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Abstract. In this paper, we introduce the concepts of (internal, external) IVI-octahedron sets, and study some of their properties and give some examples. Also, we define Type i-order, Type i-intersection, Type i-union ($i = 1,\ 2,\ 3,\ 4$) and obtain their some properties. Second, we define an IVI-octahedron point and deal with the characterizations of Type i-union (Type i-intersection). Third, we define the image and preimage of an IVI-octahedron set under a mapping and obtain some of their properties. Finally, we define i-IVIGP and i-IVILI [resp. i-IVIRI and i-IVII], and investigate their some properties.

2010 AMS Classification: 03E72

Keywords: (Internal, External) IVI-octahedron set, Union (Intersection) of IVI-octahedron sets, IVI-octahedron point, Level set, IVIGP [resp. IVILI, IVIRI and IVII].

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1. Introduction

In the real world, we encounter with decision making circumstances involving ambiguities and uncertainties. To overcome such problems, Zadeh [27] (1965) proposed the notion of fuzzy sets as the generalization of crisp sets. After then, many researchers have been trying to find a mathematical expression of ambiguities and uncertainties which can be applied to enguneering, medicine, and social sciences, etc. For examples, interval-valued fuzzy set theory Zadeh [28] (1975) (See [8]), intuitionistic fuzzy set theory Atanassov [3] (1983), interval-valued intuitionistic fuzzy set theory Atanassov and Gargov [4] (1989), vague set theory Gau and Buchrer [7] (1993), neutrosophic set theory Smarandache [25] (1998), bipolar fuzzy set theory Zhang [29], rough set theory Pawlak [22] (1982), soft set theory Molodtsov [19]

(1999), etc. Recently, Jun et al. [11] defined a cubic set as a pair of an interval-valued fuzzy set and a fuzzy set and studied some of its properties. N. Abughazalah and Yaqoob [1] applied cubic sets to subsystems of finite state machines. Yaqoob and Abughazalah [26] dealt with Finite switchboard state machines based on cubic sets. Smarandache et al. [12] extended the concept of cubic sets to neutrosophic sets (called a neutrosophic cubic set) and investigated some of its properties. Ali et al. [2] applied such concept to pattern recognitions. Moreover, Jun [10] defined a cubic intuionistic set composed of an interval-valued intuitionistic fuzzy set and an intuitionistic fuzzy set, and applied it to BCI/BCK-algebra. In particular, Kaur and Garg [14] studied multi-attribute decision making problems based on cubic intuionistic fuzzy sets. Also they [15] introduced cubic intuitionistic fuzzy aggregation operators. Kim et al. [16] introduced the concept of octahedron sets composed of three components: interval-valued fuzzy set, intuitionistic fuzzy set and fuzzy set, which will provide more information about ambiguity and uncertainty common in everyday life, and dealt with its various properties. Of course, the octahedron set can reduce information loss about ambiguity and uncertainty than cubic sets and cubic interval-valued intuitionistic fuzzy sets (See [11] and [13]).

Since the interval-valued intuionistic fuzzy set provides more ambiguity and uncertainty than the interval-valued fuzzy set, a new concept (will be called an interval-valued intuionistic fuzzy octahedron set) is needed to replace the first component of the octahedron set, the interval-valued fuzzy set, with the interval-valued intuitionistic fuzzy set. In particular, we expect this concept to be used as a tool to deal with multi-attribute decision making problems. Then this paper is formed of the followings: in Section 2, we list some definitions needed next sections: in Section 3, we define an IVI-octahedron set as a triple of interval-valued intuitionistic fuzzy set, an intuitionistic fuzzy set and a fuzzy set, and studied some related properties and give some examples; In Section 4, we introduce the octahedron point and the level set of an IVI-octahedron set, and find some of their properties; In Section 5, we define the image and preimage of an IVI-octahedron set under a mapping and investigate some of their properties; In Section 6, we apply IVI-octahedron sets to groupoid theory.

2. Preliminaries

In this section, we list some basic definitions needed in the next sections.

For a set X, let I^X denotes the set of all fuzzy sets in X and members of I^X will write λ, μ, ν, etc., where $I = [0, 1]$. In particular, 0 and 1 denote the fuzzy empty set and the fuzzy whole set in X, respectively (See [27]). Also, refer to [27] for the inclusion, intersection, union of two fuzzy sets and the complement of a fuzzy set.

Each member of a set $I \oplus I = \{(a^{\in}, a^{\notin}) : (a^{\in}, a^{\notin}) \in I \times I \text{ and } a^{\in} + a^{\notin} \leq 1\}$ is called an intuitionistic fuzzy number, and $(0, 1)$ and $(1, 0)$ are denoted by $\bar{0}$ and $\bar{1}$, respectively (See [6]). We will denote intuitionistic fuzzy numbers $(a^{\in}, a^{\notin})$, $(b^{\in}, b^{\notin})$, $(c^{\in}, c^{\notin})$, etc. as $\bar{a}$, $\bar{b}$, $\bar{c}$, etc. It is well-known (Theorem 2.1 in [6]) that $(I \oplus I, \leq)$ is a complete distributive lattice with the greatest element $\bar{1}$ and the least element $\bar{0}$ satisfying De-Morgan's laws.

Definition 2.1 ([3]). For a nonempty set X, a mapping $A : X \to I \oplus I$ is called an intuitionistic fuzzy set (briefly, IF set) in X, where for each $x \in X$, $A(x) = (A^{\in}(x), A^{\notin}(x))$, and $A^{\in}(x)$ and $A^{\notin}(x)$ represent the degree of membership and the degree of nonmembership of an element x to A, respectively. Let $(I \oplus I)^X$ denote the set of all IF sets in X and for each $A \in (I \oplus I)^X$, we write $A = (A^{\in}, A^{\notin})$. In particular, $\bar{\mathbf{0}}$ and $\bar{\mathbf{1}}$ denote the IF empty set and the IF whole set in X defined by, respectively: for each $x \in X$,

$$\bar{\mathbf{0}}(x) = \bar{0} \text{ and } \bar{\mathbf{1}}(x) = \bar{1}.$$

Refer to [3] for the inclusion, intersection, union of two IF sets and the complement of an IF set, and operators $[\ ]A$,$\diamond A$ for an IF set A.

The set of all closed subintervals of I is denoted by $[I]$, and members of $[I]$ are called interval numbers and denoted by $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, etc., where $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$. In particular, if $a^- = a^+$, then we write as $\widetilde{a} = \mathbf{a}$ (See [16]).

Definition 2.2 ([8, 20]). For a nonempty set X, a mapping $A : X \to [I]$ is called an interval-valued fuzzy set (briefly, an IVF set) in X. Let $[I]^X$ denote the set of all IVF sets in X. For each $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the degree of membership of an element x to A, where $A^-, A^+ \in I^X$ are called a lower fuzzy set and an upper fuzzy set in X, respectively. For each $A \in [I]^X$, we write $A = [A^-, A^+]$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy empty whole set in X.

Also, refer to [8, 20] for the inclusion, intersection, union of two IVF sets and the complement of an IVF set

Let $[I] \oplus [I] = \{(\widetilde{a}^{\in}, \widetilde{a}^{\notin}) : (\widetilde{a}^{\in}, \widetilde{a}^{\notin}) \in [I] \times [I] \text{ and } a^{\in,+} + a^{\notin,+} \leq 1\}$, where

$$\widetilde{a}^{\in} = [a^{\in,-}, a^{\in,+}],\ \widetilde{a}^{\notin} = [a^{\notin,-}, a^{\notin,+}] \in [I].$$

Each member of $[I] \oplus [I]$ is called an interval-valued intuitionistic fuzzy number. In particular, we write as $\widetilde{\widetilde{0}} = (\mathbf{0}, \mathbf{1})$ and $\widetilde{\widetilde{1}} = (\mathbf{1}, \mathbf{0})$. We will interval-valued intuitionistic fuzzy numbers $(\widetilde{a}^{\in}, \widetilde{a}^{\notin})$, $(\widetilde{b}^{\in}, \widetilde{b}^{\notin})$, $(\widetilde{c}^{\in}, \widetilde{c}^{\notin})$, etc. as $\widetilde{\widetilde{a}}$, $\widetilde{\widetilde{b}}$, $\widetilde{\widetilde{c}}$, etc.

