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Abstract. Let R be a commutative ring and B be an intuitionistic fuzzy submodule (IFSM) of an R-module M. Then an IFSM A of M is called B-essential in M provided for each IFSM C of M, $A \cap C \subseteq B$ implies that $C \subseteq B$. Further, for IFSMs A, B, C of M, the IFSM C is called B-complement to A if C is maximal with respect to the property that $A \cap C \subseteq B$. We study these mentioned notations which are generalization of the intuitionistic fuzzy essential (compliment) submodules, introduced by Basnet in [5]. Here we shall study some related results.

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1. Introduction

The theory of intuitionistic fuzzy sets introduced by Atanassov [3, 4] plays an important role in modern mathematics. It is a generalization to the theory of fuzzy sets given by Zadeh. Biswas was the first one to apply the theory of intuitionistic fuzzy sets in algebra and introduced the notion of intuitionistic fuzzy subgroup of a group in [6]. Later on Hur and others in [9] and [10] introduced the notion of intuitionistic fuzzy subring and ideals. Davaaz and others in [7] introduced the notion of intuitionistic fuzzy submodule of a module. Later on many mathematicians contributed much to the study of intuitionistic fuzzy submodules see [8, 11, 13, 14, 15]. Authors in [5] and [14] have studied the concept of essential submodule (complement) and superfluous submodule in intuitionistic fuzzy environment respectively.

In this paper, we shall study the intuitionistic fuzzification of the notion of essential submodule (complement) with respect to another submodule as introduced by Safaeeyan and Saboori Shirazi in [12], and Al-Dhaheri and Al-Bahrani in [1].

2. Preliminaries

In this section, we list some basic concepts and well known results on essential (complement) submodule of a module and intuitionistic fuzzy essential (complement) submodule of a module for the sake of completeness of the topic under study. Throughout the paper R is a commutative ring with identity, M a unitary R-module with zero element θ. We denote the closed interval $[0, 1]$ by I and for all $a, b \in I$, we write $a \wedge b = min\{a, b\}$; $a \vee b = max\{a, b\}$. The material of this section and some related results can be found in [1, 2, 3, 4, 5] and [11, 12, 13, 14].

Definition 2.1 ([2])**.** A submodule K of an R-module M is said to be an essential (or large) submodule of M, if for every submodule L of M with $K \cap L = \{0\}$ implies that $L = \{0\}$ (or equivalently if $L \neq \{0\}$ implies $K \cap L \neq \{0\}$). This notion is symbolized by $K \trianglelefteq M$.

Equivalently, a submodule K of an R-module M is essential in M if and only if for each $\theta \neq x \in M$ there exists an $r \in R$ such that $0 \neq rx \in K$ (See [2], Lemma 5.19).

Definition 2.2 ([1, 12])**.** Let T be a proper submodule of an R-module M. A submodule K of M is called T-essential (in M), provided that $K \nsubseteq T$ and for each submodule L of M, $L \cap K \subseteq T$ implies that $L \subseteq T$. In this case K is denoted by $K \trianglelefteq_T M$.

Proposition 2.3 ([12])**.** *Let K, L, T be submodule of an R-module M.*
(1) *If $K \trianglelefteq_T M$ and $L \trianglelefteq_T M$ then $K \cap L \trianglelefteq_T M$.*
(2) *Let $K \leq L \leq M$, then $K \trianglelefteq_T M$ if and only if $K \trianglelefteq_T L$ and $L \trianglelefteq_T M$.*

Proposition 2.4 ([12])**.** *Let $T_1 \leq K_1 \leq M_1 \leq M$ and $T_2 \leq K_2 \leq M_2 \leq M$ such that $M_1 \cap M_2 = T_1 \cap T_2$. Then $K_1 + K_2 \trianglelefteq_{T_1+T_2} M_1 + M_2$ if and only if $K_1 \trianglelefteq_{T_1} M_1$ and $K_2 \trianglelefteq_{T_2} M_2$.*

Lemma 2.5 ([12])**.** *Let M and N be R-modules T and K be submodules of N and $f \in Hom_R(M, N)$. If $K \trianglelefteq_T N$, then $f^{-1}(K) \trianglelefteq_{f^{-1}(T)} M$.*

Definition 2.6 ([1, 12])**.** A submodule K of an R-module M is called a complement of submodule N in M, if K is maximal with respect to the property that $N \cap K = \{\theta\}$.

Definition 2.7 ([1, 12])**.** A submodule K of an R-module M is called a complement of a submodule N relative to another submodule T in M, if K is maximal with respect to the property that $N \cap K \subseteq T$. Then we say that the submodule K is T-complement of the submodule N in M.

Clearly, when $T = \{\theta\}$ then K is T-complement to N if and only if K is complement relative to N in M.

A mapping $A = (\mu_A, \nu_A) : X \to [0, 1] \times [0, 1]$ is called an intuitionistic fuzzy set (IFS) on X, if $\mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$, where the mapping $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denotes the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to A. When

$\mu_A(x) + \nu_A(x) = 1$ for each $x \in X$, then A is called a fuzzy set. The class of intuitionistic fuzzy subsets of X is denoted by IFS(X).

For $A, B \in IFS(X)$ we say $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$. Also, $A \subset B$ if and only if $A \subseteq B$ and $A \neq B$.

