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Abstract. In this paper, we define a cubic bipolar subalgebra, BCK-ideal and Q-ideal of a Q-algebra, and obtain some of their properties and give some examples. Also we define a cubic bipolar fuzzy point, cubic bipolar fuzzy topology, cubic bipolar fuzzy base and for each concept obtained some of its properties.

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1. Introduction

Imai and Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras (See [1, 2]). It is known that the class of BCK-algebras is proper subclass of the class of BCI-algebras. In [3, 4], Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Neggers et al. [5] introduced a new notion, called Q-algebra, which is a generalization of $BCH/BCI/BCK$-algebras and generalize some theorems discussed in BCI-algebras. Mostafa et al. [6, 7, 8] discussed fuzzy, intuitionistic fuzzy and interval valued fuzzy of Q-ideals in Q-algebra. In 1965, Zadeh [9] introduced the concept of fuzzy sets as a generalization of a crisp sets. The notion of a fuzzy topology was established by Chang [10] in 1968. Lee [11] found the idea of bipolar fuzzy sets which is an extension of traditional, fuzzy sets whose membership degree range is extended from [0, 1] to [-1, 1] and also in 1994, Zhang [12] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. In [13, 14, 15, 16], the authors introduced bipolar-valued fuzzy set on different structures. Azhagappan and Kamaraj [17] investigated bipolar fuzzy topological spaces. Jun and Hur [18] generalized the fuzzy points and quasi-coincidence in fuzzy sets, introduced the notion of bipolar-valued fuzzy point and bipolar quasi-coincidence in bipolar-valued fuzzy sets. Lee et al. [19] defined concepts of a bipolar fuzzy topology, bipolar fuzzy base, sub-base, bipolar fuzzy point and find some properties of each concept. Jun et al. [20, 21, 22] introduced the concept of a cubic set and applied it to the cubic set to BCK/BCI-algebras. Mostafa et al. [23] established the notion of cubic bipolar BCC-ideal of BCC-algebras and investigate several properties. In 2017, extending the concept of a cubic set, Jun [24] established the idea of a cubic intuitionistic set.

In this paper, we present another extension of fuzzy set theory which is called cubic bipolar structures as the generalization of cubic set and bipolar set and apply it to Q-algebra. In order to apply it to Q-algebra, we deal with the followings: in Section 2, we recall definitions of BCK, BCI, BG, QS, Q-algebra; in Section 3, we define the inclusion, the equality and operations of cubic bipolar sets and the image and preimage of a cubic bipolar set under a mapping, and obtain their some properties; in Section 4, we define a cubic bipolar point and find some of its properties; in Section 5, we define a cubic bipolar subalgebra, BCK-ideal and Q-ideal of a Q-algebra, and obtain some of their properties and give some examples; in Section 6, we discuss with the image and the preimage of a Q-ideal under a mapping; in Section 7, we define a cubic bipolar fuzzy topology and a cubic bipolar fuzzy base and for each concept obtain some of its properties.

2. Preliminaries

We review some definitions and properties that will be useful in the next sections.

Definition 2.1 ([1, 2, 25])**.** Let X be a set with a binary operation "*" and a constant 0. Then $(X, *, 0)$ is called a *BCI-algebra*, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,
(BCI-1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCI-2) $(x * (x * y)) * y = 0$,
(BCI-3) $x * x = 0$,
(BCI-4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

If a BCI-algebra X satisfies the identity $0 * x = 0$ for all $x \in X$, then X is called a *BCK-algebra*. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

Definition 2.2 ([25])**.** Let $(X, *, 0)$ be a BCK-algebra, and let S be a non-empty subset of X. Then S is called a *subalgebra* of X, if $x * y \in S$ for all $x,\ y \in S$, i.e., S is closed under the binary operation $*$ of X.

Definition 2.3 ([5])**.** An algebraic system $(X, *, 0)$ of type $(2, 0)$ is called a *Q-algebra*, if it satisfying the following axioms: for all $x,\ y,\ z \in X$,
(i) $x * x = 0$, (ii) $x * 0 = x$, (iii) $(x * y) * z = (x * z) * y$.

In X, we can define a binary relation $\leq$ by

$$x \leq y \text{ if and only if } x * y = 0.$$

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Table 2.1

Example 2.4 ([5]). Let $X = \{0, 1, 2\}$ be a set with a binary operation $*$ defined by the following table:
Then $(X, *, 0)$ is a Q-algebra.

Theorem 2.5 ([5]). *Every BCK-algebra is a Q-algebra, but the converse is not true* (See Example 2.6).

Example 2.6 ([5]). Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation $*$ defined by the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Table 2.2

Then $(X, *, 0)$ is a Q-algebra but not a BCK-algebra.

Definition 2.7 (See [26]). An algebraic system $(X, *, 0)$ of type $(2, 0)$ is called a *BG-algebra*, if it satisfying the following axioms: for all $x,\ y \in X$,
(BG1) $x * x = 0$, (BG2) $x * 0 = x$, (BG3) $(x * y) * (0 * y) = x$.

We can confirm that BG-algebras and Q-algebra are different notions as the following Example.

Example 2.8 ([5]). Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation $*$ defined by the following table:

$*$	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

Table 2.3

Then $(X, *, 0)$ is a Q-algebra but not a BG-algebra, since $(2 * 3) * (0 * 3) = 0 \neq 2$.

Proposition 2.9 ([5]). *If $(X, *, 0)$ is a Q-algebra, then*

$$(x * (x * y)) * y = 0 \text{ for any } x,\ y \in X.$$

Theorem 2.10 ([5]). *Every BCH-algebra is a Q-algebra. Every Q-algebra satisfying the condition (BCI-4) is a BCH-algebra.*

Theorem 2.11 ([5]). *Every Q-algebra satisfying the condition (BCI-4) and the following axiom*

$$((x * y) * (x * z)) * (z * y) = 0 \text{ for all } x,\ y,\ z \in X. \tag{2.1}$$

is a BCI-algebra.

Theorem 2.12 ([5]). *Every Q-algebra satisfying the conditions (BCI-4), (2.1)* and the following axiom

$$(x * y) * x = 0 \text{ for any } x,\ y \in X. \tag{2.2}$$

is a BCK-algebra.

Theorem 2.13 ([5]). *Every Q-algebra X satisfying $x * (x * y) = x * y$ for any $x,\ y \in X$, is a trivial algebra.*

Definition 2.14 ([27]). A *QS-algebra* $(X, *, 0)$ is a non-empty set with a constant 0 and a binary operation $*$ satisfying the following axioms: for all $x,\ y,\ z \in X$,
(QS-1) $(x * y) * z = (x * z) * y$,
(QS-2) $x * 0 = x$,
(QS-3) $x * x = 0$,
(QS-4) $(x * y) * (x * z) = z * y$.

Remark 2.15. Every QS-algebra is a Q-algebra. Every Q-algebra satisfying the condition

$$(x * y) * (x * z) = z * y \text{ for any } x,\ y,\ z \in X. \tag{2.3}$$

is a QS-algebra.

Definition 2.16 ([25]). A non-empty subset I of a BCK-algebra X is called an *ideal* of X, if it satisfies the following conditions:
(I_1) $0 \in I$,
(I_2) $x \in I$ and $y * x \in I$ imply $y \in I$, for all $x,\ y \in X$.

Definition 2.17 ([20, 7]). A non-empty subset I of a Q-algebra X is called a *Q-ideal* of X, if it satisfies the following conditions:
(Q_1) $0 \in I$,
(Q_2) $(x * y) * z \in I$ and $y \in I$ imply $x * z \in I$, for all $x,\ y,\ z \in X$.

Definition 2.18 ([9]). Let X be a non-empty set. Then a mapping $\lambda : X \to [0, 1]$ is called a *fuzzy set* in X.

Definition 2.19 ([11, 12]). Let X be a non-empty set. Then a pair $A = (A^N, A^P)$ is called a *bipolar-valued fuzzy set* (or bipolar fuzzy set) (briefly, BPFS) in X , if $A^P : X \to [0, 1]$ and $A^N : X \to [-1, 0]$ are mappings.

For each $x \in X$, we use the positive membership degree A^P to denote the satisfaction degree of the element x to the property corresponding to the bipolar fuzzy set A and the negative membership degree A^N to denote the satisfaction degree of the element x to some implicit counter-property corresponding to the bipolar fuzzy set A. We will denote the set of all bipolar fuzzy sets in X as BPF(X). Now, we list some concepts related to bipolar fuzzy sets (for examples, the inclusion between two bipolar fuzzy sets ,the complement of a bipolar fuzzy set, the intersection of bipolar fuzzy set and the union of a bipolar fuzzy set.

Definition 2.20 ([19])**.** Let X be a nonempty set and let $A,\ B \in BPF(X)$.

(i) We say that A is a *subset* of B, denoted by $A \subset B$, if for each $x \in X$,

$$A^P(x) \leq B^P(x) \text{ and } A^N(x) \geq B^N(x).$$

(ii) We say that A is *equal to* B, denoted by $A = B$, if $A \subset B$ and $B \subset A$.

(iii) The complement of A, denoted by $A^c = (A^{N^c}, A^{P^c})$, is a bipolar fuzzy set in X defined as: for each $x \in X$, $A^c(x) = (-1 - A^N(x), 1 - A^P(x))$, i.e.,

$$(A^{N^c})(x) = -1 - A^N(x),\ (A^{P^c})(x) = 1 - A^P(x).$$

(iv) The intersection of A and B, denoted by $A \cap B$, is a bipolar fuzzy set in X defined as: for each $x \in X$,

$$(A \cap B)(x) = (A^N(x) \vee B^N(x), A^P(x) \wedge B^P(x)).$$

(v) The union of A and B, denoted by $A \cup B$, is a bipolar fuzzy set in X defined as: for each $x \in X$,

$$(A \cup B)(x) = (A^N(x) \wedge B^N(x), A^P(x) \vee B^P(x)).$$

Result 2.21 ([19], Proposition 3.5)**.** *Let $A,\ B,\ C \in BPF(X)$. Then*

(1) (Idempotent laws)*: $A \cap A = A,\ A \cup A = A$,*
(2) (Commutative laws)*: $A \cap B = B \cap A,\ A \cup B = B \cup A$,*
(3) (Associative laws)*: $(A \cap B) \cap C = A \cap (B \cap C),\ (A \cup B) \cup C = A \cup (B \cup C)$,*
(4) (Distributive laws)*: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,*
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(5) (Absorption laws)*: $A \cup (A \cap B) = A,\ A \cap (A \cup B) = A$,*
(6) (DeMorgan's laws)*: $(A \cap B)^c = A^c \cup B^c,\ (A \cup B)^c = A^c \cap B^c$,*
(7) *$(A^c)^c = A$,*
(8) *$A \cap B \subset A,\ A \cap B \subset B$,*
(9) *$A \subset A \cup B,\ B \subset A \cup B$,*
(10) *if $A \subset B$ and $B \subset C$, then $A \subset C$,*
(11) *if $A \subset B$, then $A \cap C \subset B \cap C,\ A \cup C \subset B \cup C$.*

Now, we begin with the concepts of interval-valued fuzzy sets.

