = 1 - 0 = 1$ in (X, T). So there exists an fuzzy set λ such that $cl(\lambda) = 1$ and $cl(1 - \lambda) = 1$. Hence

$$(X, T) \text{ is a fuzzy resolvable space.} \tag{3.6}$$

Therefore from (3.5) and (3.6), (X, T) is a fuzzy Baire and fuzzy resolvable space. □

Proposition 4.2. *If there exists a fuzzy set λ having the property of fuzzy Baire in a fuzzy topological space (X, T) and if $int(\gamma) = 0$, for each fuzzy F_σ-set γ in (X, T), then (X, T) is a fuzzy Baire space.*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then by Proposition 3.25, there exist fuzzy F_σ-sets δ_1 and δ_2 such that $\lambda \wedge (1 - \mu) \le \delta_1$ and $\mu \wedge (1 - \lambda) \le \delta_2$, where $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets and $\mu \in T$ in (X, T). By hypothesis, $int(\delta_1) = 0$ and $int(\delta_2) = 0$. This implies that $int[\lambda \wedge (1 - \mu)] = 0$ and $int[\mu \wedge (1 - \lambda)] = 0$. Thus for the fuzzy first category sets $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ in (X, T), $int[\lambda \wedge (1 - \mu)] = 0$ and $int[\mu \wedge (1 - \lambda)] = 0$. So by Theorem 2.8, (X, T) is a fuzzy Baire space. □

The following proposition shows that fuzzy sets having the property of fuzzy Baire in the fuzzy Baire spaces are fuzzy second category sets.

Proposition 4.3. *If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy Baire space (X, T), then λ is a fuzzy second category set in (X, T).*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then there exists an fuzzy open set μ in (X, T) such that $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets in (X, T). Now, $\lambda \wedge (1 - \mu) \le \lambda$ and thus $int[\lambda \wedge (1 - \mu)] \le int(\lambda) \le \lambda$. So $int[\lambda \wedge (1 - \mu)] \le \lambda$. Suppose that λ is a fuzzy first category set in (X, T). Then by Theorem 2.7, $int[\lambda \wedge (1 - \mu)]$ is a fuzzy first category set in (X, T) which is a contradiction, since no fuzzy open set is an fuzzy first category set in a fuzzy Baire space. Thus λ is not a fuzzy first category set. So λ is a fuzzy second category set in (X, T). □

The following proposition establishes that there are no fuzzy sets having the property of fuzzy Baire in the fuzzy hyperconnected spaces.

Proposition 4.4. *If there exists a fuzzy set having the property of fuzzy Baire in a fuzzy topological space (X, T), then (X, T) is not a fuzzy hyperconnected space.*

Proof. Suppose that there exists a fuzzy set λ having the property of fuzzy Baire in (X, T). Then by Proposition 3.16, there exists a fuzzy open set μ in (X, T) such that $[\mu \wedge (1 - \lambda)] \le \mu \le \{1 - [\lambda \wedge (1 - \mu)]\}$, where $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets in (X, T). Then $cl(\mu) \le cl\{1 - [\lambda \wedge (1 - \mu)]\}$. Thus $cl(\mu) \le 1 - int[\lambda \wedge (1 - \mu)]$ and $1 - int[\lambda \wedge (1 - \mu)]$ is a fuzzy closed set in (X, T). So the fuzzy open set μ is not a fuzzy dense set in (X, T). Hence (X, T) is not a fuzzy hyperconnected space. □

Proposition 4.5. *If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy submaximal and fuzzy P-space (X, T), then $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category and fuzzy closed sets in (X, T), where $\mu \in T$.*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then there exists a fuzzy open set μ in (X, T) such that $\lambda \wedge (1 - \mu)$ and $\mu \wedge (1 - \lambda)$ are fuzzy first category sets in (X, T). Then $1 - [\lambda \wedge (1 - \mu)]$ and $1 - [\mu \wedge (1 - \lambda)]$ are fuzzy residual sets in (X, T). Since (X, T) is a fuzzy submaximal and fuzzy P-space, by Theorem 2.10, $1 - [\lambda \wedge (1 - \mu)]$ and $1 - [\mu \wedge (1 - \lambda)]$ are fuzzy open sets in (X, T).

