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Abstract. In this paper, we have proposed a refinement for the concept of soft mapping along with a new definition of the soft element of soft sets to discuss the soft continuity of soft mapping over soft sets. Moreover, we have introduced the concept of soft modules over soft rings and studied the soft continuity of soft module operations over soft topological spaces to proceed with the main goal of this paper. Introducing the concept of soft topological soft modules.

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Keywords: Soft sets, Soft topology, Soft modules, Soft topological modules, Soft topological soft modules.

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1. Introduction

In 1999, Molodtsov [17] proposed the soft set theory as a new approach to managing uncertainties as he introduced the concept of the soft set to be a set associated with a set of parameters applied in several directions, that makes it a comprehensive extension for the theories of fuzzy sets [28] and rough sets [19].

Shabir and Naz [23] studied the soft topological structures by adding the notion of soft topology, which has been extensively studied and investigated by several authors like [6, 8, 18, 20, 21, 22].

In [11], the authors defined new types of belong and non-belong relations and utilized them to define strong types of soft separation axioms. Then Al-shami and El-Shafei [4, 10] presented two applications of these relations on the fields of soft separation axioms and decision-making problems. The interrelations between soft topological space and its parametric topological spaces were investigated in [5]. Recently, the concepts of soft topological ordered spaces and sum of soft topological spaces have been introduced in [3] and [6], respectively.

Some of authors went to study the algebraic side of soft set theory. For instance, [1] studied the soft algebraic structures of rings by introducing the idea of soft rings, which studied later by [27]. And others went to examine the connection between the soft topological structures and the soft algebraic structures such as the concept of soft topological soft groups and rings [25] and the concept of soft topological rings [26]. To promote this type of study, our motive is to complete the gaping in the studies of the connections between the soft topological space and the group, rings, and module theories by studying the combination between the modules and the soft topological spaces and introduce the concept of soft topological soft modules. To contribute to the completion of the vision of the main basic concepts in the topological algebra field in the sense of the soft theory.

In Section 2, we present well-known results of the essential preliminaries related to soft set and soft topological spaces. In Section 3, we propose a refinement for the concept of soft mapping along with a new definition of the soft element of soft sets to discuss the soft continuity of soft mapping over soft sets. In Section 4, we introduce the concept of soft modules over soft rings. In Section 5, we introduce the concept of soft topological soft modules over soft topological rings and studied some of their properties.

2. Preliminaries

In this section, we recall some basic concepts and results which we will use in this paper. Throughout this paper, R, X, Y, and Z are assumed to be initially universal sets and E assumed to be the fixed set of parameters.

Definition 2.1 ([16])**.** A soft set F_A over X is defined to be a mapping $F_A : A \to P(X)$, where A is a subset of the fixed parameters set E. If $A = E$, we put $F_E = F$.

Let $S(X)$ denotes the class of all soft sets over X. Let $F_A \in S(X)$. We may write $F_A = \{(a, F_A(a)) \mid a \in A\}$. If F_A is defined such that $F_A(a) = \phi, \forall a \in A$, then F_A is called a null soft set over X and denoted by $\tilde{\phi_A}$. And if F_A is defined such that $F_A(a) = X, \forall a \in A$, then F_A is called an absolute soft set over X, and denoted by $\tilde{X_A}$ ([16]). If $F_A(a)$ is a singleton set for each $a \in A$, then F_A is called a singleton soft set over X. We denote the class of all singleton soft sets over X by $\hat{X}$. Note that if $X \subseteq Y$, then $\hat{X} \subseteq \hat{Y}$.

We will use the notations $\hat{n}, \hat{m}, \hat{k}, \cdots$ to denote singleton soft sets.

Definition 2.2 ([12, 16])**.** Let $F_A, G_B \in S(X)$.

(i) F_A is called a soft subset of G_B, denoted by $F_A \subseteq G_B$, if $A \subseteq B$ and $F_A(a) \subseteq G_B(a)$, $\forall a \in A$.
(ii) F_A is called equal to G_B, denoted by $F_A = G_B$, if $F_A \subseteq G_B$ and $G_B \subseteq F_A$.
(iii) The intersection of F_A and G_B is defined to be the soft set $(F \cap G)_{A\cap B} \in S(X)$ such that $(F \cap G)_{A\cap B}(a) = F_A(a) \cap G_B(a), \forall a \in A \cap B$.
(iv) The union of F_A and G_B is defined to be the soft set $(F \cup G)_{A\cup B} \in S(X)$ such that for all $a \in A \cup B$

$$(F \cup G)_{A\cup B}(a) = \begin{cases} F_A(a) \cup G_B(a), & \text{if } a \in A \cap B \\ F_A(a), & \text{if } a \in A \setminus B \\ G_B(a), & \text{if } a \in B \setminus A. \end{cases}$$

Definition 2.3 ([7])**.** Let $F_A \in S(X)$ and $G_B \in S(Y)$. Then the Cartesian product of F_A and G_B is defined to be the soft set $(F \times G)_{A\times B} \in S(X \times Y)$ such that $(F \times G)_{A\times B}(a, b) = F_A(a) \times G_B(b)$, $\forall (a, b) \in A \times B$.

Definition 2.4.

(i) Let G be a group. A soft set $F_A \in S(G)$ is called a soft group (over G), if $F_A(a) \leqslant G$, $\forall a \in A$. [2]
(ii) Let R be a ring. A soft set $F_A \in S(R)$ is called a soft ring (over R), if $F_A(a)$ is a subring of R, $\forall a \in A$. [1]
(iii) Let M be a left (right) R-module. A soft set $F_A \in S(M)$ is called a left (right) soft R-module (over M), if $F_A(a)$ is a R-submodule of M, for all $a \in A$. [24]

Definition 2.5.
(i) Let G be a group and $F_A \in S(G)$ a soft group. Then a soft subset G_B of F_A is called a soft subgroup of F_A, if $G_B(b) \leqslant F_A(b)$, $\forall b \in B$. [2]
(ii) Let R be a ring and $F_A \in S(R)$ a soft ring. Then a soft subset G_B of F_A is called a soft subring of F_A, if $G_B(b)$ is a subring of $F_A(b)$, $\forall b \in B$. [1]
(iii) Let M be a left (right) R-module and $F_A \in S(M)$ a left (right) soft R-module. Then a soft subset G_B of F_A is called a left (right) soft R-submodule of F_A ,if $G_B(b)$ is a left (right) R-submodule of $F_A(b)$, $\forall b \in B$. [24]

Definition 2.6 ([25])**.** Suppose that X is a ring and $F, H \in S(X)$. The soft sets $FH, F+H, F-H, -F \in S(X)$ are defined as follows, for all $e \in E$:
(i) $(FH)(e) = F(e)\cdot H(e) = \{xy \mid x \in F(e), y \in H(e)\}$.
(ii) $(F+H)(e) = F(e) + H(e) = \{x + y \mid x \in F(e), y \in H(e)\}$.
(iii) $(F-H)(e) = F(e) - H(e) = \{x - y \mid x \in F(e), y \in H(e)\}$.
(iv) $-F(e) = -(F(e)) = \{-x \mid x \in F(e)\}$.

Definition 2.7 ([14])**.** Let R and S be rings and let $f : R \to S$ be a mapping.
(i) f is called a ring homomorphism, if it satisfies the following conditions: for all $a, b \in R$,
(a) $f(a+b) = f(a) + f(b)$,
(b) $f(ab) = f(a)f(b)$,
(c) $f(1_R) = 1_S$.
(ii) f is called a ring isomorphism, if it is a one-to-one and onto ring homomorphism.
(iii) Let $f : R \to S$ be a ring homomorphism (isomorphism), τ a topology on R and υ a topology on S. Then f is called an open ring homomorphism (isomorphism), if $f(H) \in \upsilon, \forall H \in \tau$.

Definition 2.8 ([9])**.** Let F_A and G_B be soft rings over rings X and Y individually. A soft mapping $(\psi, \varphi) : F_A \to G_B$ is called a soft ring homomorphism (resp. isomorphism), if ψ is a ring homomorphism (resp. isomorphism) and φ is a bijection.

Similarly, one can define soft ring monomorphism (epimorphism). Note that a soft ring F_A over a ring X is called a softly homomorphic (resp. softly isomorphic) to a soft ring G_B over a ring Y if and only if there must be a soft ring homomorphism (resp. isomorphism) from F_A to G_B and it denoted by $F_A \sim G_B$ (resp. $F_A \simeq G_B$).

Definition 2.9 ([23])**.** (i) Let τ be a family of soft sets in $S(X)$ such that
(a) $\tilde{\phi}, \tilde{X} \in \tau$,
(b) τ is closed under finite intersections,
(c) τ is closed under arbitrary union.
Then τ is called a soft topology on X and the pair (X, τ) is called a soft topological space (in short S.T.S).
(ii) Let (X, τ) be a S.T.S. A soft set $H \in S(X)$ is called soft open, if $H \in \tau$.

Definition 2.10 ([13]). Let (X, τ) be an S.T.S and $G \subseteq X$. The soft topology $\tau_G = \{\tilde{G} \cap H \mid H \in \tau\}$ on G is called a soft relative topology generated by the set G and the soft topological space (G, τ_G) is called a soft subspace of (X, τ).