We define relations $\leq$ and $=$ on $[I] \oplus [I]$ as follows: for any $\widetilde{\widetilde{a}}$, $\widetilde{\widetilde{b}} \in [I] \oplus [I]$,

$$\widetilde{\widetilde{a}} \leq \widetilde{\widetilde{b}} \Longleftrightarrow a^{\in,-} \leq b^{\in,-},\ a^{\in,+} \leq b^{\in,+} \text{ and } a^{\notin,-} \geq b^{\notin,-},\ a^{\notin,+} \geq b^{\notin,+},$$

$$\widetilde{\widetilde{a}} = \widetilde{\widetilde{b}} \Longleftrightarrow \widetilde{\widetilde{a}} \leq \widetilde{\widetilde{b}} \text{ and } \widetilde{\widetilde{a}} \geq \widetilde{\widetilde{b}}.$$

Let $(\widetilde{\widetilde{a_j}})_{j \in J} \subset [I] \oplus [I]$. Then its inf and sup, denoted by $\bigwedge_{j \in J} \widetilde{\widetilde{a_j}}$ and $\bigvee_{j \in J} \widetilde{\widetilde{a_j}}$, are defined as follows:

$$\bigwedge_{j \in J} \widetilde{\widetilde{a_j}} = ([\bigwedge_{j \in J} a_j^{\in,-}, \bigwedge_{j \in J} a_j^{\in,+}], [\bigvee_{j \in J} a_j^{\notin,-}, \bigvee_{j \in J} a_j^{\notin,+}]),$$

$$\bigvee_{j \in J} \widetilde{\widetilde{a_j}} = ([\bigvee j \in J a_j^{\in,-}, \bigvee_{j \in J} a_j^{\in,+}], [\bigwedge_{j \in J} a_j^{\notin,-}, \bigwedge_{j \in J} a_j^{\notin,+}]).$$

Definition 2.3 ([4]). Let X be a nonempty set. Then a mapping $\mathbf{A} = (\mathbf{A}^{\in}, \mathbf{A}^{\notin}) : X \to [I] \oplus [I]$ is called an interval-valued intuitionistic fuzzy set (briefly, IVI set) in

X, where for each $x \in X$, $\mathbf{A}^{\in} = [A^{\in,-}(x), A^{\in,+}(x)]$, $\mathbf{A}^{\notin} = [A^{\notin,-}(x), A^{\notin,+}(x)]$ and $A^{\in,+}(x) + A^{\notin,+}(x) \leq 1$.

In particular, $\widetilde{\widetilde{\mathbf{0}}}$ (resp. $\widetilde{\widetilde{\mathbf{1}}}$) will be called an IVI empty set (resp. IVI whole set) in X. We will denote the set of all IVI sets as $([I] \oplus [I])^X$.

The relations ($\subset$, $=$), operations ($\cup$, $\cap$, ^{c}) and operators ($[\ ]$, $\diamond$) on $([I] \oplus [I])^X$ are defined as follows.

Definition 2.4 ([4]). Let $\mathbf{A} = (\mathbf{A}^{\in}, \mathbf{A}^{\notin})$, $\mathbf{B} = (\mathbf{B}^{\in}, \mathbf{B}^{\notin}) \in ([I] \oplus [I])^X$ and let $(\mathbf{A}_j)_{j\in J} = ((\mathbf{A}_j^{\in}, \mathbf{A}^{\notin})_j)_{j\in J} \subset ([I] \oplus [I])^X$. Then

(i) $\mathbf{A} \subset \mathbf{B} \Longleftrightarrow (\forall x \in X)(A^{\in,-}(x) \leq B^{\in,-}(x),\ A^{\in,+}(x) \leq B^{\in,+}(x)$
and $A^{\notin,-}(x) \geq B^{\notin,-}(x),\ A^{\notin,+}(x) \geq B^{\notin,+}(x))$,

(ii) $\mathbf{A} = \mathbf{B} \Longleftrightarrow \mathbf{A} \subset \mathbf{B}$ and $\mathbf{B} \subset \mathbf{A}$,

(iii) $\mathbf{A}^c(x) = (\mathbf{A}^{\notin}(x), \mathbf{A}^{\in}(x))$ for each $x \in X$,

(iv) $(\mathbf{A} \cup \mathbf{B})(x) = ([A^{\in,-}(x) \vee B^{\in,-}(x), A^{\in,+}(x) \vee B^{\in,+}(x)],$
$[A^{\notin,-}(x) \wedge B^{\notin,-}(x), A^{\notin,+}(x) \wedge B^{\notin,+}(x)])$ for each $x \in X$,

(v) $(\mathbf{A} \cap \mathbf{B})(x) = ([A^{\in,-}(x) \wedge B^{\in,-}(x), A^{\in,+}(x) \wedge B^{\in,+}(x)],$
$[A^{\notin,-}(x) \vee B^{\notin,-}(x), A^{\notin,+}(x) \vee B^{\notin,+}(x)])$ for each $x \in X$,

(vi) $(\bigcup_{j\in J} \mathbf{A}_j)(x) = ([\bigvee_{j\in J} A_j^{\in,-}(x), \bigvee_{j\in J} A_j^{\in,+}(x)],$
$[\bigwedge_{j\in J} A_j^{\notin,-}(x), \bigwedge_{j\in J} A_j^{\notin,+}(x)])$ for each $x \in X$,

(vii) $(\bigcap_{j\in J} \mathbf{A}_j)(x) = ([\bigwedge_{j\in J} A_j^{\in,-}(x), \bigwedge_{j\in J} A_j^{\in,+}(x)],$
$[\bigvee_{j\in J} A_j^{\notin,-}(x), \bigvee_{j\in J} A_j^{\notin,+}(x)])$ for each $x \in X$,

(viii) $[\ ]\mathbf{A}(x) = (\mathbf{A}^{\in}(x), [(A^{\notin,-}(x), 1 - A^{\in,+}(x)])$ for each $x \in X$,

(ix) $\diamond\mathbf{A}(x) = ([A^{\in,-}(x), 1 - A^{\notin,+}(x)], \mathbf{A}^{\notin}(x))$ for each $x \in X$.

Definition 2.5 ([16]). Let X be a nonempty set and let $\mathbf{A} = [A^-, A^+] \in [I]^X$, $A = (A^{\in}, A^{\notin}) \in (I \oplus I)^X$, $\lambda \in I^X$. Then the triple $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$ is called an octahedron set in X. In fact, $\mathcal{A} : X \to [I] \times (I \oplus I) \times I$ is a mapping.

In this case, $\ddot{0}$ (resp. $\ddot{1}$) is called an octahedron empty set (resp. octahedron whole set) in X. We denote the set of all octahedron sets as $\mathcal{O}(X)$.

3. Interval-valued intuitionistic octahedron sets

Members of $([I] \oplus [I]) \times (I \oplus I) \times I$ are called interval-valued intuitionistic fuzzy octahedron numbers (briefly, IVI-octahedron numbers) and we write them as

$$\bar{\widetilde{\widetilde{a}}} = \left\langle \widetilde{\widetilde{a}}, \bar{a}, a \right\rangle,\ \bar{\widetilde{\widetilde{b}}} = \left\langle \widetilde{\widetilde{b}}, \bar{b}, b \right\rangle,\ \text{etc},$$

where $\widetilde{\widetilde{a}} = (\widetilde{a}^{\in}, \widetilde{a}^{\notin}) = ([a^{\in,-}, a^{\in,+}], [a^{\notin,-}, a^{\notin,+}])$, $\bar{a} = (a^{\in}, a^{\notin})$. In particular, $\left\langle \widetilde{\widetilde{0}}, \bar{0}, 0 \right\rangle$ and $\left\langle \widetilde{\widetilde{1}}, \bar{1}, 1 \right\rangle$ as $\bar{\widetilde{\widetilde{0}}}$ and $\bar{\widetilde{\widetilde{1}}}$, respectively.