The intuitionistic fuzzy characterisitic function of X with respect to a subset Y is denoted by χ_Y and is defined as:

$$\mu_{\chi_Y}(y) = \begin{cases} 1, & \text{if } y \in Y \\ 0, & \text{if otherwise} \end{cases} ; \quad \nu_{\chi_Y}(y) = \begin{cases} 0, & \text{if } y \in Y \\ 1, & \text{otherwise.} \end{cases}$$

Let $f : X \to Y$ be a surjective mapping. Then the image of IFS A is defined as $f(A)(y) = (\vee\{\mu_A(x) : x \in f^{-1}(y)\}, \wedge\{\nu_A(x) : x \in f^{-1}(y)\})$ and the pre-image of IFS B is defined as $f^{-1}(B)(x) = (\mu_B(f(x)), \nu_B(f(x)))$.

The (α, β)-cut of IFS A of set X is a crisp subset $C_{(\alpha,\beta)}(A)$ is given by
$C_{(\alpha,\beta)}(A) = \{x \in X : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$, where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$.
The support of the A is denoted by A^* and is defined as $A^* = \{x \in X : \mu_A(x) > 0, \nu_A(x) < 1\}$.

Definition 2.8 ([5, 7])**.** Let M be an R-module. An IFS $A = (\mu_A, \nu_A)$ of M is called an intuitionistic fuzzy submodule (IFSM), if it satisfies the following conditions:
(i) $\mu_A(\theta) = 1$, $\nu_A(\theta) = 0$,
(ii) $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x + y) \leq \nu_A(x) \vee \nu_A(y)$, $\forall\ x, y \in M$,
(iii) $\mu_A(rx) \geq \mu_A(x)$ and $\nu_A(rx) \leq \nu_A(x)$, $\forall x \in M, r \in R$.

The set of all intuitionistic fuzzy submodules of M is denoted by $IFM(M)$.

Note that $\chi_{\{\theta\}}, \chi_M \in IFM(M)$ and these are called trivial IFSMs. Any IFSM other than these is called proper IFSM. Moreover, an IFS $A \in IFM(M)$ if and only if $C_{(\alpha,\beta)}(A)$ is submodule of M, for all $\alpha, \beta \in [0, 1]$ with $\alpha+\beta \leq 1$, where $\mu_A(\theta) \geq \alpha$, $\nu_A(\theta) \leq \beta$. In particular A^* is a submodule of M.

Proposition 2.9 ([5, 11])**.** *Let $f : M \to N$ be a R-module homomorphism. If A is an IFSM of M and B is an IFSM of N. Then the following result holds:*
(1) *$f(A^*) \subseteq (f(A))^*$ and equality hold when the map f is bijective,*
(2)*) $f^{-1}(B^*) = (f^{-1}(B))^*$.*

Proposition 2.10 ([5, 14])**.** *Let $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ be two IFSMs of R-module M. Then the sum $A + B = (\mu_{A+B}, \nu_{A+B})$ is an IFSM of M and is defined as follows: for each $x \in M$,*

$$\mu_{A+B}(x) = \bigvee\{\mu_A(a) \wedge \mu_B(b) : x = a + b\}$$

and

$$\nu_{A+B}(x) = \bigwedge\{\nu_A(a) \vee \nu_B(b) : x = a + b\}.$$

The IFSM C of M is called the direct sum of IFSMs A and B, if $C = A+B$ and $A \cap B = \chi_{\{\theta\}}$. This is expressed as $C = A \bigoplus B$. Further, $(A + B)^* = A^* + B^*$ and $(A \cap B)^* = A^* \cap B^*$.

Definition 2.11 ([5, 11])**.** Let $A, B \in IFM(M)$ are such that $A \subseteq B$. Then the quotient of B with respect to A, denoted by B/A, is an IFSM of M/A^* defined as:

$$B/A(x + A^*) = (\mu_{B/A}(x + A^*), \nu_{B/A}(x + A^*)),$$

where $x \in B^*$,

$$\mu_{B/A}(x + A^*) = \bigvee\{\mu_B(x + y) : y \in A^*\}$$

and

$$\nu_{B/A}(x + A^*) = \bigwedge\{\nu_B(x + y) : y \in A^*\}.$$

From [5] we can see that $(B/A)^* = B^*/A^*$.

Definition 2.12 ([5])**.** An IFSM A of an R-module M is called an intuitionistic fuzzy essential submodule of M, denoted by $A \trianglelefteq M$ (or $A \trianglelefteq \chi_M$), if for any IFSM $C(\neq \chi_{\{\theta\}})$ of M, $A \cap C \neq \chi_{\{\theta\}}$. Equivalently, $A \trianglelefteq M$ if for any IFSM C of M such that $A \cap C = \chi_{\{\theta\}} \Rightarrow C = \chi_{\{\theta\}}$.

Theorem 2.13. *A submodule N of an R-module M is essential in M if and only intuitionistic fuzzy characteristic function χ_N is an intuitionistic fuzzy essential submodule of M.*

Proof. The proof is obvious. □

Theorem 2.14 ([5])**.** *Let $A, B(\neq \chi_{\{\theta\}}) \in IFM(M)$. Then $A \trianglelefteq B$ if and only if $A^* \trianglelefteq B^*$.*

Theorem 2.15 ([5])**.** *If $f : M \to K$ is a module homomorphism and $A \trianglelefteq K$, then $f^{-1}(A) \trianglelefteq M$.*

Definition 2.16 ([5])**.** Let A and B are two IFSMs of an R-module M such that $A \subseteq B$. An intuitionistic fuzzy relative complement for A in M is an IFSM C of M such that $C \subseteq B$ and C is maximal with respect to the property that $A \cap C = \chi_{\{\theta\}}$.