Throughout this paper we will deal with soft topological spaces defined over a soft set $F \in S(X)$. Thus, we will recall the following definition for soft topology.

Definition 2.11 ([8]). Let $F \in S(X)$ and τ be a family of soft subsets of F.
(i) τ is called a soft topology on F, if it satisfies the following conditions:
(a) $F, \tilde{\phi} \in \tau$,
(b) τ is closed under the finite intersection,
(c) τ is closed under the arbitrary union.
(ii) The soft topology τ and soft set F together called the soft topological space (in short, S.T.S) and denoted by (F, τ).
(iii) $V \in S(X)$ is called a soft open set, if $V \in \tau$ and τ is a soft topology on F.

Let $F \in S(X)$ and the set $G = \{F(e) \mid \forall e \in E\}$. Note that $\tilde{G} = F$ and $\tau_F = \tau_{\tilde{G}}$ which is a soft subspace of τ. In other words, the soft topological space (F, τ) defined in (Çağman et al 2011, [8]) is equivalent to the soft topological subspace $(\tilde{G}, \tau_{\tilde{G}})$ of (X, τ).

Note that, the existence of tow definitions for the same concept of soft topology that conflicts the readier. To avoid this confliction from now on, we will refer the definition of (Çağman et al 2011, [8]) for the soft topology τ defined over a soft set F, as a soft subspace generated by an arbitrary soft set F defined over the universal set X, denoted by (F, τ_F).

Proposition 2.12 ([25]). *Suppose that (X, τ) is an S.T.S and $H, G \in S(X)$ such that $H \subseteq G$. Then $\tau_H = (\tau_G)_H$.*

Note that if τ and υ are two two soft topological spaces on X, then the soft intersection $\tau \cap \upsilon$ of τ and υ is defined by $\tau \cap \upsilon = \{V \cap W \mid V \in \tau, W \in \upsilon\}$.

Proposition 2.13 ([25]). *Suppose that (X, τ) and (X, υ) are soft topological spaces and $F, H \in S(X)$. If $U \in \tau_F$ and $V \in \upsilon_H$, then $U \cap V \in (\tau \cap \upsilon)_{F \cap H}$.*

Definition 2.14 ([23]). Let (X, τ) be an S.T.S. If $\tau = \{\tilde{\phi}, \tilde{X}\}$, then τ is called soft indiscreet topology on X and if τ is the collection of all soft subsets $F \in S(X)$, then τ is called the soft discrete topology on X.

Proposition 2.15. *Let (X, τ) be an S.T.S.*
(1) [23] $\tau^e = \{H(e) \mid H \in \tau\}$ *is a topology on X for each $e \in E$.*
(2) [18] $\tau^* = \{F \in S(X) \mid F(e) \in \tau^e, \forall e \in E\}$ *is a soft topology on X and* $(\tau^*)^e = \tau^e \; \forall e \in E$.

Definition 2.16 ([13]). Let (X, τ) be an S.T.S and $G \subseteq X$. The soft topology $\tau_G = \{\tilde{G} \cap H \mid H \in \tau\}$ on G is called a soft relative topology generated by the set G and the soft topological space (G, τ_G) is called a soft subspace of (X, τ).

Definition 2.17 ([27]). (i) Let G be an additive group. Then the topological space (G, τ) is called a topological group, if the mapping $(x, y) \mapsto x - y$ from $(G \times G, \tau \times \tau)$ to (G, τ) is continuous.
(ii) Let R be a ring. Then the topological space (R, τ) is called a topological ring (in short T.R), if it satisfies the following conditions:

(a) The mapping $(x, y) \mapsto x - y$ from $(R \times R, \tau \times \tau)$ to (R, τ) is continuous, and
(b) The mapping $(x, y) \mapsto xy$ from $(R \times R, \tau \times \tau)$ to (R, τ) is continuous.

Definition 2.18 ([27]). Let R be a ring, M a left (right) R-module and (R, ν) a topological ring. Then a topological space (M, τ) is called left (right) topological R-module, if it satisfies the following conditions:

(i) The mapping $(x, y) \mapsto x + y$ from $(M \times M, \tau \times \tau)$ to (M, τ) is continuous,
(ii) The mapping $x \mapsto -x$ from (M, τ) to (M, τ) is continuous.
(iii) The mapping $(r, x) \mapsto rx$ from $(R \times M, \nu \times \tau)$ to (M, τ) is continuous.

Definition 2.19 ([25]). Suppose that $F \in S(X)$ is a soft ring and (X, τ) is a soft topological space. Then (F, τ_F) is called soft topological soft ring over X and denoted by S.T.S.R if the following conditions are satisfied:

(i) The soft mapping $\tilde{f} : (F \times F, \tau_F \times \tau_F) \to (F, \tau_F)$ is soft continuous, where

$$\begin{aligned} f : X \times X &\to X \\ (x, y) &\mapsto x + y \end{aligned}$$

(ii) The soft mapping $\tilde{j} : (F, \tau_F) \to (F, \tau_F)$ is soft continuous, where

$$\begin{aligned} j : X &\to X \\ x &\mapsto -x \end{aligned}$$

(iii) The soft mapping $\tilde{g} : (F \times F, \tau_F \times \tau_F) \to (F, \tau_F)$ is soft continuous, where

$$\begin{aligned} g : X \times X &\to X \\ (x, y) &\mapsto xy \end{aligned}$$

3. On soft mapping

In literature, Kharal and Ahmad [15] introduced the mapping between two classes of a soft set where the image of the soft set is soft set, but in their work, the behavior of the mapping is unpredictable, where the co-domain of mapping can not be restricted on a particular soft set. In this section, we will propose a refinement for the soft mapping that has been suggested by Kharal and Ahmad [15].

Definition 3.1. Let $F_A \in S(X)$ and $G_B \in S(Y)$. Let $\varphi : A \to B$ and $\psi : X \to Y$. The pair (ψ, φ) is called a soft mapping from F_A to G_B, if

$$\psi\left(F_A(a)\right) = G_B\left(\varphi(a)\right) \text{ and } \psi^{-1}\left(G_B\left(\varphi(a)\right)\right) = \bigcup_{\varphi(a')=\varphi(a)} F_A(a') \quad , \forall a \in A.$$

Definition 3.2. Let $F_A \in S(X)$ and $G_B \in S(Y)$ and $(\psi, \varphi) : F_A \to G_B$. Let $H_C \in S(X)$ and $K_D \in S(Y)$ such that $H_C \subseteq F_A$ and $K_D \subseteq G_B$. Then

(i) the image of H_C under (ψ, φ) is the soft set $((\psi, \varphi)(H_C))_{\varphi(C)} \in S(Y)$ such that

$$((\psi, \varphi)(H_C))_{\varphi(C)}(\varphi(c)) = \bigcup_{\varphi(c')=\varphi(c)} \psi(H_C(c')), \forall c \in C,$$

(ii) the inverse image of K_D under (ψ, φ) is the soft set $\left((\psi, \varphi)^{-1}(K_D)\right)_{\varphi^{-1}(D)} \in S(X)$ such that

$$\left((\psi, \varphi)^{-1}(K_D)\right)_{\varphi^{-1}(D)}(a) = \psi^{-1}\left(K_D\left(\varphi(a)\right)\right), \forall a \in \varphi^{-1}(D).$$

Definition 3.3. Let $F_A \in S(X)$ and $G_B \in S(Y)$ and $(\psi, \varphi) : F_A \to G_B$ be a soft mapping.

(i) $(\psi, \varphi) : F_A \to G_B$ is called surjective, if φ and ψ are surjective.
(ii) (ψ, φ) is called injective, if φ and ψ are injective.
(iii) $(\psi, \varphi) : F_A \to G_B$ is called bijective, if it is both injective and surjective.

Proposition 3.4. *Let $F_A \in S(X)$ and $G_B \in S(Y)$. Then $(\psi, \varphi) : F_A \to G_B$ is injective if and only if*

(1) *φ is injective,*
(2) $\hat{x}, \hat{y} \subseteq F_A; (\psi, \varphi)(\hat{x}) = (\psi, \varphi)(\hat{y}) \Rightarrow \hat{x} = \hat{y}$.

Proof. Let (ψ, φ) be an injective. Let $\hat{x}, \hat{y} \subseteq F_A$ such that $(\psi, \varphi)(\hat{x}) = (\psi, \varphi)(\hat{y})$. Then $\psi(x_a) = \psi(y_a), \forall a \in A$. Thus $x_a = y_a, \forall a \in A$. So $\hat{x} = \hat{y}$.

Suppose (1) and (2) are valid. Let $x, y \in X$ such that $\psi(x) = \psi(y)$. Let $\hat{u}, \hat{v} \subseteq F_A$ such that $u_a = x,\ v_a = y, a \in A$ and $u_{a'} = v_{a'}, \forall a' \in A \setminus \{a\}$. Then $\psi(u_{a''}) = \psi(v_{a''})$, $\forall a'' \in A$. Since φ is injective, $(\psi, \varphi)(\hat{u}) = (\psi, \varphi)(\hat{v})$. Thus $\hat{u} = \hat{v}$. So $u_a = v_a$. Hence $x = y$. Therefore ψ is injective. □

Note that if $A = B = E$ and $\varphi = id_E$, then $(\psi, \varphi) : F \to G$ is injective if and only if $\bar{x}, \bar{y} \in F; (\psi, \varphi)(\bar{x}) = (\psi, \varphi)(\bar{y}) \Rightarrow \bar{x} = \bar{y}$.