Definition 2.22 ([28])**.** Let $D[0, 1]$ denote the set of all closed subintervals of $[0, 1]$, i.e.,

$$D[0, 1] = \{\widetilde{a} = [a^-, a^+] : \widetilde{a} \subset [0, 1] \text{ and } 0 \leq a^- \leq a^+ \leq 1\}.$$

Each member of $D[0, 1]$ is called an *interval-valued number*. We define the operations $\leq, \geq, =$, min (denoted by $\wedge$) and max (denoted by $\vee$) between two elements in $D[0, 1]$ as follows: for any $\widetilde{a},\ \widetilde{b} \in D[0, 1]$,

(i) $\widetilde{a} \leq \widetilde{b}$ iff $a^- \leq b^-$, $a^+ \leq b^+$,
(ii) $\widetilde{a} \geq \widetilde{b}$ iff $a^- \geq b^-$, $a^+ \geq b^+$,
(iii) $\widetilde{a} = \widetilde{b}$ iff $a^- = b^-$, $a^+ = b^+$,
(iv) $\widetilde{a} \wedge \widetilde{b} = [a^- \wedge b^-, a^+ \wedge b^+]$,
(v) $\widetilde{a} \vee \widetilde{b} = [a^- \vee b^-, a^+ \vee b^+]$.

Here we consider that $\widetilde{0} = [0, 0]$ as the least element and $\widetilde{1} = [1, 1]$ as the greatest element. Now let $\{\widetilde{a}_j : j \in J\} \subset D[0, 1]$. Then its inf and sup are defined as follows:

$$\inf_{j\in J}\widetilde{a}_j = [\bigwedge_{j\in J} a_j^-, \bigwedge_{j\in J} a_j^+],\ \sup_{j\in J}\widetilde{a}_j = [\bigvee_{j\in J} a_j^-, \bigvee_{j\in J} a_j^+].$$

Let $D[-1, 0]$ denote the set of all closed subintervals of $[-1, 0]$. Then min, max, inf and sup of members of $D[-1, 0]$ are defined similarly to the above.

Definition 2.23 (See [28, 29, 30]). For a nonempty set X, a mapping $A : X \to D[0, 1]$ is called an *interval-valued fuzzy set* (briefly, an IVF set) in X. Let $D[0, 1]^X$ denote the set of all IVF sets in X. For each $A \in D[0, 1]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree of membership* of an element x to A, where $A^-, A^+ \in I^X$ are called a *lower fuzzy set* and an *upper fuzzy set* in X, respectively. For each $A \in [I]^X$, we write $A = [A^-, A^+]$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy empty whole set in X.

Also, refer to [28, 30] for the inclusion, the equality, the intersection, the union of two IVF sets and the complement of an IVF set.

Result 2.24 ([28], Proposition 3.5). *Let $A,\ B,\ C \in D[0, 1]^X$. Then*
(1) (Idempotent laws)*: $A \cap A = A,\ A \cup A = A,$*
(2) (Commutative laws)*: $A \cap B = B \cap A,\ A \cup B = B \cup A,$*
(3) (Associative laws)*: $(A \cap B) \cap C = A \cap (B \cap C),\ (A \cup B) \cup C = A \cup (B \cup C),$*
(4) (Distributive laws)*: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$*
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$
(5) (Absorption laws)*: $A \cup (A \cap B) = A,\ A \cap (A \cup B) = A,$*
(6) (DeMorgan's laws)*: $(A \cap B)^c = A^c \cup B^c,\ (A \cup B)^c = A^c \cap B^c,$*
(7) *$(A^c)^c = A,$*
(8) *$A \cap B \subset A,\ A \cap B \subset B,$*
(9) *$A \subset A \cup B,\ B \subset A \cup B,$*
(10) *if $A \subset B$ and $B \subset C$, then $A \subset C$,*
(11) *if $A \subset B$, then $A \cap C \subset B \cap C,\ A \cup C \subset B \cup C$.*

Definition 2.25 ([21, 22]). For a nonempty set X, a pair $\mathcal{A} = \langle A, \lambda\rangle$ is called a *cubic set* in X, where A is an IVF set in X and λ is a fuzzy set in X.

Definition 2.26 ([14]). For a non-empty set X, a pair $\mathbf{A} = (\mathbf{A}^N, \mathbf{A}^P)$ is called an *interval-valued bipolar fuzzy set* (briefly, IVBPFS) in X, if $\mathbf{A}^N : X \to D[-1, 0]$ and $\mathbf{A}^P : X \to D[0, 1]$. In this case for each $x \in X$, $\mathbf{A}^N(x) = [A^{N,-}(x), A^{N,+}(x)]$ and $\mathbf{A}^P(x) = [A^{P,-}(x), A^{P,+}(x)]$ are called the *interval-valued positive* and *negative membership degree* of x. In fact, for each $x \in X$,

$$\mathbf{A}(x) = ([A^{N,-}(x), A^{N,+}(x)], [A^{P,-}(x), A^{P,+}(x)]).$$

We will denote the set of all IVBPFSs as $IVBP(X)$.

For any $\mathbf{A},\ \mathbf{B} \in IVBP(X)$,

(i) the intersection of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \cap \mathbf{B}$, is an IVBPFS in X defined as: for each $x \in X$,

$$(\mathbf{A} \cap \mathbf{B})(x) = (\mathbf{A}^N(x) \vee \mathbf{B}^N(x), A^P(x) \wedge \mathbf{B}^P(x)),$$

where $\mathbf{A}^N(x) \vee \mathbf{B}^N(x) = [A^{N,-}(x) \vee B^{N,-}(x), A^{N,+}(x) \vee B^{N,+}(x)]$,

$\mathbf{A}^P(x) \wedge \mathbf{B}^P(x) = [A^{P,-}(x) \wedge B^{P,-}(x), A^{P,+}(x) \wedge B^{P,+}(x)]$.

(ii) the union of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \cup \mathbf{B}$, is an IVBPFS in X defined as: for each $x \in X$,

$$(\mathbf{A} \cup \mathbf{B})(x) = (\mathbf{A}^N(x) \wedge \mathbf{B}^N(x), A^P(x) \vee \mathbf{B}^P(x)),$$

where $\mathbf{A}^N(x) \wedge \mathbf{B}^N(x) = [A^{N,-}(x) \wedge B^{N,-}(x), A^{N,+}(x) \wedge B^{N,+}(x)]$,

$\mathbf{A}^P(x) \vee \mathbf{B}^P(x) = [A^{P,-}(x) \vee B^{P,-}(x), A^{P,+}(x) \vee B^{P,+}(x)]$.

3. Cubic bipolar sets

In this section, we will introduce a new notion called cubic bipolar set and study its several properties.

Definition 3.1 ([23])**.** Let X be a non-empty set. Then a pair $\mathcal{A} = \langle \mathbf{A}, A \rangle$ is called a *cubic bipolar set* in X, if $\mathbf{A} = (\mathbf{A}^N, \mathbf{A}^P)$ is an interval-valued bipolar fuzzy set and $A = (A^N, A^P)$ is an bipolar fuzzy set in X, where

$\mathbf{A}^N : X \to D[-1, 0]$, $\mathbf{A}^P : X \to D[0, 1]$

and

$A^N : X \to [-1, 0]$, $A^P : X \to [0, 1]$.

We will denote the set of all cubic bipolar sets in X as $CBP(X)$. In particular, we will denote the cubic fuzzy set and the cubic bipolar whole set as 0^X and 1^X, and are defined as follows, respectively:

$$0^X = (\widetilde{\mathbf{0}}, \bar{0}),\ 1^X = (\widetilde{\mathbf{1}}, \bar{1}),$$

where $\widetilde{\mathbf{0}} = ([0, 0], [0, 0])$, $\bar{0} = (0, 0)$, $\widetilde{\mathbf{1}} = ([-1, -1], [1, 1])$, $\bar{1} = (-1, 1)$.

Example 3.2. Let $X = \{a, b, c\}$ and consider the IVBPFS $\mathbf{A}$ and BPFS A in X, respectively given by:

$$\mathbf{A}(a) = ([-0.5, -0.3], [0.5, 0.6]),\ A(a) = (-0.1, 0.1),$$
$$\mathbf{A}(b) = ([-0.7, -0.5], [0.7, 0.8]),\ A(a) = (-0.5, 0.3),$$
$$\mathbf{A}(c) = ([-0.8, -0.5], [0.4, 0.7]),\ A(a) = (-0.4, 0.6).$$

Then we can easily see that $\mathcal{A} = \langle \mathbf{A}, A \rangle \in CBP(X)$

Definition 3.3. Let X be a non-empty set and let $\mathcal{A}$, $\mathcal{B} \in CBP(X)$.

(i) We say that $\mathcal{A}$ is a *subset* $\mathcal{B}$, denoted by $\mathcal{A} \subset \mathcal{B}$, if for each $x \in X$,

$$\mathbf{A}(x) \leq \mathbf{B}(x),\ A(x) \leq B(x),\ \text{i.e.,}$$
$$A^{N,-}(x) \geq B^{N,-}(x),\ A^{N,+}(x) \geq B^{N,+}(x),\ A^{P,-}(x) \leq B^{P,-}(x),\ A^{P,+}(x) \leq B^{P,+}(x),$$
$$A^N(x) \geq B^N(x),\ A^P(x) \leq B^P(x).$$

(ii) We say that $\mathcal{A}$ is *equal to* $\mathcal{B}$, denoted by $\mathcal{A} = \mathcal{B}$, if $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A}$.

(iii) The complement of $\mathcal{A}$, denoted by $\mathcal{A}^c$, is a cubic bipolar set in X defined by: for each $x \in X$,

$$\mathcal{A}^c(x) = (\mathbf{A}^c(x), A^c(x)),$$

where $\mathbf{A}^c(x) = ([-1 - A^{N,+}(x), -1 - A^{N,-}(x)], [1 - A^{P,+}(x), 1 - A^{P,-}(x)])$,

$A^c(x) = (-1 - A^N(x), 1 - A^P(x))$.

(iv) The intersection of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \cap \mathcal{B}$, is a cubic bipolar set in X defined by: for each $x \in X$,

$$(\mathcal{A} \cap \mathcal{B})(x) = (\mathbf{A}(x) \wedge \mathbf{B}(x), A(x) \wedge B(x)),$$

where $\mathbf{A}(x) \wedge \mathbf{B}(x) = ([A^{N,-}(x) \vee B^{N,-}(x)], [A^{P,+}(x) \wedge B^{P,+}(x)])$,
$A(x) \wedge B(x) = (A^N(x) \vee B^N(x), A^P(x) \wedge B^P(x))$.

