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Abstract. In this paper, we define an interval-valued set and an interval-valued (vanishing) point, and study some of their properties. In particular, we get the characterization of inclusions, intersections and unions of interval-valued sets. Also, we deal with some properties of the images and the preimages under a mapping. Moreover, we introduce the concept of interval-valued ideals and some of its properties. Next, we define an interval-valued topology, an interval-valued base [rep. subbase] and an interval-valued neighborhood, and find their various properties. Finally, we define an Interval-valued closure [resp. interior] and obtain some of each properties. Moreover, we show that that there is a unique IVT for interval-valued interior [resp. closure] operators.

2010 AMS Classification: 54A40

Keywords: Interval-valued set, Interval-valued (vanishing) point, Interval-valued ideal, Interval-valued topological space, Interval-valued base [resp. subbase], Interval-valued neighborhood, Interval-valued closure [resp. interior].

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1. Introduction

In 1983, Atanassove [1] define an intuitionisic fuzzy set as the generalization of fuzzy sets (the generalization of classical sets) introduced by Zadeh [18]. In 1996, Çoker [5] introduced the notion of intuitionistic sets as the generalization of classical sets and the special case of intuitionistic fuzzy sets. After that time, many researchers [2, 3, 4, 6, 7, 8, 15, 16] applied it to topologies. Lee and Chu [13] formed the category **ITop** and studied some relationships between **ITop** and **Top**. Recently, Kim et al. [10] investigated the category **ISet** composed of intuitionistic sets and morphisms between them in the sense of a topological universe. Also, Kim et al. [11] defined an intuitionistic topology, intuitionistic interior and closure, and studied some of their properties. Furthermore, Kim et al. [12] discussed intuitinistic hyperspaces based on intuitionistic topological spaces.

Yao [17] defined an interval set (which will called an interval-valued set by us) giving a tool for approximating undefinable or complex concepts and discussed their algebraic structures. We can easily see that this concept is the generalization of classical sets and the special case of interval-valued fuzzy sets proposed by Zadeh [19]. One motivation for the introduction of these sets is to provide a point-set based setting for the classical sets and for these sets to apply to topology. In order to accomplish our research, this paper is composed of six sections. In Section 2, we recall some definitions of intuitionistic sets introduced by Çoker [5]. In Section 3, we list some definitions and results for algebraic structures of interval-valued set who Yao has dealt with. Also, we define interval-valued points of two types and discuss with the characterizations of inclusions, intersections and unions of interval-valued sets. Furthermore, we introduce the concept of interval-valued ideals and obtain some of its properties. In Section 4, we define an interval-valued topology, an interval-valued base and subbase, and investigate some of their properties. In Section 5, we introduce the notions of interval-valued neighborhoods of two types and find some of their properties. In particular, we show that there is an IVT under the hypothesis satisfying some properties of interval-valued neighborhoods. In Section 6, We define interval-valued interiors and closures an obtain some of their properties. Also, we prove that there is a unique IVT for interval-valued interior [resp. closure] operators.

2. Preliminaries

In this section, we recall the concepts of an intuitionistic set, an intuitionistic point, an intuitionistic vanishing point and intuitionistic topological space.

Definition 2.1 ([5])**.** Let X be a non-empty set. Then A is called an intuitionistic set (briefly, IS) of X, if it is an object having the form

$$A = (A^{\in}, A^{\not\in}),$$

such that $A^{\in} \cap A^{\not\in} = \varnothing$, where $A^{\in}$ [resp. $A^{\not\in}$] represents the set of memberships [resp. non-memberships] of elements of X to A. In fact, $A^{\in}$ [resp. $A^{\not\in}$] is a subset of X agreeing or approving [resp. refusing or opposing] for a certain opinion, view, suggestion or policy. The intuitionistic empty set [resp. the intuitionistic whole set] of X, denoted by $\bar{\varnothing}$ [resp. $\bar{X}$], is defined by $\bar{\varnothing} = (\varnothing, X)$ [resp. $\bar{X} = (X, \varnothing)$].

We will denote the set of all ISs of X as $IS(X)$. The inclusion, the equality, the intersection and the union of ISs, the complement of an IS, and the operations intersection [ ] and $<>$ on $IS(X)$ refer to [5].

It is obvious that $A = (A, \phi) \in IS(X)$ for each ordinary subset A of X. Then we can consider an IS of X as the generalization of an ordinary subset of X.

Remark 2.2. Let X be a set and let $A \in IS(X)$. We define the mappings $\mu,\ \nu : X \to [0, 1]$ as follows: for each $x \in X$,

$$\mu(x) = \chi_{_{A^{\in}}}(x),\ \nu(x) = \chi_{_{A^{\not\in}}}(x).$$

Then we can easily see that (μ, ν) is an intuitionistic fuzzy set in X introduced by Atanassov [1]. Thus by identifying A with (μ, ν), we can consider an intuitionistic set A in X as the specilization of an intuitionistic fuzzy set in X.

Definition 2.3 ([5]). Let X be a non-empty set, $a \in X$ and let $A \in IS(X)$.

(i) The form $(\{a\}, \{a\}^c)$ [resp. $(\phi, \{a\}^c)$]is called an intuitionistic point [resp. vanishing point] of X and denoted by $a_{_I}$ [resp. $a_{_{IV}}$].

(ii) We say that $a_{_I}$ [resp. $a_{_{IV}}$] is contained in A, denoted by $a_{_I} \in A$ [resp. $a_{_{IV}} \in A$], if $a \in A^{\in}$ [resp. $a \notin A^{\notin}$].

It is clear that for each $A \in IS(X)$ and each $x \in X$, $x_{_I} \in A \Leftrightarrow x_{_I} \subset A$ and $x_{_{IV}} \in A \Leftrightarrow x_{_{IV}} \subset A$.

We will denote the set of all intuitionistic points and intuitionistic vanishing points in X as $I_P(X)$.

Definition 2.4 ([6, 11]). Let X be a non-empty set and let $\tau \subset IS(X)$. Then τ is called an intuitionistic topology (in short IT) on X, if it satisfies the following axioms:

(i) $\bar{\varnothing}, \bar{X} \in \tau$,

(ii) $A \cap B \in \tau$, for any $A, B \in \tau$,

(iii) $\bigcup_{j \in J} A_j \in \tau$, for each $(A_j)_{j \in J} \subset \tau$.

In this case, the pair (X, τ) is called an intuitionistic topological space (in short, ITS) and each member O of τ is called an intuitionistic open set (in short, IOS) in X. An IS F of X is called an intuitionistic closed set (in short, ICS) in X, if $F^c \in \tau$.

It is obvious that $\{\bar{\varnothing}, \bar{X}\}$ is the smallest IT on X and will be called the intuitionistic indiscreet topology and denoted by $\tau_{I,0}$. Also $IS(X)$ is the greatest IT on X and will be called the intuitionistic discreet topology and denoted by $\tau_{I,1}$. The pair $(X, \tau_{I,0})$ [resp. $(X, \tau_{I,1})$] will be called the intuitionistic indiscreet [resp. discreet] space.

We will denote the set of all ITs on X as $IT(X)$. For an ITS X, we will denote the set of all IOSs [resp. ICSs] on X as $IO(X)$ [resp. $IC(X)$].

3. Interval-valued sets

In this section, we list some definitions and some results interval-valued sets who Yao has proposed and has obtained. Next, We define interval-valued points of two types and obtain the characterizations of inclusions, intersections and unions of interval-valued sets. Also, we deal with properties for the images and preimages under a mapping. finally, we introduce the concept of interval-valued ideals and study some of their properties.

Definition 3.1 (See [17]). Let X be an non-empty set. Then the form

$$[A^-, A^+] = \{B \subset X : A^- \subset B \subset A^+\}$$

is called an interval-valued set (briefly, IVS) or interval set in X, if A^-, $A^+ \subset X$ and $A^- \subset A^+$. In this case, A^- [resp. A^+] represents the set of minimum [resp. maximum] memberships of elements of X to A. In fact, A^- [resp. A^+] is a minimum [resp. maximum] subset of X agreeing or approving for a certain opinion, view, suggestion or policy. $[\varnothing, \varnothing]$ [resp. $[X, X]$] is called the interval-valued empty [resp. whole] set in X and denoted by $\widetilde{\varnothing}$ [resp. $\widetilde{X}$]. We will denote the set of all IVSs in X as $IVS(X)$.

It is obvious that $[A, A] \in IVS(X)$ for classical subset A of X. Then we can consider an IVS in X as the generalization of a classical subset of X. Furthermore, if $A = [A^-, A^+] \in IVS(X)$, then

$$\chi_A = [\chi_{A^-}, \chi_{A^+}]$$

is an interval-valued fuzzy set in X introduced by Zadeh [19]. Thus we can consider an interval-valued fuzzy set as the generalization of an IVS.

Example 3.2. Let $X = \{a, b, c\}$. Then

$$\begin{aligned} IVS(X) = \{&\widetilde{\varnothing}, [\varnothing, \{a\}], [\varnothing, \{b\}], [\varnothing, \{c\}], [\varnothing, \{a, b\}], [\varnothing, \{b, c\}], [\varnothing, \{a, c\}], [\varnothing, X], \\ &[\{a\}, \{a\}], [\{a\}, \{a, b\}], [\{a\}, \{a, c\}], [\{a\}, X], , [\{b\}, \{a, b\}], [\{b\}, \{b, c\}], \\ &[\{b\}, X], [\{c\}, \{a, c\}], [\{c\}, \{b, c\}], [\{c\}, X], [\{a, b\}, \{a, b\}], [\{a, b\}, X], \\ &[\{a, c\}, \{a, c\}], [\{a, c\}, X], [\{b, c\}, \{b, c\}], [\{b, c\}, X], \widetilde{X}\}. \end{aligned}$$

Definition 3.3 (See [17])**.** Let X be a non-empty set and let $A,\ B \in IVS(X)$. Then

(i) we say that A contained in B, denoted by $A \subset B$, if $A^- \subset B^-$ and $A^+ \subset B^+$,
(ii) we say that A equal to B, denoted by $A = B$, if $A \subset B$ and $B \subset A$,
(iii) the complement of A, denoted A^c, is an interval-valued set in X defined by:

$$A^c = [(A^+)^c, (A^-)^c],$$

(iv) the union of A and B, denoted by $A \cup B$, is an interval-valued set in X defined by:

$$A \cup B = [A^- \cup B^-, A^+ \cup B^+],$$

(v) the intersection of A and B, denoted by $A \cap B$, is an interval-valued set in X defined by:

$$A \cap B = [A^- \cap B^-, A^+ \cap B^+].$$

Example 3.4. Let $X = \{a, b, c\}$. Consider $A = [\{a\}, \{a, b\}],\ B = [\{b\}, \{b, c\}] \in IVS(X)$. Then clearly we have

$$A^c = [\{c\}, \{b, c\}],\ A \cup B = [\{a, b\}, X],\ A \cap B = [\varnothing, \{b\}].$$

The followings are (i1), (i2), (i3), (k1), (k2) and (k3) in [17].