Definition 3.5. Let $F_A \in S(X)$, $G_B \in S(Y)$ and $H_C \in S(Z)$. Let $(\psi, \varphi) : F_A \to G_B$ and $(\alpha, \beta) : G_B \to H_C$. Then the composition of (ψ, φ) and (α, β), denoted by $(\alpha, \beta)(\psi, \varphi)$ is defined to be the soft mapping $(\alpha\,\psi, \beta\,\varphi)$ from F_A to H_C.

Definition 3.6. Let $F_A \in S(X)$, $G_B \in S(Y)$. Let $(\psi, \varphi) : F_A \to G_B$ be a bijection. Then the inverse of (ψ, φ), denoted by $(\psi, \varphi)^{-1}$, is defined to be the soft map $(\psi^{-1}, \varphi^{-1})$ from G_B to F_A.

If $A = B = E$, $\varphi = id_E$ and $\psi : X \to Y$ is bijective, then $\tilde{\psi} : F \to G$ has the inverse $\widetilde{\psi^{-1}} : G \to F$. Note that $\widetilde{\psi^{-1}} = \tilde{\psi}^{-1}$, $\tilde{\psi}^{-1}\tilde{\psi} = \tilde{1}_X$ (from F to F) and $\tilde{\psi}\tilde{\psi}^{-1} = \tilde{1}_Y$ (from G to G).

Remark 3.7. Let $F_A \in S(X)$, $G_B \in S(Y)$ and $(\psi, \varphi) : F_A \to G_B$.

(1) If $H_C \subseteq F_A$, then $((\psi, \varphi)(H_C))_{\varphi(C)} \subseteq G_B$ and $(\psi, \varphi)(F_A) = G_B\ |_{\varphi(A)}$.
(2) If $K_D \subseteq G_B$ and φ is injective, then$\left((\psi, \varphi)^{-1}(K_D)\right)_{\varphi^{-1}(D)} \subseteq F_A$.
(3) We denote the restriction of ψ over $\bigcup \{F_A(a') \mid a' \in A, \varphi(a') = \varphi(a)\}$ by $\psi_{\varphi(a)}$. So, it is clear that $\psi_{\varphi(a)} : \bigcup\limits_{\varphi(a')=\varphi(a)} F_A(a') \to G_B(\varphi(a))$ is a mapping, $\forall a \in A$.

Definition 3.8. Let $F_A \in S(X)$ such that $F_A(a) \neq \phi$, $\forall a \in A$. A soft element in F_A is defined to be a collection of elements x_a, where $a \in A$ such that $x_a \in F_A(a)$ for all $a \in A$, and written as $\bar{x} = (x_a)_{a \in A} \in F_A$. If $x \in X$ and $(x_a)_{a \in A} \in F_A$ such that $x_a = x \in F_A(a)$ for all $a \in A$, then we put $\tilde{x} \in F_A$.

Note that each soft element $(x_a)_{a \in A}$ in $F_A \in S(X)$ corresponds to a singleton soft set $\hat{x} = \{(a, \{x_a\}) \mid a \in A\}$ of F_A.

Let $F_A \in S(X)$, $G_B \in S(Y)$. Let $(\psi, \varphi) : F_A \to G_B$ such that φ is a bijection. Then the soft image of $\bar{x} = (x_a)_{a \in A} \in F_A$ under (ψ, φ) is defined to be the soft element $(\psi, \varphi)(\bar{x}) = (\psi(x_a))_{\varphi(a) \in B} \in G_B$. Which corresponds to the soft subset $\{(\varphi(a), \{\psi(x_a)\}) \mid a \in A\}$ of G_B. If $A = B = E$ and $\varphi = id_E$, then $\tilde{\psi}(\bar{x}) = (\psi(x_e))_{e \in E} \in G$, $\forall \bar{x} \in F$ and the inverse image of $\bar{y} \in G$ is the soft set $\tilde{\psi}^{-1}(\bar{y}) \in S(X)$ such that $(\psi, \varphi)^{-1}(\bar{y})(e) = \psi^{-1}(y_e), \forall e \in E$.

Definition 3.9. Let (X,τ) be an S.T.S. Let $\bar{x}=(x_e)_{e\in E}\in\tilde{X}$. A soft subset $U_{\bar{x}}$ of $\tilde{X}$ is called a soft neighborhood of $\bar{x}$ if there exists $H\in\tau$ such that $\bar{x}\in H\subseteq U_{\bar{x}}$. A soft neighborhood $U_{\bar{x}}$ of $\bar{x}\in\tilde{X}$ is called open if $U_{\bar{x}}\in\tau$.

Proposition 3.10. *Let (X,τ) be an S.T.S. Let $\bar{x}=(x_e)_{e\in E}\in\tilde{X}$ and $U\in S(X)$. Then*

(1) *U is a soft (open) neighborhood of $\bar{x}$ in $(X,\tau)\Rightarrow U(e)$ is an (open) neighborhood of x_e in (X,τ^e), for all $e\in E$,*

(2) *$U(e)$ is an (open) neighborhood of x_e in (X,τ^e), for all $e\in E\Rightarrow U$ is a soft (open) neighborhood of $\bar{x}$ in (X,τ^*).*

Proof. The proof come out directly from definition of soft neighborhood and the constructions of τ^* and τ^e □

Definition 3.11. Let (X,τ) and (Y,ν) be two S.T.S's and let $\tilde{f}:(X,\tau)\to(Y,\nu)$ be a soft mapping. Then $\tilde{f}$ is called soft continuous, if for every $\bar{x}\in\tilde{X}$ and every soft open neighborhood $U_{\tilde{f}(\bar{x})}$ of $\tilde{f}(\bar{x})$, there exists a soft open neighborhood $U_{\bar{x}}$ of $\bar{x}$ such that $\tilde{f}(\bar{x})\in\tilde{f}(U_{\bar{x}})\subseteq U_{\tilde{f}(\bar{x})}$.

Definition 3.12. Let $F\in S(X)$ and $G\in S(Y)$. Let (F,τ) and (G,ν) be soft subspace of (X,ω) and (Y,Ω),respectively. Let $\tilde{\psi}:F\to G$. Then

(i) $\tilde{f}$ is called soft continuous, if for every $\bar{x}\in F$ and every soft open neighborhood $U_{\tilde{f}(\bar{x})}$ of $\tilde{f}(\bar{x})$, there exists a soft open neighborhood $U_{\bar{x}}$ of $\bar{x}$ such that $\tilde{f}(\bar{x})\in\tilde{f}(U_{\bar{x}})\subseteq U_{\tilde{f}(\bar{x})}$,

(ii) $\tilde{f}$ is called soft homeomorphism, if $\tilde{f}$ is a bijection and $\tilde{f}$ and $\tilde{f}^{-1}$ are soft continuous,

(iii) $\tilde{f}$ is called soft open, if $\tilde{f}(H)\in\nu$ for every $H\in\tau$.

Proposition 3.13. *Let $F\in S(X)$ and $G\in S(Y)$. Let (F,τ) and (G,ν) be soft subspace of (X,ω) and (Y,Ω),respectively.*

(1) *If the soft mapping $\tilde{f}:(F,\tau)\to(G,\nu)$ is soft continuous, then $f_e:(F(e),\tau^e)\to(G(e),\nu^e)$ is continuous for every $e\in E$.*

(2) *If the mapping $f_e:(F(e),\tau^e)\to(G(e),\nu^e)$ is continuous for every $e\in E$, then $\tilde{f}:(F,\tau^*)\to(G,\nu^*)$ is soft continuous.*

Proof. (1) Let $\tilde{f}:(F,\tau)\to(G,\nu)$ be a soft continuous and $\bar{x}\in F$. Then for every $\bar{x}\in F$ and every soft open neighborhood $U_{\tilde{f}(\bar{x})}$ of $\tilde{f}(\bar{x})$, there exists a soft open neighborhood $U_{\bar{x}}$ of $\bar{x}$ such that $\tilde{f}(\bar{x})\in\tilde{f}(U_{\bar{x}})\subseteq U_{\tilde{f}(\bar{x})}$. Thus $f(x_e)\in f(U_{\bar{x}}(e))\subseteq U_{f(x_e)}\quad\forall e\in E$. But by Proposition 3.10 (1), $U_{\bar{x}}(e)$ is an open neighborhood of x_e in $(F(e),\tau^e)$. So $U_{f(x_e)}$ is an open neighborhood of $f(x_e)$ in (G,ν) for all $e\in E$. Hence f_e is continuous for all $e\in E$.