Theorem 2.17 ([5])**.** *Let A and B are two IFSMs of an R-module M such that B is an intuitionistic fuzzy relative complement for A. Then $A \bigoplus B \trianglelefteq M$.*

Theorem 2.18 ([5])**.** *Let A and B are two IFSMs of an R-module M such that $A \subseteq B$. If A is an intuitionistic fuzzy relative complement for some IFSM $C \subseteq B$, then A is an intuitionistic fuzzy closed submodule in B.*

3. Intuitionistic fuzzy essential submodule with respect to another intuitionistic fuzzy submodule

Definition 3.1. Let B be a proper intuitionistic fuzzy submodule of an R-module M. An IFSM A of M is called B-essential in M, written as $A \trianglelefteq_B M$, provided that $A \nsubseteq B$ and for any IFSM C of M, $A \cap C \subseteq B$ implies that $C \subseteq B$.

Remark 3.2. (1) If $B = \chi_M$, then every IFSM A of M is B-essential in M.

(2) If $B = \chi_{\{\theta\}}$, then $A \trianglelefteq_B M$ if and only if $A \trianglelefteq M$.

Example 3.3. Consider $M = Z_{16}$ under addition modulo 16 over the ring of integer Z. Define IFSs A and B of M as follows:

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = \{0\} \\ 0.7, & \text{if } x = \{2,4,6,8,10,12,14\} \\ 0, & \text{otherwise} \end{cases} ; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x = \{0\} \\ 0.2, & \text{if } x = \{2,4,6,8,10,12,14\} \\ 1, & \text{otherwise} \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 1, & \text{if } x = \{0\} \\ 0.5, & \text{if } x = \{4,8,12\} \\ 0, & \text{otherwise} \end{cases} ; \quad \nu_B(z) = \begin{cases} 0, & \text{if } x = \{0\} \\ 0.3, & \text{if } x = \{4,8,12\} \\ 1, & \text{otherwise.} \end{cases}$$

Then A is an intuitionistic fuzzy B-essential submodule of M.

Remark 3.4. If A is an IFSM of an R-module M. Then $A \trianglelefteq M$ if and only if $A \trianglelefteq_{\chi_{\{\theta\}}} M$.

Theorem 3.5. *Let $A, C \in IFM(M)$ be such that $A \subseteq C$. Then $A \trianglelefteq_B C$ if and only if for each $\theta \neq x \in M$ with $x \in C^*$, there exists $0 \neq r \in R$ such that $rx \neq \theta$ and $rx \in (A \cap B)^*$.*

Proof. Firstly assume that for each $\theta \neq x \in M$ with $x \in C^*$. Then there exists $0 \neq r \in R$ such that $rx \neq \theta$ and $rx \in (A \cap B)^*$. To show that $A \trianglelefteq_B C$. Take any $D \in IFM(M)$ with $D \subseteq C$ such that $A \cap D \subseteq B$. Then we show that $D \subseteq B$.

Consider $\theta \neq x \in M$ such that $x \in D^*$. Then for each $r \in R$, we have $\mu_D(rx) \geq \mu_D(x) > 0$ and $\nu_D(rx) \leq \nu_D(x) < 1$. Thus $rx \in D^*$. Further as $D \subseteq C$ implies $D^* \subseteq C^*$. So $x \in C^*$. By given hypothesis, there exists $0 \neq r \in R$ such that $rx \neq \theta$ and $rx \in (A \cap B)^*$. But $(A \cap B)^* = A^* \cap B^*$. This implies that $rx \in A^*$ and $rx \in B^*$. Hence $D^* \subseteq B^*$. Therefore $D \subseteq B$.

Conversely, suppose that $A \trianglelefteq_B C$. Let $A \in IFM(M)$ such that $A \subseteq C$. Let $\theta \neq x \in M$ with $x \in A^*$. Then $x \in C^*$, since $A^* \subseteq C^*$. Thus for every $r \in R$, we have $\mu_A(rx) \geq \mu_A(x) > 0$ and $\nu_A(rx) \leq \nu_A(x) < 1$. So $rx \in A^*$.

Now consider a non-zero submodule $N = Rx$ of M. Define an $D \in IFS(M)$ as follows:

$$\mu_D(y) = \begin{cases} \mu_C(y), & \text{if } y \in N \\ 0, & \text{if otherwise} \end{cases} ; \quad \nu_D(y) = \begin{cases} \nu_C(y), & \text{if } y \in N \\ 1, & \text{otherwise.} \end{cases}$$

Then clearly, $D \in IFM(M)$ and $D \subseteq C$ with $D^* = N$.
As $A \trianglelefteq_B C$, $A \nsubseteq B$ and if $A \cap D \subseteq B$, then $D \subseteq B$. As $N \neq \{\theta\}$, for each $\theta \neq y \in N$, there exists $0 \neq r \in R$ such that $y = rx$. Thus $rx \in A^* \cap D^* = (A \cap D)^* \subseteq B^*$. So $rx \in B^*$. Hence $rx \in A^* \cap B^* = (A \cap B)^*$. This complete the proof. □

For the next proposition, follow the definition of intuitionistic fuzzy ideal and intuitionistic fuzzy prime ideal as in [15].