(v) The union of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \cup \mathcal{B}$, is a cubic bipolar set in X defined by: for each $x \in X$,

$$(\mathcal{A} \cup \mathcal{B})(x) = (\mathbf{A}(x) \vee \mathbf{B}(x), A(x) \vee B(x)),$$

where $\mathbf{A}(x) \vee \mathbf{B}(x) = ([A^{N,-}(x) \wedge B^{N,-}(x)], [A^{P,+}(x) \vee B^{P,+}(x)])$,
$A(x) \vee B(x) = (A^N(x) \wedge B^N(x), A^P(x) \vee B^P(x))$.

Example 3.4. Let $X = \{a, b, c\}$ be a set.

(1) Consider two cubic bipolar sets $\mathcal{A}$, $\mathcal{B}$ in X given by:

$$\mathcal{A}(a) = \langle([-0.5, -0.3], [0.5, 0.6]), (-0.1, 0.1)\rangle,$$
$$\mathcal{A}(b) = \langle([-0.7, -0.5], [0.6, 0.8]), (-0.5, 0.3)\rangle,$$
$$\mathcal{A}(c) = \langle([-0.8, -0.3], [0.4, 0.7]), (-0.4, 0.6)\rangle,$$
$$\mathcal{B}(a) = \langle([-0.6, -0.5], [0.6, 0.8]), (-0.3, 0.4)\rangle,$$
$$\mathcal{B}(b) = \langle([-0.8, -0.6], [0.7, 0.8]), (-0.4, 0.4)\rangle,$$
$$\mathcal{B}(c) = \langle([-0.9, -0.7], [0.5, 0.9]), (-0.6, 0.8)\rangle.$$

Then we can easily check that $\mathcal{A} \subset \mathcal{B}$.

(2) Consider a cubic bipolar sets $\mathcal{A}$ in X given by:

$$\mathcal{A}(a) = \langle([-0.80, -0.50], [0.20, 0.50]), (-0.60, 0.68)\rangle,$$
$$\mathcal{A}(b) = \langle([-0.90, -0.70], [0.40, 0.60]), (-0.60, 0.55)\rangle,$$
$$\mathcal{A}(c) = \langle([-0.80, -0.60], [0.20, 0.60]), (-0.45, 0.45)\rangle.$$

Then we obtain easily the $\mathcal{A}^c$ given by:

$$\mathcal{A}^c(a) = \langle([-0.50, -0.20], [0.50, 0.80]), (-0.32, 0.40)\rangle,$$
$$\mathcal{A}^c(b) = \langle([-0.30, -0.10], [0.40, 0.60]), (-0.55, 0.45)\rangle,$$
$$\mathcal{A}^c(c) = \langle([-0.40, -0.20], [0.40, 0.80]), (-0.55, 0.55)\rangle.$$

(3) Consider two cubic bipolar sets $\mathcal{A}$, $\mathcal{B}$ in X given by:

$$\mathcal{A}(a) = \langle([-0.80, -0.40], [0.28, 0.59]), (-0.62, 0.38)\rangle,$$
$$\mathcal{A}(b) = \langle([-0.50, -0.30], [0.46, 0.71]), (-0.61, 0.53)\rangle,$$
$$\mathcal{A}(c) = \langle([-0.40, -0.20], [0.24, 0.40]), (-0.47, 0.25)\rangle,$$
$$\mathcal{B}(a) = \langle([-0.80, -0.20], [0.35, 0.87]), (-0.51, 0.43)\rangle,$$
$$\mathcal{B}(b) = \langle([-0.70, -0.20], [0.53, 0.82]), (-0.70, 0.40)\rangle,$$
$$\mathcal{B}(c) = \langle([-0.60, -0.10], [0.33, 0.73]), (-0.71, 0.33)\rangle.$$

Then we have easily $\mathcal{A} \cap \mathcal{B}$ and $\mathcal{A} \cup \mathcal{B}$, respectively given by:

$$(\mathcal{A} \cap \mathcal{B})(a) = \langle([-0.80, -0.20], [0.28, 0.59]), (-0.51, 0.38)\rangle,$$
$$(\mathcal{A} \cap \mathcal{B})(b) = \langle([-0.50, -0.20], [0.46, 0.71]), (-0.61, 0.40)\rangle,$$
$$(\mathcal{A} \cap \mathcal{B})(c) = \langle([-0.40, -0.10], [0.24, 0.40]), (-0.47, 0.25)\rangle$$

and

$$(\mathcal{A} \cup \mathcal{B})(a) = \langle([-0.80, -0.40], [0.35, 0.87]), (-0.62, 0.43)\rangle,$$
$$(\mathcal{A} \cup \mathcal{B})(b) = \langle([-0.70, -0.30], [0.53, 0.82]), (-0.70, 0.53)\rangle,$$

$$(\mathcal{A} \cup \mathcal{B})(c) = \langle([-0.60, -0.20], [0.33, 0.73]), (-0.71, 0.33)\rangle .$$

From Results 2.21 and 2.24, we can easily prove the following.

Theorem 3.5. *Let* $\mathcal{A},\ \mathcal{B},\ \mathcal{C} \in CBP(X)$. *Then*
(1) (Idempotent laws)*:* $\mathcal{A} \cap \mathcal{A} = \mathcal{A},\ \mathcal{A} \cup \mathcal{A} = \mathcal{A}$,
(2) (Commutative laws)*:* $\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A},\ \mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$,
(3) (Associative laws)*:* $(\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C} = \mathcal{A} \cap (\mathcal{B} \cap \mathcal{C}),\ (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C} = \mathcal{A} \cup (\mathcal{B} \cup \mathcal{C})$,
(4) (Distributive laws)*:* $\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$,
$\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$,
(5) (Absorption laws)*:* $\mathcal{A} \cup (\mathcal{A} \cap \mathcal{B}) = \mathcal{A},\ \mathcal{A} \cap (\mathcal{A} \cup \mathcal{B}) = \mathcal{A}$,
(6) (DeMorgan's laws)*:* $(\mathcal{A} \cap \mathcal{B})^c = \mathcal{A}^c \cup \mathcal{B}^c,\ (\mathcal{A} \cup \mathcal{B})^c = \mathcal{A}^c \cap \mathcal{B}^c$,
(7) $(\mathcal{A}^c)^c = \mathcal{A}$,
(8) $\mathcal{A} \cap \mathcal{B} \subset \mathcal{A},\ \mathcal{A} \cap \mathcal{B} \subset \mathcal{B}$,
(9) $\mathcal{A} \subset \mathcal{A} \cup \mathcal{B},\ \mathcal{B} \subset \mathcal{A} \cup \mathcal{B}$,
(10) *if* $\mathcal{A} \subset \mathcal{B}$ *and* $\mathcal{B} \subset \mathcal{C}$, *then* $\mathcal{A} \subset \mathcal{C}$,
(11) *if* $\mathcal{A} \subset \mathcal{B}$, *then* $\mathcal{A} \cap \mathcal{C} \subset \mathcal{B} \cap \mathcal{C},\ \mathcal{A} \cup \mathcal{C} \subset \mathcal{B} \cup \mathcal{C}$,
(12) *the followings are equivalent:*
(*a*) $\mathcal{A} \subset \mathcal{B}$, (*b*) $\mathcal{A} \cap \mathcal{B} = \mathcal{A}$, (*c*) $\mathcal{A} \cup \mathcal{B} = \mathcal{B}$,
(13) $0^X \subset \mathcal{A} \subset 1^X$.

Definition 3.6. Let $X,\ Y$ be two non-empt sets, let $\mathcal{A} \in CBP(X)$, $\mathcal{B} \in CBP(y)$ and let $f : X \to Y$ be a mapping.

(i) The *image of* $\mathcal{A}$ *under* f, denoted by $f(\mathcal{A}) = \langle f(\mathbf{A}), f(A)\rangle$, is a cubic bipolar set in Y defined as follows: for each $y \in Y$,

$$f(\mathcal{A})(y) = \begin{cases} \left\langle \bigvee_{x \in f^{-1}(y)} \mathbf{A}(x), \bigvee_{x \in f^{-1}(y)} A(x) \right\rangle & \text{if } f^{-1}(y) \neq \varnothing \\ \langle([0,0],[0,0]),(0,0)\rangle & \text{otherwise.} \end{cases}$$

(ii) The *preimage of* $\mathcal{B}$ *under* f, denoted by $f^{-1}(\mathcal{B}) = \left\langle f^{-1}(\mathbf{B}), f^{-1}(B)\right\rangle$, is a cubic bipolar set in X defined as follows: for each $x \in X$,

$$f^{-1}(\mathcal{B})(x) == \langle \mathbf{B}(f(x)), B(f(x))\rangle .$$

Proposition 3.7. *Let* $f : X \to Y$ *be a mapping, let* $\mathcal{A},\ \mathcal{A}_1,\ \mathcal{A}_2 \in CBP(X)$ *and let* $\mathcal{B},\ \mathcal{B}_1,\ \mathcal{B}_2 \in CBP(Y)$.
(1) *If* $\mathcal{A}_1 \subset \mathcal{A}_2$, *then* $f(\mathcal{A}_1) \subset f(\mathcal{A}_2)$.
(2) $f(\mathcal{A}_1 \cup \mathcal{A}_2) = f(\mathcal{A}_1) \cup f(\mathcal{A}_2)$.
(3) $f(\mathcal{A}_1 \cap \mathcal{A}_2) \subset f(\mathcal{A}_1) \cap f(\mathcal{A}_2)$.
(4) *If* $\mathcal{B}_1 \subset \mathcal{B}_2$, *then* $f^{-1}(\mathcal{B}_1) \subset f^{-1}(\mathcal{B}_2)$.
(5) $f^{-1}(\mathcal{B}_1 \cup \mathcal{B}_2) = f^{-1}(\mathcal{B}_1) \cup f^{-1}(\mathcal{B}_2)$.
(6) $f^{-1}(\mathcal{B}_1 \cap \mathcal{B}_2) = f^{-1}(\mathcal{B}_1) \cap f^{-1}(\mathcal{B}_2)$.
(7) $\mathcal{A} \subset f^{-1} \circ f(\mathcal{A})$ *and* $\mathcal{A} = f^{-1} \circ f(\mathcal{A})$, *if* f *is injective.*
(8) $f \circ f^{-1}(\mathcal{B}) \subset \mathcal{B}$ *and* $f \circ f^{-1}(\mathcal{B}) = \mathcal{B}$, *if* f *is surjective.*
(9) $f^{-1}(\mathcal{B}^c) = (f^{-1}(\mathcal{B}))^c$.
(10) *If* $g : Y \to Z$ *is a mapping, then* $(g \circ f)(\mathcal{A}) = g(f(\mathcal{A}))$.
Moreover, for each $\mathcal{C} \in CBP(Z)$, *we have*

$$(g \circ f)^{-1}(\mathcal{C}) = f^{-1}(g^{-1}(\mathcal{C})).$$

Proof. The proofs are straightforward. □

4. Cubic bipolar point

Definition 4.1. Let X be a non-empty set, let $\bar{a} = (a^N, a^P) \in [-1, 0) \times (0, 1]$, $\widetilde{\mathbf{a}} = (\mathbf{a}^N, \mathbf{a}^P) \in D[-1, 0) \times D(0, 1]$ and let $\mathcal{A} \in CBP(X)$. Then $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle}$ is called a *cubic bipolar point* in X with the value $\langle \widetilde{\mathbf{a}}, \bar{a}\rangle$ and the support $x \in X$, if for each $y \in X$,

$$[x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle}](y) = \begin{cases} \langle \widetilde{\mathbf{a}}, \bar{a}\rangle & \text{if } y = x, \\ \langle ([0,0],[0,0]),(0,0)\rangle & \text{otherwise.} \end{cases}$$

We say that $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle}$ *belongs to* $\mathcal{A}$, denoted by $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{A}$, if the following conditions hold:

(i) $\mathbf{A}^N(x) \leq \mathbf{a}^N,\ \mathbf{A}^P(x) \geq \mathbf{a}^P$, i.e.,

$$A^{N,-}(x) \leq a^{N,-},\ A^{N,+}(x)A^{N,+}(x) \leq A^{N,+}(x),\ A^{P,-}(x) \geq a^{P,-},\ A^{P,-}(x) \geq a^{P,-},$$

(ii) $A^N(x) \leq a^N,\ A^P(x) \geq a^P$.