Thus $\lambda \wedge (1-\mu)$ and $\mu \wedge (1-\lambda)$ are fuzzy first category and fuzzy closed sets in (X, T). □

Proposition 4.6. *If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy Baire and fuzzy P-space (X, T), then there exist fuzzy second category sets β_1 and β_2 in (X, T) such that $\beta_1 \leq \{1 - [\lambda \wedge (1-\mu)]\}$ and $\beta_2 \leq \{1 - [\mu \wedge (1-\lambda)]\}$, where $\mu \in T$.*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then by Proposition 3.24, there exist fuzzy G_δ-sets β_1 and β_2 in (X, T) such that $\beta_1 \leq \{1 - [\lambda \wedge (1-\mu)]\}$ and $\beta_2 \leq \{1 - [\mu \wedge (1-\lambda)]\}$, where $\mu \in T$. Since (X, T) is a fuzzy Baire and fuzzy P-space, by Theorem 2.12, the fuzzy G_δ-sets β_1 and β_2 are fuzzy second category sets in (X, T). Thus there exist fuzzy second category sets β_1 and β_2 in (X, T) such that $\beta_1 \leq \{1 - [\lambda \wedge (1-\mu)]\}$ and $\beta_2 \leq \{1 - [\mu \wedge (1-\lambda)]\}$, where $\mu \in T$. □

Proposition 4.7. *If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy submaximal and fuzzy P-space (X, T), then $\lambda \geq (\delta \vee \gamma)$, where δ is a fuzzy open set and γ is a fuzzy first category set in (X, T).*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then by Proposition 3.22, $\lambda \geq (\delta \vee \gamma)$, where δ is a fuzzy Baire set and γ is a fuzzy first category set in (X, T). Since (X, T) is a fuzzy submaximal and fuzzy P-space, by Theorem 2.13, the fuzzy Baire set δ is a fuzzy open set in (X, T). Thus for the fuzzy set λ having the property of fuzzy Baire in (X, T), $\lambda \geq (\delta \vee \gamma)$, where $\delta \in T$ and γ is a fuzzy first category set in (X, T). □

Proposition 4.8. *If λ is a fuzzy set having the property of fuzzy Baire in a fuzzy submaximal and fuzzy ∂-space (X, T), then there exists a fuzzy simply open set γ in (X, T) such that $\lambda \leq \gamma$.*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then by Proposition 3.20, there exists fuzzy first category sets δ and η in (X, T) such that $\delta \leq \lambda \leq (1-\eta)$. Since (X, T) is a fuzzy submaximal and fuzzy ∂-space, by Theorem 2.14, the fuzzy residual set $1-\eta$ is a fuzzy simply open set in (X, T). Let $\gamma = 1-\eta$. Then, for the fuzzy set λ having the property of fuzzy Baire in (X, T), there exists a fuzzy simply open set γ in (X, T) such that $\lambda \leq \gamma$. □

Proposition 4.9. *If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy P-space (X, T), then $\lambda \wedge (1-\mu)$ and $\mu \wedge (1-\lambda)$ are not fuzzy dense sets in (X, T).*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X, T). Then there exists a fuzzy open set μ in (X, T) such that $\lambda \wedge (1-\mu)$ and $\mu \wedge (1-\lambda)$ are fuzzy first category sets in (X, T). Now $1 - [\lambda \wedge (1-\mu)]$ and $1 - [\mu \wedge (1-\lambda)]$ are fuzzy residual sets in (X, T). Since (X, T) is a fuzzy P-space, by Theorem 2.15, $int\{1 - [\lambda \wedge (1-\mu)]\} \neq 0$ and $int\{1 - [\mu \wedge (1-\lambda)]\} \neq 0$. Then $1 - cl[\lambda \wedge (1-\mu)] \neq 0$ and $1 - cl[\mu \wedge (1-\lambda)] \neq 0$. Thus $cl[\lambda \wedge (1-\mu)] \neq 1$ and $cl[\mu \wedge (1-\lambda)] \neq 1$ in (X, T). So $\lambda \wedge (1-\mu)$ and $\mu \wedge (1-\lambda)$ are not fuzzy dense sets in (X, T). □

Proposition 4.10. *If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy P-space (X, T), then*

(1) *$1-[\lambda \wedge (1-\mu)]$ and $1-[\mu \wedge (1-\lambda)]$ are fuzzy G_δ-sets in (X,T), where $\mu \in T$,*
(2) *$\lambda \wedge (1-\mu)$ and $\mu \wedge (1-\lambda)$ are fuzzy F_σ-sets in (X,T), where $\mu \in T$.*

Proof. (1) Let λ be a fuzzy set having the property of fuzzy Baire in (X,T). Then there exists a fuzzy open set μ in (X,T) such that $\lambda \wedge (1-\mu)$ and $\mu \wedge (1-\lambda)$ are fuzzy first category sets in (X,T). Now $1-[\lambda \wedge (1-\mu)]$ and $1-[\mu \wedge (1-\lambda)]$ are fuzzy residual sets in (X,T). Since (X,T) is a fuzzy submaximal space, by Theorem 2.16, the fuzzy residual sets $1-[\lambda \wedge (1-\mu)]$ and $1-[\mu \wedge (1-\lambda)]$ are fuzzy G_δ-sets in (X,T).