Proposition 3.6. *Let A, B be intuitionistic fuzzy ideals of the ring R. Let B be an intuitionistic fuzzy prime ideal of R such that $A \nsubseteq B$. Then $A \trianglelefteq_B R$.*

Proof. Let C be an intuitionistic fuzzy ideal of R such that $A \cap C \subseteq B$. Since $AC \subseteq A \cap C \subseteq B$, $AC \subseteq B$. As B is intuitionistic fuzzy prime ideal of R, either

$A \subseteq B$ or $C \subseteq B$. But $A \nsubseteq B$. Then $C \subseteq B$ which implies that $A \trianglelefteq_B R$. This complete the proof. □

Theorem 3.7. *Let $A, B \in IFM(M)$. Then $A \trianglelefteq_B M$ if and only if $A^* \trianglelefteq_{B^*} M$.*

Proof. Assume that $A \trianglelefteq_B M$. Let N be a submodule M such that $A^* \cap N \subseteq B^*$. Then $(A \cap \chi_N)^* \subseteq B^*$. Thus $A \cap \chi_N \subseteq B$. But $A \trianglelefteq_B M$. So we have $\chi_N \subseteq B$. But this gives $N \subseteq B^*$. Hence $A^* \trianglelefteq_{B^*} M$.

Conversely, assume that $A^* \trianglelefteq_{B^*} M$. Let C be an IFSM of M such that $A \cap C \subseteq B$. Then $(A \cap C)^* \subseteq B^*$. Thus $A^* \cap C^* \subseteq B^*$. But $A^* \trianglelefteq_{B^*} M$. So we have $C^* \subseteq B^*$. But $C \subseteq B$. Hence $A \trianglelefteq_B M$. □

Theorem 3.8. *Let $P, Q \in IFM(M)$. If $P \trianglelefteq_B M$ and $Q \trianglelefteq_B M$, then $P \cap Q \trianglelefteq_B M$.*

Proof. Let $S \in IFM(M)$ such that $(P \cap Q) \cap S \subseteq B$. Then $P \cap (Q \cap S) \subseteq B$. As $P \trianglelefteq_B M$, we get $Q \cap S \subseteq B$. Again as $Q \trianglelefteq_B M$, $S \subseteq B$. Thus $P \cap Q \trianglelefteq_B M$. □

Definition 3.9. Let $A, C \in IFM(M)$ be such that $A \subseteq C$. Then A is called B-essential of C, denoted by $A \trianglelefteq_B C$, provided that $A \nsubseteq B$ and for any IFSM D of M satisfying $D \subseteq C$ and $A \cap D \subseteq B$ implies $D \subseteq B$.

Theorem 3.10. *Let $A, C \in IFM(M)$. Then $A \trianglelefteq_B C$ if and only if $A^* \trianglelefteq_{B^*} C^*$.*

Proof. Assume that $A \trianglelefteq_B C$. Let N be a submodule of C^* such that $A^* \cap N \subseteq C^*$. Define an IFS D of M as follow:

$$\mu_D(x) = \begin{cases} \mu_C(x), & \text{if } x \in N \\ 0, & \text{if } x \notin N \end{cases}; \quad \nu_D(x) = \begin{cases} \nu_C(x), & \text{if } x \in N \\ 1, & \text{if } x \notin N. \end{cases}$$

Then clearly, $D^* = N$. Thus we have $A^* \cap D^* \subseteq C^*$. So $(A \cap D)^* \subseteq C^*$. Hence $A \cap D \subseteq C$. Since $A \trianglelefteq_B C$, we have $D \subseteq C$ which further implies that $D^* \subseteq C^*$. Therefore $A^* \trianglelefteq_{B^*} C^*$.

Conversely, assume that $A^* \trianglelefteq_{B^*} C^*$. Let $E \subseteq C$ such that $A \cap E \subseteq C$. Then $(A \cap E)^* \subseteq C^*$. Thus $A^* \cap E^* \subseteq C^*$. As $A^* \trianglelefteq_{B^*} C^*$, $E^* \subseteq C^*$. So $E \subseteq C$. Hence $A \trianglelefteq_B C$. □

Theorem 3.11. *Let $A, C, D \in IFM(M)$ such that $A \subseteq C \subseteq D$. Then $A \trianglelefteq_B D$ if and only if $A \trianglelefteq_B C$ and $C \trianglelefteq_B D$.*

Proof. Let $A \trianglelefteq_B D$ and $E \subseteq C$ be such that

$$A \cap E \subseteq B. \tag{3.1}$$

As $E \subseteq C$ and $C \subseteq D$, $E \subseteq D$. Then from (3.1) and using $A \trianglelefteq_B D$, we get $E \subseteq B$. Thus $A \trianglelefteq_B C$.

Again, let $F \subseteq D$ be such that

$$F \cap C \subseteq B. \tag{3.2}$$

As $A \subseteq C$, $A \cap F \subseteq C \cap F$. Then (3.2) becomes $A \cap F \subseteq B$. But $A \trianglelefteq_B D$ implies that $F \subseteq B$. Thus $C \trianglelefteq_B D$. So we get $A \trianglelefteq_B C$ and $C \trianglelefteq_B D$.

Conversely, assume that $A \trianglelefteq_B C$ and $C \trianglelefteq_B D$. Let $G \subseteq D$ such that $A \cap G \subseteq B$. Consider $(A \cap C) \cap G \subseteq B$. Then $A \cap (C \cap G) \subseteq B$. As $C \cap G \subseteq C$ and $A \trianglelefteq_B C$, $C \cap G \subseteq B$. Since $C \trianglelefteq_B D$, $G \subseteq B$. Thus $A \trianglelefteq_B D$. □

Theorem 3.12. *Let $B_1, B_2, C_1, C_2 \in IFM(M)$ such that $C_1 \cap C_2 = B_1 \cap B_2$. Also, $A_1, A_2 \in IFM(M)$ be such that $B_1 \subseteq A_1 \subseteq C_1$ and $B_2 \subseteq A_2 \subseteq C_2$. Then $A_1 + A_2 \trianglelefteq_{B_1+B_2} C_1 + C_2$ if and only if $A_1 \trianglelefteq_{B_1} C_1$ and $A_2 \trianglelefteq_{B_2} C_2$.*