We will denote the set of all cubic bipolar points in X as $CBP_P(X)$.

Lemma 4.2. *For any $\mathcal{A} \in CBP(X)$, $\mathcal{A} = \bigcup\{x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in CBP_P(X) : x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{A}\}$.*

Proof. The proof is straightforward. □

Proposition 4.3. *Let $\mathcal{A},\ \mathcal{B} \in CBP(X)$. If $\mathcal{A} \subset \mathcal{B}$, then $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{B}$, for each $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{A}$.*

Proof. Suppose $\mathcal{A} \subset \mathcal{B}$ and let $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{A}$. Then clearly, we have

$$\mathbf{A}^N(x) \leq \mathbf{a}^N,\ \mathbf{A}^P(x) \geq \mathbf{a}^P \text{ and } A^N(x) \leq a^N,\ A^P(x) \geq a^P.$$

Since $\mathcal{A} \subset \mathcal{B}$, we have

$$\mathbf{B}^N(x) \leq \mathbf{A}^N(x),\ \mathbf{B}^P(x) \geq \mathbf{A}^P(x) \text{ and } B^N(x) \leq A^N(x),\ B^P(x) \geq A^P(x).$$

Thus $\mathbf{B}^N(x) \leq \mathbf{a}^N,\ \mathbf{B}^P(x) \geq \mathbf{a}^P$ and $B^N(x) \leq a^N,\ B^P(x) \geq a^P$. So $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{B}$. □

Theorem 4.4. *Let $\mathcal{A},\ \mathcal{B} \in CBP(X)$ and let $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in CBP_P(X)$.*

(1) *$x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{A}$ and $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{B}$ if and only if $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{A} \cap \mathcal{B}$.*

(2) *If $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{A}$ or $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{B}$, then $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{A} \cup \mathcal{B}$.*

Proof. Suppose $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{A} \cap \mathcal{B}$. Then clearly, we have

$$(\mathbf{A}^N \cap \mathbf{B}^N)(x) = \mathbf{A}^N(x) \vee \mathbf{B}^N(x) \leq \mathbf{a}^N,$$
$$(\mathbf{A}^P \cap \mathbf{B}^P)(x) = \mathbf{A}^P(x) \wedge \mathbf{B}^P(x) \geq \mathbf{a}^P,$$
$$(A^N \cap B^N)(x) = A^N(x) \vee B^N(x) \leq a^N,$$
$$(A^P \cap B^P)(x) = A^P(x) \wedge B^P(x) \geq a^P.$$

Thus we have

$$\mathbf{A}^N(x) \leq \mathbf{a}^N,\ \mathbf{A}^P(x) \geq \mathbf{a}^P,\ A^N(x) \leq a^N,\ A^P(x) \geq a^P$$

and

$$\mathbf{B}^N(x) \leq \mathbf{a}^N,\ \mathbf{B}^P(x) \geq \mathbf{a}^P,\ B^N(x) \leq a^N,\ B^P(x) \geq a^P.$$

So $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{A}$ and $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{B}$.

The converse is proved similarly.

(2) Suppose $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{A}$ or $x_{\langle \widetilde{\mathbf{a}}, \bar{a}\rangle} \in \mathcal{B}$. Then we have

$$\mathbf{A}^P(x) \geq \mathbf{a}^P,\ A^P(x) \geq a^P \text{ and } \mathbf{A}^N(x) \leq \mathbf{a}^N,\ A^N(x) \leq a^N$$

or

$$\mathbf{B}^P(x) \geq \mathbf{a}^P,\ B^P(x) \geq a^P \text{ and } \mathbf{B}^N(x) \leq \mathbf{a}^N,\ B^N(x) \leq a^N.$$

Thus we have

$$\mathbf{A}^P(x) \geq \mathbf{a}^P,\ A^P(x) \geq a^P \text{ or } \mathbf{B}^P(x) \geq \mathbf{a}^P,\ B^P(x) \geq a^P$$

and

$$\mathbf{A}^N(x) \leq \mathbf{a}^N,\ A^N(x) \leq a^N \text{ or } \mathbf{B}^N(x) \leq \mathbf{a}^N,\ B^N(x) \leq a^N.$$

So $\quad \mathbf{A}^P(x) \vee \mathbf{B}^P(x) \geq \mathbf{a}^P,\ A^P(x) \vee B^P(x) \geq a^P$

and

$$\mathbf{A}^N(x) \wedge \mathbf{B}^N(x) \leq \mathbf{a}^N,\ A^N(x) \wedge B^N(x) \leq a^N.$$

Hence $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{A} \cup \mathcal{B}$. □

Theorem 4.5. *Let $(\mathcal{A})_{j \in J}$ be a family of cubic bipolar sets in X and let $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in CBP_P(X)$.*

(1) *$x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \bigcap_{j \in J} \mathcal{A}_j$ if and only if $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{A}_j$ for each $j \in J$.*

(2) *If there is a $j \in J$ such that $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{A}_j$, then $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \bigcup_{j \in J} \mathcal{A}_j$.*

Proof. Suppose $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \bigcap_{j \in J} \mathcal{A}_j$. Then clearly, we have

$$(\bigcap_{j \in J} \mathbf{A}_j^P)(x) = \bigwedge_{j \in J} \mathbf{A}_j^P(x) \geq \mathbf{a}^P,\ (\bigcap_{j \in J} A_j^P)(x) = \bigwedge_{j \in J} A_j^P(x) \geq a^P$$

and

$$(\bigcap_{j \in J} \mathbf{A}_j^N)(x) = \bigvee_{j \in J} \mathbf{A}_j^N(x) \leq \mathbf{a}^N,\ (\bigcap_{j \in J} A_j^N)(x) = \bigvee_{j \in J} A_j^N(x) \leq a^N.$$

Thus we have for each $j \in J$,

$$\mathbf{A}_j^P(x) \geq \mathbf{a}^P,\ A_j^P(x) \geq a^P,\ \mathbf{A}_j^N(x) \leq \mathbf{a}^N,\ A_j^N(x) \leq a^N.$$

So $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{A}_j$ for each $j \in J$.

The converse is proved similarly.

(2) From the fact that $\mathcal{A}_j \subset \bigcup_{j \in J} \mathcal{A}_j$, the proof is immediate. □

5. Cubic bipolar Q-ideal in Q-algebras

In this section, we present the idea another extension of fuzzy set theory which is called cubic bipolar structures and it application on Q-algebra, which generalizations of cubic set and bipolar set.

Definition 5.1. Let X be a Q-algebra and let $\mathcal{A} \in CBP(X)$. Then $\mathcal{A}$ is called a *cubic bipolar subalgebra* of X, if it satisfies the following conditions: for any $x,\ y \in X$,

(CBPS1) $\mathbf{A}^P(x * y) \geq \mathbf{A}^P(x) \wedge \mathbf{A}^P(y),\ \mathbf{A}^N(x * y) \leq \mathbf{A}^N(x) \vee \mathbf{A}^N(y)$,

(CBPS2) $A^P(x * y) \geq A^P(x) \wedge A^P(y),\ A^N(x * y) \leq A^N(x) \vee A^N(y)$.

Example 5.2. Let $X = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $*$ defined by the following table:

Then we can easily check that $(X, *, 0)$ is a Q-algebra. Consider the CBPS $\mathcal{A}$ in X given by:

$$\mathbf{A}(x) = \begin{cases} ([-0.9, -0.5], [0.3, 0.9]) & \text{if } x \in \{0, 1\} \\ ([-0.8, -0.2], [0.1, 0.6]) & \text{otherwise} \end{cases}$$

and

$*$	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	0	3	0	4
4	4	4	4	4	0

Table 5.1

$A(0) = (-0.5, 0.8),\ A(1) = (-0.4, 0.7),\ A(2) = (-0.3, 0.6),$
$A(3) = (-0.2, 0.5),\ A(4) = (-0.1, 0.4).$

Then it is easy to check that $\mathcal{A}$ is a cubic bipolar subalgebra of X.