Result 3.5. *Let X be a non-empty set and let $A,\ B,\ C \in IVS(X)$. Then*

(1) $\widetilde{\varnothing} \subset A \subset \widetilde{X}$,
(2) *if* $A \subset B$ *and* $B \subset C$, *then* $A \subset C$,
(3) $A \subset A \cup B$ *and* $B \subset A \cup B$,
(4) $A \cap B \subset A$ *and* $A \cap B \subset B$,
(5) $A \subset B$ *if and only if* $A \cap B = A$,
(6) $A \subset B$ *if and only if* $A \cup B = B$.

The followings are (I1)–(I8) in [17].

Result 3.6. *Let X be a non-empty set and let $A,\ B,\ C \in IVS(X)$. Then*

(1) (Idempotent laws) $A \cup A = A,\ A \cap A = A$,
(2) (Commutative laws) $A \cup B = B \cup A,\ A \cap B = B \cap A$,
(3) (Associative laws) $A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C) = (A \cap B) \cap C$,
(4) (Distributive laws) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(5) (Absorption laws) $A \cup (A \cap B) = A,\ A \cap (A \cup B) = A$,
(6) (DeMorgan's laws) $(A \cup B)^c = A^c \cap B^c,\ (A \cap B)^c = A^c \cup B^c$,
(7) $(A^c)^c = A$,
(8) (8_a) $A \cup \widetilde{\varnothing} = A,\ A \cap \widetilde{\varnothing} = \widetilde{\varnothing}$,
(8_b) $A \cup \widetilde{X} = \widetilde{X},\ A \cap \widetilde{X} = A$,
(8_c) $\widetilde{X}^c = \widetilde{\varnothing},\ \widetilde{\varnothing}^c = \widetilde{X}$,
(8_d) $A \cup A^c \neq \widetilde{X},\ A \cap A^c \neq \widetilde{\varnothing}$ *in general (See Example 3.7).*

Example 3.7. Let $X = \{a, b, c\}$. Consider $A = [\{a\}, \{a, b\}] \in IVS(X)$. Then clearly, $A^c = [\{c\}, \{b, c\}]$. Thus $A \cap A^c = [\varnothing, \{b\}] \neq \widetilde{\varnothing}$ and $A \cup A^c = [\{a, c\}, X] \neq \widetilde{X}$.

Definition 3.8. Let $(A_j)_{j\in J}$ be a family of members of $IVS(X)$. Then

(i) the intersection of $(A_j)_{j\in J}$, denoted by $\bigcap_{j\in J} A_j$, is an IVS in X defined by:

$$\bigcap_{j\in J} A_j = [\bigcap_{j\in J} A_j^-, \bigcap_{j\in J} A_j^+],$$

(ii) the union of $(A_j)_{j\in J}$, denoted by $\bigcup_{j\in J} \widetilde{A}_j$, is an IVS in X defined by:

$$\bigcup_{j\in J} A_j = [\bigcup_{j\in J} A_j^-, \bigcup_{j\in J} A_j^+].$$

The following is the immediate result of Definition 3.8.

Proposition 3.9. *Let $A \in [X]$ and let $(A_j)_{j\in J}$ be a family of members of $IVS(X)$. Then*

(1) $(\bigcap_{j\in J} A_j)^c = \bigcup_{j\in J} A_j^c,\ (\bigcup_{j\in J} A_j)^c = \bigcap_{j\in J} A_j^c$,
(2) $A \cap (\bigcup_{j\in J} A_j) = \bigcup_{j\in J}(A \cap A_j),\ A \cup (\bigcap_{j\in J} A_j) = \bigcap_{j\in J}(A \cup A_j)$.

Definition 3.10. Let X be a non-empty set, let $a \in X$ and let $A \in IVS(X)$. Then the form $[\{a\}, \{a\}]$ [resp. $[\varnothing, \{a\}]$] is called an interval-valued [resp. vanishing] point in X and denoted by a_{IVP} [resp. a_{IVVP}]. We will denote the set of all interval-valued points in X as $IV_P(X)$.

(i) We say that a_{IVP} belongs to A, denoted by $a_{IVP} \in A$, if $a \in A^-$.
(ii) We say that a_{IVVP} belongs to A, denoted by $a_{IVVP} \in A$, if $a \in A^+$.

Proposition 3.11. *Let X be a non-empty set and let $A \in IVS(X)$. Then*

$$A = A_{IVP} \cup A_{IVVP},$$

where $A_{IVP} = \bigcup_{a_{IVP}\in A} a_{IVP}$ and $A_{IVVP} = \bigcup_{a_{IVVP}\in A} a_{IVVP}$.

In fact, $A_{IVP} = [A^-, A^-]$ and $A_{IVVP} = [\varnothing, A^+]$

Proof. From Definition 3.10 and the definitions of A_{IVP} and A_{IVVP}, we have

$$A_{IVP} = [\bigcup_{a_{IVP}\in A} \{a\}, \bigcup_{a_{IVP}\in A} \{a\}] \text{ and } A_{IVVP} = [\varnothing, \bigcup_{a_{IVVP}\in A} \{a\}].$$

Thus $A_{_{IVP}} = [A^-, A^-]$ and $A_{_{IVVP}} = [\varnothing, A^+]$. So $A = A_{_{IVP}} \cup A_{_{IVVP}}$. □

Example 3.12. Let $X = \{a, b, c, d, e, f, g, h, i\}$. Consider $A = [\{a, b, c, d\}, \{a, b, c, d, e, f\}]$. Then clearly, $a,\ b,\ c,\ d \in A^-$ and $e,\ f \in A^+ \setminus A^-$. Thus we have

$$a_{_{IVP}},\ b_{_{IV}},\ c_{_{IVP}},\ d_{_{IVP}} \in A$$

and

$$a_{_{IVVP}},\ b_{_{IVVP}},\ c_{_{IVVP}},\ d_{_{IVVP}},\ e_{_{IVVP}},\ f_{_{IVVP}} \in A.$$

So $A_{_{IVP}} = [\{a, b, c, d\}, \{a, b, c, d\}] = [A^-, A^-]$ and $A_{_{IVVP}} = [\varnothing, \{a, b, c, d, e, f\}] = [\varnothing, A^+]$. Hence we can confirm that Proposition 3.11 holds.

Proposition 3.13. *Let X be a non-empty set. We two mappings $f : IS(X) \to IVS(X)$ and $g : IVS(X) \to IS(X)$, respectively as follows: for each $(A^{\in}, A^{\not\in}) \in IS(X)$ and $[A^-, A^+] \in IVS(X)$,*

$$f((A^{\in}, A^{\not\in})) = [A^{\in}, A^{\not\in c}],\ g([A^-, A^+]) = (A^-, A^{+c}).$$

Then $g \circ f = id_{IS(X)}$ and $f \circ g = id_{IVS(X)}$ such that $f(a_{_I}) = a_{_{IVP}},\ f(a_{_{IV}}) = a_{_{IVVP}}$ and $g(a_{_{IVP}}) = a_{_I},\ g(a_{_{IVVP}}) = a_{_{IV}}$ for each $a \in X$.

Proof. Straightforward. □

Theorem 3.14. *Let $(A_j)_{j \in J} \subset IVS(X)$ and let $a \in X$.*

(1) *$a_{_{IVP}} \in \bigcap A_j$ [resp. $a_{_{IVVP}} \in \bigcap A_j$] if and only if $a_{_{IVP}} \in A_j$ [resp. $a_{_{IVVP}} \in A_j$], for each $j \in J$.*

(2) *$a_{_{IVP}} \in \bigcup A_j$ [resp. $a_{_{IVVP}} \in \bigcup A_j$] if and only if there exists $j \in J$ such that $a_{_{IVP}} \in A_j$ [resp. $a_{_{IVVP}} \in A_j$.*

Proof. Straightforward. □

Theorem 3.15. *Let $A, B \in IVS(X)$. Then*

(1) *$A \subset B$ if and only if $a_{_{IVP}} \in A \Rightarrow a_{_{IVP}} \in B$ [resp. $a_{_{IVVP}} \in A \Rightarrow a_{_{IVVP}} \in B$] for each $a \in X$.*

(2) *$A = B$ if and only if $a_{_{IVP}} \in A \Leftrightarrow a_{_{IVP}} \in B$ [resp. $a_{_{IVVP}} \in A \Leftrightarrow a_{_{IVVP}} \in B$] for each $a \in X$.*

Proof. Straightforward. □

Definition 3.16. Let $X,\ Y$ be two non-empty sets, let $f : X \to Y$ be a mapping and let $A \in IVS(X),\ B \in IVS(Y)$.

(i) The image of A under f, denoted by $f(A)$, is an IVS in Y defined as:

$$f(A) = [f(A^-), f(A^+)].$$

(ii) The preimage of B under f, denoted by $f^{-1}(B)$, is an IVS in X defined as:

$$f^{-1}(B) = [f^{-1}(B^-), f^{-1}(B^+)].$$

It is obvious that $f(a_{_{IVP}}) = f(a)_{_{IVP}}$ and $f(a_{_{IVVP}}) = f(a)_{_{IVVP}}$ for each $a \in X$.

Proposition 3.17. *Let $X,\ Y$ be two non-empty sets, let $f : X \to Y$ be a mapping, let $A,\ A_1,\ A_2 \in IVS(X)$, $(A_j)_{j \in J} \subset IVS(X)$ and let $B,\ B_1,\ B_2 \in IVS(Y)$, $(A_j)_{j \in J} \subset IVS(Y)$. Then*

(1) *if $A_1 \subset A_2$, then $f(A_1) \subset f(A_2)$,*

(2) *if* $B_1 \subset B_2$*, then* $f^{-1}(B_1) \subset f^{-1}(B_2)$,
(3) $A \subset f^{-1}(f(A))$ *and if* f *is injective, then* $A = f^{-1}(f(A))$,
(4) $f(f^{-1}(B)) \subset B$ *and if* f *is surjective,* $f(f^{-1}(B)) = B$,
(5) $f^{-1}(\bigcup_{j\in J} B_j) = \bigcup_{j\in J} f^{-1}(B_j)$,
(6) $f^{-1}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} f^{-1}(B_j)$,
(7) $f(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} f(A_j)$,
(8) $f(\bigcap_{j\in J} A_j) \subset \bigcap_{j\in J} f(A_j)$ *and if* f *is injective, then* $f(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} f(A_j)$,
(9) *if* f *is surjective, then* $f(A)^c \subset f(A^c)$.
(10) $f^{-1}(B^c) = f^{-1}(B)^c$.
(11) $f^{-1}(\widetilde{\varnothing}) = \widetilde{\varnothing}$, $f^{-1}(\widetilde{X}) = \widetilde{X}$,
(12) $f(\widetilde{\varnothing}) = \widetilde{\varnothing}$ *and if* f *is surjective, then* $f(\widetilde{X}) = \widetilde{X}$,
(13) *if* $g : Y \to Z$ *is a mapping, then* $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$*, for each* $C \in IVS(Z)$.