(2) Suppose $f_e:(F(e),\tau^e)\to(G(e),\nu^e)$ be continuous for all $e\in E$. Let $\bar{x}$ and $U_{\tilde{f}(\bar{x})}$ be a soft open neighborhood of $\tilde{f}(\bar{x})$ in (G,ν^*). Then it follows, from Proposition 3.10 (1), that $U_{\tilde{f}(\bar{x})}(e)$ is an open neighborhood of $f(x_e)$ in $(G(e),\nu^e)$ for all $e\in E$. Put $U_{\tilde{f}(\bar{x})}(e)=U_{f(x_e)}$ for all $e\in E$. Then there exists an open neighborhood U_{x_e} of x_e in $(F(e),\tau^e)$ such that

$$f_e(x_e)\in f_e(U_{x_e})\subseteq U_{f_e(x_e)},\quad\forall e\in E.$$

Let $U\in S(X)$ such that $U(e)=U_{x_e}\ \forall e\in E$. Then U is a soft open neighborhood of $\bar{x}$ in (F,τ^*) and $\tilde{f}(\bar{x})\in\tilde{f}(U)\subseteq U_{\tilde{f}(\bar{x})}$. Thus $\tilde{f}$ is soft continuous. □

Proposition 3.14. *Let $F \in S(X)$ and $G \in S(Y)$. Let (F, τ) and (G, ν) be soft subspace of (X, ω) and (Y, Ω),respectively. Let $\tilde{\psi} : F \to G$. Let $\tilde{\psi}(\tau) = \{\tilde{\psi}(H) \mid H \in \tau\}$ and $\tilde{\psi}^{-1}(\nu) = \{\tilde{\psi}^{-1}(D) \mid D \in \nu\}$.*

(1) If $\tilde{\psi}$ is injective, then $\tilde{\psi}(\tau)$ is a soft topology on $\tilde{\psi}(F)$.

(2) $\tilde{\psi}^{-1}(\nu)$ is a soft topology on F.

(3) The soft mapping $\tilde{\psi} : F \to G$ is soft continuous if and only if for each $\bar{x} \in F$ and each soft neighborhood $U_{\tilde{\psi}(\bar{x})}$ of $\tilde{\psi}(\bar{x})$ in ν, the soft set $\tilde{\psi}^{-1}(U_{\tilde{\psi}(\bar{x})})$ is a soft open in τ.

(4) Let $\bar{x} \in F$ and $U_{\bar{x}} \in S(X)$. Let $\tilde{\psi}$ be injective. Then $U_{\bar{x}}$ is a soft open neighborhood of $\bar{x}$ in (F, τ) if and only if $\tilde{\psi}(U_{\bar{x}})$ is a soft open neighborhood of $\tilde{\psi}(\bar{x})$ in $\left(\tilde{\psi}(F), \tilde{\psi}(\tau)\right)$. (So in this case, we write in this case $\tilde{\psi}(U_{\bar{x}}) = U_{\tilde{\psi}(\bar{x})}$).

(5) Let $\bar{y} \in G$ and $U_{\bar{y}} \in S(Y)$. Let $\tilde{\psi}$ be surjective. Then $U_{\bar{y}}$ is a soft open neighborhood of $\bar{y}$ in (G, ν) if and only if $\tilde{\psi}^{-1}(U_{\bar{y}})$ is a soft open neighborhood of $\bar{x} \in F$ with $\tilde{\psi}(\bar{x}) = \bar{y}$ in $(F, \tilde{\psi}^{-1}(\nu))$. (So in this case, we write in this case $\tilde{\psi}^{-1}(U_{\bar{y}}) = U_{\bar{x}}$ such that $\tilde{\psi}(\bar{x}) = \bar{y}$).

Proof. (1) and (2) can easily be proved by cheeking the condition of soft topology.

(3) $\tilde{\psi} : F \to G$ is soft continuous if and only if for every $\bar{x} \in F$ and every soft open neighborhood $U_{\tilde{f}(\bar{x})}$ of $\tilde{f}(\bar{x})$, there exists a soft open neighborhood $U_{\bar{x}}$ of $\bar{x}$ such that $\tilde{f}(\bar{x}) \in \tilde{f}(U_{\bar{x}}) \subseteq U_{\tilde{f}(\bar{x})}$. That implies $\tilde{\psi} : F \to G$ is soft continuous if and only if for every soft open neighborhood $U_{\tilde{f}(\bar{x})}$ of $\tilde{f}(\bar{x})$, there exists a soft open neighborhood $U_{\bar{x}}$ of $\bar{x}$, such that $\bar{x} \in U_{\bar{x}} \subseteq \tilde{\psi}^{-1}(U_{\tilde{f}(\bar{x})})$. That implies $\tilde{\psi} : F \to G$ is soft continuous if and only $\tilde{\psi}^{-1}(U_{\tilde{f}(\bar{x})})$ is soft open τ.

(4) ($\Rightarrow$) Let $U_{\bar{x}}$ be a soft open neighborhood of $\bar{x}$ in (F, τ). Then there exists $U \in \tau$ such that $\bar{x} \in U \subseteq U_{\bar{x}}$. Thus $\tilde{\psi}(\bar{x}) \in \tilde{\psi}(U) \subseteq \tilde{\psi}(U_{\bar{x}})$. But $\tilde{\psi}(U) \in \tilde{\psi}(\tau)$. So $\tilde{\psi}(U_{\bar{x}})$ is a soft open neighborhood of $\tilde{\psi}(\bar{x})$ in $\left(\tilde{\psi}(F), \tilde{\psi}(\tau)\right)$.

($\Leftarrow$) Let $\tilde{\psi}(U_{\bar{x}})$ a soft open neighborhood of $\tilde{\psi}(\bar{x})$ in $\left(\tilde{\psi}(F), \tilde{\psi}(\tau)\right)$. Then there exists $\tilde{\psi}(V) \in \tilde{\psi}(\tau)$, where $V \in \tau$ such that $\tilde{\psi}(\bar{x}) \in \tilde{\psi}(V) \subseteq \tilde{\psi}(U_{\bar{x}})$. Since $\tilde{\psi}$ is injective, $\bar{x} \in V \subseteq U_{\bar{x}}$. Since $\tilde{\psi}(U_{\bar{x}}) \in \tilde{\psi}(\tau)$ and $\tilde{\psi}$ is injective, we have $U_{\bar{x}} \in \tau$. Thus $U_{\bar{x}}$ is a soft open neighborhood of $\bar{x}$ in (F, τ).

(5) ($\Rightarrow$) Let $U_{\bar{y}}$ be a soft open neighborhood of $\bar{y}$ in (G, ν). Then $U_{\bar{y}} \in \nu$ and there exists $K \in \nu$ such that $\bar{y} \in K \subseteq U_{\bar{y}}$. Thus $\bar{x} \in \tilde{\psi}^{-1}(K) \subseteq \tilde{\psi}^{-1}(U_{\bar{y}})$, where $\bar{x} \in F$ such that $\tilde{\psi}(\bar{x}) = \bar{y}$. But $\tilde{\psi}^{-1}(K), \tilde{\psi}^{-1}(U_{\bar{y}}) \in \tilde{\psi}^{-1}(\nu)$. So $\tilde{\psi}^{-1}(U_{\bar{y}})$ is a soft open neighborhood of $\bar{x}$ in $(F, \tilde{\psi}^{-1}(\nu))$ such that $\tilde{\psi}(\bar{x}) = \bar{y}$.

($\Leftarrow$) Let $\tilde{\psi}^{-1}(U_{\bar{y}})$ be a soft open neighborhood of $\bar{x}$ with $\tilde{\psi}(\bar{x}) = \bar{y}$ in $(F, \tilde{\psi}^{-1}(\nu))$. Since $\tilde{\psi}^{-1}(U_{\bar{y}}) \in \tilde{\psi}^{-1}(\nu)$ and $\tilde{\psi}$ is surjective, we have $U_{\bar{y}} \in \nu$. Then there exists $\tilde{\psi}^{-1}(D) \in \tilde{\psi}^{-1}(\nu)$, where $D \in \nu$ such that $\bar{x} \in \tilde{\psi}^{-1}(D) \subseteq \tilde{\psi}^{-1}(U_{\bar{y}})$. Thus $\tilde{\psi}(\bar{x}) \in D \subseteq U_{\bar{y}}$. So $U_{\bar{y}}$ is a soft open neighborhood of $\tilde{\psi}(\bar{x}) = \bar{y}$ in (F, ν). $\square$

4. Soft Modules over soft rings

In algebra, the construction of modules mainly depends on the rings. In literature (Sun et al, [24]) defined the modules in soft setting by assuming the initial universal set X is a module over an arbitrary ring R and any soft set $F \in S(X)$ is called a soft modules if $F(e)$ is submodule of X for each parameter $e \in E$. To be more convenient with the soft set theory we will introduce a new definition for the soft module that provides a natural extension of modules in algebra where we will use the soft ring instead of rings to define soft modules.

From now on, R assumed to be a ring and Y, Z be left R-modules.

Definition 4.1. Let $F \in S(R)$ be a soft ring and $M \in S(Y)$. Then M is called a soft left F-module (over Y), if $M(e)$ is a left $F(e)$-submodule of Y for all $e \in E$.