Lemma 5.3. *If $\mathcal{A}$ is a a cubic bipolar subalgebra of a Q-algebra X, then for each $x \in X$,*

$$\mathbf{A}^P(0) \geq \mathbf{A}^P(x),\ \mathbf{A}^N(0) \leq \mathbf{A}^N(x),\ \text{and}\ A^P(0) \geq A^P(x),\ A^N(0) \leq A^N(x).$$

Proof. Let $x \in X$. Then from (CBPS1) and Definition 2.3 (i), we have

$$\mathbf{A}^P(0) = \mathbf{A}^P(x * x) \geq \mathbf{A}^P(x) \wedge \mathbf{A}^P(x) = \mathbf{A}^P(x),$$

$$\mathbf{A}^N(0) = \mathbf{A}^N(x * x) \leq \mathbf{A}^N(x) \vee \mathbf{A}^N(x) = \mathbf{A}^N(x).$$

Also, by (CBPS2) and Definition 2.3 (i), we have

$$A^P(0) = A^P(x * x) \geq A^P(x) \wedge A^P(x) = A^P(x),$$

$$A^N(0) = A^N(x * x) \leq A^N(x) \vee A^N(x) = A^N(x).$$

Thus the sufficient conditions hold. □

Definition 5.4. Let X be a Q-algebra and let $\mathcal{A} \in CBP(X)$. Then $\mathcal{A}$ is called a cubic bipolar BCK-ideal of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

(CBPBCKI0) $A^{P,-}(0) \geq A^{P,-}(x),\ A^{P,+}(0) \geq A^{P,+}(x),\ A^{N,-}(0) \leq A^{N,-}(x),$
$A^{N,+}(0) \geq A^{N,+}(x),\ A^P(0) \geq A^P(x),\ A^N(0) \leq A^N(x),$

(CBPBCKI1) $\mathbf{A}^P(x) \geq \mathbf{A}^P(x * y) \wedge \mathbf{A}^P(y),\ \mathbf{A}^N(x) \leq \mathbf{A}^N(x * y) \vee \mathbf{A}^N(y),$

(CBPBCKI2) $A^P(x) \geq A^P(x * y) \wedge A^P(y),\ A^N(x) \leq A^N(x * y) \vee A^N(y).$

Definition 5.5. Let X be a Q-algebra and let $\mathcal{A} \in CBP(X)$. Then $\mathcal{A}$ is called a *cubic bipolar Q-ideal* of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

(CBPQI0) $A^{P,-}(0) \geq A^{P,-}(x),\ A^{P,+}(0) \geq A^{P,+}(x),\ A^{N,-}(0) \leq A^{N,-}(x),$
$A^{N,+}(0) \geq A^{N,+}(x),\ A^P(0) \geq A^P(x),\ A^N(0) \leq A^N(x),$

(CBPQI1) $\mathbf{A}^P(x*z) \geq \mathbf{A}^P((x*y)*z) \wedge \mathbf{A}^P(y),\ \mathbf{A}^N(x*z) \leq \mathbf{A}^N((x*y)*z) \vee \mathbf{A}^N(y),$

(CBPQI2) $A^P(x*z) \geq A^P((x*y)*z) \wedge A^P(y),\ A^N(x*z) \leq A^N((x*y)*z) \vee A^N(y).$

Example 5.6. Let $X = \{0, 1, 2, 3, 4\}$ be the Q-algebra given in Example 5.2. Consider the CBPS $\mathcal{A}$ in X given by:

$$\mathbf{A}(x) = \begin{cases} ([-0.9, -0.5], [0.2, 0.8]) & \text{if } x \in \{0, 1\} \\ ([-0.8, -0.2], [0.1, 0.5]) & \text{otherwise} \end{cases}$$

and

$A(0) = (-0.5, 0.6),\ A(1) = (-0.4, 0.5),\ A(2) = (-0.3, 0.3),$

$A(3) = (-0.2, 0.3),\ A(4) = (-0.1, 0.2)$.
Then it is easy to check that $\mathcal{A}$ is a cubic bipolar Q-ideal of X.

Proposition 5.7. *Every cubic bipolar Q-ideal of a Q-algebra is cubic bipolar BCK-ideal of X. But the converse is not true* (See Example 5.8).

Proof. The proof is straightforward. □

Example 5.8. Let $X = \{0, 1, 2, 3\}$ be the Q-algebra with a binary operation $*$ defined by the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	2	0

Table 5.2

Consider the CBPS $\mathcal{A}$ in X given by:

$$\mathbf{A}(x) = \begin{cases} ([-0.7, -0.4], [0.2, 0.8]) & \text{if } x \in \{0, 1\} \\ ([-0.7, -0.3], [0.1, 0.5]) & \text{otherwise} \end{cases}$$

and

$A(0) = (-0.5, 0.9),\ A(1) = (-0.4, 0.8),\ A(2) = (-0.3, 0.6),\ A(3) = (-0.2, 0.5)$.
Then we can easily check that $\mathcal{A}$ is a cubic bipolar BCK-ideal of X. On the other hand,

$$A^P(3) = A^P(3 * 1) \geq A^P((3 * 1) * 1) \wedge A^P(1) = A^P(3) \wedge A^P(1) = 0.5$$

But $0.5 = A^P(3 * 1) \ngeq A^P(3 * 2) * 1) \wedge A^P(2) = A^P(2 * 1) \wedge A^P(2) = A^P(2) = 0.6$.
Thus $\mathcal{A}$ is not a cubic bipolar Q-ideal of X.

Lemma 5.9. *If $\mathcal{A}$ is a a cubic bipolar Q-ideal of a Q-algebra X, then for each $x \in X$,*

$$\mathbf{A}^P(0) \geq \mathbf{A}^P(x),\ \mathbf{A}^N(0) \leq \mathbf{A}^N(x) \text{ and } A^P(0) \geq A^P(x),\ A^N(0) \leq A^N(x). \tag{5.1}$$

Proof. The proof is straightforward. □

Lemma 5.10. *Let $\mathcal{A}$ be a cubic bipolar Q-ideal of a Q-algebra X. If $x \leq y$ in X, then we have*

$$\mathbf{A}^P(y) \geq \mathbf{A}^P(x),\ \mathbf{A}^N(y) \leq \mathbf{A}^N(x) \text{ and } A^P(y) \geq A^P(x),\ A^N(y) \leq A^N(x).$$

Proof. Let $x,\ y \in X$ such that $x \leq Y$. Then clearly, $x * y = 0$. Thus we have

$$\begin{aligned} \mathbf{A}^P(x) &= \mathbf{A}^P(x * 0) \text{ [By Definition 2.3 (ii)]} \\ &\geq \mathbf{A}^P((x * y) * 0) \wedge \mathbf{A}^P(y) \text{ [Since } \mathcal{A} \text{ is a cubic bipolar } Q\text{-ideal of } X] \\ &= \mathbf{A}^P(0 * 0) \wedge \mathbf{A}^P(y) = \mathbf{A}^P(0) \wedge \mathbf{A}^P(y) \\ &= \mathbf{A}^P(y), \text{ [By 5.1)]} \\ \mathbf{A}^N(x) &= \mathbf{A}^P(x * 0) \\ &\leq \mathbf{A}^N((x * y) * 0) \vee \mathbf{A}^N(y) \\ &= \mathbf{A}^N(0 * 0) \vee \mathbf{A}^N(y) = \mathbf{A}^N(0) \vee \mathbf{A}^N(y) \end{aligned}$$

$= \mathbf{A}^N(y)$.
Similarly we have $A^P(x) \geq A^P(y)$ and $A^N(x) \leq A^N(y)$. This completes the proof. □

Lemma 5.11. *Let $\mathcal{A}$ be a cubic bipolar Q-ideal of a Q-algebra X. If $x * y \leq z$ in X, then we have*

$$\mathbf{A}^P(x) \geq \mathbf{A}^P(z) \wedge \mathbf{A}^P(y),\ \ \mathbf{A}^N(x) \leq \mathbf{A}^N(z) \vee \mathbf{A}^N(y),$$

$$A^P(x) \geq A^P(z) \wedge A^P(y),\ \ A^N(x) \leq A^N(z) \vee A^N(y).$$

Proof. Let $x,\ y,\ z \in X$ such that $x * y \leq z$. Then clearly, $(x * y) * z = 0$. Thus we have

$\mathbf{A}^P(x) = \mathbf{A}^P(x * 0)$ [By Definition 2.3 (ii)]
$\geq \mathbf{A}^P((x * y) * 0) \wedge \mathbf{A}^P(y)$ [Since $\mathcal{A}$ is a cubic bipolar Q-ideal of X]
$= \mathbf{A}^P(x * y) \wedge \mathbf{A}^P(y)$ [By Definition 2.3 (ii)]
$\geq \mathbf{A}^P(z) \wedge \mathbf{A}^P(y)$. [By Lemma 5.10]

Similarly we have

$$\mathbf{A}^N(x) \leq \mathbf{A}^N(z) \vee \mathbf{A}^N(y),\ \ A^P(x) \geq A^P(z) \wedge A^P(y),\ \ A^N(x) \leq A^N(z) \vee A^N(y).$$

This completes the proof. □

Proposition 5.12. *Let X be a Q-algebra satisfying the condition (2.2). Then every cubic bipolar BCK-ideal of X is a cubic bipolar subalgebra of X. But the converse is not true* (See Example 5.13).

Proof. Let $\mathcal{A}$ be a cubic bipolar BCK-ideal of X and let $x,\ y \in X$. Then we have

$\mathbf{A}^P(x * y) \geq \mathbf{A}^P((x * y) * x) \wedge \mathbf{A}^P(x)$ [By (CBPBCKI1)]
$= \mathbf{A}^P(0) \wedge \mathbf{A}^P(x)$ [Since $(x * y) * x = 0$ by (2.2)]
$\geq \mathbf{A}^P(y) \wedge \mathbf{A}^P(x)$. [By (CBPBCKI0)]

Also, by conditions (CBPBCKI2), (vi) and (CBPBCKI0), we have

$$A^P(x * y) \geq A^P(y) \wedge A^P(x).$$

Similarly, we have $\mathbf{A}^N(x*y) \leq \mathbf{A}^N(y) \vee \mathbf{A}^N(x),\ A^N(x*y) \leq A^N(y) \vee A^N(x)$. Thus $\mathcal{A}$ is a cubic bipolar subalgebra of X. □

Example 5.13. Let $(X, *, 0)$ be the Q-algebra given in Example 5.8. Then we can see that X satisfies the condition (2.2). Consider the cubic bipolar set $\mathcal{B}$ in X defined by:

$$\mathbf{B}(0) = ([-0.9, -0.8], [0.8, 0.9]),\ \ \mathbf{B}(1) = ([-0.6, -0.5], [0.7, 0.8]),$$

$$\mathbf{B}(2) = ([-0.7, -0.6], [0.5, 0.6]),\ \ \mathbf{B}(3) = ([-0.5, -0.4], [0.6, 0.7]),$$

$$B(0) = (-0.9, 0.8),\ B(1) = (-0.6, 0.7),\ B(2) = (-0.7, 0.5),\ B(3) = (-0.5, 0.6).$$

Then we can easily check that $\mathcal{B}$ is a cubic bipolar subalgebra of X. But

$$\mathbf{B}^N(3) = [-0.5, -0.4] \not\leq [-0.7, -0.6] = \mathbf{B}^N(2) = \mathbf{B}^N(3 * 2) \vee \mathbf{B}^N(2).$$

Thus $\mathcal{B}$ is a cubic bipolar BCK-ideal of X.

Proposition 5.14. *Let $\mathcal{A}$ be a cubic bipolar subalgebra of a Q-algebra X. Suppose for all $x,\ y,\ z \in X$ such that $x * y \leq z$, the following conditions:*

$$\mathbf{A}^P(x) \geq \mathbf{A}^P(y) \wedge \mathbf{A}^P(z),\ \ \mathbf{A}^N(x) \leq \mathbf{A}^N(y) \vee \mathbf{A}^N(z),$$

$$A^P(x) \geq A^P(y) \wedge A^P(z),\ \ A^N(x) \leq A^N(y) \vee A^N(z).$$

Then $\mathcal{A}$ is a cubic bipolar Q-ideal of X.