Proof. The proofs are straightforward. □

Definition 3.18. Let X be a non-empty sets and let L be a non-empty family of IVSs in X. Then L is called an interval-valued ideal (briefly, IVI) on X, provided that it satisfies the following conditions: for any $A,\ B \in IVS(X)$,
(i) (Heredity) if $A \in L$ and $B \subset A$, then $B \in L$,
(ii) (Finite additivity) if $A,\ B \in L$, then $A \cup B \in L$.

An interval-valued ideal L is called a σ-interval-valued ideal (briefly, σ-IVI), provided that it satisfies the following condition:
(Countable additivity) if $(A_n)_{n\in\mathbb{N}} \subset L$, then $\bigcup_{n\in\mathbb{N}} A_n \in L$.

In particular, an IVI L is said to be proper [resp. improper], if $\widetilde{X} \notin L$ [resp. $\widetilde{X} \in L$].

It is obvious that $\widetilde{\varnothing} \in L$ and for each $\widetilde{\varnothing} \neq A \in IVS(X)$,

$$\{B \in IVS(X) : B \subset A\}$$

is an IVI on X. In this case, We will write $\{B \in IVS(X) : B \subset A\} = IVI(A)$ and call it as the principal IVI of A, and A is called a base of $IVI(A)$.

We will denote the interval-valued ideal of IVSs in X having finite [resp. countable] support of X as IVI_f [resp. IVI_c], and the set of all IVIs on X as $IVI(X)$.

Remark 3.19. Let L be an IVI on X and let $L^- = \{A^- \subset X : A \in L\}$, $L^+ = \{A^+ \subset X : A \in L\}$. Then L^- and L^+ are ordinary ideals on X. In this case, L^- [resp. L^+] will be called the lower [resp. upper] interval-valued ideal of L.

Example 3.20. Let $X = \{a, b, c\}$ and consider IVSs in X given by:

$$A = [\{a\}, \{a\}],\ B = [\varnothing, \{a\}],\ C = [\{a\}, \{a, b\}],$$
$$D = [\varnothing, \{a, b\}],\ E = [\varnothing, \{b\}],\ F = [\{c\}, \{c\}],$$
$$G = [\{c\}, \{a, c\}],\ H = [\{c\}, \{b, c\}],\ I = [\{a, c\}, \{a, c\}],\ J = [\{a, c\}, X]$$
$$K = [\varnothing, \{c\}],\ M = [\varnothing, \{a, c\}],\ N = [\varnothing, \{b, c\}],\ O = [\varnothing, X].$$

Let $L = \{\widetilde{\varnothing}, A, B, C, D, E, F, G, H, I, J, K, M, N, O\}$. Then we can easily check that L is an IVI on X. Moreover, consider the following sets:
$L^- = \{\varnothing, \{a\}, \{c\}, \{a, c\}\}$ and $L^+ = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$.
Then we can easily confirm that L^- and L^+ are ordinary ideals on X.

Definition 3.21. Let $L_1,\ L_2$ be two IVIs on a non-empty set X. Then
(i) L_2 is said to be finer than L_1 or L_1 is coarser then L_2, if $L_1 \subset L_2$,
(i) L_2 is said to be strictly finer than L_1 or L_1 is strictly coarser then L_2, if $L_1 \subset L_2$ and $L_1 \neq L_2$,
(iii) L_1 and L_2 are said to be comparable, if one is finer than the other.

It is clear that $(IVI(X), \subset)$ is a poset. Furthermore, $\{\widetilde{\varnothing}\}$ [resp. $IVS(X)$] is the smallest [resp. largest] IVI on X.

The following is the immediate result of Definitions 3.3 and 3.18.

Proposition 3.22. *Let X be a non-empty set and let $(L_j)_{j\in J}$ be a non-empty family of IVIs on X. Then $\bigcap_{j\in J} L_j,\ \bigcup_{j\in J} L_j \in IVI(X)$.*

In fact, $\bigcap_{j\in J} L_j = \inf_{j\in J} L_j$ and $\bigcup_{j\in J} L_j = \sup_{j\in J} L_j$.

The following is the immediate result of Definition 3.18.

Theorem 3.23. *Let X be a non-empty set, $A \in IVS(X)$ and let $L \in IVI(X)$. Then A is a base of L if and only if $B \subset A$ for each $B \in L$*

Theorem 3.24. *Let X be a non-empty set and $A,\ B \in IVS(X)$. Let L_1 be an IVI on X with a base A and let L_2 be an IVI on X with a base B. Then L_1 is finer than L_2 if and only if $B \subset A$ for each $C \in IVS(X)$ such that $C \subset B$.*

Proof. The proof is straightforward from Definition 3.21. □

The following is the immediate result of Theorem 3.24.

Corollary 3.25. *Let X be a non-empty set and $A,\ B \in IVS(X)$. Let L_1 be an IVI on X with a base A and let L_2 be an IVI on X with a base B. Then A and B are equivalent if and only if $C \subset A$ for each $C \in IVS(X)$ such that $C \subset B$ and $D \subset B$ for each $D \in IV(X)$ such that $D \subset A$.*

Proposition 3.26. *Let X be a non-empty set and let $\eta = (A_j)_{j\in J}$ be a non-empty family of IVSs on X. Then there is an IVI $L(\eta)$ on X, where*

$$L(\eta) = \{A \in IV(X) : A \subset \bigcup_{j\in J} A_j,\ J \text{ is finite}\}.$$

Proof. The proof is straightforward from Definition 3.18. □

4. Interval-valued topological spaces

In this section, we define an interval-valued topology on a non-empty set X, and study some of its properties and an interval-valued intuitionistic set combined by an intuitionistic set and an interval-valued set, and give some examples. Also, we introduce the concepts of an interval-valued base and subbase , and a family of IVSs gets the necessary and sufficient conditions to become IVB and gives some examples. Moreover, we define some interval-valued interval in $\mathbb{R}$.

Definition 4.1. Let X be a non-empty set and let τ be a non-empty family of IVSs on X. Then τ is called an interval-valued topology (briefly, IVT) on X, if it satisfies the following axioms:
(IVO$_1$) $\widetilde{\varnothing},\ \widetilde{X} \in \tau$,
(IVO$_2$) $A \cap B \in \tau$ for any $A,\ B \in \tau$,
(IVO$_3$) $\bigcup_{j \in J} A_j \in \tau$ for any family $(A_j)_{j \in J}$ of members of τ.
In this case, the pair (X, τ) is called an interval-valued topological space (briefly, IVTS) and each member of τ is called an interval-valued open set (briefly, IVOS) in X. A IVS A is called an interval-valued closed set (briefly, IVCS) in X, if $A^c \in \tau$.

It is obvious that $\{\widetilde{\varnothing}, \widetilde{X}\}$ is an IVT on X, and will be called the interval-valued indiscrete topology on X and denoted by $\tau_{_{IV,0}}$. Also $IVS(X)$ is an IVT on X, and will be called the interval-valued discrete topology on X and denoted by $\tau_{_{IV,1}}$. The pair $(X, \tau_{_{IV,0}})$ [resp. $(X, \tau_{_{IV,1}})$] will be called the interval-valued indiscrete [resp. discrete] space.

We will denote the set of all IVTs on X as $IVT(X)$. for an IVTS X, we will denote the set of all IVOs [resp. IVCSs] in X as $IVO(X)$ [resp. $IVC(X)$].

We can easily see that for each $\tau \in IVT(X)$, the family

$$\chi_{_\tau} = \{\chi_{_A} : \chi_{_A} = [\chi_{_{A^-}}, \chi_{_{A^+}}],\ A \in \tau\}$$

is an interval-valued fuzzy topology on X proposed by Mondal and Samanta [14]. Then an IVT is a special case of an an interval-valued fuzzy topology on X.

Remark 4.2. (1) For each $\tau \in IVT(X)$, consider two families of subsets of X:

$$\tau^- = \{A^- \subset X : A \in \tau\} \text{ and } \tau^+ = \{A^+ \subset X : A \in \tau\}.$$

Then we can easily check that τ^- and τ^+ are ordinary topologies on X.
In this case, τ^- [resp. τ^+] will be called the lower [resp. upper] topology of τ and we will write $\tau = (\tau^-, \tau^+)$. In fact, we can consider (X, τ^-, τ^+) as a bitopological space on X introduced by Kelly [9]. Then an IVT is a generalization of a classical topology on X.
(2) Let (X, τ) be an ordinary topological space such that τ is not indiscrete. Then there are two IVTs on X given by:

$$\tau^1 = \{\widetilde{\varnothing}, \widetilde{X}\} \bigcup \{[G, G] : G \in \tau\},\ \tau^2 = \{\widetilde{\varnothing}, \widetilde{X}\} \bigcup \{[\varnothing, G] : G \in \tau\}.$$

(3) Let (X, τ_I) be an intuitionistic topological space and consider the family τ of IVSs in X defined as follows:

$$\tau = \{[A^{\in}, A^{\notin^c}] : A = (A^{\in}, A^{\notin}) \in \tau_I\}.$$

Then we can easily see that τ_{IV} is an IVTS from Proposition 3.13 and Definition 4.1.

Example 4.3. (1) Let $X = \{a, b\}$. Then clearly, we have

$$\tau_{IV,1} = \{\widetilde{\varnothing}, a_{IVP}, b_{IVP}, a_{IVVP}, b_{IVVP}, [\varnothing, X], [\{a\}, X], [\{b\}, X], \widetilde{X}\}.$$

(2) Let X be a set and let $A \in IVS(X)$. Then A is said to be finite, if A^+ is finite. Consider the family $\tau = \{U \in IV(X) : U = \widetilde{\varnothing}$ or U^c is finite$\}$. Then we can easily check that $\tau \in IVT(X)$.

In this case, τ will be called an interval-valued cofinite topology (briefly, IVCFT) on X and denoted by $IVCof(X)$.

(3) Let X be a set and let $A \in IVS(X)$. Then A is said to be countable, if A^+ is countable. Consider the family $\tau = \{U \in IV(X) : U = \widetilde{\varnothing}$ or U^c is countable$\}$. Then we can easily prove that $\tau \in IVT(X)$.

In this case, τ will be called an interval-valued cocountable topology (briefly, IVCCT) on X and denoted by $IVCoc(X)$.