We define relations $\leq_i$ $(i = 1,\ 2,\ 3,\ 4)$ and $=$ on $([I] \oplus [I]) \times (I \oplus I) \times I$ as follows: for any $\bar{\widetilde{\widetilde{a}}}$, $\bar{\widetilde{\widetilde{b}}} \in ([I] \oplus [I]) \times (I \oplus I) \times I$,

$$\bar{\widetilde{\widetilde{a}}} \leq_1 \bar{\widetilde{\widetilde{b}}} \Longleftrightarrow \widetilde{\widetilde{a}} \leq \widetilde{\widetilde{b}},\ \bar{a} \leq \bar{b},\ a \leq b,\ \bar{\widetilde{\widetilde{a}}} \leq_2 \bar{\widetilde{\widetilde{b}}} \Longleftrightarrow \widetilde{\widetilde{a}} \leq \widetilde{\widetilde{b}},\ \bar{a} \leq \bar{b},\ a \geq b,$$

$$\bar{\widetilde{\widetilde{a}}} \leq_3 \bar{\widetilde{\widetilde{b}}} \Longleftrightarrow \widetilde{\widetilde{a}} \leq \widetilde{\widetilde{b}},\ \bar{a} \geq \bar{b},\ a \leq b,\ \bar{\widetilde{\widetilde{a}}} \leq_4 \bar{\widetilde{\widetilde{b}}} \Longleftrightarrow \widetilde{\widetilde{a}} \leq \widetilde{\widetilde{b}},\ \bar{a} \geq \bar{b},\ a \geq b,$$

$$\tilde{\tilde{a}} = \tilde{\tilde{b}} \Longleftrightarrow \tilde{\tilde{a}} \leq_i \tilde{\tilde{b}},\ \tilde{\tilde{b}} \leq_i \tilde{\tilde{a}} (i = 1,\ 2,\ 3,\ 4).$$

For any $(\tilde{\tilde{a}}_j)_{j\in J} \subset ([I] \oplus [I]) \times (I \oplus I) \times I$, its inf $\bigwedge^i_{j\in J} \tilde{\tilde{a}}_j$ and sup $\bigvee^i_{j\in J} \tilde{\tilde{a}}_j$ $(i = 1,\ 2,\ 3,\ 4)$ are defined as follows:

$$\bigwedge^1_{j\in J} \tilde{\tilde{a}}_j = \left\langle \bigwedge_{j\in J} \tilde{a}_j, \bigwedge_{j\in J} \bar{a}_j, \bigwedge_{j\in J} a_j \right\rangle,\ \bigwedge^2_{j\in J} \tilde{\tilde{a}}_j = \left\langle \bigwedge_{j\in J} \tilde{a}_j, \bigwedge_{j\in J} \bar{a}_j, \bigvee_{j\in J} a_j \right\rangle,$$

$$\bigwedge^3_{j\in J} \tilde{\tilde{a}}_j = \left\langle \bigwedge_{j\in J} \tilde{a}_j, \bigvee_{j\in J} \bar{a}_j, \bigwedge_{j\in J} a_j \right\rangle,\ \bigwedge^4_{j\in J} \tilde{\tilde{a}}_j = \left\langle \bigwedge_{j\in J} \tilde{a}_j, \bigvee_{j\in J} \bar{a}_j, \bigvee_{j\in J} a_j \right\rangle,$$

$$\bigvee^1_{j\in J} \tilde{\tilde{a}}_j = \left\langle \bigvee_{j\in J} \tilde{a}_j, \bigvee_{j\in J} \bar{a}_j, \bigvee_{j\in J} a_j \right\rangle,\ \bigvee^2_{j\in J} \tilde{\tilde{a}}_j = \left\langle \bigvee_{j\in J} \tilde{a}_j, \bigvee_{j\in J} \bar{a}_j, \bigwedge_{j\in J} a_j \right\rangle,$$

$$\bigvee^3_{j\in J} \tilde{\tilde{a}}_j = \left\langle \bigvee_{j\in J} \tilde{a}_j, \bigwedge_{j\in J} \bar{a}_j, \bigvee_{j\in J} a_j \right\rangle,\ \bigvee^4_{j\in J} \tilde{\tilde{a}}_j = \left\langle \bigvee_{j\in J} \tilde{a}_j, \bigwedge_{j\in J} \bar{a}_j, \bigwedge_{j\in J} a_j \right\rangle.$$

Definition 3.1. Let X be a nonempty set and let $\mathbf{A} = ([A^{\in,-}, A^{\in,+}], [A^{\notin,-}, A^{\notin,+}]) \in ([I] \oplus [I])^X,\ A = (A^{\in}, A^{\notin}) \in (I \oplus I)^X,\ \lambda \in I^X$. Then the triple $\mathcal{A} =< \mathbf{A}, A, \lambda >$ is called an interval-valued intuitionistic fuzzy octahedron set (briefly, IVI-octahedron set) in X. In fact, $\mathcal{A} : X \to ([I] \oplus [I]) \times (I \oplus I) \times I$ is a mapping.

We can consider following special IVI-octahedron sets in X:

$< \tilde{\tilde{\mathbf{0}}}, \bar{\mathbf{0}}, 0 >= \ddot{\mathbf{0}}$,
$< \tilde{\tilde{\mathbf{0}}}, \bar{\mathbf{0}}, 1 >,\ < \tilde{\tilde{\mathbf{0}}}, \bar{\mathbf{1}}, 0 >,\ < \tilde{\tilde{\mathbf{1}}}, \bar{\mathbf{0}}, 0 >$,
$< \tilde{\tilde{\mathbf{0}}}, \bar{\mathbf{1}}, 1 >,\ < \tilde{\tilde{\mathbf{1}}}, \bar{\mathbf{0}}, 1 >,\ < \tilde{\tilde{\mathbf{1}}}, \bar{\mathbf{1}}, 0 >$,
$< \tilde{\tilde{\mathbf{1}}}, \bar{\mathbf{1}}, 1 >= \ddot{\mathbf{1}}$.

In this case, $\ddot{\mathbf{0}}$ (resp. $\ddot{\mathbf{1}}$) will be called an IVI-octahedron empty set (resp. IVI-octahedron whole set) in X. We will denote the set of all IVI-octahedron sets as $IVIO(X)$.

It is obvious that for each $A \in 2^X$, $\chi_{\mathcal{A}} =< ([\chi_A, \chi_A], [\chi_{A^c}, \chi_{A^c}]), (\chi_A, \chi_{A^c}), \chi_A >\in IVIO(X)$ and then $2^X \subset IVIOX$, where 2^X denotes the set of all subsets of X and χ_A denotes the characteristic function of A.

Example 3.2. (1) Let $X = \{a, b, c\}$ be a set and let $\mathcal{A} =< \mathbf{A}, A, \lambda >: X \to ([I] \oplus [I]) \times (I \oplus I) \times I$ be the mapping given by:

$$\mathcal{A}(a) =< ([0.3, 0.6], [0.2, 0.3]), (0.7, 0.2), 0.5 >,$$

$$\mathcal{A}(b) =< ([0.2, 0.4], [0.4, 0.5]), (0.6, 0.3), 0.7 >,$$

$$\mathcal{A}(c) =< ([0.4, 0.7], [0.1, 0.2]), (0.5, 0.4), 0.3 > .$$

Then we can easily see that $\mathcal{A}$ is an IVI-octahedron set in X.

(2) Let $X = I$ and let $\mathcal{A} =< \mathbf{A}, A, \lambda >: X \to ([I] \oplus [I]) \times (I \oplus I) \times I$ be the mapping defined as follows: for each $x \in X$,

$$\mathcal{A}(x) =< ([\frac{x}{4}, \frac{1+x}{3}], [\frac{x}{5}, \frac{x}{4}]), (\frac{x}{3}, \frac{1+x}{5}), x > .$$

Then we can easily calculate that $\mathcal{A}$ is an IVI-octahedron set in X.