Proof. It follows from Theorem 3.10, Lemma 2.5 and Proposition 2.10. □

Theorem 3.13. *Let $f : M \to N$ be a module epimorphism, where M and N are R-modules. Let A, B are IFSMs of N. If $A \trianglelefteq_B N$, then $f^{-1}(A) \trianglelefteq_{f^{-1}(B)} M$.*

Proof. From Theorem 2.15, we have

$$f^{-1}(A^*) = (f^{-1}(A))^*. \tag{3.3}$$

As $A \trianglelefteq_B N$, by Theorem 3.7, $A^* \trianglelefteq_{B^*} N$. Then by Lemma 2.5, $f^{-1}(A^*) \trianglelefteq_{f^{-1}(B^*)} M$. Thus from (3.3), $(f^{-1}(A))^* \trianglelefteq_{(f^{-1}(B))^*} M$. So by Theorem 3.7, $f^{-1}(A) \trianglelefteq_{f^{-1}(B)} M$. □

Theorem 3.14. *Let $f : M \to N$ be a module epimorphism, where M and N are R-modules. Let A is IFSM of N. Then $f^{-1}(A) \trianglelefteq_{\chi_{kerf}} M$ if and only if $A \trianglelefteq N$.*

Proof. Assume $f^{-1}(A) \trianglelefteq_{\chi_{kerf}} M$. From Theorem **??**, we have

$$f^{-1}(A^*) = (f^{-1}(A))^*. \tag{3.4}$$

As $f^{-1}(A) \trianglelefteq_{\chi_{kerf}} M$, by Theorem 3.7, $(f^{-1}(A))^* \trianglelefteq_{kerf} M$. Then by (3.4), we have $f^{-1}(A^*) \trianglelefteq_{\chi_{kerf}} M$. Thus by Corollary 2.6 of [12], we get $A^* \trianglelefteq N$. So $A \trianglelefteq N$.

Conversely, assume that $A \trianglelefteq N$ implies $A^* \trianglelefteq N$. Then by Corollary 2.5 of [12], we have $(f^{-1}(A^*)) \trianglelefteq_{kerf} M$. Thus by (3.4) we have $(f^{-1}(A))^* \trianglelefteq_{kerf} M$. So by Theorem 3.7, we get $f^{-1}(A) \trianglelefteq_{\chi_{kerf}} M$. □

Theorem 3.15. *Let $A, B \in IFM(M)$. If $A \trianglelefteq_B M$. Then $(A + B)/B \trianglelefteq \chi_M/B$.*

Proof. Let C be an IFSM of M such that $B \subseteq C$ and C/B be an IFSM of χ_M/B such that $C/B \cap (A+B)/B = \chi_M/B^*$. Then $(C/B)^* \cap ((A+B)/B)^* = \{B^*\}$, i.e.,

$$(C^*/B^*) \cap (A^* + B^*)/B^* = \{B^*\}. \tag{3.5}$$

Let $x \in A^* \cap C^*$ be any element. Then $x + B^* \in (A^* \cap C^*) + B^* = (C^* \cap A^*) + B^*$. Since $B \subseteq C$, $B^* \subseteq C^*$. By modular identities, $x + B^* \in C^* \cap (B^* \cap A^*)$. Thus $x + B^* \in (C^*/B^*) \cap (B^* + A^*)/B^* = \{B^*\}$. By (3.5), $x + B^* \in B^*$. So $x \in B^*$. Hence $A^* \cap C^* \subseteq B^* \Rightarrow (A \cap C)^* \subseteq B^*$. This implies $A \cap C \subseteq B$. But $A \trianglelefteq_B M$ implies $C \subseteq B$. Therefore $(A + B)/B \trianglelefteq \chi_M/B$. □

Remark 3.16. The converse of above theorem is true when $B \subseteq A$ (See the next Theorem).

Theorem 3.17. *Let $A, B \in IFM(M)$ be such that $B \subseteq A$. Then $A \trianglelefteq_B M$ if and only if $A/B \trianglelefteq \chi_M/B$.*

Proof. Assume that $A, B \in IFM(M)$ and $B \subseteq A$. Let C/B be an IFSM of χ_M/B such that $(C/B) \cap (A/B) = \chi_M/B^*$. Then we have

$$(C/B)^* \cap (A/B)^* = \{B^*\}. \tag{3.6}$$

Let $x + B^* \in (C^*/B^*) \cap (A^*/B^*) \Rightarrow x + B^* \in C/B) \cap (A/B) = \chi_M/B^*$. Then From (3.6), $x + B^* = B^*$. Thus $x \in B^*$. This gives that

$$C^* \cap A^* \subseteq B^*. \tag{3.7}$$

But it is given that $A \trianglelefteq_B M$. By Theorem 3.5, $A^* \trianglelefteq_{B^*} M$. From (3.7), $C^* \subseteq B^*$. So $C = B$. Hence $C/B = \chi_{B^*}$. Therefore $A/B \trianglelefteq \chi_M/B$.

Conversely, assume that $A/B \trianglelefteq \chi_M/B$ implies that

$$(A/B)^* \trianglelefteq (\chi_M/B)^*. \tag{3.8}$$

Let D be any IFSM of M such that $A \cap D \subseteq B$. Then $(A \cap D)^* \subseteq B^*$. Thus $A^* \cap D^* \subseteq B^*$. So $A^*/B^* \cap D^*/B^* \subseteq \{B^*\}$. Hence $(A/B)^* \cap (D/B)^* \subseteq \{B^*\}$. Therefore from (3.8), we get $(D/B)^* \subseteq \{B^*\}$.