(2) The proof follows from (1). □

Proposition 4.11. *If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy P-space (X,T), then $\lambda \geq \vee_{i=1}^{\infty}(\mu_i)$, where (μ_i)'s are fuzzy closed sets in (X,T).*

Proof. Let λ be a fuzzy set having the property of fuzzy Baire in (X,T). Then there exists a fuzzy open set μ in (X,T) such that $\lambda \wedge (1-\mu)$ and $\mu \wedge (1-\lambda)$ are fuzzy first category sets in (X,T). Since (X,T) is a fuzzy P-space, by Proposition 4.10, $\lambda \wedge (1-\mu)$ and $\mu \wedge (1-\lambda)$ are fuzzy F_σ-sets in (X,T). Thus $\lambda \wedge (1-\mu) = \vee_{i=1}^{\infty}(\mu_i)$, where (μ_i)'s are fuzzy closed sets in (X,T). Since $\lambda \wedge (1-\mu) \leq \lambda$ in (X,T), $\lambda \geq \vee_{i=1}^{\infty}(\mu_i)$, where (μ_i)'s are fuzzy closed sets in (X,T). □

Proposition 4.12. *If the fuzzy set λ is having the property of fuzzy Baire in a fuzzy topological space (X,T) in which fuzzy closed sets are having zero interior, then (X,T) is a fuzzy second category space.*

Proof. Let λ be an fuzzy set having the property of fuzzy Baire in (X,T). Then by Proposition 3.26, there exists a fuzzy closed set η in (X,T) such that $(1-\lambda)\wedge(1-\eta)$ and $\lambda \wedge \eta$ are fuzzy first category sets in (X,T). By the hypothesis, for the fuzzy closed set, $int(\eta) = 0$ in (X,T). Now $(\lambda \wedge \eta) \leq \eta$ implies that $int(\lambda \wedge \eta) \leq int(\eta)$. Thus $int(\lambda \wedge \eta) = 0$ in (X,T). So by Theorem 2.8, (X,T) is an fuzzy Baire space. Hence by Theorem 2.17, the fuzzy Baire space (X,T) is a fuzzy second category space. □

5. Conclusions

In this paper, fuzzy sets having the property of fuzzy Baire in the fuzzy topological spaces are introduced by means of fuzzy first category sets. The conditions for the existence of fuzzy Baireness and fuzzy resolvability of fuzzy topological spaces are established by means of fuzzy sets having the property of fuzzy Baire. It is established that there are no fuzzy sets having the property of fuzzy Baire in fuzzy hyperconnected spaces and fuzzy sets having the property of fuzzy Baire in fuzzy Baire spaces are fuzzy second category sets. The existence of denseness of an fuzzy set which is having the property of fuzzy Baire, ensures the fuzzy Baireness of fuzzy topological space. It is established that fuzzy open and fuzzy first category sets are the fuzzy sets having the property of fuzzy Baire in fuzzy topological spaces.

References

[1] L. A. Zadeh, Fuzzy sets, Inform and Control, 8 (1965) 338–353.

[2] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. 24 (1968) 182–190.
[3] G. Thangaraj and G. Balasubramanian, On somewhat fuzzy continuous functions, J. Fuzzy Math. 11 (2) (2003) 725–736.
[4] G. Thangaraj and S. Anjalmose, On fuzzy Baire spaces, J. Fuzzy Math. 21 (3) (2013) 667–676.
[5] K. K. Azad, On fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly contiunity, J. Math. Anal. Appl. 52 (1981) 14–32.
[6] M. Athar and B. Ahmad, Fuzzy boundary and fuzzy semiboundary, Adv. Fuzzy Sys. 2008 Article ID: 586893, 9 Pages. doi:10.1155/2008/586893.
[7] G. Balasubramanian, Maximal fuzzy topologies, Kybernetika 31 (5) (1995) 459–464.
[8] R. Palani, Contributions to the study on some aspects of fuzzy Baire spaces, Ph.D. Thesis, Thiruvalluvar University, Tamil Nadu, India 2017.
[9] G. Thangaraj and K. Dinakaran, On fuzzy simply continuous functions, J. Fuzzy Math. 24 (4) (2016) 99–124.
[10] G. Thangaraj and G. Balasubramanian, On fuzzy resolvable and fuzzy irresolvable spaces, Fuzzy Sets, Rough Sets and Multivalued Operations and Appl. 1 (2) (2009) 173–180.
[11] Miguel Caldas, Govindappa Navalagi and Ratnesh Saraf, On fuzzy weakly semi-open function, Proyecciones, Universidad Catolica del Norte, Antofagasta- Chile 21 (1) (2002) 51–63.
[12]] G. Thangaraj and C. Anbazhagan, Some remarks on fuzzy P-spaces, Gen. Math. Notes 26 (1) (2015) 8–16.
[13] G. Thangaraj and J. Premkumar, On fuzzy ∂-spaces, Inter. J. Appl. Engin. Research 13 (15) (2018) 12417–12421.

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