The following is the immediate result of Definition 4.1

Proposition 4.4. *Let X be an IVTS. Then*
(1) $\widetilde{\varnothing},\ \widetilde{X} \in IVC(X)$,
(2) $A \cup B \in IVC(X)$ *for any* $A,\ B \in IVC(X)$,
(3) $\bigcap_{j\in J} A_j \in IVC(X)$ *for any* $(A_j)_{j\in J} \subset IVC(X)$.

Definition 4.5. Let X be a non-empty set and let $\tau_1,\ \tau_2 \in IVT(X)$. Then we say that τ_1 is contained in τ_2 or τ_1 is coarser than τ_2 or τ_2 is finer than τ_1, if $\tau_1 \subset \tau_2$, i.e., $A \in \tau_2$ for each $A \in \tau_1$.

It is obvious that $\tau_{IV,0} \subset \tau \subset \tau_{IV,1}$ for each $\tau \in IVT(X)$.

The following is the immediate result of Definitions 3.8 and 4.1.

Proposition 4.6. *Let $(\tau_j)_{j\in J} \subset IVT(X)$. Then $\bigcap_{j\in J} \tau_j \in IVT(X)$.*
In fact, $\bigcap_{j\in J} \tau_j$ is the coarsest IVT on X containing each τ_j.

Proposition 4.7. *Let $\tau,\ \gamma \in IVT(X)$. We define $\tau \wedge \gamma$ and $\tau \vee \gamma$ as follows:*

$$\tau \wedge \gamma = \{W : W \in \tau,\ W \in \gamma\},$$

$$\tau \vee \gamma = \{W : W = U \cup V,\ U \in \tau,\ V \in \gamma\}.$$

Then we have
(1) *$\tau \wedge \gamma$ is an IVT on X which is the finest IVT coarser than both τ and γ,*
(2) *$\tau \vee \gamma$ is an IVT on X which is the coarsest IVT finer than both τ and γ,*

Proof. (1) It is clear that $\tau \wedge \gamma \in IVT(X)$. Let η be any IVT on X which is coarser than both τ and γ, and let $W \in \eta$. Then clearly, $W \in \tau$ and $W \in \gamma$. Thus $W \in \tau \wedge \gamma$. So η is coarser than $\tau \wedge \gamma$.

(2) The proof is similar to (1). □

Definition 4.8. Let (X, τ) be an IVTS.

(i) A subfamily β of τ is called an interval-valued base (briefly, IVB) for τ, if for each $A \in \tau$, $A = \widetilde{\varnothing}$ or there is $\beta' \subset \beta$ such that $A = \bigcup \beta'$.

(ii) A subfamily σ of τ is called an interval-valued subbase (briefly, IVSB) for τ, if the family $\beta = \{\bigcap \sigma' : \sigma'$ is a finite subset of $\sigma\}$ is an IVB for τ.

Remark 4.9. (1) Let β be an IVB for an IVT τ on a non-empty set X and let $\beta^- = \{A^- : A \in \beta\}$, $\beta^+ = \{A^+ : A \in \beta\}$. Then β^- is an ordinary base for τ^- and β^+ is an ordinary base for τ^+.

(2) Let σ be an IVSB for an IVT τ on a non-empty set X and let $\sigma^- = \{A^- \subset X : A \in \sigma\}$, $\sigma^+ = \{A^+ \subset X : A \in \sigma\}$. Then σ^- is an ordinary subbase for τ^- and σ^+ is an ordinary subbase for τ^+.

Example 4.10. (1) Let $\sigma = \{[(a, b), (a, \infty)] : a,\ b \in \mathbb{R}\}$ be the family of IVSs in $\mathbb{R}$. Then σ generates an IVT τ on $\mathbb{R}$ which will be called the "usual left interval-valued topology (briefly, ULIVT)" on $\mathbb{R}$. In fact, the IVB β for τ can be written in the form:

$$\beta = \{\widetilde{\mathbb{R}}\} \cup \{\cap_{\gamma \in \Gamma} S_\gamma : S_\gamma \in \sigma,\ \Gamma \text{ is finite}\}$$

and τ consists of the following IVSs in $\mathbb{R}$:

$$\tau = \{\widetilde{\varnothing}, \widetilde{\mathbb{R}}, [\cup(a_j, b_j), (c, \infty)], [\cup(a_k, b_k), \mathbb{R}]\},$$

where $a_j, b_j, c \in \mathbb{R}$, $\{a_j : j \in J\}$ is bounded from below, $c < inf\{a_j : j \in J\}$ and $a_k, b_k \in \mathbb{R}$, $\{a_k : k \in K\}$ is not bounded from below.

Similarly, one can define the "usual right interval-valued topology (briefly, URIVT)" on $\mathbb{R}$ using an analogue construction.

(2) Consider the family σ of IVSs in $\mathbb{R}$

$$\sigma = \{[(a, b), (a_1, \infty) \cap (-\infty, b_1)] : a,\ b,\ a_1,\ b_1 \in \mathbb{R},\ a_1 \leq a,\ b_1 \geq b\}.$$

Then σ generates an IVT τ on $\mathbb{R}$ which will be called the "usual interval-valued topology (briefly, UIVT)" on $\mathbb{R}$. In fact, the IVB β for τ can be written in the form:

$$\beta = \{\widetilde{\mathbb{R}}\} \cup \{\cap_{\gamma \in \Gamma} S_\gamma : S_\gamma \in \sigma,\ \Gamma \text{ is finite}\}$$

and the elements of τ can be easily written down as in (1).

(3) Consider the family $\sigma_{[0,1]}$ of IVSs in $\mathbb{R}$

$$\sigma_{[0,1]} = \{[[a, b], [a, \infty) \cap (-\infty, b]] : a, b \in \mathbb{R} \text{ and } 0 \leq a \leq b \leq 1\}.$$

Then $\sigma_{[0,1]}$ generates an IVT $\tau_{[0,1]}$ on $\mathbb{R}$ which will be called the "usual unit closed interval interval-valued topology" on $\mathbb{R}$. In fact, the IVB $\beta_{[0,1]}$ for $\tau_{[0,1]}$ can be written in the form:

$$\beta_{[0,1]} = \{\widetilde{\mathbb{R}}\} \cup \{\cap_{\gamma \in \Gamma} S_\gamma : S_\gamma \in \sigma_{[0,1]},\ \Gamma \text{ is finite}\}$$

and the elements of τ can be easily written down as in (1).

In this case, $([0, 1], \tau_{[0,1]})$ is called the "interval-valued usual unit closed interval" and will be denoted by $[0, 1]_{IVI}$, where $[0, 1]_{IVI} = [[0, 1], [0, \infty) \cup (-\infty, 1]]$.

(4) Let X be a non-empty set and let $\beta = \{a_{_{IVP}} : a \in X\} \cup \{a_{_{IVVP}} : a \in X\}$. Then β is an IVB for the interval-valued discrete topology $\tau_{_{IV,1}}$ on X.

(5) Let $X = \{a, b, c\}$ and let $\beta = \{[\{a, b\}, X], [\{b, c\}, X], \widetilde{X}\}$. Assume that β is an IVB for an IVT τ on X. Then by the definition of base, $\beta \subset \tau$. Thus $[\{a, b\}, X]$, $[\{b, c\}, X] \in \tau$. So $[\{a, b\}, X] \cap [\{b, c\}, X] = [\{b\}, X] \in \tau$. But for any $\beta' \subset \beta$, $[\{b\}, X] \neq \bigcup \beta'$. Hence β is not an IVB for an IVT on X.

From (1), (2) and (3) in Example 4.10, we can define interval-valued intervals as following.

Definition 4.11. Let $a,\ b \in \mathbb{R}$ such that $a \leq b$. Then

(i) (the closed interval) $[a,b]_{IVI} = [[a,b],[a,-\infty) \cap (-\infty,b]]$,

(ii) (the open interval) $(a,b)_{IVI} = [(a,b),(a,-\infty) \cap (-\infty,b)]$,

(iii) (the half open interval or the half closed interval)

$$(a,b]_{IVI} = [(a,b],(a,-\infty) \cap (-\infty,b]],\ [a,b)_{IVI} = [[a,b),[a,-\infty) \cap (-\infty,b)],$$

(iv) (the half interval-valued real line)

$$(-\infty,a]_{IVI} = [(-\infty,a],(-\infty,a]],\ (-\infty,a)_{IVI} = [(-\infty,a),(-\infty,a)],$$

$$[a,\infty)_{IVI} = [[a,\infty),[a,\infty)],\ (a,\infty)_{IVI} = [(a,\infty),(a,\infty)],$$

(v) (the interval-valued real line) $(-\infty,\infty)_{IVI} = [(-\infty,\infty),(-\infty,\infty)] = \widetilde{\mathbb{R}}$.

Theorem 4.12. *Let X be a non-empty set and let $\beta \subset IVS(X)$. Then β is an IVB for an IVT τ on X if and only if it satisfies the followings:*

(1) $\widetilde{X} = \bigcup \beta$,

(2) *if $B_1, B_2 \in \beta$ and $a_{IVP} \in B_1 \cap B_2$ [resp. $a_{IVVP} \in B_1 \cap B_2$], then there exists $B \in \beta$ such that $a_{IVP} \in B \subset B_1 \cap B_2$ [resp. $a_{IVVP} \in B \subset B_1 \cap B_2$].*

Proof. The proof is the same as one in ordinary topological spaces. $\square$

Example 4.13. Let $X = \{a,b,c\}$ and let $\beta = \{[\{a\},\{a\}],[\{a,b\},\{a,b\}],[\{a,c\},\{a,c\}]\}$. Then clearly, β satisfies two conditions of Theorem 4.12. Thus β is an IVB for an IVT τ on X. Furthermore, $\tau = \{\widetilde{\varnothing},[\{a\},\{a\}],[\{a,b\},\{a,b\}],[\{a,c\},\{a,c\}],\widetilde{X}\}$.

Proposition 4.14. *Let X be a non-empty set and let $\sigma \subset IVS(X)$ such that $\widetilde{X} = \bigcup \sigma$. Then there exists a unique IVT τ on X such that σ is an IVSB for τ.*

Proof. Let $\beta = \{B \in IVS(X) : B = \bigcup_{i=1}^{n} S_i$ and $S_i \in \sigma\}$. Let $\tau = \{U \in IVS(X) : U = \widetilde{\varnothing}$ or there is a subcollection $\beta^{'}$ of β such that $U = \bigcup \beta^{'}\}$. Then we can show that τ is the unique IVT on X such that σ is an IVSB for τ. $\square$

In Proposition 4.14, τ is called the IVT on X generated by σ.