Now, let $x \notin B^*$. Then $(D/B)(x + B^*) = (0, 1)$. Thus

$$\bigvee\{\mu_D(x + y) : y \in B^*\} = 0 \text{ and } \bigwedge\{\nu_D(x + y) : y \in B^*\} = 1.$$

So $\mu_D(x+\theta) = \mu_D(x) = 0$ and $\nu_D(x+\theta) = \mu_D(x) = 1$, i.e., $x \notin D^*$. Hence $D^* \subseteq B^*$ and thus $D \subseteq B$. Therefore $A \trianglelefteq_B M$. □

4. Intuitionistic fuzzy complement with respect to another intuitionistic fuzzy submodule

Definition 4.1. Let A, B are two IFSMs of an R-module M. Then an IFSM C of M is called B-complement to A, if C is maximal with respect to the property that $A \cap C \subseteq B$.

Theorem 4.2. *If an IFSM C is B-complement of an IFSM A in M. Then the submodule C^* of M is complement of A^* relative to B^* in M.*

Proof. Since C is B-complement of A in M, C is maximal IFSM of M with the property that $A \cap C \subseteq B$. Then $A^* \cap C^* \subseteq B^*$. It remain to show that C^* is maximal with this property. Let N be a submodule of M such that $C^* \subseteq N$ and $A^* \cap N \subseteq B^*$. Since $\mu_C(x) > 0, \nu_C(x) < 1$ for all $x \in C^*$. Now let $p = Inf\{\mu_C(x) : x \in C^*\}$ and $q = Sup\{\nu_C(x) : x \in C^*\}$. Then $p, q \in [0.1]$ such that $p + q \leq 1$. Choose $\alpha, \beta \in [0.1]$ such that $0 < \alpha \leq p$ and $q \leq \beta < 1$ and define an IFS D on M as follows:

$$\mu_D(x) = \begin{cases} \mu_C(x), & \text{if } x \in C^* \\ \alpha, & \text{if } x \in N - C^* \\ 0, & \text{if } x \notin N \end{cases}; \quad \nu_D(x) = \begin{cases} \nu_C(x), & \text{if } x \in C^* \\ \beta, & \text{if } x \in N - C^* \\ 1, & \text{if } x \notin N. \end{cases}$$

Then clearly, D is an IFSM of M such that $C \subseteq D$. Thus $D^* = N$. Since $A^* \cap N \subseteq B^*$, $A^* \cap D^* \subseteq B^*$. This implies $(A \cap D)^* \subseteq B^* \Rightarrow A \cap D \subseteq B$. But C is maximal with this property that $A \cap C \subseteq B$. So $C = D$. Hence $C^* = D^* = N$. Therefore C^* is B^*-complement of A^* in M. □

Remark 4.3. The converse of the above theorem is not true, i.e., if for any IFSMs A, B, C of M, the submodule C^* is B^*-complement of A^* in M, then C need not be B-complement of A in M.

Example 4.4. Consider the module $M = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ over the ring $\mathbb{Z}$ and define IFSs A, B, C of M as follows: for each $x \in \mathbb{Z}$,

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x = 3 \\ 0, & \text{if } x \in \{1, 2, 4, 5\} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.3, & \text{if } x = 3 \\ 1, & \text{if } x \in \{1, 2, 4, 5\} \end{cases}$$

$$\mu_B(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \neq 0 \end{cases}$$

$$\mu_C(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.6, & \text{if } x \in \{2,4\} \\ 0, & \text{if } x \in \{1,3,5\} \end{cases}; \quad \nu_C(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.3, & \text{if } x \in \{2,4\} \\ 1, & \text{if } x \in \{1,3,5\}. \end{cases}$$

Then it is easy to check that A, B, C are IFSMs of M such that $A^* = \{0,3\}$, $B^* = \{0\}$ and $C^* = \{0,2,4\}$. Thus clearly, $A^* \cap C^* = \{0\} = B^*$ and C^* is maximal with this property. So C^* is B^*-complement of A^*. But C is not B-complement to A, for if we define the IFS D on M as follows: for each $x \in \mathbb{Z}$,

$$\mu_D(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.7, & \text{if } x \in \{2,4\} \\ 0, & \text{if } x \in \{1,3,5\} \end{cases}; \quad \nu_D(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.1, & \text{if } x \in \{2,4\} \\ 1, & \text{if } x \in \{1,3,5\}, \end{cases}$$

then D is an IFSM of M with $C \subseteq D$ and $D \cap A \subseteq B$. This shows that C is not maximal with the property that $C \cap A \subseteq B$.

Theorem 4.5. *Let $A, C \in IFM(M)$ and $B = A \cap C$. Then C is B-complement to A if and only if $(A + C)/C \trianglelefteq \chi_M/C$.*

Proof. Let $D \in IFM(M)$ such that $C \subseteq D$ and $D/C \cap (A + C)/C = \chi_{C^*}$. Then $(D/C)^* \cap ((A + C)/C)^* = \{C^*\}$ implies

$$D^*/C^* \cap (A^* + C^*)/C^* = \{C^*\}. \tag{4.1}$$

Let $x \in D^* \cap A^*$ be any element. By the modular law,

$$\therefore x + C^* \in (D^* \cap A^*) + C^* = D^* \cap (A^* + C^*).$$

From (4.1), $x + C^* \in D^*/C^* \cap (A^* + C^*)/C^* = \{C^*\}$. Then $x + C^* = C^*$. Thus $x \in C^*$. So $D^* \cap A^* \subseteq C^*$, i.e., $(D \cap A)^* \subseteq C^*$, i.e., $(D \cap A) \subseteq C$. This implies $D \cap A \subseteq C \cap A = B$. As C is B-complement to A in M, C is maximal with the property that $C \cap A \subseteq B$. So we get $C = D$. Hence $(D/C)^* = \{C^*\}$. Therefore $(A + C)/C \trianglelefteq \chi_M/C$.