Similarly, one can define soft right F-module. Note that $M \in S(Y)$ is a soft $\tilde{R}$-module if and only if M is a soft R-module.

Example 4.2. Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$, $E = \{e_1, e_2\}$, $F(e_1) = R$ and $F(e_2) = \{(0,0), (1,1), (2,2)\}$. Then F is a soft ring over R. Let $Y = R$ and $M \in S(Y)$ such that $M(e_1) = R$ and $M(e_2) = F(e_2)$. Then M is a soft F-module over Y.

Let $M, N \in S(Y)$ be soft F-modules. Since $M(e) \cap N(e)$ is a $F(e)$-submodule of Y for all $e \in E$, then $M \cap N$ is a soft F-module over Y.

Definition 4.3. Let $M, N \in S(Y)$ be soft F-modules. Then M is called a soft F-submodule of N, if $M(e)$ is a $F(e)$-submodule of $N(e)$ for all $e \in E$.

Remark 4.4. If M is a R-submodule of N, we write $M \leqslant_R N$ and if M is a soft F-submodule of N, we write $M \leqslant_F N$.

Definition 4.5. Let $M \in S(Y)$ and $N \in S(Z)$ be soft F-modules. The soft mapping $\tilde{\psi} : M \to N$ is called soft F-homomorphism if $\psi : Y \to Z$ is R-homomorphism.

Similarly, one can define F-monomorphisms, F-epimorphisms and F-isomorphisms.

Remark 4.6. Let $M \in S(Y)$ and $N \in S(Z)$ be soft F-modules and $\tilde{\psi} : M \to N$ be a soft F-homomorphism. Then $\psi_e : M(e) \to N(e)$ is $F(e)$-homomorphism for all $e \in E$.

Proposition 4.7. *Let $M, M' \in S(Y)$ and $N, N' \in S(Z)$ be soft F-modules. Let $\tilde{\psi} : M \to N$ be a soft F-homomorphism. If $M' \leqslant_F M$ and $N' \leqslant_F N$, then $\tilde{\psi}(M') \leqslant_F N$ and $\tilde{\psi}^{-1}(N') \leqslant_F M$.*

Proof. Let $e \in E$. Then we have

$$\left(\tilde{\psi}(M')\right)(e) = \psi\left(M'(e)\right) \leqslant_{F(e)} \psi\left(M(e)\right) \leqslant_{F(e)} N(e).$$

Then $\tilde{\psi}(M') \leqslant_F N$. Also, we have

$$\left(\tilde{\psi}^{-1}(N')\right)(e) = \psi^{-1}\left(N'(e)\right) \leqslant_{F(e)} \psi^{-1}\left(N(e)\right) = M(e).$$

Thus $\tilde{\psi}^{-1}(N') \leqslant_F M$. □

Remark 4.8. (1) Let $M \in S(Y)$ be a soft F-module. Let $\bar{r}, \bar{s} \in F$ and $\bar{x}, \bar{y} \in M$. Then

- (i) The addition of $\bar{x}$ and $\bar{y}$ is defined to be the soft element $\bar{x} + \bar{y} = (x_e + y_e)_{e\in E} \in M$,
- (ii) The multiplication of $\bar{r}$ and $\bar{s}$ is defined to be the soft element $\bar{r}\bar{s} = (r_e s_e)_{e\in E} \in F$,
- (iii) The multiplication of $\bar{r}$ and $\bar{x}$ is defined to be the soft element $\bar{r}\bar{x} = (r_e x_e)_{e\in E} \in M$ and
- (iv) The additive inverse of $\bar{x}$ is defined to be the soft element $-\bar{x}$ such that $-\bar{x} = (-x_e)_{e\in E}$.

(2) Let $M \in S(Y)$. Then M is a soft F-module if and only if M is closed with respect to addition of soft elements and multiplication by soft elements in F.

(3) Let $M \in S(Y)$ and $N \in S(Z)$ be soft F-modules and let $\tilde{\psi} : M \to N$ be a soft F-homomorphism. Then $\tilde{\psi}(\bar{x} + \bar{y}) = \tilde{\psi}(\bar{x}) + \tilde{\psi}(\bar{y})$ and $\tilde{\psi}(\bar{r}\bar{x}) = \bar{r}\tilde{\psi}(\bar{x})$, $\forall \bar{x}, \bar{y} \in M, \bar{r} \in F$.

5. Soft topological soft modules

In literature, the concept of a topological module is a module that is defined in a topological setting under certain conditions. This section will introduce the concept of the soft topological soft module as a hybrid of soft algebraic and soft topological structures.

Let R be a ring, Y and Z be left R-modules.

Definition 5.1. Let (F, ν_F) be an S.T.S.R over (R, ν), $M \in S(Y)$ be a soft F-module and (M, τ_M) a soft subspace of (Y, τ). Then (M, τ_M) is called (left) soft topological soft module over (F, ν_F) (or soft topological soft F-module), denoted by S.T.S.M, if the following conditions are satisfied:

(i) $\tilde{f} : (M \times M, \tau_M \times \tau_M) \to (M, \tau_M)$ is soft continuous, where
$$f : Y \times Y \to Y$$
$$(x, y) \mapsto x + y$$

(ii) $\tilde{j} : (M, \tau_M) \to (M, \tau_M)$ is soft continuous, where
$$j : Y \to Y$$
$$x \mapsto -x$$

(iii) $\tilde{g} : (F \times M, \nu_F \times \tau_M) \to (M, \tau_M)$ is soft continuous, where
$$g : R \times Y \to Y$$
$$(r, y) \mapsto ry$$

Similarly, one can define right soft topological soft module over (F, ν_F).

Remark 5.2. (1) Every S.T.S.M is an S.T.S.G.

(2) Let G be a soft subset of F and K, D be soft subsets of M, where $M \in S(Y)$ is a soft F-module over Y. Then the addition of K and D and the multiplication of G and K are defined to be the soft subsets $K + D, GK$, respectively such that $(K + D)(e) = K(e) + D(e), (GK)(e) = G(e)K(e)$, $\forall e \in E$. Note that if $\tilde{r} \in G$, $\tilde{x} \in K$ and $\tilde{y} \in D$, then $\bar{x} + \bar{y} \in K + D$ and $\bar{r}\bar{x} \in GK$.

(3) Let K be a soft subset of M. Then $-K$ is defied to be a soft subset of M such that $(-K)(e) = -(K(e))$, $\forall e \in E$.

(4) Let $\tilde{\psi} : M \to N$ be a soft F-homomorphism with $M \in S(Y)$, $N \in S(Z)$. Let $G \subseteq F$ and $U, V \subseteq M$. Then $\tilde{\psi}(U + V) = \tilde{\psi}(U) + \tilde{\psi}(V)$, $\tilde{\psi}(-U) = -\tilde{\psi}(U)$ and $\tilde{\psi}(GU) = G\tilde{\psi}(U)$.

Proposition 5.3. *Let (F, ν_F) be an S.T.S.R over (R, ν), $M \in S(Y)$ be a soft F-module and (M, τ_M) a soft subspace of (Y, τ). Then (M, τ_M) is an S.T.S.M over (F, ν_F) if and only if the following conditions are satisfied:*

(i) For every $\bar{x}, \bar{y} \in M$ and every soft open neighborhood $U_{\bar{x}+\bar{y}}$ of $\bar{x} + \bar{y}$, there exists a soft open neighborhood $U_{\bar{x}}$ of $\bar{x}$ and a soft open neighborhood $U_{\bar{y}}$ of $\bar{y}$ such that $U_{\bar{x}} + U_{\bar{y}} \subseteq U_{\bar{x}+\bar{y}}$,

(ii) For every $\bar{x} \in M$ and every soft open neighborhood $U_{-\bar{x}}$ of $-\bar{x}$, there exists a soft open neighborhood $U_{\bar{x}}$ of $\bar{x}$ such that $-U_{\bar{x}} \subseteq U_{-\bar{x}}$,

(iii) For every $\bar{r} \in F$, $\bar{x} \in M$ and every soft open neighborhood $U_{\bar{r}\bar{x}}$ of $\bar{r}\bar{x}$, there exists a soft open neighborhood $U_{\bar{x}}$ of $\bar{x}$ and a soft open neighborhood $U_{\bar{r}}$ of $\bar{r}$ such that $U_{\bar{r}}U_{\bar{x}} \subseteq U_{\bar{r}\bar{x}}$.

Proof. Let (F, ν_F) be an S.T.S.R over (R, ν), $M \in S(Y)$ be a soft F-module.