Example 4.15. Let $X = \{a,b,c,d,e\}$ and let

$$\sigma = \{[\{a\},\{a\}],[\{a,b,c\},\{a,b,c\}],[\{b,c,e\},\{b,c,e\}],[\{c,d\},\{c,d\}]\}.$$

Then clearly, $\bigcup \sigma = \widetilde{X}$. Let β be the collection of all finite intersections of members of σ. Then

$$\beta = \{\widetilde{\varnothing},[\{a\},\{a\}],[\{c\},\{c\}],[\{b,c\},\{b,c\}],[\{a,b,c\},\{a,b,c\}], [\{b,c,e\},\{b,c,e\}],[\{c,d\},\{c,d\}]\}.$$

Thus the generated IVT τ by σ is

$$\tau = \{\widetilde{\varnothing},[\{a\},\{a\}],[\{c\},\{c\}],[\{a,c\},\{a,c\}],[\{b,c\},\{b,c\}], [\{c,d\},\{c,d\}],[\{a,b,c\},\{a,b,c\}],[\{b,c,d\},\{b,c,d\}], [\{b,c,e\},\{b,c,e\}],[\{a,b,c,e\},\{a,b,c,e\}],\widetilde{X}\}.$$

In fact, $\tau^{-} = \{\varnothing,\{a\},\{c\},\{a,c\},\{b,c\},\{c,d\},\{a,b,c\}, \{b,c,d\},\{b,c,e\},\{a,b,c,e\},X\} = \tau^{+}$.

5. Interval-valued neighborhoods

In this section, we introduce the concept of interval-valued neighborhoods of IVPs of two types, and find their various properties and give some examples.

Definition 5.1. Let X be an IVTS, $a \in X$ and let $N \in IVS(X)$. Then

(i) N is called an interval-valued neighborhood (briefly, IVN) of $a_{_{IVP}}$, if there exists a $U \in IVO(X)$ such that

$$a_{_{IVP}} \in U \subset N, \text{ i.e., } a \in U^- \subset N^-,$$

(ii) N is called an interval-valued vanishing neighborhood (briefly, IVVN) of $a_{_{IVVP}}$, if there exists a $U \in IVO(X)$ such that

$$a_{_{IVVP}} \in U \subset N, \text{ i.e., } a \in U^+ \subset N^+.$$

We will denote the set of all IVNs [resp. IVVNs] of $a_{_{IVP}}$ [resp. $a_{_{IVVP}}$] by $N(a_{_{IVP}})$ [resp. $N(a_{_{IVVP}})$].

Example 5.2. Let $X = \{a, b, c, d\}$ and let τ be the IVT on X given by:

$\tau = \{\widetilde{\varnothing}, [\varnothing, \{a\}], [\{a\}, \{a\}], [\{b\}, \{b\}], [\{b\}, \{a, b\}], [\{a, b\}, \{a, b\}],$
$[\{b, c\}, \{b, c, d\}], [\{a, b, c\}, X], \widetilde{X}\},$

where $A_1 = [\varnothing, \{a\}],\ A_2 = [\{a\}, \{a\}],\ A_3 = [\{b\}, \{b\}],\ A_4 = [\{b\}, \{a, b\}],$
$A_5 = [\{a, b\}, \{a, b\}],\ A_6 = [\{b, c\}, \{b, c, d\}],\ A_7 = [\{a, b, c\}, X].$

Let $N = [\{a, b\}, \{a, b, d\}]$. Then we can easily see that

$$N \in N(a_{_{IVP}}) \cap N(a_{_{IVVP}}),\ N \in N(b_{_{IVP}}) \cap N(b_{_{IVVP}}),\ N \in N(d_{_{IVVP}}).$$

Proposition 5.3. *Let X be an IVTS and let $a \in X$.*

[IVN1] *If $N \in N(a_{_{IVP}})$, then $a_{_{IVP}} \in N$.*

[IVN2] *If $N \in N(a_{_{IVP}})$ and $N \subset M$, then $M \in N(a_{_{IVP}})$.*

[IVN3] *If $N, M \in N(a_{_{IVP}})$, then $N \cap M \in N(a_{_{IVP}})$.*

[IVN4] *If $N \in N(a_{_{IVP}})$, then there exists $M \in N(a_{_{IVP}})$ such that $N \in N(b_{_{IVP}})$ for each $b_{_{IVP}} \in M$.*

Proof. From Definition 5.1, the proofs of [IVN1] and [IVN2] are easy.

[IVN3] Suppose $N, M \in N(a_{_{IVP}})$. Then there are $U,\ V \in IVO(X)$ such that

$$a_{_{IVP}} \in U \subset N \text{ and } a_{_{IVP}} \in V \subset M.$$

Let $W = U \cap V$. Then clearly, $W \in IVO(X)$ and $a_{_{IVP}} \in W \subset N \cap M$. Thus $N \cap M \in N(a_{_{IVP}})$.

[IVN4] The proof is obvious from Definition 5.1 and the condition [IVN1]. □

Proposition 5.4. *Let X be an IVTS and let $a \in X$.*

[IVVN1] *If $N \in N(a_{_{IVVP}})$, then $a_{_{IVVP}} \in N$.*

[IVVN2] *If $N \in N(a_{_{IVVP}})$ and $N \subset M$, then $M \in N(a_{_{IVVP}})$.*

[IVVN3] *If $N, M \in N(a_{_{IVVP}})$, then $N \cap M \in N(a_{_{IVVP}})$.*

[IVVN4] *If $N \in N(a_{_{IVVP}})$, then there exists $M \in N(a_{_{IVVP}})$ such that $N \in N(b_{_{IVVP}})$ for each $b_{_{IVVP}} \in M$.*

Proof. The proof is similar to one of Proposition 5.3. □

Proposition 5.5. *Let (X, τ) be an IVTS and let us define two families:*

$$\tau_{IVP} = \{U \in IVS(X) : U \in N(a_{IVP}) \text{ for each } a_{IVP} \in U\}$$

and

$$\tau_{IVVP} = \{U \in IVS(X) : U \in N(a_{IVVP}) \text{ for each } a_{IVVP} \in U\}.$$

Then we have

(1) $\tau_{IVP},\ \tau_{IVVP} \in IVT(X)$,
(2) $\tau \subset \tau_{IVP}$ *and* $\tau \subset \tau_{IVVP}$.

Proof. (1) We only prove that $\tau_{IVVP} \in IVT(X)$.

(IVO_1) From the definition of τ_{IVVP}, we have $\widetilde{\varnothing},\ \widetilde{X} \in \tau_{IVVP}$.

(IVO_2) Let $U\ , V \in IVS(X)$ such that $U\ , V \in \tau_{IVVP}$ and let $a_{IVVP} \in U \cap V$. Then clearly, $U,\ V \in N(a_{IVVP})$. Thus by [IVVN3], $U \cap V \in N(a_{IVVP})$. So $U \cap V \in \tau_{IVVP}$.

(IVO_3) Let $(U_j)_{j \in J}$ be any family of IVSs in τ_{IVVP}, let $U = \bigcup_{j \in J} U_j$ and let $a_{IVVP} \in U$. Then by Theorem 3.14 (2), there is $j_0 \in J$ such that $a_{IVVP} \in U_{j_0}$. Since $U_{j_0} \in \tau_{IVVP}$, $U_{j_0} \in N(a_{IVVP})$ by the definition of τ_{IVVP}. Since $U_{j_0} \subset U$, $U \in N(a_{IVVP})$ by [IVVN2]. So by the definition of τ_{IVVP}, $U \in \tau_{IVVP}$.

(2) Let $U \in \tau$. Then clearly, $U \in N(a_{IVP})$ and $U \in N(a_{IVVP})$ for each $a_{IVP} \in G$ and $a_{IVVP} \in G$, respectively. Thus $U \in \tau_{IVP}$ and $U \in \tau_{IVVP}$. So the results hold. □

Remark 5.6. (1) From the definitions of τ_{IVP} and τ_{IVVP}, we can easily have:

$$\tau_{IVP} = \tau \cup \{[U^-, S] \in IVS(X) : U^+ \subset S,\ U \in \tau\}$$

and

$$\tau_{IVVP} = \tau \cup \{S \in IVS(X) : \varnothing \neq S^- \subset X \setminus U^+,\ S^+ = S^- \cup U^+,\ U = [\varnothing, U^+] \in \tau\}.$$

In fact, if $U^- \neq \varnothing$ for each $U \in \tau$, then $\tau_{IVVP} = \tau$.

(2) For any IVT τ on a set X, we can have four ordinary topologies on X given by:

$$\tau^-_{IVP} = \{U^- \subset X : U \in \tau_{IVP}\},\ \tau^+_{IVP} = \{U^+ \subset X : U \in \tau_{IVP}\}$$

and

$$\tau^-_{IVVP} = \{U^- \subset X : U \in \tau_{IVVP}\},\ \tau^+_{IVVP} = \{U^+ \subset X : U \in \tau_{IVVP}\}.$$

Example 5.7. (1) Let $X = \{a, b, c, d\}$ and consider the family τ of IVSs in X given by:

$$\tau = \{\widetilde{\varnothing}, \widetilde{X}, A_1, A_2, A_3, A_4, A_5, A_6, A_7\},$$

where $A_1 = [\{a, b\}, \{a, b, c\}],\ A_2 = [\{c\}, \{b, c\}],\ A_3 = [\varnothing, \{a, c\}]$,
$A_4 = [\{a, b, c\}, \{a, b, c\}],\ A_5 = [\varnothing, \{b, c\}],\ A_6 = [\varnothing, \{c\}],\ A_7 = [\{c\}, \{a, b, c\}]$.
Then we can easily check that (X, τ) is an IVTS. Thus we have

$$\tau_{IVP} = \tau \cup \{A_8, A_9, A_{10}\},$$

where $A_8 = [\{a, b\}, X],\ A_9 = [\{c\}, X],\ A_{10} = [\{a, b, c\}, X]$.

Also, we have

$$\tau_{IVVP} = \tau \cup \{A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16}, A_{17}, A_{18}, A_{19}, A_{20}, A_{21}, A_{22}\},$$

where $A_{11} = [\{b\}, \{a, b, c\}],\ A_{12} = [\{d\}, \{a, c, d\}],\ A_{13} = [\{b, d\}, X]$,
$A_{14} = [\{a\}, \{a, b, c\}],\ A_{15} = [\{d\}, \{b, c, d\}],\ A_{16} = [\{b, d\}, X]$,
$A_{17} = [\{a\}, \{a, c\}],\ A_{18} = [\{b\}, \{b, c\}],\ A_{19} = [\{d\}, \{c, d\}]$,

$A_{20} = [\{a, d\}, \{a, c, d\}],\ A_{21} = [\{b, d\}, \{b, c, d\}],\ A_{22} = [\{a, b, d\}, X]$.
So we can confirm that Proposition 5.5 holds.