Conversely, suppose $(A + C)/C \trianglelefteq \chi_M/C$ and $B = A \cap C$. To show that C is B-complement to A, for this we show that $A \cap C \subseteq B$ and C is maximal with this property. If possible, let $D \in IFM(M)$ such that $C \subseteq D$ and $D \cap A \subseteq B$. As $(A+C)/C \trianglelefteq \chi_M/C$, $A+C \trianglelefteq_B M$ by Theorem 3.15. Then by Definition 3.1, we have $(A+C) \cap D \subseteq B \Rightarrow D \subseteq B$, i.e., $(A \cap D) + (C \cap D) \subseteq B \Rightarrow D \subseteq B$. As $D \cap A \subseteq B$, $D \subseteq B$. But $B = A \cap C$. Thus we get $D \subseteq B = A \cap C \subseteq C$, i.e., $D \subseteq C$. But $C \subseteq D$ implies that $C = D$. So C is B-complement to A. □

Theorem 4.6. *Let $A, B, C \in IFM(M)$. If C/B is intuitionistic fuzzy complement to A/B in χ_M/B, then C is B-complement to A in M. The converse is true if $B \subseteq C \cap A$.*

Proof. Let C/B is intuitionistic fuzzy complement to A/B in χ_M/B. Then

$$\begin{aligned} C/B \cap A/B = \chi_{B^*} &\Rightarrow (C/B)^* \cap (A/B)^* = \{B^*\} \\ &\Rightarrow C^*/B^* \cap A^*/B^* = \{B^*\} \end{aligned}$$

$$\Rightarrow C^* \cap A^* = \{B^*\}$$
$$\Rightarrow C \text{ is } B\text{-complement of } A \text{ in } M.$$

Suppose $C \subseteq D \subseteq B$ such that $D \cap A \subseteq B$. Then we have

$$\begin{aligned} D^* \cap A^* \subseteq B^* &\Rightarrow D^*/B^* \cap A^*/B^* \subseteq \{B^*\} \\ &\Rightarrow (D/B)^* \cap (A/B)^* \subseteq \{B^*\} \\ &\Rightarrow (D/B) \cap (A/B) \subseteq \{\chi_{B^*}\} \\ &\Rightarrow D/B \text{ is an intuitionistic fuzzy complement of } A/B \text{ in } \chi_M/B. \end{aligned}$$

This is a contradiction to the assumption. Thus we get

$$\begin{aligned} C/B = D/B &\Rightarrow (C/B)^* = (D/B)^* \\ &\Rightarrow C^*/B^* = D^*/B^* \\ &\Rightarrow C^* = D^* \\ &\Rightarrow C = D. \end{aligned}$$

So C is B-complement to A in M.

Conversely, let C is B-complement to A in M and $B \subseteq C \cap A$. Then

$$\begin{aligned} C \cap A = B &\Rightarrow (C \cap A)^* = B^* \\ &\Rightarrow C^* \cap A^* = B^* \\ &\Rightarrow C^*/B^* \cap A^*/B^* = \{B^*\} \\ &\Rightarrow (C/B)^* \cap (A/B)^* = \{B^*\} \\ &\Rightarrow (C/B \cap A/B)^* = \{B^*\} \\ &\Rightarrow (C/B \cap A/B) = \chi_{B^*} \\ &\Rightarrow C/B \text{ is intuitionistic fuzzy complement of } A/B \text{ in } \chi_M/B. \end{aligned}$$

Suppose that $C/B \subseteq D/B \subseteq \chi_M/B$ such that $D/B \cap A/B = \chi_{\{B^*\}}$. Then

$$\begin{aligned} (D/B)^* \cap (A/B)^* = \{B^*\} &\Rightarrow D^*/B^* \cap A^*/B^* = \{B^*\} \\ &\Rightarrow D^* \cap A^* = B^* \\ &\Rightarrow D \cap A = B. \end{aligned}$$

Since C is B-complement to A in M, $C = D$. Thus $C/B = D/B$. So C/B is an intuitionistic fuzzy complement to A/B in χ_M/B. □

Corollary 4.7. *Let $A, B, C \in IFM(M)$ such that $\chi_M/B = C/B \bigoplus A/B$. Then C is B-complement to A in M.*

Proof. As $\chi_M/B = C/B \bigoplus A/B$, $C/B \cap A/B = \chi_{\{B^*\}}$. Let $C/B \subseteq D/B \subseteq \chi_M/B$ such that $D/B \cap A/B = \chi_{\{B^*\}}$. Since $\chi_M/B = C/B \bigoplus A/B$ and $C/B \subseteq D/B$, $\chi_M/B = D/B \bigoplus A/B$. Then $C/B = D/B$. Thus C/B is intuitionistic fuzzy complement to A/B in χ_M/B. So by Theorem 4.2, C is B-complement to A in M. □

Theorem 4.8. *Let $A, B, C \in IFM(M)$. Then B is complement to A in M if and only if $A \bigoplus B \trianglelefteq_B M$.*

Proof. Assume that B is complement to A in M. Then by Theorem 2.17, $A \bigoplus B \trianglelefteq M$. But by Theorem 2.18, B is closed in M. Thus $(A \bigoplus B)/B \trianglelefteq \chi_M/B$ by Theorem 4.4.6 of [5]. So by Theorem 3.17, $A \bigoplus B \trianglelefteq_B M$.