Proof. Let $\mathcal{A}$ be a cubic bipolar subalgebra of X and let $x \in X$. Then by Lemma 5.3, we have

$$\mathbf{A}^P(0) \geq \mathbf{A}^P(x),\ \ \mathbf{A}^N(0) \leq \mathbf{A}^N(x),\ \ A^P(0) \geq A^P(x),\ \ A^N(0) \leq A^N(x).$$

Now let $x,\ y,\ z \in X$ such that $x * y \leq z$. Then
$0 = ((x * y) * z) * ((x * y) * z)$ [By Definition 2.3 (i)]
$= ((x * z) * y) * ((x * y) * z)$ [By Definition 2.3 (iii)]
$= ((x * z) * ((x * y) * z)) * y.$ [By Definition 2.3 (iii)]
Thus $(x * z) * ((x * y) * z)) \leq y$. So by Lemma 5.11, we have

$$\mathbf{A}^P(x * z) \geq \mathbf{A}^P((x * y) * z) \wedge \mathbf{A}^P(y),\ \ \mathbf{A}^N(x * z) \leq \mathbf{A}^N((x * y) * z) \vee \mathbf{A}^N(y),$$

$$A^P(x * z) \geq A^P((x * y) * z) \wedge A^P(y),\ \ A^N(x * z) \leq A^N((x * y) * z) \vee A^N(y).$$

Hence $\mathcal{A}$ is a cubic bipolar Q-ideal of X. This completes the proof. □

Definition 5.15. Let X be a Q-algebra.
(i) $A = (A^N, A^P) \in BPF(X)$ is called a *bipolar fuzzy Q-ideal* of X, if it satisfies the following conditions: for all $x,\ y,\ z \in X$,
(BPFQI_1) $A^P(0) \geq A^P(x),\ \ A^N(0) \leq A^N(x)$,
(BPFQI_2) $A^P(x * z) \geq A^P((x * y) * z) \wedge A^P(y)$,
$A^N(x * z) \leq A^N((x * y) * z) \vee A^N(y)$.
(ii) $\mathbf{A} = (\mathbf{A}^N, \mathbf{A}^P) \in IVBP(X)$ is called an interval-valued bipolar fuzzy Q-ideal of X, if it satisfies the following conditions: for all $x,\ y,\ z \in X$,
(IVBPQI_1) $\mathbf{A}^P(0) \geq \mathbf{A}^P(x),\ \ \mathbf{A}^N(0) \leq \mathbf{A}^N(x)$,
(IVBPQI_2) $\mathbf{A}^P(x * z) \geq \mathbf{A}^P((x * y) * z) \wedge \mathbf{A}^P(y)$,
$\mathbf{A}^N(x * z) \leq \mathbf{A}^N((x * y) * z) \vee \mathbf{A}^N(y)$.

Theorem 5.16. *$\mathcal{A}$ is a cubic bipolar Q-ideal of a Q-algebra X if and only if $\mathbf{A}$ is an interval-valued bipolar fuzzy Q-ideal and A is a bipolar fuzzy Q-ideal of X.*

Proof. Suppose $\mathbf{A}$ is an interval-valued bipolar fuzzy Q-ideal and A is a bipolar fuzzy Q-ideal of X and let $x \in X$. Then clearly, we have

$$\mathbf{A}^P(0) \geq \mathbf{A}^P(x),\ \ \mathbf{A}^N(0) \leq \mathbf{A}^N(x),\ \ A^P(0) \geq A^P(x),\ \ A^N(0) \leq A^N(x).$$

Now let $x,\ y,\ z \in X$. Since $\mathbf{A}$ is an interval-valued bipolar fuzzy Q-ideal, we have
$\mathbf{A}^P(x * z) \geq \mathbf{A}^P((x * y) * z) \wedge \mathbf{A}^P(y)$
$= [A^{P,-}((x * y) * z), A^{P,+}((x * y) * z) \wedge [A^{P,-}(y), A^{P,+}(y)]$
$= [A^{P,-}((x * y) * z) \wedge A^{P,-}(y), A^{P,+}((x * y) * z) \wedge A^{P,+}(y)]$
$= \mathbf{A}^P((x * y) * z) \wedge \mathbf{A}^P(y)$,
$\mathbf{A}^N(x * z) \leq \mathbf{A}^N((x * y) * z) \vee \mathbf{A}^N(y)$
$= [A^{N,-}((x * y) * z), A^{N,+}((x * y) * z) \vee [A^{N,-}(y), A^{N,+}(y)]$
$= [A^{N,-}((x * y) * z) \vee A^{N,-}(y), A^{N,+}((x * y) * z) \vee A^{N,+}(y)]$

$= \mathbf{A}^N((x*y)*z) \vee \mathbf{A}^N(y)$.

Since A is a bipolar fuzzy Q-ideal, we have

$$A^P(x*z) \geq A^P((x*y)*z) \wedge A^P(y),\ A^N(x*z) \leq A^N((x*y)*z) \vee A^N(y).$$

Thus $\mathcal{A} = \langle \mathbf{A}, A \rangle$ is a cubic bipolar Q-ideal.

Conversely, suppose $\mathcal{A}$ is a cubic bipolar Q-ideal of X and let $x \in X$. Then we have

$$\mathbf{A}^P(0) \geq \mathbf{A}^P(x),\ \mathbf{A}^N(0) \leq \mathbf{A}^N(x),\ A^P(0) \geq A^P(x),\ A^N(0) \leq A^N(x).$$

Now let $x,\ y,\ z \in X$. Since $\mathbf{A}$ is an interval-valued bipolar fuzzy Q-ideal, we have

$$\begin{aligned}\mathbf{A}^P(x*z) &\geq \mathbf{A}^P((x*y)*z) \wedge \mathbf{A}^P(y)\\ &= [A^{P,-}((x*y)*z), A^{P,+}((x*y)*z) \wedge [A^{P,-}(y), A^{P,+}(y)]\\ &= [A^{P,-}((x*y)*z) \wedge A^{P,-}(y), A^{P,+}((x*y)*z) \wedge A^{P,+}(y)],\\ \mathbf{A}^N(x*z) &\leq \mathbf{A}^N((x*y)*z) \vee \mathbf{A}^N(y)\\ &= [A^{N,-}((x*y)*z), A^{N,+}((x*y)*z) \vee [A^{N,-}(y), A^{N,+}(y)]\\ &= [A^{N,-}((x*y)*z) \vee A^{N,-}(y), A^{N,+}((x*y)*z) \vee A^{N,+}(y)].\end{aligned}$$

Since $\mathbf{A}^P(x*z) = ([A^{N,-}(x*z), A^{N,+}(x*z)], [A^{P,-}(x*z), A^{P,+}(x*z)])$, we have

$$[A^{P,-}(x*z), A^{P,+}(x*z)] \geq [A^{P,-}((x*y)*z) \wedge A^{P,-}(y), A^{P,+}((x*y)*z) \wedge A^{P,+}(y)], \tag{5.2}$$

$$[A^{N,-}(x*z), A^{N,+}(x*z)] \leq [A^{N,-}((x*y)*z) \vee A^{N,-}(y), A^{N,+}((x*y)*z) \vee A^{N,+}(y)]. \tag{5.3}$$

Then from (5.2) and (5.3), we have

$$A^{P,-}(x*z) \geq A^{P,-}((x*y)*z) \wedge A^{P,-}(y),\ A^{P,+}(x*z) \geq A^{P,+}((x*y)*z) \wedge A^{P,+}(y), \tag{5.4}$$

$$A^{N,-}(x*z) \leq A^{N,-}((x*y)*z) \vee A^{N,-}(y),\ A^{N,+}(x*z) \leq A^{N,+}((x*y)*z) \vee A^{N,+}(y). \tag{5.5}$$

Similarly, we have

$$A^P(x*z) \geq A^P((x*y)*z) \wedge A^P(y), \tag{5.6}$$

$$A^N(x*z) \leq A^N((x*y)*z) \wedge A^N(y). \tag{5.7}$$

Thus from (5.4) and (5.5), $\mathbf{A}$ is an interval-valued bipolar fuzzy Q-ideal of X and from (5.6) and (5.7), A is a bipolar fuzzy Q-ideal of X. This completes the proof. □

Proposition 5.17. *Let $(\mathcal{I}_j)_{j \in J}$ be a family cubic bipolar Q-ideals of a Q-algebra X. Then $\bigcap_{j \in J} \mathcal{I}_j$ is a cubic bipolar Q-ideal of X.*

Proof. The proof is straightforward. □

6. The image and the preimage of a cubic bipolar Q-ideal under a homomorphism of Q-algebras

Definition 6.1. Let $(X, *, 0)$ and $(Y, *^{'}, 0^{'})$ be two Q-algebras. Then a mapping $f : X \to Y$ is called a *homomorphism*, if $f(x*y) = f(x) *^{'} f(y)$ for any $x,\ y \in X$.

It is clear that $f(0) = 0^{'}$.

Proposition 6.2. *Let $f : X \to Y$ be a homomorphism of Q-algebras and let $\mathcal{B} \in CBP(Y)$. If $\mathcal{B}$ is a cubic bipolar Q-ideal of Y, then $f^{-1}(\mathcal{B})$ is a cubic bipolar Q-ideal of X.*

Proof. Suppose $\mathcal{B}$ is a cubic bipolar Q-ideal of Y and let $x \in X$. Then

$$\begin{aligned}
[f^{-1}(B^P)](x) &= B^P(f(x)) \\
&\leq B^P(0^{'}) \text{ [By Theorem 5.16 and Definition 5.15 (i)]} \\
&= B^P(f(0)) \text{ [Since } f \text{ is a homomorphism]} \\
&= [f^{-1}(B^P)](0), \\
[f^{-1}(\mathbf{B}^P)](x) &= \mathbf{B}^P(f(x)) \\
&\leq \mathbf{B}^P(0^{'}) \text{ [By Theorem 5.16 and Definition 5.15 (ii)]} \\
&= \mathbf{B}^P(f(0)) \text{ [Since } f \text{ is a homomorphism]} \\
&= [f^{-1}(\mathbf{B}^P)](0).
\end{aligned}$$

Similarly, we have $[f^{-1}(B^N)](x) \geq [f^{-1}(B^N)](0)$ and $[f^{-1}(\mathbf{B}^N)](x) \geq [f^{-1}(\mathbf{B}^N)](0)$. Thus the condition (CBPQI0) holds.