(3) Let $\mathbf{A} = ([A^{\in,-}, A^{\in,+}], [A^{\notin,-}, A^{\notin,+}]) \in ([I] \oplus [I])^X$. Then clearly, $< \mathbf{A}, \bar{\mathbf{0}}, 0 >$ (resp. $< \mathbf{A}, \bar{\mathbf{1}}, 0 >$, $< \mathbf{A}, \bar{\mathbf{0}}, 1 >$, $< \mathbf{A}, \bar{\mathbf{1}}, 1 >$) is an IVI-octahedron set in X. In this case, we will denote $< \mathbf{A}, \bar{\mathbf{0}}, 0 >$ (resp. $< \mathbf{A}, \bar{\mathbf{1}}, 0 >$, $< \mathbf{A}, \bar{\mathbf{0}}, 1 >$, $< \mathbf{A}, \bar{\mathbf{1}}, 1 >$) as $\mathcal{IO}_{\bar{\mathbf{0}},0}$ (resp. $\mathcal{IO}_{\bar{\mathbf{1}},0}$, $\mathcal{IO}_{\bar{\mathbf{0}},1}$, $\mathcal{IO}_{\bar{\mathbf{1}},1}$).

Now let us $A : X \to I \oplus I$ and $\lambda : X \to I$ be the mappings defined as follows, respectively: for each $x \in X$,

$$A(x) = (A^{\in}(x), A^{\notin}(x)) = (\frac{A^{\in,-}(x) + A^{\in,+}(x)}{2}, \frac{A^{\notin,-}(x) + A^{\notin,+}(x)}{2}),$$

$$\lambda(x) = \frac{A^{\in,-}(x) + A^{\in,+}(x)}{2}.$$

Then we can easily see that $< \mathbf{A}, A, \lambda >$ is an IVI-octahedron set in X. In this case, $< \mathbf{A}, A, \lambda >$ will be called the IVI-octahedron set in X induced by $\mathbf{A}$ and will be denoted by $\mathcal{IO}_{\mathbf{A}}$.

(4) Let $A = (A^{\in}, A^{\notin}) \in (I \oplus I)^X$. Then clearly $< \bar{\bar{\mathbf{0}}}, A, 0 >$ (resp. $< \bar{\bar{\mathbf{1}}}, A, 0 >$, $< \bar{\bar{\mathbf{0}}}, A, 1 >$, $< \bar{\bar{\mathbf{1}}}, A, 1 >$) is an IVI-octahedron set in X. In this case, $< \bar{\bar{\mathbf{0}}}, A, 0 >$ (resp. $< \bar{\bar{\mathbf{1}}}, A, 0 >$, $< \bar{\bar{\mathbf{0}}}, A, 1 >$, $< \bar{\bar{\mathbf{1}}}, A, 1 >$) will be denoted by $\mathcal{IO}_{\bar{\bar{\mathbf{0}}},0}$ (resp. $\mathcal{IO}_{\bar{\bar{\mathbf{1}}},0}$, $\mathcal{IO}_{\bar{\bar{\mathbf{0}}},1}$, $\mathcal{IO}_{\bar{\bar{\mathbf{1}}},1}$).

Now let us $\mathbf{A} : X \to [I] \oplus [I]$ and $\lambda : X \to I$ be the mappings defined as follows, respectively: for each $x \in X$,

$$\mathbf{A}(x) = ([A^{\in}(x), 1 - A^{\notin}(x)], [A^{\notin}(x), 1 - A^{\in}(x)]),$$

$$\lambda(x) = \frac{A^{\in}(x) + 1 - A^{\notin}(x)}{2}.$$

Then clearly $< \mathbf{A}, A, \lambda >$ is an IVI-octahedron set in X. In this case, $< \mathbf{A}, A, \lambda >$ will be called the IVI-octahedron set in X induced by A and will be denoted by $\mathcal{IO}_A$.

(5) Let $\mathcal{A} =< \mathbf{A}, A, \lambda >$ be an IVI-octahedron set in X. Then clearly, $< [\ ]\mathbf{A}, A, \lambda >$, $< \mathbf{A}, [\ ]A, \lambda >$, $< [\ ]\mathbf{A}, [\ ]A, \lambda >$, $< \diamond\mathbf{A}, A, \lambda >$, $< \mathbf{A}, \diamond A, \lambda >$ and $< \diamond\mathbf{A}, \diamond A, \lambda >$ are IVI-octahedron sets in X.

From orders of IVI-octahedron numbers, we can define the following.

Definition 3.3. Let X be a nonempty set and let $\mathcal{A} =< \mathbf{A}, A, \lambda >$, $\mathcal{B} =< \mathbf{B}, B, \mu >\in IVIO(X)$. Then we can define following order relations between $\mathcal{A}$ and $\mathcal{B}$:

(i) (Equality) $\mathcal{A} = \mathcal{B} \Leftrightarrow \mathbf{A} = \mathbf{B},\ A = B,\ \lambda = \mu$,
(ii) (Type 1-order) $\mathcal{A} \subset_1 \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B},\ A \subset B,\ \lambda \leq \mu$,
(iii) (Type 2-order) $\mathcal{A} \subset_2 \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B},\ A \subset B,\ \lambda \geq \mu$,
(iv) (Type 3-order) $\mathcal{A} \subset_3 \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B},\ A \supset B,\ \lambda \leq \mu$,
(v) (Type 4-order) $\mathcal{A} \subset_4 \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B},\ A \supset B,\ \lambda \geq \mu$.

Definition 3.4. Let X be a nonempty set and let $(\mathcal{A}_j)_{j\in J} = (< \mathbf{A}_j, A_j, \lambda_j >)_{j\in J}$ be a family of IVI-octahedron sets in X. Then the Type i-union $\cup^i$ and Type i-intersection $\cap^i$ of $(\mathcal{A}_j)_{j\in J}$, $(i = 1,\ 2,\ ,3, 4)$, are defined as follows, respectively:

(i) (Type i-union) $\bigcup^1_{j\in J} \mathcal{A}_j =< \bigcup_{j\in J} \mathbf{A}_j, \bigcup_{j\in J} A_j, \bigcup_{j\in J} \lambda_j >$,
$\bigcup^2_{j\in J} \mathcal{A}_j =< \bigcup_{j\in J} \mathbf{A}_j, \bigcup_{j\in J} A_j, \bigcap_{j\in J} \lambda_j >$,

$$\bigcup^3_{j\in J}\mathcal{A}_j =< \bigcup_{j\in J}\mathbf{A}_j, \bigcap_{j\in J}A_j, \bigcup_{j\in J}\lambda_j >,$$
$$\bigcup^4_{j\in J}\mathcal{A}_j =< \bigcup_{j\in J}\mathbf{A}_j, \bigcap_{j\in J}A_j, \bigcap_{j\in J}\lambda_j >,$$

(ii) (Type i-intersection) $\bigcap^1_{j\in J}\mathcal{A}_j =< \bigcap_{j\in J}\mathbf{A}_j, \bigcap_{j\in J}A_j, \bigcap_{j\in J}\lambda_j >,$

$$\bigcap^2_{j\in J}\mathcal{A}_j =< \bigcap_{j\in J}\mathbf{A}_j, \bigcap_{j\in J}A_j, \bigcup_{j\in J}\lambda_j >,$$
$$\bigcap^3_{j\in J}\mathcal{A}_j =< \bigcap_{j\in J}\mathbf{A}_j, \bigcup_{j\in J}A_j, \bigcap_{j\in J}\lambda_j >,$$
$$\bigcap^4_{j\in J}\mathcal{A}_j =< \bigcap_{j\in J}\mathbf{A}_j, \bigcup_{j\in J}A_j, \bigcup_{j\in J}\lambda_j >.$$

The followings are the immediate results of Definitions 3.3 and 3.4.