Conversely, let $A \bigoplus B \trianglelefteq_B M$. Then $A \cap B = \chi_{\{\theta\}}$. By Theorem 3.12, $(A \bigoplus B)/B \trianglelefteq \chi_M/B$. Let $C \in IFM(M)$ such that $B \subseteq C$ and $C \cap A = \chi_{\{\theta\}}$. Now $C/B \subseteq \chi_M/B$ and by the modular law,

$$(A^* \bigoplus B^*) \cap C^* = A^* \cap C^* + B^* \cap C^* = A^* \cap C^* + B^* = \{\theta\} + B^* = B^*.$$

Then we have

$$(A^* \bigoplus B^*)/B^* \cap C^*/B^* = \{B^*\} \;\Rightarrow (A \bigoplus B/B)^* \cap (C/B)^* = \{B^*\}.$$

Thus $(A \bigoplus B/B) \cap (C/B) = \chi_{\{B^*\}}$. But $(A \bigoplus B)/B \trianglelefteq \chi_M/B$ gives $C/B = \chi_{\{B^*\}}$. So $(C/B)^* = \{B^*\}$.

Let $x \notin B^*$. Then $\mu_{C/B}(x + B^*) = 1$ and $\nu_{C/B}(x + B^*) = 0$. Thus

$$\begin{aligned} &\bigvee\{\mu_C(x + y) : y \in B^*\} = 1 \text{ and } \bigwedge\{\nu_C(x + y) : y \in B^*\} = 0 \\ \Rightarrow\ &\mu_C(x + \theta) = \mu_C(x) = 1 \text{ and } \nu_C(x + \theta) = \nu_C(x) = 0 \\ \Rightarrow\ &x \notin C^*. \end{aligned}$$

So $B^* \subseteq C^*$, i.e., $B \subseteq C$. Hence B is complement to A in M. □

Theorem 4.9. *Let $A, B, C \in IFM(M)$. If $A \trianglelefteq M$, then C is intuitionistic fuzzy complement for B in M if and only if $A + C \trianglelefteq_B M$.*

Proof. Firstly, let $A \trianglelefteq M$ and C is intuitionistic fuzzy complement for B in M. To show that $A + C \trianglelefteq_B M$, let $D \in IFM(M)$ such that $C \subseteq D$. Also let D/C be an intuitionistic fuzzy submodule of χ_M/C such that $(A \bigoplus C)/C \cap (D/C) = \chi_{\{C^*\}}$. Then we get

$$\begin{aligned} &(A \bigoplus C/C)^* \cap (D/C)^* = \{C^*\} \\ \Rightarrow\ &(A \bigoplus C)^*/C^* \cap D^*/C^* = \{C^*\} \\ \Rightarrow\ &(A \bigoplus C)^* \cap D^* = \{C^*\} \\ \Rightarrow\ &(A^* \bigoplus C^*) \cap D^* = \{C^*\} \\ \Rightarrow\ &(A^* \cap D^*) + (C^* \cap D^*) = C^* \\ \Rightarrow\ &(A^* \cap D^*) + C^* = C^* \\ \Rightarrow\ &(A^* \cap D^*) \subseteq C^* \\ \Rightarrow\ &A \cap D \subseteq C. \end{aligned}$$

Thus $(A \cap D) \cap B \subseteq C \cap B = \chi_{\{\theta\}}$ because B is an intuitionistic fuzzy complement for C in M implies $A \cap (D \cap B) = \chi_{\{\theta\}}$. But $A \trianglelefteq M$ gives that $D \cap B = \chi_{\{\theta\}}$. Since C is intuitionistic fuzzy complement for B in M, $C = D$. So $(A \bigoplus C)/C \trianglelefteq \chi_M/C$. Hence by Theorem 3.17, we have $A + C \trianglelefteq_B M$.

Conversely, let $A \trianglelefteq M$ and $A + C \trianglelefteq_B M$. To show that C is intuitionistic fuzzy complement for B in M, we show that $C \neq \chi_{\{\theta\}} \in IFM(M)$ be such that $C \cap B = \chi_{\{\theta\}}$ and C is maximal with this property. If possible, let $D \in IFM(M)$ such that $D \cap B = \chi_{\{\theta\}}$ and $C \subsetneq D$. As $A \trianglelefteq M$ and $A + C \trianglelefteq_B M$, by Theorem 3.17, we have $(A + C)/C \trianglelefteq \chi_M/C$. Also, by Theorem 4.5, we get C is B-complement to A, where $B = A \cap C$. Then we have $C \cap A \subseteq B$ and C is maximum with this property. Thus $D \cap (C \cap A) \subseteq B \cap D = \chi_{\{\theta\}}$. So $(D \cap C) \cap A = \chi_{\{\theta\}}$. As $A \trianglelefteq M$, we get $D \cap C = \chi_{\{\theta\}}$. But $C \subsetneq D \Rightarrow C = \chi_{\{\theta\}}$. This is not possible. Hence C is intuitionistic fuzzy complement for B in M. □

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Conclusion

It is well-known fact that the three concepts, direct summand, essential submodule, and superfluous submodule, are reminiscent of the topological concepts of connected component, dense, and nowhere dense. Moreover the concept of direct summand also leads to the notion of complement submodules. Also the study of these concepts paves the way to other important concepts like closed submodules, uniform modules etc. The study of these concepts forms an important area both in module theory as well as in topology. A sincere attempt has been made to generalise these concepts in the intuitionistic fuzzy environment with an eye towards the possible applications.

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