Now let $x,\ y,\ z \in X$. Then

$$\begin{aligned}
[f^{-1}(B^P)](x * z) &= B^P(f(x * z)) \\
&= B^P(f(x) *^{'} f(z)) \text{ [Since } f \text{ is a homomorphism]} \\
&\geq B^P((f(x) *^{'} f(y)) *^{'} f(z)) \wedge B^P(f(y)) \\
&\quad \text{[By Theorem 5.16 and Definition 5.15 (i)]} \\
&= B^P((f(x * y) * z)) \wedge B^P(f(y)) \\
&\quad \text{[Since } f \text{ is a homomorphism]} \\
&= [f^{-1}(B^P)]((x * y) * z) \wedge [f^{-1}(B^P)](y), \\
[f^{-1}(\mathbf{B}^P)](x * z) &= \mathbf{B}^P(f(x * z)) \\
&= \mathbf{B}^P(f(x) *^{'} f(z)) \text{ [Since } f \text{ is a homomorphism]} \\
&\geq \mathbf{B}^P((f(x) *^{'} f(y)) *^{'} f(z)) \wedge \mathbf{B}^P(f(y)) \\
&\quad \text{[By Theorem 5.16 and Definition 5.15 (ii)]} \\
&= [f^{-1}(\mathbf{B}^P)]((x * y) * z) \wedge [f^{-1}(\mathbf{B}^P)](y). \\
&\quad \text{[Since } f \text{ is a homomorphism]}
\end{aligned}$$

Similarly, we have the following inequalities:

$$[f^{-1}(B^N)](x * z) \leq [f^{-1}(B^N)]((x * y) * z) \vee [f^{-1}(B^N)](y),$$

$$[f^{-1}(\mathbf{B}^N)](x * z) \leq [f^{-1}(\mathbf{B}^N)]((x * y) * z) \vee [f^{-1}(\mathbf{B}^N)](y).$$

Thus the conditions (CBPQI1) and (CBPQI2) hold. So $f^{-1}(\mathcal{B})$ is a cubic bipolar Q-ideal of X. □

Proposition 6.3. *Let $f : X \to Y$ be an epimomorphism of Q-algebras and let $\mathcal{B} \in CBP(Y)$. If $f^{-1}(\mathcal{B})$ is a cubic bipolar Q-ideal of X, $\mathcal{B}$ is a cubic bipolar Q-ideal of Y.*

Proof. Suppose $f^{-1}(\mathcal{B})$ is a cubic bipolar Q-ideal of X and let $a \in Y$. Since f is surjective, there is $x \in X$ such that $f(x) = a$. Then

$$\begin{aligned}
B^P(a) &= B^P(f(x)) = [f^{-1}(B^P)](x) \\
&\leq [f^{-1}(B^P)](0) \text{ [By Theorem 5.16 and Definition 5.15 (i)]} \\
&= B^P(f(0)) = B^P(0^{'}), \text{ [Since } f \text{ is a homomorphism]} \\
\mathbf{B}^P(a) &= \mathbf{B}^P(f(x)) = [f^{-1}(\mathbf{B}^P)](x) \\
&\leq [f^{-1}(\mathbf{B}^P)](0) \text{ [By Theorem 5.16 and Definition 5.15 (ii)]}
\end{aligned}$$

$= \mathbf{B}^P(f(0)) = \mathbf{B}^P(0^{'})$.
Similarly, we have $B^N(b) \geq B^N(0^{'})$, $\mathbf{B}^N(b) \geq \mathbf{B}^N(0^{'})$. Thus the condition (CBPQI0) holds.

Now let $a,\ b,\ c \in Y$. Then clearly, there are $x,\ y,\ z \in X$ such that

$$f(x) = a,\ f(y) = b,\ f(z) = c.$$

Thus

$$\begin{aligned}
B^P(a *^{'} c) &= B^P(f(x * z)) \text{ [Since } f \text{ is a homomorphism]}\\
&= [f^{-1}(B^P)](x * z)\\
&\geq [f^{-1}(B^P)]((x * y) * z) \wedge [f^{-1}(B^P)](y)\\
&\quad \text{[By Theorem 5.16 and Definition 5.15 (i)]}\\
&= B^P(((f(x) *^{'} f(y)) *^{'} f(z)) \wedge B^P(f(y))\\
&\quad \text{[Since } f \text{ is a homomorphism]}\\
&= B^P((a *^{'} b) *^{'} c) \wedge B^P(b),\\
\mathbf{B}^P(a *^{'} c) &= \mathbf{B}^P(f(x * z))\\
&= [f^{-1}(\mathbf{B}^P)](x * z)\\
&\geq [f^{-1}(\mathbf{B}^P)]((x * y) * z) \wedge [f^{-1}(\mathbf{B}^P)](y)\\
&\quad \text{[By Theorem 5.16 and Definition 5.15 (ii)]}\\
&= \mathbf{B}^P(((f(x) *^{'} f(y)) *^{'} f(z)) \wedge \mathbf{B}^P(f(y))\\
&\quad \text{[Since } f \text{ is a homomorphism]}\\
&= \mathbf{B}^P((a *^{'} b) *^{'} c) \wedge \mathbf{B}^P(b).
\end{aligned}$$

Similarly, we have the following inequalities:

$$B^N(a *^{'} c) \leq B^N((a *^{'} b) *^{'} c) \vee B^N(b),$$

$$\mathbf{B}^N(a *^{'} c) \leq \mathbf{B}^N((a *^{'} b) *^{'} c) \vee \mathbf{B}^N(b).$$

Thus the conditions (CBPQI1) and (CBPQI2) hold. So $\mathcal{B}$ is a cubic bipolar Q-ideal of Y. □

7. Cubic bipolar topological spaces

Definition 7.1. Let X be a set and let τ be a family of cubic bipolar sets in X. Then τ is called a *cubic bipolar topology* (briefly, CBPT) on X, if it satisfies the following axioms:

(CBPO1) $0^X,\ 1^X \in \tau$,
(CBPO2) $\mathcal{A} \cap \mathcal{B} \in \tau$ for any $\mathcal{A},\ \mathcal{B} \in \tau$,
(CBPO3) $\bigcup_{j \in J} \mathcal{A}_j \in \tau$,

where $(\mathcal{A}_j)_{j \in J}$ is a family of members of τ and J denotes an index set.

The pair (X, τ) is called a *cubic bipolar topological space* (briefly, CBPTS) and each member of τ is called a *cubic bipolar open set* (briefly, CBPOS) in X. We will denote the set of all CBPTs on X as $CBPT(X)$ and the set of all CBPOSs in X as $CBPO(X)$.

A cubic bipolar set $\mathcal{A}$ in X is called a *cubic bipolar closed set* (briefly, CBPCS) in X, if $\mathcal{A}^c \in CBPO(X)$. We will denote the set of all CBPCSs in X as $CBPC(X)$.

Example 7.2. (1) Let $X = \{a, b.c\}$ and two cubic bipolar sets $\mathcal{A},\ \mathcal{B}$ in X given by:

$$\mathcal{A}(a) \left\langle ([-0.40, -0.20], [0.50, 0.60]), (-0.62, 0.38) \right\rangle,$$

$$\mathcal{A}(b)\,\langle([-0.50,-0.40],[0.40,0.50]),(-0.61,0.53)\rangle\,,$$
$$\mathcal{A}(c)\,\langle([-0.60,-0.50],[0.20,0.30]),(-0.47,0.25)\rangle\,,$$
$$\mathcal{B}(a)\,\langle([-0.30,-0.10],[0.20,0.70]),(-0.51,0.43)\rangle\,,$$
$$\mathcal{B}(b)\,\langle([-0.50,-0.40],[0.30,0.50]),(-0.70,0.40)\rangle\,,$$
$$\mathcal{B}(c)\,\langle([-0.50,-0.50],[0.10,0.40]),(-0.47,0.25)\rangle\,.$$

Consider the family $\tau=\{0^X,1^X,\mathcal{A},\mathcal{B},\mathcal{A}\cap\mathcal{B},\mathcal{A}\cup\mathcal{B}\}$. Then we can easily check that $\tau\in CBPT(X)$.

(2) Let $\mathcal{A}\in CBP(X)$ and let $\tau\in CBPT(X)$. Consider the family $\tau_{\mathcal{A}}$ cubic bipolar sets in X given by:

$$\tau_{\mathcal{A}}=\{\mathcal{A}\cap\mathcal{U}:\mathcal{U}\in\tau\}.$$

Then we can easily see that $\tau_{\mathcal{A}}$ is a cubic bipolar topology on $\mathcal{A}$. In this case, $\tau_{\mathcal{A}}$ is called the induced cubic bipolar topology on $\mathcal{A}$ by τ and the pair $(\mathcal{A},\tau_{\mathcal{A}})$ is called a cubic bipolar subspace of (X,τ).

(3) For a non-empty set X, $\{0^X,1^X\}$ and $CBP(X)$ are CBPTs on X. In this case, $\{0^X,1^X\}$ [resp. $CBP(X)$] is called the cubic bipolar indiscrete [resp. discrete] topology on X and we will denote $\{0^X,1^X\}$ $\{0^X,1^X\}$ [resp. $CBP(X)$] as τ^0 [resp. τ^1].

Let X be a non-empty set, let A be a fuzzy set in X and let τ be a fuzzy topology on X. Then it is well-known [31] that $\tau_A=\{A\cap U:U\in\tau\}$ is a fuzzy topology (called the induced fuzzy topology by τ) on A. In this case, the pair (A,τ_A) is called a fuzzy subspace of (X,τ).

Let (X,τ) and (y,σ) be two fuzzy topological spaces. Then a mapping $f:X\to Y$ is called a fuzzy continuous [10], if $f^{-1}(V)\in\tau$ for each $V\in\sigma$.

Proposition 7.3. *Let X be a non-empty set and let $(\tau_j)_{j\in J}\subset CBPT(X)$. Then $\bigcap_{j\in J}\tau_j\in CBPT(X)$.*

Proof. The proof is straightforward. □

Remark 7.4. From Proposition 7.3, we can easily see that $CBPT(X)$ forms a meet complete lattice with respect to the set inclusion relation of which τ^0 is the smallest element and τ^1 is the largest element.

Proposition 7.5. *Let (X,τ) be a CBPTS. Then we have the followings:*
(i) $0^X,\ 1^X\in CBPC(X)$,
(ii) $\mathcal{A}\cup\mathcal{B}\in CBPC(X)$ for any $\mathcal{A},\ \mathcal{B}\in CBPC(X)$,
(iii) $\bigcap_{j\in J}\mathcal{A}_j\in CBPC(X)$ for each $(\mathcal{A}_j)_{j\in J}\subset CBPC(X)$.

Proof. The proof is straightforward. □

Definition 7.6. Let (X,τ) be a CBPTS, let B [resp. S] is a subfamily of τ.

(i) δ is called a *base* for τ, if every member of τ can be expressed as a union of members of δ.

(ii) η is called a *subbase* for τ, if the family of all finite intersections of members of η forms a base for τ.