Proposition 3.5. *Let X be a nonempty set and let $\mathcal{A} =< \mathbf{A}, A, \lambda_{\mathcal{A}} >$, $\mathcal{B} =< \mathbf{B}, B, \lambda_{\mathcal{B}} >$, $\mathcal{C} =< \mathbf{C}, C, \lambda_{\mathcal{C}} >$ and $\mathcal{D} =< \mathbf{D}, D, \lambda_{\mathcal{D}} >$ be IVI-octahedron sets in X. Then for each $i = 1,\ 2,\ 3,\ 4$,*
(1) *if $\mathcal{A} \subset_i \mathcal{B}$ and $\mathcal{B} \subset_i \mathcal{C}$, then $\mathcal{A} \subset_i \mathcal{C}$,*
(2) *if $\mathcal{A} \subset_i \mathcal{B}$ and $\mathcal{A} \subset_i \mathcal{C}$, then $\mathcal{A} \subset_i \mathcal{B} \cap^i \mathcal{C}$,*
(3) *if $\mathcal{A} \subset_i \mathcal{B}$ and $\mathcal{C} \subset_i \mathcal{B}$, then $\mathcal{A} \cup^i \mathcal{C} \subset_i \mathcal{B}$, .*
(4) *if $\mathcal{A} \subset_i \mathcal{B}$ and $\mathcal{C} \subset_i \mathcal{D}$, then $\mathcal{A} \cup^i \mathcal{C} \subset_i \mathcal{B} \cup^i \mathcal{D}$ and $\mathcal{A} \cap^i \mathcal{C} \subset_i \mathcal{B} \cap^i \mathcal{D}$.*

Definition 3.6. Let X be a nonempty set and let $\mathcal{A} =< \mathbf{A}, A, \lambda >$ be an IVI-octahedron set in X. Then the complement $\mathcal{A}^c$, operators [ ] and $\diamond$ of $\mathcal{A}$ are defined as follows, respectively: for each $x \in X$,
(i) $\mathcal{A}^c =< \mathbf{A}^c, A^c, \lambda^c >$,
(ii) $[\ ]\mathcal{A} =< [\ ]\mathbf{A}, [\ ]A, \lambda >$,
(iii) $\diamond\mathcal{A} =< \diamond\mathbf{A}, \diamond A, \lambda >$.

From the above Definition (i), we can easily see that the followings hold:

$$\ddot{\mathbf{0}}^c = \ddot{1},\ \ddot{\mathbf{1}}^c = \ddot{0},$$
$$< \bar{\tilde{\underline{\mathbf{0}}}}, \bar{\mathbf{0}}, 1 >^c =< \bar{\tilde{\underline{\mathbf{1}}}}, \bar{\mathbf{1}}, 0 >,\ < \bar{\tilde{\underline{\mathbf{1}}}}, \bar{\mathbf{1}}, 0 >^c =< \bar{\tilde{\underline{\mathbf{0}}}}, \bar{\mathbf{0}}, 1 >,$$
$$< \bar{\tilde{\underline{\mathbf{0}}}}, \bar{\mathbf{1}}, 0 >^c =< \bar{\tilde{\underline{\mathbf{1}}}}, \bar{\mathbf{0}}, 1 >,\ < \bar{\tilde{\underline{\mathbf{1}}}}, \bar{\mathbf{0}}, 1 >^c =< \bar{\tilde{\underline{\mathbf{0}}}}, \bar{\mathbf{1}}, 0 >,$$
$$< \bar{\tilde{\underline{\mathbf{1}}}}, \bar{\mathbf{0}}, 0 >^c =< \bar{\tilde{\underline{\mathbf{0}}}}, \bar{\mathbf{1}}, 1 >,\ < \bar{\tilde{\underline{\mathbf{0}}}}, \bar{\mathbf{1}}, 1 >^c =< \bar{\tilde{\underline{\mathbf{1}}}}, \bar{\mathbf{0}}, 0 >,$$
$$< \bar{\tilde{\underline{\mathbf{0}}}}, \bar{\mathbf{1}}, 1 >^c =< \bar{\tilde{\underline{\mathbf{1}}}}, \bar{\mathbf{0}}, 0 >,\ < \bar{\tilde{\underline{\mathbf{1}}}}, \bar{\mathbf{0}}, 0 >^c =< \bar{\tilde{\underline{\mathbf{0}}}}, \bar{\mathbf{1}}, 1 >,$$
$$< \bar{\tilde{\underline{\mathbf{1}}}}, \bar{\mathbf{0}}, 1 >^c =< \bar{\tilde{\underline{\mathbf{0}}}}, \bar{\mathbf{1}}, 0 >,\ < \bar{\tilde{\underline{\mathbf{0}}}}, \bar{\mathbf{1}}, 0 >^c =< \bar{\tilde{\underline{\mathbf{1}}}}, \bar{\mathbf{0}}, 1 >,$$
$$< \bar{\tilde{\underline{\mathbf{1}}}}, \bar{\mathbf{1}}, 0 >^c =< \bar{\tilde{\underline{\mathbf{0}}}}, \bar{\mathbf{0}}, 1 >,\ < \bar{\tilde{\underline{\mathbf{0}}}}, \bar{\mathbf{0}}, 1 >^c =< \bar{\tilde{\underline{\mathbf{1}}}}, \bar{\mathbf{1}}, 0 > .$$

The followings are the immediate results of Definitions 3.3 and 3.6 (i).

Proposition 3.7. *Let X be a nonempty set and let $\mathcal{A} =< \mathbf{A}, A, \lambda >$ and $\mathcal{B} =< \mathbf{B}, B, \mu >$ be IVI-octahedron sets in X. If $\mathcal{A} \subset_i \mathcal{B}$, then $\mathcal{B}^c \subset_i \mathcal{A}^c$, for each $i = 1,\ 2,\ 3,\ 4$.*

The followings are the immediate results of Definitions 3.4 and 3.6 (i).

Proposition 3.8. *Let $\mathcal{A} \in IVIO(X)$ and let $(\mathcal{A}_j)_{j\in J} \subset IVIO(X)$. Then*
(1) *$(\mathcal{A}^c)^c = \mathcal{A}$,*
(2) *for each $i = 1,\ 2,\ 3,\ 4$,*

$$(\bigcup^i_{j\in J}\mathcal{A}_j)^c = \bigcap^i_{j\in J}\mathcal{A}^c_j,\ (\bigcap^i_{j\in J}\mathcal{A}_j)^c = \bigcup^i_{j\in J}\mathcal{A}^c_j.$$

Remark 3.9. For any $\mathcal{A} =< \mathbf{A}, A, \lambda >\in IVIO(X)$ and each $i = 1,\ 2,\ 3,\ 4$, the followings do not hold, in general:

$$\mathcal{A} \cup^i \mathcal{A}^c = \ddot{\mathbf{1}} \text{ and } \mathcal{A} \cap^i \mathcal{A}^c = \ddot{\mathbf{0}}.$$

Example 3.10. Consider the IVI set $\mathbf{A}$, the IF set A and the fuzzy set λ in a nonempty set X given by respectively: for each $x \in X$,

$$\mathbf{A}(x) = ([0.5, 0.5], [0.5, 0.5]),\ A(x) = (0.5, 0.5) \text{ and } \lambda = 0.5.$$

Then clearly, $\mathcal{A} =< \mathbf{A}, A, \lambda >$ is an IVI-octahedron set in X. Moreover,

$$(\mathbf{A} \cup^i \mathbf{A}^c)(x) =< ([0.5, 0.5], [0.5, 0.5]), (0.5, 0.5), 0.5 >\neq \ddot{\mathbf{1}}(x)$$

and

$$(\mathbf{A} \cap^i \mathbf{A}^c)(x) =< [([0.5, 0.5], [0.5, 0.5]), (0.5, 0.5), 0.5 >\neq \ddot{\mathbf{0}}(x).$$

Thus $\mathcal{A} \cup^i \mathcal{A}^c \neq \ddot{\mathbf{1}}$ and $\mathcal{A} \cap^i \mathcal{A}^c \neq \ddot{\mathbf{0}}$.

The followings are the immediate results of Definition 3.4.