($\Rightarrow$) Suppose (M, τ_M) be an S.T.S.M over (F, ν_F). Let $\bar{x}, \bar{y} \in M$ and $U_{\bar{x}+\bar{y}}$ be a soft open neighborhood of $\bar{x} + \bar{y}$. Since (M, τ_M) is an S.T.S.M, the soft mapping $\tilde{f} : (M \times M, \tau_M \times \tau_M) \to (M, \tau_M)$ is soft continuous, where

$$\begin{aligned} f : Y \times Y &\to Y \\ (x, y) &\mapsto x + y. \end{aligned}$$

Thus by Definition 3.12, there exists a soft open neighborhood $U_{(\bar{x},\bar{y})}$ of $(\bar{x}, \bar{y})$ such that $\tilde{f}(\bar{x}, \bar{y}) \in \tilde{f}(U_{(\bar{x},\bar{y})}) \subseteq U_{\tilde{f}(\bar{x},\bar{y})} = U_{\bar{x}+\bar{y}}$. Since $U_{(\bar{x},\bar{y})} \in \tau_M \times \tau_M$, there exists $V, W \in \tau_M$, such that $U_{(\bar{x},\bar{y})} = V \times W$. Note that $\bar{x} \in V$, $\bar{y} \in W$. Now, we put $V = U_{\bar{x}}$ and $W = U_{\bar{y}}$. Then $U_{(\bar{x},\bar{y})} = U_{\bar{x}} \times U_{\bar{y}}$. Note that $\tilde{f}(U_{(\bar{x},\bar{y})}) = \tilde{f}(U_{\bar{x}} \times U_{\bar{y}}) = U_{\bar{x}} + U_{\bar{y}}$. Thus we have $U_{\bar{x}} + U_{\bar{y}} \subseteq U_{\bar{x}+\bar{y}}$. So we have (i)

Also, since (M, τ_M) is an S.T.S.M, the soft mapping $\tilde{j} : (M, \tau_M) \to (M, \tau_M)$ is soft continuous, where

$$\begin{aligned} j : Y &\to Y \\ x &\mapsto -x. \end{aligned}$$

It follows from Definition 3.12 that for every $x \in M$ and for every soft open neighborhood $U_{\tilde{j}(\bar{x})}$ of $\tilde{j}(\bar{x})$, there must be a soft open neighborhood $U_{\bar{x}}$ of $\bar{x}$ such that $\bar{x} \in \tilde{j}(U_{(\bar{x})}) \subseteq U_{\tilde{j}(\bar{x})}$. Then $-U_{\bar{x}} \subseteq U_{-\bar{x}}$. Thus we have (ii).

Moreover, since (M, τ_M) is an S.T.S.M, the soft mapping $\tilde{g} : (F \times M, \nu_F \times \tau_M) \to (M, \tau_M)$ is soft continuous, where

$$\begin{aligned} g : R \times Y &\to Y \\ (r, x) &\mapsto rx. \end{aligned}$$

Let $\bar{r} \in F$ and $U_{\bar{r}\bar{x}}$ be a soft open neighborhood of $\bar{r}\bar{x} = \tilde{g}(\tilde{r}, \tilde{x})$. Then by Definition 3.12, there exists a soft open neighborhood $U_{(\bar{r},\bar{x})}$ of $(\bar{r}, \bar{x})$, such that

$$\tilde{g}(\tilde{r}, \tilde{x}) \in \tilde{g}(U_{(\bar{r},\bar{x})}) \subseteq U_{\tilde{g}(\bar{r},\bar{x})} = U_{\bar{r}\bar{x}}$$

Since $U_{(\bar{r},\bar{x})} \in \nu_F \times \tau_M$, there exists $V \in \nu_F$ and $W \in \tau_M$ such that $U_{(\bar{r},\bar{x})} \in V \times W$. Since $\bar{r} \in V$, $\bar{x} \in W$, we put $V = U_{\bar{r}}$ and $W = U_{\bar{x}}$. Then $\tilde{g}(U_{(\bar{r},\bar{x})}) = U_{\bar{r}}U_{\bar{x}}$. Thus $U_{\bar{r}}U_{\bar{x}} \subseteq U_{\bar{r}\bar{x}}$. So we have (iii).

($\Leftarrow$) The proof Follows directly by Definition 3.12. ◻

Theorem 5.4. *Let (F, ν_F) be an S.T.S.R over (R, ν), $M \in S(Y)$ be a soft F-module and (M, τ_M) a soft subspace of (Y, τ). If (M, τ_M) is an S.T.S.M over (F, ν_F), then for every $e \in E$, $\big(M(e), (\tau^e)_{M(e)}\big)$ is a T.M over the ring $F(e)$.*

Proof. Suppose (M, τ_M) is an S.T.S.M over (F, ν_F). Then the soft mappings $\tilde{f} : (M \times M, \tau_M \times \tau_M) \to (M, \tau_M)$, $\tilde{g} : (F \times M, \nu_F \times \tau_M) \to (M, \tau_M)$ and $\tilde{j} : (M, \tau_M) \to (M, \tau_M)$ are soft continuous, where

$$\begin{aligned} f : Y \times Y &\to Y \\ (x, y) &\mapsto x + y \end{aligned}, \quad \begin{aligned} g : R \times Y &\to Y \\ (r, y) &\mapsto ry \end{aligned} \quad \text{and} \quad \begin{aligned} j : Y &\to Y \\ x &\mapsto -x. \end{aligned}$$

Thus by Propositions 3.13 and (3.13), the mappings

$$f_e : \big(M(e) \times M(e), (\tau^e)_{M(e)} \times (\tau^e)_{M(e)}\big) \to (M(e), (\tau^e)_{M(e)}),$$

$$g_e : \left(F(e) \times M(e), (\upsilon^e)_{F(e)} \times (\tau^e)_{M(e)}\right) \to (M(e), (\tau^e)_{M(e)}),$$
$$j_e : \left(M(e), (\tau^e)_{M(e)}\right) \to \left(M(e), (\tau^e)_{M(e)}\right)$$

are continuous for all $e \in E$, where

$$\begin{array}{r} f_e : M(e) \times M(e) \to M(e) \\ (x, y) \mapsto x + y \end{array}, \quad \begin{array}{r} g_e : F(e) \times M(e) \to M(e) \\ (r, y) \mapsto ry \end{array} \quad \text{and} \quad \begin{array}{r} j_e : M(e) \to M(e) \\ x \mapsto -x. \end{array}$$

So $\left(M(e), (\tau^e)_{M(e)}\right)$ is a T.M over the ring $F(e)$. □

Proposition 5.5. *Let (F, ν_F) be an S.T.S.R over (R, ν), $M \in S(Y)$ be a soft F-module and (Y, τ) be an S.T.S. If the subspace $\left(M(e), (\tau^e)_{M(e)}\right)$ of (Y, τ^e) is a T.M over the ring $F(e)$, for every $e \in E$, then the soft subspace (M, τ_M^*) is an S.T.S.M over (F, ν_F).*

Proof. Suppose $\left(M(e), (\tau^e)_{M(e)}\right)$ is a T.M over the ring $F(e)$. Then the mappings

$$f_e : \left(M(e) \times M(e), (\tau^e)_{M(e)} \times (\tau^e)_{M(e)}\right) \to (M(e), (\tau^e)_{M(e)}),$$
$$g_e : \left(F(e) \times M(e), (\upsilon^e)_{F(e)} \times (\tau^e)_{M(e)}\right) \to (M(e), (\tau^e)_{M(e)}),$$
$$j_e : \left(M(e), (\tau^e)_{M(e)}\right) \to \left(M(e), (\tau^e)_{M(e)}\right)$$

are continuous for all $e \in E$, where

$$\begin{array}{r} f_e : M(e) \times M(e) \to M(e) \\ (x, y) \mapsto x + y \end{array}, \quad \begin{array}{r} g_e : F(e) \times M(e) \to M(e) \\ (r, y) \mapsto ry \end{array} \quad \text{and} \quad \begin{array}{r} j_e : M(e) \to M(e) \\ x \mapsto -x. \end{array}$$

Then by Propositions 3.13 and (3.13), the soft mappings $\tilde{f} : (M \times M, \tau_M^* \times \tau_M^*) \to (M, \tau_M^*)$, $\tilde{g} : (F \times M, \nu_F^* \times \tau_M^*) \to (M, \tau_M^*)$, and $\tilde{j} : (M, \tau_M^*) \to (M, \tau_M^*)$ are soft continuous, where

$$\begin{array}{r} f : Y \times Y \to Y \\ (x, y) \mapsto x + y \end{array}, \quad \begin{array}{r} g : R \times Y \to Y \\ (r, y) \mapsto ry \end{array} \quad \text{and} \quad \begin{array}{r} j : Y \to Y \\ x \mapsto -x. \end{array}$$

Thus (M, τ_M^*) is an S.T.S.M over (F, ν_F). □

Proposition 5.6. *Let (F, ν_F) be an S.T.S.R over (R, ν), $M \in S(Y)$ and $N \in S(Z)$ be soft F-modules. Let $\tilde{\psi} : M \to N$ be a soft F-monomorphism. If and (M, τ_M) a soft subspace of (Y, τ) such that (M, τ_M) is an S.T.S.M over (F, ν_F), then $(\tilde{\psi}(M), \tilde{\psi}(\tau_M))$ is an S.T.S.M over (F, ν_F).*