Proposition 7.7. *Let X be a non-empty set and let $\delta \subset CBP(X)$ such that $0^X,\ 1^X \in \delta$. Suppose for any $\mathcal{A}_1,\ \mathcal{A}_2 \in \delta$ and for each $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{A}_1 \cap \mathcal{A}_2$, there is $\mathcal{A} \in \delta$ such that $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{A} \subset \mathcal{A}_1 \cap \mathcal{A}_2$. Then δ is a base for some CBPT on X.*

Proof. Let τ be the collection of all unions of members of δ.

(CBPO1) It is clear that $0^X,\ 1^X \in \tau$.

(CBPO3) The proof is obvious.

(CBPO2) Let $\mathcal{U},\ \mathcal{V} \in \tau$. Then clearly there are $(\mathcal{A}_j)_{j \in J},\ (\mathcal{B}_k)_{k \in K} \subset \delta$ such that

$$\mathcal{U} = \bigcup_{j \in J} \mathcal{A}_j \text{ and } \mathcal{V} = \bigcup_{k \in K} \mathcal{B}_k.$$

Thus by Theorem 3.5 (4), we have

$$\mathcal{U} \cap \mathcal{V} = \bigcup_{(j,k) \in J \times K} (\mathcal{A}_j \cap \mathcal{B}_k). \tag{7.1}$$

Now let $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{A}_j \cap \mathcal{B}_k$. Then by the hypothesis, there is $\mathcal{W} \in \delta$ such that

$$x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{W} \subset \mathcal{A}_j \cap \mathcal{B}_k. \tag{7.2}$$

By Lemma 4.2, $\mathcal{A}_j \cap \mathcal{B}_k = \bigcup \{x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} : x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{A}_j \cap \mathcal{B}_k\}$. Thus by (7.2), $\mathcal{U} \cap \mathcal{V}$ is expressed as a union of members of δ. So $\mathcal{U} \cap \mathcal{V} \in \tau$. Hence τ is a CBPT on X for which δ is a base. □

Proposition 7.8. *Let (X, τ) be a CBPTS and let $\mathcal{A} \int CBP(X)$. Suppose for each $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{A}$, there is $\mathcal{U}_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \tau$ such that $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{U}_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \subset \mathcal{A}$. Then $\mathcal{A} \in \tau$.*

Proof. Suppose the sufficient condition holds. Then by Lemma 4.2, we have

$$\mathcal{A} = \bigcup_{x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{A}} \mathcal{U}_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle}.$$

Thus by Definition 7.1 (CBPO3), $\mathcal{A} \in \tau$. □

Definition 7.9. Let (X, τ) be a CBPTS, let $\mathcal{A} \int CBP(X)$ and let $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in CBP_P(X)$. Then $\mathcal{A}$ is called a *cubic bipolar neighborhood* (briefly, CBPN) of $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle}$, if there is $\mathcal{U} \in \tau$ such that $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{U} \subset \mathcal{A}$.

We will denote the set of all CBPNs of $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle}$ as $CBPN(x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle})$.

Theorem 7.10. *Let (X, τ) be a CBPTS and let $\mathcal{A} \in CBP(X)$. Then $\mathcal{A} \in \tau$ if and only if for each $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{A}$, there is $\mathcal{U} \in CBPN(x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle})$ such that $\mathcal{U} \subset \mathcal{A}$.*

Proof. Suppose $\mathcal{A} \in \tau$ and let $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{A}$. Then clearly, we have $\mathcal{A} \in CBPN(x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle})$. Thus the necessary condition holds.

Conversely, suppose the necessary condition holds and let $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{A}$. Then there is $\mathcal{U}_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \tau$ such that $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in \mathcal{U}_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \subset \mathcal{A}$. Thus by Proposition 7.8, $\mathcal{A} \in \tau$. □

Definition 7.11. Let $(X, \tau_1),\ (Y, \tau_2)$ be two CBPTSs. Then a mapping $f : X \to Y$ is said to be *continuous*, if $f^{-1}(\mathcal{V}) \in \tau_1$ for each $\mathcal{V} \in \tau_2$.

The following is an immediate result of the above definition.

Proposition 7.12. *The identity mapping $I_X : (X, \tau) \to (X, \tau)$ is continuous.*

Proposition 7.13. *If the mappings $f : (X, \tau_1) \to (Y, \tau_2)$ and $g : (Y, \tau_2) \to (Z, \tau_3)$ are continuous, then $g \circ f : (X, \tau_1) \to (Z, \tau_3)$ is continuous.*

Proof. From Proposition 3.7 (10) and Definition 7.11, the proof is obvious. □

Remark 7.14. (1) Let **CBPTop** be the collection of all CBPTSs and continuous mappings. Then we can easily see that **CBPTop** forms a concrete category, from Propositions 7.12 and 7.13.

(2) From Definitions 3.6 and 4.1, it is obvious that for a mapping $f : X \to Y$ and each $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in CBP_P(X)$, $f(x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle}) \in CBP_P(Y)$.

Definition 7.15. Let (X, τ_1), (Y, τ_2) be two CBPTSs. Then a mapping $f : X \to Y$ is said to be continuous at $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in CBP_P(X)$, if $f^{-1}(\mathcal{V}) \in CBPN(x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle})$ for each $\mathcal{V} \in CBPN(f(x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle}))$.

Theorem 7.16. *A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is continuous if and only if f is continuous at each $x_{\langle \widetilde{\mathbf{a}}, \bar{a} \rangle} \in CBP_P(X)$.*

Proof. The proof is easy. □

Definition 7.17 ([31], Proposition 3.1)**.** Let (A, τ_A), (B, δ_B) be fuzzy subspaces of fts's (X, τ), (Y, δ), respectively and let f be a mapping of (A, τ_A) into (B, δ_B).

(i) f is said to be *relatively fuzzy continuous*, if for each open fuzzy set V in δ_B, $f^{-1}(V) \cap A \in \tau_A$.

(ii) f is said to be *relatively fuzzy open*, if for each open fuzzy set U in τ_B, $f(U) \in \delta_B$.

Result 7.18 ([31])**.** *Let (X, τ) and (Y, δ) be two fuzzy topological spaces, let A [resp. B] be a fuzzy set in X [resp. Y] and let (A, τ_A) [resp. (B, δ_B)] be a fuzzy subspace of (X, τ) [resp. (Y, σ)]. Suppose $f : X \to Y$ is fuzzy continuous such that $f(A) \subset B$. Then f is a relatively continuous mapping of (A, τ_A) into (B, δ_B).*

Lemma 7.19. *Let (X, τ) be a cubic bipolar topological space and let $\mathcal{A} \in CBP(X)$. Then the family of cubic bipolar sets $\tau_{\mathcal{A}}$ in X given by:*

$$\tau_{\mathcal{A}} = \{\mathcal{U} \cap \mathcal{A} : \mathcal{U} \in \tau\}$$

is a cubic bipolar topology on $\mathcal{A}$.

In this case, the pair $(\mathcal{A}, \tau_{\mathcal{A}})$ will be called a cubic bipolar subspace of (X, τ).

Proof. The proof is obvious from Theorem 3.5 and Definition 7.1. □

Lemma 7.20. *Let X, Y be two Q-algebras and let $f : X \to Y$ be a Q-homomorphism. If $\mathcal{B}$ is a subalgebra of Y, then $f^{-1}(\mathcal{B})$ is a subalgebra of X.*

Proof. Suppose $\mathcal{B}$ is a subalgebra of Y and let $x,\ y \in X$. Then

$$\begin{aligned}
[f^{-1}(\mathbf{A}^P)](x * y) &= \mathbf{A}^P(f(x * y)) \\
&= \mathbf{A}^P(f(x) * f(y)) \text{ [Since } f \text{ is a homomorphism]} \\
&\geq \mathbf{A}^P(f(x)) \wedge \mathbf{A}^P(f(y)) \text{ [Since } \mathcal{B} \in SA(Y)] \\
&= [f^{-1}(\mathbf{A}^P)](x) \wedge [f^{-1}(\mathbf{A}^P)](y).
\end{aligned}$$

Similarly, we have $[f^{-1}(\mathbf{A}^N)](x * y) \leq [f^{-1}(\mathbf{A}^N)](x) \vee [f^{-1}(\mathbf{A}^N)](y)$. Also, we have

$$[f^{-1}(A^P)](x * y) \geq [f^{-1}(A^P)](x) \wedge [f^{-1}(A^P)](y),$$

$$[f^{-1}(A^N)](x * y) \leq [f^{-1}(A^N)](x) \vee [f^{-1}(A^P)](y).$$

Thus $f^{-1}(\mathcal{B})$ is a subalgebra of X. □

Definition 7.21. Let (A, τ_A), (B, δ_B) be cubic bipolar subspaces of cubic bipolar topological spaces (X, τ) and (Y, δ), respectively and let f be a mapping of (A, τ_A) into (B, δ_B). Then

(i) f is said to be *relatively cubic bipolar continuous*, if for each cubic bipolar open set V in δ_B, $f^{-1}(V) \cap A \in \tau_A$.

(ii) f is said to be *relatively cubic bipolar open*, if for each cubic bipolar open set U in τ_B, $f(U) \in \delta_B$.

From Lemmas 7.19 and 7.20, and Definition 7.21, we get a similar property to Result 7.18.

Proposition 7.22. *Let X, Y be two Q-algebras and let $f : X \to Y$ be a Q-homomorphism. Let (X, τ) and (Y, δ) be two cubic bipolar topological spaces, let $\mathcal{A}$ [resp. $\mathcal{B}$] be a cubic bipolar subalgebra of X [resp. Y] and let $(\mathcal{A}, \tau_{\mathcal{A}})$ [resp. (Y, δ)]. Suppose $f : X \to Y$ is cubic bipolar continuous such that $f(\mathcal{A}) \subset \mathcal{B}$. Then f is a relatively cubic bipolar continuous mapping of $(\mathcal{A}, \tau_{\mathcal{A}})$ into $(\mathcal{B}, \delta_{\mathcal{B}})$.*

8. Conclusions

We dealt with basic properties of cubic bipolar sets and cubic bipolar points, cubic bipolar base and fore each concept obtained some of its properties. Also we applied the concept of cubic bipolar sets to Q-algebra. Moreover, we discussed with the image and the preimage of a cubic bipolar set under Q-homorphism. Finally, we defined a cubic bipolar topology and a continuity, and dealt with some of their properties.

In the future, we expect that one will study the octahedron bipolar foldedness, octahedron bipolar topology and octahedron m-polar topology. Furthermore, in our future study of fuzzy structures of $BCK/BCI/KU$-algebras, the following topics should be considered:

(i) cubic bipolar positive implicative ideals in a $BCK/BCI/KU$-algebra and its application to topology,

(ii) cubic bipolar commutative ideals in a $BCK/BCI/KU$-algebra and its application to topology,

(iii) the relationship between cubic bipolar implicative ideals, positive implicative ideals and commutative ideals in a $BCK/BCI/KU$-algebra and its application to topology,

(IV) cubic bipolar ideals in a semigroup and its application to topology.

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