Proposition 3.11. *Let X be a nonempty set, let $\mathcal{A} =< \mathbf{A}, A, \lambda_{\mathcal{A}} >$, $\mathcal{B} =< \mathbf{B}, B, \lambda_{\mathcal{B}} >$, $\mathcal{C} =< \mathbf{C}, C, \lambda_{\mathcal{C}} >\in O^X$ and let $(\mathcal{A}_j)_{j\in J} = (< \mathbf{A}_j, A_j, \lambda_j >)_{j\in J} \subset O^X$. Then each $i = 1,\ 2,\ 3,\ 4$,*
(1) $\mathcal{A} \cup^i \mathcal{A} = \mathcal{A},\ \ \mathcal{A} \cap^i \mathcal{A} = \mathcal{A}$,
(2) $\mathcal{A} \cup^i \mathcal{B} = \mathcal{B} \cup^i \mathcal{A},\ \ \mathcal{A} \cap^i \mathcal{B} = \mathcal{B} \cap^i \mathcal{A}$,
(3) $\mathcal{A} \cup^i (\mathcal{B} \cup^i \mathcal{C}) = (\mathcal{A} \cup^i \mathcal{B}) \cup^i \mathcal{C},\ \ \mathcal{A} \cap^i (\mathcal{B} \cap^i \mathcal{C}) = (\mathcal{A} \cap^i \mathcal{B}) \cap^i \mathcal{C}$,
(4) $\mathcal{A} \cup^i (\mathcal{B} \cap^i \mathcal{C}) = (\mathcal{A} \cup^i \mathcal{B}) \cap^i (\mathcal{A} \cup^i \mathcal{C}),\ \ \mathcal{A} \cap^i (\mathcal{B} \cup^i \mathcal{C}) = (\mathcal{A} \cap^i \mathcal{B}) \cup^i (\mathcal{A} \cap^i \mathcal{C})$,
$(4)'$ $\mathcal{A} \cup^i (\bigcap^i_{j\in J} \mathcal{A}_j) = \bigcap^i_{j\in J}(\mathcal{A} \cup^i \mathcal{A}_j),\ \ \mathcal{A} \cap^i (\bigcup^i_{j\in J} \mathcal{A}_j) = \bigcup^i_{j\in J}(\mathcal{A} \cap^i \mathcal{A}_j)$.

From the above Propositions 3.8 and 3.11, we can see that $(IVIO(X), \cup^i, \cap^i, \ddot{\mathbf{0}}, \ddot{\mathbf{1}})$ forms a Boolean algebra except the property of Remark 3.9.

From Definition 3.6, we have the similar results to Theorem 1 in [4].

Proposition 3.12. *Let $\mathcal{A} =< \mathbf{A}, A, \lambda >$ be an IVI-octahedron set in a nonempty set X. Then*
(1) $([\ ]\mathcal{A}^c)^c = \diamond\mathcal{A},\ (\diamond\mathcal{A}^c)^c = [\ ]\mathcal{A}$,
(2) $[\ ]\mathcal{A} \subset_i \mathcal{A} \subset_i \diamond\mathcal{A}$ *for each* $i = 1,\ 2,\ 3,\ 4$,
(3) $[\ ][\ ]\mathcal{A} = [\ ]\mathcal{A}$,
(4) $[\ ] \diamond \mathcal{A} = \diamond\mathcal{A}$,
(5) $\diamond[\ ]\mathcal{A} = [\ ]\mathcal{A}$,
(6) $\diamond \diamond \mathcal{A} = \diamond\mathcal{A}$.

Also, we obtain the similar results to Theorems 2 and 3 in [4].

Proposition 3.13. *Let $\mathcal{A} =< \mathbf{A}, A, \lambda >$ and $\mathcal{B} =< \mathbf{B}, B, \mu >$ be IVI-octahedron sets in a nonempty set X and let $i = 1,\ 2,\ 3,\ 4$. Then*
(1) $(\mathcal{A}^c \cup^i \mathcal{B}^c) = \mathcal{A}^c \cap^i \mathcal{B}^c,\ (\mathcal{A}^c \cap^i \mathcal{B}^c) = \mathcal{A}^c \cup^i \mathcal{B}^c$,
(2) $[\ ](\mathcal{A} \cup^i \mathcal{B}) = [\ ]\mathcal{A} \cup^i [\ ]\mathcal{B},\ \ [\ ](\mathcal{A} \cap^i \mathcal{B}) = [\ ]\mathcal{A} \cap^i [\ ]\mathcal{B}$,
(3) $\diamond(\mathcal{A} \cup^i \mathcal{B}) = \diamond\mathcal{A} \cup^i \diamond\mathcal{B},\ \ \diamond(\mathcal{A} \cap^i \mathcal{B}) = \diamond\mathcal{A} \cap^i \diamond\mathcal{B}$.

Definition 3.14. Let X be a nonempty set and let $\mathcal{A} =< \mathbf{A}, A, \lambda >\in IVIO(X)$. Then $\mathcal{A}$ is called:

(i) an internal IVI-octahedron set (briefly, IIVI-octahedron set) in X, if for each $x \in X$,

$$A^{\in}(x),\ \lambda(x) \in \mathbf{A}^{\in}(x) = [A^{\in,-}(x), A^{\in,+}(x)] \text{ and } A^{\notin}(x) \in \mathbf{A}^{\notin}(x) = [A^{\notin,-}(x), A^{\notin,+}(x)],$$

(ii) a external IVI-octahedron set (briefly, EIVI-octahedron set) in X, if for each $x \in X$,

$$A^{\in}(x),\ \lambda(x) \notin (A^{\in,-}(x), A^{\in,+}(x)) \text{ and } A^{\notin}(x) \notin (A^{\notin,-}(x), A^{\notin,+}(x)).$$

Example 3.15. (1) Let $\mathcal{A}_1 =< \mathbf{A}_1, A_1, \lambda_1 >$ be the IVI-octahedron set in I given by: for each $x \in I$,

$$\mathcal{A}_1(x) =< ([\frac{1+x}{6}, \frac{1+x}{4}], [\frac{1+x}{7}, \frac{1+x}{5}]), (\frac{1+x}{5}, \frac{1+x}{6}), \frac{1+x}{5} > .$$

Then we can easily calculate that $\mathcal{A}_1$ is an IIVI-octahedron set in X.

(2) Let $\mathcal{A}_2 =< \mathbf{A}_6, A_6, \lambda_6 >$ be the IVI-octahedron set in I given by: for each $x \in I$,

$$\mathcal{A}_2(x) =< ([\frac{1+x}{6}, \frac{1+x}{4}], [\frac{1+x}{7}, \frac{1+x}{5}]), (\frac{1+x}{3}, \frac{1+x}{8}), \frac{1+x}{3} > .$$

Then we can easily see that $\mathcal{A}_2$ is an EIVI-octahedron set in X.

The following is the immediate result of Definition 3.14.

Proposition 3.16. *Let X be a nonempty set and let $\mathcal{A} =< \mathbf{A}, A, \lambda >\in IVIO(X)$. If $\mathcal{A}$ is not external, then there is $x \in X$ such that $A^{\in}(x) \in \mathbf{A}^{\in}(x)$, $\lambda(x) \in \mathbf{A}^{\in}(x)$ or $A^{\notin}(x) \in \mathbf{A}^{\notin}(x)$.*

For $\mathcal{A} =< \mathbf{A}, A, \lambda >\in IVIO(X)$, $\mathcal{A}$ is internal (resp. external) but $\mathcal{A}^c$ is not internal (resp. external), in general as shown as the following examples.

Example 3.17. (1) Consider the IIVI-octahedron set $\mathcal{A}_1$ in X in Example 3.15 (1). Then we can easily calculate that $1 - \lambda_1(x) = \frac{4-x}{5} > \frac{1+x}{5}$. Thus $1 - \lambda_1(x) \notin (\frac{1+x}{7}, \frac{1+x}{5}) = (\mathbf{A}^c)^{\in}$. So $\mathcal{A}_1^c$ is not internal.

(2) Consider the EIVI-octahedron set $\mathcal{A}_2$ in X in Example 3.15 (2). Then we can easily see that $\frac{1+x}{7} \leq 1 - \lambda_2(x) = \frac{2-x}{3} \leq \frac{1+x}{5}$. Thus $1 - \lambda_2(x) \in [\frac{1+x}{7}, \frac{1+x}{5}] = (\mathbf{A}^c)^{\in}$. So $\mathcal{A}_2^c$ is not external.