Furthermore, we obtain six ordinary topologies on X for the IVT τ:

$$\tau^- = \{\varnothing, X, \{c\}, \{a, b\}, \{a, b, c\}\},$$
$$\tau^+ = \{\varnothing, X, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$
$$\tau^-_{_{IV}} = \{\varnothing, X, \{c\}, \{a, b\}, \{a, b, c\}\} = \tau^-,$$
$$\tau^+_{_{IV}} = \{\varnothing, X, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} = \tau^+,$$
$$\tau^-_{_{IVV}} = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}\},$$
$$\tau^+_{_{IVV}} = \{\varnothing, X, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}.$$

(2) $X = \{a, b, c, d\}$ and η be the IVT on X given by:

$$\eta = \{\widetilde{\varnothing}, \widetilde{X}, A_1, A_2, A_3, A_4\},$$

where $A_1 = [\{b\}, \{b, c\}],\ A_2 = [\{a, b\}, \{a, b, c\}],\ A_3 = [\{b, c\}, \{b, c, d\}],\ A_3 = [\{a, b, c\}, X]$.
Then we easily check that $\eta_{_{IVV}} = \eta$.

The following is the immediate result of Proposition 5.5 (2).

Corollary 5.8. *Let (X, τ) be an IVTS and let IVC_τ [resp. $IVC_{\tau_{_{IVP}}}$ and $IVC_{\tau_{_{IVVP}}}$] be the set of all IVCSs w.r.t. τ [resp. $\tau_{_{IVP}}$ and $\tau_{_{IVVP}}$]. Then*

$$IVC_\tau \subset IVC_{\tau_{_{IVP}}},\ \text{and}\ IVC_\tau \subset IVC_{\tau_{_{IVVP}}}.$$

Example 5.9. Let (X, τ) be the IVTS given in Example 5.7. Then we have:
$IVC_\tau = \{\widetilde{\varnothing}, \widetilde{X}, A^c_1, A^c_2, A^c_3, A^c_4, A^c_5, A^c_6, A^c_7, A^c_8\}$,
$IVC_{\tau_{_{IVP}}} = IVC_\tau \cup \{A^c_8, A^c_9, A^c_{10}\}$,
$IVC_{\tau_{_{IVVP}}} = IVC_\tau \cup \{A^c_{11}, A^c_{12}, A^c_{13}, A^c_{14}, A^c_{15}, A^c_{16}, A^c_{17}, A^c_{18}, A^c_{19}, A^c_{20}, A^c_{21}, A^c_{22}\}$,
where $A^c_1 = [\{d\}, \{c, d\}],\ A^c_2 = [\{a, b\}, \{a, b, d\}],\ A^c_3 = [\{b, d\}, X]$,
$A^c_4 = [\{d\}, \{d\}],\ A^c_5 = [\{a, d\}, X],\ A^c_6 = [\{a, b, d\}, X]$,
$A^c_7 = [\{d\}, \{a, b, d\}],\ A^c_8 = [\varnothing, \{c, d\}],\ A^c_9 = [\varnothing, \{a, b, d\}]$,
$A^c_{10} = [\varnothing, \{d\}], A^c_{11} = [\{d\}, \{a, c, d\}],\ A^c_{12} = [\{b\}, \{a, b, c\}],\ A^c_{13} = [\varnothing, \{a, c\}]$,
$A^c_{14} = [\{d\}, \{b, c, d\}],\ A^c_{15} = [\{a\}, \{a, b, c\}],\ A^c_{16} = [\varnothing, \{a, c\}]$,
$A^c_{17} = [\{b, d\}, \{b, c, d\}],\ A^c_{18} = [\{a, d\}, \{a, c, d\}],\ A^c_{19} = [\{a, b\}, \{a, b, c\}]$,
$A^c_{20} = [\{b\}, \{a, b, c\}],\ A^c_{21} = [\{a\}, \{a, c\}],\ A^c_{22} = [\varnothing, \{c\}]$.
Thus we can confirm that Corollary 5.8 holds.

Now let us consider the converses of Propositions 5.3 and 5.4.

Proposition 5.10. *Let X be a non-empty set. Suppose to each $a \in X$, there corresponds a set $N^*(a_{_{IVVP}})$ of IVSs in X satisfying the conditions [IVVN1], [IVVN2], [IVVN3] and [IVVN4] in Proposition 5.4. Then there is an IVT on X such that $N^*(a_{_{IVVP}})$ is the set of all IVVNs of $a_{_{IVVP}}$ in this IVT for each $a \in X$.*

Proof. Let

$$\tau_{_{IVVP}} = \{U \in IVS(X) : U \in N(a_{_{IVVP}}) \text{ for each } a_{_{IVVP}} \in U\},$$

where $N(a_{_{IVVP}})$ denotes the set of all IVVNs in τ.
Then clearly, $\tau_{_{IVVP}} \in IVT(X)$ by Proposition 5.4. we will prove that $N^*(a_{_{IVVP}})$ is the set of all IVVNs of $a_{_{IVVP}}$ in $\tau_{_{IVVP}}$ for each $a \in X$.

Let $V \in IVS(X)$ such that $V \in N^*(a_{IVVP})$ and let U be the union of all the IVVPs b_{IVVP} in X such that $U \in N^*(a_{IVVP})$. If we can prove that

$$a_{IVVP} \in U \subset V \text{ and } U \in \tau_{IVVP},$$

then the proof will be complete.

Since $V \in N^*(a_{IVVP})$, $a_{IVVP} \in U$ by the definition of U. Moreover, $U \subset V$. Suppose $b_{IVVP} \in U$. Then by [IVVN4], there is an IVS $W \in N^*(b_{IVVP})$ such that $V \in N^*(c_{IVVP}))$ for each $c_{IVVP} \in W$. Thus $c_{IVVP} \in U$. By Theorem 3.15, $W \subset U$. So by [IVVN2], $U \in N^*(b_{IVVP})$ for each $b_{IVVP} \in U$. Hence by the definition of τ_{IVV}, $U \in \tau_{IVV}$. This completes the proof. □

Proposition 5.11. *Let X be a non-empty set. Suppose to each $a \in X$, there corresponds a set $N^*(a_{IVP})$ of IVSs in X satisfying the conditions [IVN1], [IVN2], [IVN3] and [IVN4] in Proposition 5.3. Then there is an IVT on X such that $N^*(a_{IVP})$ is the set of all IVNs of a_{IVP} in this IVT for each $a \in X$.*

Proof. The proof is similar to Proposition 5.10. □

Theorem 5.12. *Let (X, τ) be an IVTS and let $A \in IVS(X)$. Then $A \in \tau$ if and only if $A \in N(a_{IVP})$ and $A \in N(a_{IVVP})$ for each $a_{IVP},\ a_{IVVP} \in A$.*

Proof. Suppose $A \in N(a_{IVP})$ and $A \in N(a_{IVVP})$ for each $a_{IVP},\ a_{IVVP} \in A$. Then there are $U_{a_{IVP}},\ V_{a_{IVVP}} \in \tau$ such that $a_{IVP} \in U_{a_{IVP}} \subset A$ and $a_{IVVP} \in V_{a_{IVVP}} \subset A$. Thus

$$A = (\bigcup_{a_{IVP} \in A} a_{IVP}) \cup (\bigcup_{a_{IVVP} \in A} a_{IVVP}) \subset (\bigcup_{a_{IVP} \in A} U_{a_{IVP}}) \cup (\bigcup_{a_{IVVP} \in A} V_{a_{IVVP}}) \subset A.$$

So $A = (\bigcup_{a_{IVP} \in A} U_{a_{IVP}}) \cup (\bigcup_{a_{IVVP} \in A} V_{a_{IVVP}})$. Since $U_{a_{IVP}},\ V_{a_{IVVP}} \in \tau$, $A \in \tau$.

The proof of the necessary condition is easy. □

Now we will give the relation among three IVTs, τ, τ_{IVP} and τ_{IVVP}.

Proposition 5.13. $\tau = \tau_{IVP} \cap \tau_{IVVP}$.

Proof. From Proposition 5.5 (2), it is clear that $\tau \subset \tau_{IVP} \cap \tau_{IVVP}$.

Conversely, let $U \in \tau_{IVP} \cap \tau_{IVVP}$. Then clearly, $U \in \tau_{IVP}$ and $U \in \tau_{IVVP}$. Thus U is an IVN of each of its IVPs a_{IVP} and an IVVN of each of its IVVPs a_{IVVP}. Thus there are $U_{a_{IVP}},\ U_{a_{IVVP}} \in \tau$ such that $a_{IVP} \in U_{a_{IVP}} \subset U$ and $a_{IVVP} \in U_{a_{IVVP}} \subset U$. So we have

$$U_{IVP} = \bigcup_{a_{IVP} \in U} a_{IVP} \subset \bigcup_{a_{IVP} \in U} U_{a_{IVP}} \subset U$$

and

$$U_{IVVP} = \bigcup_{a_{IVVP} \in U} a_{IVVP} \subset \bigcup_{a_{IVVP} \in U} U_{a_{IVVP}} \subset U.$$

By Proposition 3.11, we get

$$U = U_{IVP} \cup U_{IVVP} \subset (\bigcup_{a_{IVP} \in U} U_{a_{IVP}}) \cup (\bigcup_{a_{IVVP} \in U} U_{a_{IVVP}}) \subset U, \text{ i.e.,}$$

$$U = (\bigcup_{a_{IVP} \in U} U_{a_{IVP}}) \cup (\bigcup_{a_{IVVP} \in U} U_{a_{IVVP}}).$$

It is obvious that $(\bigcup_{a_{IVP} \in U} U_{a_{IVP}}) \cup (\bigcup_{a_{IVVP} \in U} U_{a_{IVVP}}) \in \tau$. Hence $U \in \tau$. Therefore $\tau_{IVP} \cap \tau_{IVVP} \subset \tau$. This completes the proof. □

The following is the immediate result of Proposition 5.13.

Corollary 5.14. *Let* (X, τ) *be an IVTS. Then*

$$IVC_\tau = IVC_{\tau_{IVP}} \cap IVC_{\tau_{IVVP}}.$$

Example 5.15. In Example 5.7, we can easily check that Corollary 5.14 holds.

6. Interiors and closures of IVSs

In this section, we define interval-valued interiors and closures, and investigate some of their properties and give some examples. In particular, we will show that there is a unique IVT on a set X from the interval-valued closure [resp. interior] operator.

Definition 6.1. Let (X, τ) be an IVTS and let $A \in IVS(X)$.