Proof. Let $\tilde{\psi} : M \to N$ be a soft F-monomorphism. Let $\bar{x}, \bar{y} \in M$. Then $\tilde{\psi}(\bar{x}), \tilde{\psi}(\bar{y}) \in \tilde{\psi}(M)$. Let $U_{\tilde{\psi}(\bar{x})+\tilde{\psi}(\bar{y})}$ be a soft open neighborhood of $\tilde{\psi}(\bar{x}) + \tilde{\psi}(\bar{y})$ in $\tilde{\psi}(\tau_M)$. Then there exists $U \in \tau_M$, such that $U_{\tilde{\psi}(\bar{x})+\tilde{\psi}(\bar{y})} = \tilde{\psi}(U)$. Note that $\tilde{\psi}(\bar{x}) + \tilde{\psi}(\bar{y}) = \tilde{\psi}(\bar{x} + \bar{y})$. Since, $\tilde{\psi}(U)$ is a soft open neighborhood of $\tilde{\psi}(\bar{x} + \bar{y})$ in $\tilde{\psi}(\tau_M)$, there exists $V \in \tau_M$ such that $\tilde{\psi}(\bar{x} + \bar{y}) \in \tilde{\psi}(V) \subseteq \tilde{\psi}(U)$. Then $\bar{x} + \bar{y} \in V \subseteq U$. Thus U is a soft open neighborhood of $\bar{x} + \bar{y}$ in τ_M. But (M, τ_M) is an S.T.S.M. By Proposition 5.3 (i), there exist soft open neighborhoods $U_{\bar{x}}$ and $U_{\bar{y}}$ of $\bar{x}$ and $\bar{y}$, respectively in τ_M such that $U_{\bar{x}} + U_{\bar{y}} \subseteq U_{\bar{x}+\bar{y}} := U$. So $\tilde{\psi}(U_{\bar{x}}) + \tilde{\psi}(U_{\bar{y}}) \subseteq \tilde{\psi}(U) = U_{\tilde{\psi}(\bar{x}+\bar{y})}$. Hence $U_{\tilde{\psi}(\bar{x})} + U_{\tilde{\psi}(\bar{y})} \subseteq U_{\tilde{\psi}(\bar{x})+\tilde{\psi}(\bar{y})}$, where $U_{\tilde{\psi}(\bar{x})} = \tilde{\psi}(U_{\bar{x}})$, $U_{\tilde{\psi}(\bar{y})} = \tilde{\psi}(U_{\bar{y}})$ (see Proposition 3.14).

Let $U_{-\tilde{\psi}(\bar{x})}$ be a soft open neighborhood of $-\tilde{\psi}(\bar{x})$ in $\tilde{\psi}(\tau_M)$. Then there exists $U' \in \tau_M$ such that $U_{-\tilde{\psi}(\bar{x})} = \tilde{\psi}(U')$. Also, there exists $V' \in \tau_M$ such that $-\tilde{\psi}(\bar{x}) \in \tilde{\psi}(V') \subseteq \tilde{\psi}(U')$. Thus $-\bar{x} \in V' \subseteq U'$. So U' is a soft open neighborhood of $-\bar{x}$ in τ_M. Put $U' = U_{-\bar{x}}$. But (M, τ_M) is an S.T.S.M. Then by Proposition 5.3 (ii), there exists a soft open neighborhood $U_{\bar{x}}$ of $\bar{x}$ in τ_M such that $-U_{\bar{x}} \subseteq U_{-\bar{x}}$. Thus $-\tilde{\psi}(U_{\bar{x}}) \subseteq \tilde{\psi}(U_{-\bar{x}})$. So it follows from Proposition 3.14 that $-U_{\tilde{\psi}(\bar{x})} \subseteq U_{-\tilde{\psi}(\bar{x})}$.

Let $\bar{r} \in F$ and $U_{\bar{r}\tilde{\psi}(\bar{x})}$ be a soft open neighborhood of $\bar{r}\tilde{\psi}(\bar{x})$ in $\tilde{\psi}(\tau_M)$. Then $U_{\bar{r}\tilde{\psi}(\bar{x})} = \tilde{\psi}(U'')$, $U'' \in \tau_M$ and there exists $V'' \in \tau_M$ such that $\bar{r}\tilde{\psi}(\bar{x}) \in \tilde{\psi}(V'') \subseteq \tilde{\psi}(U'')$. Note that $\bar{r}\tilde{\psi}(\bar{x}) = \tilde{\psi}(\bar{r}\bar{x})$. Then $\bar{r}\bar{x} \in V'' \subseteq U''$. Thus U'' is a soft open neighborhood of $\bar{r}\bar{x}$ in τ_M. Put $U'' = U_{\bar{r}\bar{x}}$. But (M, τ_M) is an S.T.S.M. Then by Proposition 5.3 (iii), there exist a soft open neighborhood $U_{\bar{r}}$ of $r$in nu_F and a soft open neighborhood $U_{\bar{x}}$ of $\bar{x}$ in τ_M such that $U_{\bar{r}}U_{\bar{x}} \subseteq U_{\bar{r}\bar{x}}$. Thus $U_{\bar{r}}\tilde{\psi}(U_{\bar{x}}) \subseteq \tilde{\psi}(U_{\bar{r}\bar{x}})$. So by Proposition 3.14, we have $U_{\bar{r}}U_{\tilde{\psi}(\bar{x})} \subseteq U_{\bar{r}\tilde{\psi}(\bar{x})}$. Hence it follows from Proposition 5.3 that $(\tilde{\psi}(M), \tilde{\psi}(\tau))$ is an S.T.S.M. □

Proposition 5.7. *Let (F, ν_F) be an S.T.S.R over (R, ν), $M \in S(Y)$ and $N \in S(Z)$ be soft F-modules. Let $\tilde{\psi} : M \to N$ be a soft F-epimorphism. If (N, τ_N) a soft subspace of (Z, τ) such that (N, τ_N) be an S.T.S.M over (F, ν_F), then $(M, \tilde{\psi}^{-1}(\tau_N))$ is an S.T.S.M over (F, ν_F).*

Proof. Let $\bar{x}, \bar{y} \in M$ and $U_{\bar{x}+\bar{y}}$ be a soft open neighborhood of $\bar{x}+\bar{y}$ in $\tilde{\psi}^{-1}(\tau_M)$. Then there exists $V \in \tau_M$ such that $U_{\bar{x}+\bar{y}} = \tilde{\psi}^{-1}(V)$ and there exists $U \in \tau_M$ such that $\bar{x}+\bar{y} \in \tilde{\psi}^{-1}(U) \subseteq \tilde{\psi}^{-1}(V)$. Since $\tilde{\psi}$ is surjective, we have $\tilde{\psi}(\bar{x}+\bar{y}) \in U \subseteq V$. Thus V is a soft open neighborhood of $\tilde{\psi}(\bar{x}+\bar{y})$ in τ_M. Since $\tilde{\psi}$ is F-homomorohism, then $\tilde{\psi}(\bar{x}+\bar{y}) = \tilde{\psi}(\bar{x})+\tilde{\psi}(\bar{y})$. Now, we put $V = U_{\tilde{\psi}(\bar{x})+\tilde{\psi}(\bar{y})}$. But (N, τ) is S.T.S.M. Then it follows from Proposition 5.3 (i) that there exist a soft open neighborhood $U_{\tilde{\psi}(\bar{x})}$ of $\tilde{\psi}(\bar{x})$ and a soft open neighborhood $U_{\tilde{\psi}(\bar{y})}$ of $\tilde{\psi}(\bar{y})$ in τ_M such that $U_{\tilde{\psi}(\bar{x})} + U_{\tilde{\psi}(\bar{y})} \subseteq U_{\tilde{\psi}(\bar{x})+\tilde{\psi}(\bar{y})}$. But $\tilde{\psi}$ is surjective. Thus $\tilde{\psi}^{-1}(U_{\tilde{\psi}(\bar{x})}) + \tilde{\psi}^{-1}(U_{\tilde{\psi}(\bar{y})}) \subseteq \tilde{\psi}^{-1}(U_{\tilde{\psi}(\bar{x})+\tilde{\psi}(\bar{y})})$. So it follows from Proposition 3.14 that $U_{\bar{x}} + U_{\bar{y}} \subseteq U_{\bar{x}+\bar{y}}$, where $U_{\bar{x}}$ and $U_{\bar{y}}$ are soft open neighborhoods of $\bar{x}$ and $\bar{y}$, respectively.

Let $U_{-\bar{x}}$ be a soft open neighborhoods of $-\bar{x}$ in $\tilde{\psi}^{-1}(\tau_M)$. Then there exists $V' \in \tau_M$ such that $U_{-\bar{x}} = \tilde{\psi}^{-1}(V')$ and there exists $U' \in \tau$ such that $-\bar{x} \in \tilde{\psi}^{-1}(U') \subseteq \tilde{\psi}^{-1}(V')$. Since $\tilde{\psi}$ is F-epimorphism, we have $-\tilde{\psi}(\bar{x}) \in U' \subseteq V'$. Thus V' is a soft open neighborhood of $-\tilde{\psi}(\bar{x})$ in τ_M. But (N, τ) is S.T.S.M. Then it follows from Proposition 5.3 (ii) that there exists a soft open neighborhood $U_{\tilde{\psi}(\bar{x})}$ of $\tilde{\psi}(\bar{x})$ in τ_M such that $-U_{\tilde{\psi}(\bar{x})} \subseteq U_{-\tilde{\psi}(\bar{x})}$. Thus $-\tilde{\psi}^{-1}(U_{\tilde{\psi}(\bar{x})}) \subseteq \tilde{\psi}^{-1}(U_{-\tilde{\psi}(\bar{x})})$. So $-U_{\bar{x}} \subseteq U_{-\bar{x}}$, where $U_{\bar{x}}$ is a soft open neighborhood of $\bar{x}$ (see Proposition 3.14).