Proposition 3.18. *Let X be a nonempty set and let $\mathcal{A} =< \mathbf{A}, A, \lambda >\in IVIO(X)$. If $\mathcal{A}$ is internal, then $< [\ ]\mathbf{A}, A, \lambda >$ and $< \diamond\mathbf{A}, A, \lambda >$ are internal.*

Proof. Suppose $\mathcal{A}$ is internal and let $x \in X$. Then

$$A^{\in}(x),\ \lambda(x) \in \mathbf{A}^{\in}(x) = [A^{\in,-}(x), A^{\in,+}(x)] \text{ and } A^{\notin}(x) \in \mathbf{A}^{\notin}(x) = [A^{\notin,-}(x), A^{\notin,+}(x)].$$

Since $[\ ]\mathbf{A} = ([A^{\in,-}, A^{\in,+}], [A^{\notin,-}, 1 - A^{\in,+}])$, $([\ ]\mathbf{A})^{\in} = \mathbf{A}^{\in}$. Thus $A^{\in}(x),\ \lambda(x) \in ([\ ]\mathbf{A})^{\in}(x)$. Since $A^{\in,+} + A^{\notin,+} \leq 1$, $A^{\notin,+} \leq 1 - A^{\in,+}$. So $\mathbf{A}^{\notin} \subset ([\ ]\mathbf{A})^{\notin} = [A^{\notin,-}, 1 - A^{\in,+}]$. Hence $A^{\notin}(x) \in ([\ ]\mathbf{A})^{\notin}(x)$. Therefore $< [\ ]\mathbf{A}, A, \lambda >$ is internal.

The proof of the second part is similar. $\square$

For $\mathcal{A} =< \mathbf{A}, A, \lambda >\in IVIO(X)$, $\mathcal{A}$ is internal but $[\ ]\mathcal{A}$ and $\diamond\mathcal{A}$ are not internal, in general as shown as the following example.

Example 3.19. Consider the IIVI-octahedron set $\mathcal{A}_1$ in X in Example 3.15 (1). Then clearly,

$$\begin{aligned}[\,]\mathcal{A}_1 &=< ([A_1^{\in,-}, A_1^{\in,+}], [A_1^{\notin,-}, 1-A_1^{\in,+}]), (A_1^{\in}, 1-A_1^{\in}), \lambda_1 > \\ &=< ([\tfrac{1+x}{6}, \tfrac{1+x}{4}], [\tfrac{1+x}{7}, 1-\tfrac{1+x}{4}]), (\tfrac{1+x}{5}, 1-\tfrac{1+x}{5}), \tfrac{1+x}{5} >, \\ \diamond\mathcal{A}_1 &=< ([A_1^{\in,-}, 1-A_1^{\notin,+}], [A_1^{\notin,-}, A_1^{\notin,+}]), (1-A_1^{\notin}, A_1^{\notin}), \lambda_1 > \\ &=< ([\tfrac{1+x}{6}, 1-\tfrac{1+x}{5}], [\tfrac{1+x}{7}, \tfrac{1+x}{5}]), (1-\tfrac{1+x}{6}, \tfrac{1+x}{6}), \tfrac{1+x}{5} > .\end{aligned}$$

Thus we have $1-\frac{1+x}{4} < 1-\frac{1+x}{5}$ and $1-\frac{1+x}{5} < 1-\frac{1+x}{6}$. So $([\,]A_1)^{\notin}(x) \notin ([\,]\mathbf{A}_1)^{\notin}(x)$ and $(\diamond A_1)^{\in}(x) \notin (\diamond \mathbf{A}_1)^{\in}(x)$. Hence $[\,]\mathcal{A}$ and $\diamond\mathcal{A}$ are not internal.

Proposition 3.20. *Let X be a nonempty set and let $\mathcal{A} =< \mathbf{A}, A, \lambda > \in IVIO(X)$. $< [\,]\mathbf{A}, A, \lambda >$ and $< \diamond\mathbf{A}, A, \lambda >$ are external.*

Proof. (1) Suppose $\mathcal{A}$ is external and let $x \in X$. Then

$$A^{\in}(x),\ \lambda(x) \notin (A^{\in,-}(x), A^{\in,+}(x)) \text{ and } A^{\notin}(x) \notin (A^{\notin,-}(x), A^{\notin,+}(x)).$$

Thus $A^{\in}(x),\ \lambda(x) \notin (([\,]\mathbf{A})^{\in,-}(x), ([\,]\mathbf{A})^{\in,+}(x)) = (A^{\in,-}(x), A^{\in,+}(x))$. Moreover,

$$A^{\notin}(x) \notin (A^{\notin,-}(x), 1-A^{\in}(x)) = (([\,]\mathbf{A})^{\notin,-}(x), ([\,]\mathbf{A})^{\notin,+}(x)).$$

So $< [\,]\mathbf{A}, A, \lambda >$ is external.

The proof of the second part is similar. □

For any EIVI-octahedron set $\mathcal{A} =< \mathbf{A}, A, \lambda >$ in a nonempty set X, $[\,]\mathcal{A}$ (resp. $\diamond\mathcal{A}$) need not be external as shown in following example.

Example 3.21. Consider the EIVI-octahedron set $\mathcal{A}_2$ in Example 3.15 (2). Then

$$[\,]\mathcal{A}_2 =< ([\frac{1+x}{6}, \frac{1+x}{4}], [\frac{1+x}{7}, 1-\frac{1+x}{4}]), (\frac{1+x}{3}, 1-\frac{1+x}{3}), \frac{1+x}{3} >,$$

$$\diamond\mathcal{A} =< ([\frac{1+x}{6}, 1-\frac{1+x}{5}], [\frac{1+x}{7}, \frac{1+x}{5}]), (1-\frac{1+x}{8}, \frac{1+x}{8}), \frac{1+x}{3} > .$$

Thus we can easily calculate that

$$([\,]A)^{\notin}(x) \in [\frac{1+x}{7}, 1-\frac{1+x}{4}] = [([\,]\mathbf{A})^{\notin,-}(x), ([\,]\mathbf{A})^{\notin,+}(x)]$$

and

$$(\diamond A)^{\in}(x) = 1-\frac{1+x}{8} \in [\frac{1+x}{6}, 1-\frac{1+x}{5}] = (\diamond\mathbf{A})^{\in}(x).$$

So $[\,]\mathcal{A}$ and $\diamond\mathcal{A}$ are not external.

Proposition 3.22. *Let X be a nonempty set and let $(\mathcal{A}_j)_{j\in J} = (< \mathbf{A}_j, A_j, \lambda_j >)_{j\in J}$ be a family of IVI-octahedron sets in X. If $\mathcal{A}_j$ is internal for each $j \in J$, then $\bigcup^1_{j\in J}\mathcal{A}_j$ and $\bigcap^1_{j\in J}\mathcal{A}_j$ are internal.*

Proof. Suppose $\mathcal{A}_j$ is internal for each $j \in J$ and let $x \in X$. Then

$$A_j^{\in}(x),\ \lambda_j^{\in}(x) \in [A_j^{\in,-}(x), A_j^{\in,+}(x)] \text{ and } A_j^{\notin}(x) \in [A_j^{\notin,-}(x), A_j^{\notin,+}(x)].$$

Thus

$$\bigvee_{j\in J} A_j^{\in}(x),\ \bigvee_{j\in J}\lambda_j(x) \in \bigvee_{j\in J}\mathbf{A}_j^{\in}(x) = (\bigcup^1_{j\in J}\mathbf{A}_j^{\in})(x)$$

and

$$\bigvee_{j\in J} A_j^{\notin}(x) \in \bigvee_{j\in J} \mathbf{A}_j^{\notin}(x) = (\bigcup_{j\in J}^{1} \mathbf{A}_j^{\notin})(x).$$

So $(\bigcup_{j\in J} A_j^{\in})(x)$, $(\bigvee_{j\in J} \lambda_j)(x) \in (\bigcup_{j\in J}^{1} \mathbf{A}_j^{\in})(x)$ and $\bigcap_{j\in J} A_j^{\notin})(x) \in (\bigcup_{j\in J}^{1} \mathbf{A}_j^{\notin})(x)$. Hence $\bigcup_{j\in J}^{1} \mathcal{A}_j$ is internal. □

We can see that Type i-union and Type i-intersection ($i = 2,\ 3,\ 4$) of IIVI-octahedron set may not be IIVI-octahedron sets as shown in the following examples.