(i) The interval-valued closure of A w.r.t. τ, denoted by $IVcl(A)$, is an IVS in X defined as:

$$IVcl(A) = \bigcap \{K : K^c \in \tau \text{ and } A \subset K\}.$$

(ii) The interval-valued interior of A w.r.t. τ, denoted by $IVint(A)$, is an IVS in X defined as:

$$IVint(A) = \bigcup \{G : G \in \tau \text{ and } G \subset A\}.$$

(iii) The interval-valued closure of A w.r.t. τ_{IVP}, denoted by $cl_{IVP}(A)$, is an IVS in X defined as:

$$cl_{IVP}(A) = \bigcap \{K : K^c \in \tau_{IVP} \text{ and } A \subset K\}.$$

(iv) The interval-valued interior of A w.r.t. τ_{IVP}, denoted by $int_{IVP}(A)$, is an IVS in X defined as:

$$int_{IVP}(A) = \bigcup \{G : G \in \tau_{IVP} \text{ and } G \subset A\}.$$

(v) The interval-valued closure of A w.r.t. τ_{IVVP}, denoted by $cl_{IVVP}(A)$, is an IVS in X defined as:

$$cl_{IVVP}(A) = \bigcap \{K : K^c \in \tau_{IVVP} \text{ and } A \subset K\}.$$

(vi) The interval-valued interior of A w.r.t. τ_{IVVP}, denoted by $int_{IVVP}(A)$, is an IVS in X defined as:

$$int_{IVVP}(A) = \bigcup \{G : G \in \tau_{IVVP} \text{ and } G \subset A\}.$$

Remark 6.2. From the above definition, it is obvious that the followings hold:

$$IVint(A) \subset int_{IVP}(A),\ IVint(A) \subset int_{IVVP}(A)$$

and

$$cl_{IVP}(A) \subset IVcl(A),\ cl_{IVVP}(A) \subset IVcl(A).$$

Example 6.3. Let (X, τ) be the IVTS given in Example 5.7. Consider two IVSs $A = [\{a, c\}, \{a, b, c\}]$ and $B = [\{d\}, \{b, d\}]$ in X. Then

$$\begin{aligned} IVint(A) &= \bigcup\{G \in \tau : G \subset A\} = A_2 \cup A_7 = [\{c\}, \{a, b, c\}], \\ int_{IVP}(A) &= \bigcup\{G \in \tau_{IVP} : G \subset A\} = [\{c\}, \{a, b, c\}], \\ int_{IVVP}(A) &= \bigcup\{G \in \tau_{IVVP} : G \subset A\} \\ &= A_2 \cup A_7 \cup A_{14} \cup A_{17} = [\{a, c\}, \{a, b, c\}] \end{aligned}$$

and

$$\begin{aligned} IVcl(B) &= \bigcap\{F : F^c \in \tau,\ B \subset F\} = A_3^c \cap A_5^c \cap A_6^c \cap A_7^c = [\{d\}, \{a, b, d\}], \\ cl_{IVP}(B) &= \bigcap\{F : F^c \in \tau_{IV},\ B \subset F\} = A_3^c \cap A_5^c \cap A_6^c \cap A_7^c = [\{d\}, \{a, b, d\}], \\ cl_{IVVP}(B) &= \bigcap\{F : F^c \in \tau_{IVV},\ B \subset F\} \\ &= A_3^c \cap A_5^c \cap A_6^c \cap A_7^c \cap A_{11}^c \cap A_{14}^c \cap A_{17}^c = [\{d\}, \{b, d\}]. \end{aligned}$$

Thus, we can confirm that Remark 6.2 holds.

Proposition 6.4. *Let (X, τ) be an IVTS and let $A \in IVS(X)$. Then*

$$IVint(A^c) = (IVcl(A))^c \text{ and } IVcl(A^c) = (IVint(A))^c.$$

Proof.
$$\begin{aligned} IVint(A^c) &= \bigcup\{U \in \tau : U \subset A^c\} \\ &= \bigcup\{U \in \tau : U^- \subset {A^+}^c,\ U^+ \subset {A^-}^c\} \\ &= \bigcup\{U \in \tau : A^+ \subset {U^-}^c,\ A^- \subset {U^+}^c\} \\ &= (\bigcap\{U^c : U \in \tau, A \subset U^c\})^c \\ &= (IVcl(A))^c. \end{aligned}$$

Similarly, we can show that $IVcl(A^c) = (IVint(A))^c$. □

Proposition 6.5. *Let (X, τ) be an IVTS and let $A \in IVS(X)$. Then*

$$IVint(A) = int_{IVP}(A) \cap int_{IVVP}(A).$$

Proof. The proof is straightforward from Proposition 5.13 and Definition 6.1. □

The following is the immediate result of Definition 6.1, and Propositions 6.4 and 6.5.

Corollary 6.6. *Let (X, τ) be an IVTS and let $A \in IVS(X)$. Then*

$$IVcl(A) = cl_{IVP}(A) \cup cl_{IVVP}(A).$$

Example 6.7. Consider two IVSs $A = [\{a, c\}, \{a, b, c\}]$ and $B = [\{d\}, \{b, d\}]$ in X given in Example 6.3. Then from Example 6.3, we have:

$$IVint(A) = [\{c\}, \{a, b, c\}] = int_{IVP}(A),\ int_{IVVP}(A) = [\{a, c\}, \{a, b, c\}]$$

and

$$IVcl(B) = [\{d\}, \{a, b, d\} = cl_{IVP}(B),\ cl_{IVVP}(B) = [\{d\}, \{b, d\}].$$

Thus $int_{IVP}(A) \cap int_{IVVP}(A) = [\{c\}, \{a, b, c\}] = IVint(A)$
and

$$cl_{IVP}(B) \cup cl_{IVVP}(B) = [\{d\}, \{a, b, d\} = IVcl(B).$$

Theorem 6.8. *Let X be an IVTS and let $A \in IVS(X)$. Then*
(1) *$A \in IVC(X)$ if and only if $A = IVcl(A)$,*
(2) *$A \in IVO(X)$ if and only if $A = IVint(A)$.*

Proof. Straightforward. □

Proposition 6.9 (Kuratowski Closure Axioms). *Let X be an IVTS and let $A, B \in IVS(X)$. Then*
[IVK0] *if $A \subset B$, then $IVcl(A) \subset IVcl(B)$,*
[IVK1] $IVcl(\widetilde{\varnothing}) = \widetilde{\varnothing}$,
[IVK2] $A \subset IVcl(A)$,
[IVK3] $IVcl(IVcl(A)) = IVcl(A)$,
[IVK4] $IVcl(A \cup B) = IVcl(A) \cup IVcl(A)$.

Proof. Straightforward. □

Let $IVcl^* : IVS(X) \to IVS(X)$ be the mapping satisfying the properties [IVK1], [IVK2],[IVK3] and [IVK4]. Then we will call the mapping $IVcl^*$ as the interval-valued closure operator(briefly, IVCO) on X.

Proposition 6.10. *Let $IVcl^*$ be the IVCO on X. Then there exists a unique IVT τ on X such that $IVcl^*(A) = IVcl(A)$, for each $A \in IVS(X)$, where $IVcl(A)$ denotes the interval-valued closure of A in the IVTS (X, τ). In fact,*

$$\tau = \{A^c \in IVS(X) : IVcl^*(A) = A\}.$$

Proof. The proof is almost similar to the case of ordinary topological spaces. □

Proposition 6.11. *Let X be an IVTS and let $A, B \in IVS(X)$. Then*
[IVI0] *if $A \subset B$, then $IVint(A) \subset IVint(B)$,*
[IVI1] $IVint(\widetilde{X}) = \widetilde{X}$,
[IVI2] $IVint(A) \subset A$,
[IVI3] $IVint(IVint(A)) = IVint(A)$,
[IVI4] $IVint(A \cap B) = IVint(A) \cap IVint(A)$.

Proof. Straightforward. □

Let $IVint^* : IVS(X) \to IVS(X)$ be the mapping satisfying the properties [IVI1], [IVI2],[IVI3] and [IVI4]. Then we will call the mapping $IVint^*$ as the interval-valued interior operator (briefly, IVIO) on X.

Proposition 6.12. *Let $IVint^*$ be the IVIO on X. Then there exists a unique IVT τ on X such that $IVint^*(A) = IVint(A)$, for each $A \in IVS(X)$, where $IVint(A)$ denotes the interval-valued interior of A in the IVTS (X, τ). In fact,*

$$\tau = \{A \in IVS(X) : IVint^*(A) = A\}.$$

Proof. The proof is similar to one of Proposition 6.10. □

Definition 6.13. Let (X, τ) be an IVTS, $a \in X$ and let $A \in IVS(X)$. Then
(i) $a_{IVP} \in A$ is called a τ_{IVP}-interior point of A, if $A \in N(a_{IVP})$,
(ii) $a_{IVVP} \in A$ is called a τ_{IVVP}-interior point of A, if $A \in N(a_{IVVP})$.
We will denote the union of all τ_{IVP}-interior points [resp. τ_{IVVP}-interior points] of A as $\tau_{IVP} - int(A)$ [resp. $\tau_{IVVP} - int(A)$]. It is clear that

$$\tau_{IVP} - int(A) = \bigcup\{a_{IVP} : A \in N(a_{IVP})\}$$
$$[\text{resp. } \tau_{IVVP} - int(A) = \bigcup\{a_{IVVP} : A \in N(a_{IVVP})\}].$$

Theorem 6.14. *Let (X, τ) be an IVTS and let $A \in IVS(X)$.*
(1) *$A \in \tau_{IVP}$ if and only if $A_{IVP} = \tau_{IVP} - int(A)$.*
(2) *$A \in \tau_{IVVP}$ if and only if $A_{IVVP} = \tau_{IVVP} - int(A)$.*

Proof. (1) Suppose $A \in \tau_{IVP}$ and let $a_{IVP} \in A_{IVP}$. Then by the definition of A_{IVP}, $a_{IVP} \in A$. Thus by the definition of τ_{IVP}, $A \in N(a_{IVP})$. So $a_{IVP} \in \tau_{IVP} - int(A)$, i.e., $A_{IVP} \subset \tau_{IVP} - int(A)$.

Now let $a_{IVP} \in \tau_{IVP} - int(A)$. Then $A \in N(a_{IVP})$. Thus $a_{IVP} \in A$. So $a_{IVP} \in A_{IVP}$, i.e., $\tau_{IVP} - int(A) \subset A_{IVP}$. Hence $A_{IVP} = \tau_{IVP} - int(A)$.

Conversely, suppose the necessary condition holds and let $a_{IVP} \in A$. Then $a_{IVP} \in A_{IVP}$. Thus by the hypothesis, $a_{IVP} \in \tau_{IVP} - int(A)$. So $A \in N(a_{IVP})$. Hence by the definition of τ_{IVP}, $A \in \tau_{IVP}$.

(2) The proof is similar to that of (1). □

Proposition 6.15. *Let X be a non-empty set, $(A_j)_{j\in J} \subset IVS(X)$ and let $A = \bigcup_{j\in J} A_j$. Then*
(1) $A_{IVP} = \bigcup_{j\in J} A_{j,IVP}$,
(2) $A_{IVVP} = \bigcup_{j\in J} A_{j,IVVP}$.