Let $\bar{r} \in F$ and $U_{\bar{r}\bar{x}}$ be a soft open neighborhood of $\bar{r}\bar{x}$ in $\tilde{\psi}^{-1}(\tau_M)$. Then there exists $V'' \in \tau_M$ such that $U_{\bar{r}\bar{x}} = \tilde{\psi}^{-1}(V'')$ and there exists $U'' \in \tau_M$ such that $\bar{r}\bar{x} \in \tilde{\psi}^{-1}(U'') \subseteq \tilde{\psi}^{-1}(V'')$. Thus $\bar{r}\tilde{\psi}(\bar{x}) \in U'' \subseteq V''$. So V'' is a soft open neighborhood of $\bar{r}\tilde{\psi}(\bar{x})$ in τ_M. Now, put $V'' = U_{\bar{r}\tilde{\psi}(\bar{x})}$. But (N, τ) is S.T.S.M. Then it follows from Proposition 5.3 (iii) that there exist a soft open neighborhood $U_{\bar{r}}$ of $\bar{r}$ in ν_F and a soft open neighborhood $U_{\tilde{\psi}(\bar{x})}$ of $\tilde{\psi}(\bar{x})$ in τ_M such that $U_{\bar{r}}U_{\tilde{\psi}(\bar{x})} \subseteq U_{\bar{r}\tilde{\psi}(\bar{x})}$. Thus it follows from Proposition 3.14 that $U_{\bar{r}}U_{\bar{x}} \subseteq U_{\bar{r}\bar{x}}$, where $U_{\bar{x}} = \tilde{\psi}^{-1}(U_{\tilde{\psi}(\bar{x})})$ and $U_{\bar{r}\bar{x}} = \tilde{\psi}^{-1}(U_{\tilde{\psi}(\bar{r}\bar{x})})$. So it follows from Proposition 5.3 that $(M, \tilde{\psi}^{-1}(\tau_M))$ is an S.T.S.M over (F, ν_F). □

5.1. **Soft topological soft F-submodule.**

Definition 5.8. Let (F, ν_F) be an S.T.S.R over (R, ν) and $M, N \in S(Y)$ be soft F-modules. If the subspace (M, τ_M) of (Y, τ) and the subspace (N, κ_N) of (Y, κ) are both an S.T.S.M over (F, ν_F), then (M, τ_M) is called soft topological soft F-submodule of (N, κ_N), denoted by $(M, \tau_M) \leqslant_F (N, \kappa_N)$, if the following conditions are satisfied:

(i) M is a soft F-submodule of N,

(ii) $\tau_M = ((\kappa)_N)_M$.

Example 5.9. Let $E = \{e_1, e_2\}$ and $R = \mathbb{Z}_8$. Let $F = \tilde{R}$ and ν be the discrete soft topology on F. Then (F, ν_F) is an S.T.S.R over $\mathbb{Z}_8$. Let $Y = \mathbb{Z}_8$ and

$M = \{(e_1, \{\bar{0}\}), (e_2, \{\bar{0}, \bar{4}\})\}$, $N = \{(e_1, \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}), (e_2, \{\bar{0}, \bar{4}\})\}$,
$F_1 = \{(e_1, \{\bar{0}\}), (e_2, \{\bar{0}\})\}$, $F_2 = \{(e_1, \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}), (e_2, \{\bar{0}, \bar{1}, \bar{4}\})\}$,
$F_3 = \{(e_1, \{\bar{4}\}), (e_2, \{\bar{4}\})\}$, $F_4 = \{(e_1, \{\bar{0}, \bar{4}\}), (e_2, \{\bar{0}, \bar{4}\})\}$
and $F_5 = \{(e_1, \{\bar{0}, \bar{4}, \bar{6}\}), (e_2, \{\bar{0}, \bar{4}\})\}$.

Then M and N are soft F-modules and $M \leqslant_F N$. Note that $\tau = \{\tilde{\phi}, \tilde{R}, F_1, F_2, F_3, F_4\}$ is a soft topology on F $(= \tilde{R})$ and $\kappa = \{\tilde{\phi}, \tilde{R}, F_1, F_2, F_3, F_4, F_5\}$ is a soft topology on F.

With a simple computations, we can find τ_M and κ_N such that
$\tau_M = \{\tilde{\phi}, M, \{(e_1, \{\bar{0}\}), (e_2, \{\bar{0}\})\}, \{(e_1, \phi), (e_2, \{\bar{4}\})\}\}$, and
$\kappa_N = \{\tilde{\phi}, N, \{(e_1, \{\bar{0}\}), (e_2, \{\bar{0}\})\}, \{(e_1, \{\bar{4}\}), (e_2, \{\bar{4}\})\}, \{(e_1, \{\bar{0}, \bar{4}\}), (e_2, \{\bar{0}, \bar{4}\})\}, \{(e_1, \{\bar{0}, \bar{4}, \bar{6}\}), (e_2, \{\bar{0}, \bar{4}\})\}\}$.
Also, $\nu = S(\mathbb{Z}_8) = \nu_F$. So, it is easy to show that (M, τ_M) is an S.T.S.M. Similarly, one can show that (N, κ_N) is an S.T.S.M. Since

$$\tau_M = ((\kappa)_N)_M = \{\tilde{\phi}, \{(e_1, \{\bar{0}\}), (e_2, \{\bar{0}\})\}, \{(e_1, \phi), (e_2, \{\bar{4}\}\}, \{(e_1, \{\bar{0}\}), (e_2, \{\bar{0}, \bar{4}\})\}.$$

Then $(M, \tau_M) \leqslant_F (N, \kappa_N)$.

Proposition 5.10. *Let (F, ν_F) be an S.T.S.R over (R, ν) and $M, M' \in S(Y)$, $N \in S(Z)$ be soft F-modules. Let $(M', \lambda_{M'})$ and (M, μ_M) be two S.T.S.M over (F, ν_F). If $(M', \lambda_{M'}) \leqslant_F (M, \mu_M)$ and $\tilde{\psi} : M \to N$ is a soft F-monomorphism, then $(\tilde{\psi}(M'), \tilde{\psi}(\lambda)) \leqslant_F (\tilde{\psi}(M), \tilde{\psi}(\mu))$.*

Proof. It follows from Proposition 5.6 that $(\tilde{\psi}(M'), \tilde{\psi}(\lambda))$ and $(\tilde{\psi}(M), \tilde{\psi}(\mu))$ are S.T.S.M over (F, ν_F). Since $M' \leqslant_F M$, $\tilde{\psi}(M') \leqslant_F \tilde{\psi}(M)$. Since $\tilde{\psi}$ is injective and $\mu_M = \lambda$, we have $(\tilde{\psi}(\mu))_{\tilde{\psi}(M')} = \tilde{\psi}(\lambda)$. Then $(\tilde{\psi}(M'), \tilde{\psi}(\lambda)) \leqslant_F (\tilde{\psi}(M), \tilde{\psi}(\mu))$. □

Proposition 5.11. *Let (F, ν_F) be an S.T.S.R over (R, ν) and $M \in S(Y)$, $N, N' \in S(Z)$ be soft F-modules. Let (N', λ) and (N, μ) be two S.T.S.M over (F, ν_F). If $(N', \lambda) \leqslant_F (N, \mu)$ and $\tilde{\psi} : M \to N$ is a soft F-epimorphism, then $(\tilde{\psi}^{-1}(N'), \tilde{\psi}^{-1}(\lambda)) \leqslant_F (\tilde{\psi}^{-1}(N), \tilde{\psi}^{-1}(\mu))$.*

Proof. It follows from Proposition 5.7 that $(\tilde{\psi}^{-1}(N'), \tilde{\psi}^{-1}(\lambda))$ and $(\tilde{\psi}^{-1}(N), \tilde{\psi}^{-1}(\mu))$ are S.T.S.M over (F, ν_F). Since $N' \leqslant_F N$, $\tilde{\psi}^{-1}(N') \leqslant_F \tilde{\psi}^{-1}(N)$. But $(\tilde{\psi}^{-1}(\mu))_{\tilde{\psi}^{-1}(N')} = \tilde{\psi}^{-1}(\mu_{N'}) = \tilde{\psi}^{-1}(\lambda)$. Then $(\tilde{\psi}^{-1}(N'), \tilde{\psi}^{-1}(\lambda)) \leqslant_F (\tilde{\psi}^{-1}(N), \tilde{\psi}^{-1}(\mu))$. □

6. Conclusion

We have proposed the concept of soft modules over soft rings. After that we have produced the concept of the soft topological soft module by analyzing the soft topological structures over the soft modules directly. Moreover, we have discussed the subsystem of the S.T.S.M by producing the concepts of soft topological soft F-submodule. The reader can work on the concept of the fuzzy soft topological module over fuzzy soft topological rings as a future studies.

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