Proof. (1) For each $j \in J$, let $A_j = [A_j^-, A_j^+]$. Then clearly, we have

$$A = \bigcup_{j\in J} A_j = [\bigcup_{j\in J} A_j^-, \bigcup_{j\in J} A_j^+].$$

Now let $a_{IVP} \in A$. Then $a_{IVP} \in \bigcup_{j\in J} A_j$. Thus $a \in \bigcup_{j\in J} A_j^-$. So there is $j_0 \in J$ such that $a \in A_{j_0}^-$. Hence $a_{IVP} \in A_{j_0,IVP}$, i.e., $a_{IVP} \in \bigcup_{j\in J} A_{j,IVP}$.

Conversely, suppose $a_{IVP} \in \bigcup_{j\in J} A_{j,IVP}$. Then there is $j_0 \in J$ such that $a_{IVP} \in A_{j_0,IVP}$. Thus $a \in A_{j_0}^-$. So $a \in \bigcup_{j\in J} A_j^-$. Hence $a_{IVP} \in A_{IVP}$. Therefore $A_{IVP} = \bigcup_{j\in J} A_{j,IV}$.

(2) The proof is similar to that of (1). □

Proposition 6.16. *Let (X, τ) be an IVTS and let $A \in IVS(X)$. Then*
(1) $\tau_{IVP} - int(A) = \bigcup_{G\subset A,\ G\in\tau_{IVP}} G_{IVP}$,
(2) $\tau_{IVVP} - int(A) = \bigcup_{G\subset A,\ G\in\tau_{IVVP}} G_{IVVP}$.

Proof. Suppose $a_{IVP} \in \bigcup_{G\subset A,\ G\in\tau_{IVP}} G_{IVP}$. Then there is $G \in \tau_{IVP}$ such that

$$G \subset A \text{ and } a_{IVP} \in G_{IVP}.$$

Thus $a_{IVP} \in G$. Since $G \in \tau_{IVP}$, $G \in N(a_{IVP})$. So $A \in N(a_{IVP})$. Hence $a_{IVP} \in \tau_{IVP} - int(A)$.

Conversely, suppose $a_{IVP} \in \tau_{IVP}$-$int(A)$. Then there is $G \in \tau$ such that

$$a_{IVP} \in G \subset A.$$

Moreover, $a_{IVP} \in G_{IVP}$ and $G \in \tau_{IVP}$. Thus $a_{IVP} \in \bigcup_{G\subset A,\ G\in\tau_{IVP}} G_{IVP}$. So the result holds.

(2) The proof is similar to that of (1). □

Remark 6.17. From Definitions 6.1 and 6.13, we have the following inclusions:

$$\tau_{IVP} - int(A) \subset int_{IVP}(A),\ \tau_{IVVP} - int(A) \subset int_{IVVP}(A).$$

But the reverse inclusions do not hold in general (See Example 6.18).

Example 6.18. Let $X = \{a, b, c, d, e\}$ and consider the IVTS (X, τ) given by:

$$\tau = \{\widetilde{\varnothing}, \widetilde{X}, A_1, A_2, A_3, A_4, A_5, A_5, A_7, A_8\},$$

where $A_1 = [\{a, b\}, \{a, b, c\}],\ A_2 = [\{a, d\}, \{a, b, d\}],\ A_3 = [\{d, e\}, X]$.
$A_4 = [\varnothing, \{a, b, c\}]\},\ A_5 = [\{a\}, \{a, b\}]\},\ A_6 = [\{d\}, \{a, b, d\}]$,
$A_7 = [\{a, b, d\}, X],\ A_8 = [\{a, c, d, e\}, X]\}$.
Then by Remark 5.6 (1), we have
$\tau_{_{IVP}} = \tau \bigcup \{A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16},$
$A_{17}, A_{18}, A_{19} A_{20}, A_{21}, A_{22}, A_{23}, A_{24}\}$
and

$$\tau_{_{IVVP}} = \tau \bigcup \{A_{25}, A_{26}, A_{27}, A_{28}, A_{29}, A_{30}\},$$

where $A_9 = [\{a, b\}, \{a, b, c, d\}],\ A_{10} = [\{a, b\}, \{a, b, c, e\}],\ A_{11} = [\{a, b\}, X]$,
$A_{12} = [\{a, d\}, \{a, b, c, d\}],\ A_{13} = [\{a, d\}, \{a, b, d, e\}],\ A_{14} = [\{a, d\}, X]$,
$A_{15} = [\{a\}, \{a, b, c\}],\ A_{16} = [\{a\}, \{a, b, d\}],\ A_{17} = [\{a\}, \{a, b, e\}]$,
$A_{18} = [\{a\}, \{a, b, c, d\}],\ A_{19} = [\{a\}, \{a, b, c, e\}],\ A_{20} = [\{a\}, \{a, b, d, e\}]$,
$A_{21} = [\{a\}, X],\ A_{22} = [\{d\}, \{a, b, c, d\}],\ A_{23} = [\{d\}, \{a, b, d, e\}]$,
$A_{24} = [\{d\}, X],\ A_{25} = [\{b\}, \{a, b, c\}],\ A_{26} = [\{c\}, \{a, b, c\}]$,
$A_{27} = [\{a, b\}, \{a, b, c\}],\ A_{28} = [\{b, c\}, \{a, b, c\}]$,
$A_{29} = [\{a, c\}, \{a, b, c\}],\ A_{30} = [\{a, b, c\}, \{a, b, c\}]$.

Now let us consider the IVS $B = [\{a, b, c\}, \{a, b, c, d\}]$ in X. Then

$$int_{_{IVP}}(B) = [\{a, b\}, \{a, b, c, d\}],\ int_{_{IVVP}}(B) = [\{a, b, c\}, \{a, b, c\}].$$

On the other hand, by Propositions 3.11 and 6.16, we have

$$\tau_{_{IVP}} - int(B) = [\{a, b\}, \{a, b\}],\ \tau_{_{IVVP}} - int(B) = [\varnothing, \{a, b, c\}].$$

Thus we can confirm Remark 6.17.

Remark 6.19. From Example 6.18, we have the following strict inclusions:

$$\tau_{_{IVP}} - int(A) \subset int_{_{IVP}}(A),\ \tau_{_{IVP}} - int(A) \neq int_{_{IVP}}(A),$$

$$\tau_{_{IVVP}} - int(A) \subset int_{_{IVVP}}(A),\ \tau_{_{IVVP}} - int(A) \neq int_{_{IVVP}}(A).$$

Proposition 6.20. *Let (X, τ) be an IVTS and let $A,\ B \in IVS(X)$. Then*
(1) $\tau_{_{IVP}} - int(A) \subset A_{IVP}$, $\tau_{_{IVVP}} - int(A) \subset A_{IVVP}$,
(2) *if* $A \subset B$, *then* $\tau_{_{IVP}} - int(A) \subset \tau_{_{IVP}} - int(B)$, $\tau_{_{IVVP}} - int(A) \subset \tau_{_{IVVP}} - int(B)$,
(3) $\tau_{_{IVP}} - int(A \cap B) = \tau_{_{IVP}} - int(A) \cap \tau_{_{IVP}} - int(B)$,
$\tau_{_{IVVP}} - int(A \cap B) = \tau_{_{IVVP}} - int(A) \cap \tau_{_{IVVP}} - int(B)$,
(4) $\tau_{_{IVP}} - int(\widetilde{X}) = \widetilde{X}$, $\tau_{_{IVVP}} - int(\widetilde{X}) = [\varnothing, X]$.

Proof. From Definition 6.13 and Proposition 6.16, the proofs of (1) and (2) are obvious. Also, the proof of (4) is clear from Proposition 6.16. we will prove only (3).

Let $a_{_{IVP}} \in \tau_{_{IVP}} - int(A \cap B)$. Then clearly, $A \cap B \in N(a_{_{IVP}})$. Thus $A \in N(a_{_{IVP}})$ and $B \in N(a_{_{IVP}})$. So $a_{_{IVP}} \in \tau_{_{IVP}} - int(A)$ and $a_{_{IVP}} \in \tau_{_{IVP}} - int(B)$, i.e.,

$$a_{_{IVP}} \in \tau_{_{IVP}} - int(A) \cap \tau_{_{IVP}} - int(B).$$

Hence $\tau_{_{IVP}} - int(A \cap B) \subset \tau_{_{IVP}} - int(A) \cap \tau_{_{IVP}} - int(B)$.

Conversely, suppose $a_{_{IVP}} \in \tau_{_{IVP}} - int(A) \cap \tau_{_{IVP}} - int(B)$. Then $A \in N(a_{_{IVP}})$ and $B \in N(a_{_{IVP}})$. Thus $A \cap B \in N(a_{_{IVP}})$. So $a_{_{IVP}}$ is a $\tau_{_{IVP}}$-interior point of $A \cap B$, i.e.,

$$a_{_{IVP}} \in \tau_{_{IVP}} - int(A \cap B).$$

Hence $\tau_{_{IVP}} - int(A) \cap \tau_{_{IVP}} - int(B) \subset \tau_{_{IVP}} - int(A \cap B)$. Therefore the equality holds.

The proof of the second part is similar to that of the first part. □

Remark 6.21. The equalities $\tau_{_{IVP}} - int(\tau_{_{IVP}} - int(A)) = \tau_{_{IVP}} - int(A)$ and $\tau_{_{IVVP}} - int(\tau_{_{IVVP}} - int(A)) = \tau_{_{IVVP}} - int(A)$ do not hold in general (See Example 6.22)

Example 6.22. Let (X, τ) be the IVTS and let B be the IVS in X given in Example 6.18. Then we can easily check that

$$\tau_{_{IVP}} - int(B) = [\{a, b\}, \{a, b\}] \text{ and } \tau_{_{IVP}} - int(\tau_{_{IVP}} - int(B)) = [\{a\}, \{a, b\}].$$

Thus $\tau_{_{IVP}} - int(B) \neq \tau_{_{IVP}} - int(\tau_{_{IVP}} - int(B))$.

7. Conclusions

By using the notion of interval-valued sets introduced by Yao [17], we defined an interval-valued (vanishing) point and obtained some of its properties. Also, we defined an interval-valued ideal and studied some of its properties. Next, we introduced the notion of interval-valued topological spaces which are considered as a bitopological space proposed by Kelly [9]. Moreover, we defined an interval-valued base and subbase and found the characterization of an interval-valued base. Finally, we introduced the concept of interval-valued neighborhoods and obtained some similar properties to classical neighborhoods. Furthermore, we defined an interval-valued closure and interior and dealt with their some properties. In the future, we expect that one can apply the notion of interval-valued sets to group and ring theory, BCK-algebra and category theory, etc.

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