Example 3.23. Consider two IIVI-octahedron sets $\mathcal{A} =< \mathbf{A}, A, \lambda >$ and $\mathcal{B} =< \mathbf{B}, B, \mu >$ in I defined as follows: for each $x \in I$,

$$\mathcal{A}(x) =< ([\frac{1+x}{4}, \frac{1+x}{2}], [\frac{1+x}{7}, \frac{1+x}{5}]), (\frac{1+x}{3}, \frac{1+x}{6}), \frac{2+x}{5} >$$

and

$$\mathcal{B} =< ([\frac{x}{5}, \frac{x}{3}], [\frac{1+x}{6}, \frac{1+x}{4}]), (\frac{x}{4}, \frac{x}{5}), \frac{x}{4} > .$$

Then we have the followings:

$$(\mathcal{A} \cup^2 \mathcal{B})(x) =< ([\frac{1+x}{4}, \frac{1+x}{2}], [\frac{1+x}{7}, \frac{2+x}{4}]), (\frac{1+x}{3}, \frac{x}{5}), \frac{x}{4} >, \tag{3.1}$$

$$(\mathcal{A} \cup^3 \mathcal{B})(x) =< ([\frac{1+x}{4}, \frac{1+x}{2}], [\frac{1+x}{7}, \frac{1+x}{5}]), (\frac{x}{4}, \frac{1+x}{6}), \frac{2+x}{5} >, \tag{3.2}$$

$$(\mathcal{A} \cup^4 \mathcal{B})(x) =< ([\frac{1+x}{4}, \frac{1+x}{2}], [\frac{1+x}{7}, \frac{1+x}{5}]), (\frac{x}{4}, \frac{1+x}{6}), \frac{x}{4} >, \tag{3.3}$$

$$(\mathcal{A} \cap^2 \mathcal{B})(x) =< ([\frac{x}{5}, \frac{x}{3}], [\frac{1+x}{6}, \frac{1+x}{4}]), (\frac{x}{4}, \frac{1+x}{6}), \frac{2+x}{5} >, \tag{3.4}$$

$$\mathcal{A} \cap^3 \mathcal{B})(x) =< ([\frac{x}{5}, \frac{x}{3}], [\frac{1+x}{6}, \frac{1+x}{4}]), (\frac{1+x}{3}, \frac{x}{4}), \frac{x}{4} >, \tag{3.5}$$

$$(\mathcal{A} \cap^4 \mathcal{B})(x) =< [([\frac{x}{5}, \frac{x}{3}], [\frac{1+x}{6}, \frac{1+x}{4}]), (\frac{1+x}{3}, \frac{x}{4}), \frac{2+x}{5} > . \tag{3.6}$$

Thus we can see the followings.

In (3.1), $(\lambda \wedge \mu)(1) = \frac{1}{4} \notin [\frac{1}{2}, 1] = (\mathbf{A} \cup \mathbf{B})^{\in}(1)$.
In (3.2), $(A \cap B)^{\in}(1) = \frac{1}{4} \notin [\frac{1}{2}, 1] = (\mathbf{A} \cup \mathbf{B})^{\in}(1)$.
In (3.3), $(A \cap B)^{\in}(1) = \frac{1}{4} \notin [\frac{1}{2}, 1] = (\mathbf{A} \cup \mathbf{B})^{\in}(1)$.
In (3.4), $(\lambda \vee \mu)(1) = \frac{1}{5} \notin [\frac{1}{5}, \frac{1}{3}] = (\mathbf{A} \cap \mathbf{B})^{\in}(1)$.
In (3.5), $(A \cup B)^{\notin}(1) = \frac{1}{4} \notin [\frac{1}{3}, \frac{1}{2}] = (\mathbf{A} \cap \mathbf{B})^{\notin}(1)$.
In (3.6),$(A \cup B)^{\notin}(1) = \frac{1}{4} \notin [\frac{1}{3}, \frac{1}{2}] = (\mathbf{A} \cap \mathbf{B})^{\notin}(1)$.

So $\mathcal{A} \cup^i \mathcal{B}$ and $\mathcal{A} \cap^i \mathcal{B}$ are not IIVI-octahedron set in I, for $i = 2,\ 3,\ 4$.

Remark 3.24. Type i-union and Type i-intersection ($i = 1,\ 2,\ 3,\ 4$) of two EIVI-octahedron sets may not be external, in general.

Example 3.25. Consider the EIVI-octahedron set $\mathcal{A}_2$ in Example 3.15 (2) given by: for each $x \in I$,

$$\mathcal{A}_2(x) =< ([\frac{1+x}{6}, \frac{1+x}{4}], [\frac{1+x}{7}, \frac{1+x}{5}]), (\frac{1+x}{3}, \frac{1+x}{8}), \frac{1+x}{3} > .$$

Let $\mathcal{A} =< \mathbf{A}, A, \lambda >$ be the IVI-octahedron set in I defined as follows: for each $x \in I$,

$$\mathcal{A}(x) =< ([\frac{1+x}{5}, \frac{1+x}{3}], [\frac{1+x}{6}, \frac{1+x}{5}]), (\frac{1+x}{6}, \frac{1+x}{6}), \frac{1+x}{2} > .$$

Then we can easily see that $\mathcal{A}$ is an EIVI-octahedron set in I.

(Case 1) Type 1-union and Type 1-intersection: for each $x \in I$,

$$(\mathcal{A} \cup^1 \mathcal{A}_2)(x) =< ([\frac{1+x}{5}, \frac{1+x}{3}], [\frac{1+x}{7}, \frac{1+x}{5}]), (\frac{1+x}{3}, \frac{1+x}{8}), \frac{1+x}{2} >$$

and

$$(\mathcal{A} \cap^1 \mathcal{A}_2)(x) =< ([\frac{1+x}{6}, \frac{1+x}{4}], [\frac{1+x}{6}, \frac{1+x}{5}]), (\frac{1+x}{6}, \frac{1+x}{6}), \frac{1+x}{3} > .$$

Then $(A \cup A_2)^{\in}(x) = \frac{1+x}{3} \in [\frac{1+x}{5}, \frac{1+x}{3}] = (\mathbf{A} \cup \mathbf{A}_2)^{\in}(x)$
and

$(A \cap A_2)^{\notin}(x) = \frac{1+x}{6} \in [\frac{1+x}{6}, \frac{1+x}{5}] = (\mathbf{A} \cap \mathbf{A}_2)^{\notin}(x)$,

for each $x \in I$. Thus $\mathcal{A} \cup^1 \mathcal{A}_2$ and $\mathcal{A} \cap^1 \mathcal{A}_2$ are not EIVI-octahedron sets in I.

(Case 2) Type 2-union and Type 2-intersection: for each $x \in I$,

$$(\mathcal{A} \cup^2 \mathcal{A}_2)(x) =< ([\frac{1+x}{5}, \frac{1+x}{3}], [\frac{1+x}{7}, \frac{1+x}{5}]), (\frac{1+x}{3}, \frac{1+x}{8}), \frac{1+x}{3} >$$

and

$$(\mathcal{A} \cap^2 \mathcal{A}_2)(x) =< ([\frac{1+x}{6}, \frac{1+x}{4}], [\frac{1+x}{7}, \frac{1+x}{5}]), (\frac{1+x}{6}, \frac{1+x}{6}), \frac{1+x}{2} > .$$

Then we can easily calculate that $\mathcal{A} \cup^2 \mathcal{A}_2$ and $\mathcal{A} \cap^2 \mathcal{A}_2$ are not not EIVI-octahedron sets in I.

Similarly, we can easily see that $\mathcal{A} \cup^3 \mathcal{A}_2$, $\mathcal{A} \cap^4 \mathcal{A}_2$, $\mathcal{A} \cup^4 \mathcal{A}_2$ and $\mathcal{A} \cap^4 \mathcal{A}_2$ are not not EIVI-octahedron sets in I.

4. Interval-valued intuitionistic octahedron points and level sets

Definition 4.1 ([17]). $A \in (I \oplus I)^X$ is called an intuitionistic fuzzy point (briefly, an IF point) with the support $x \in X$ and the value $\bar{a} \in I \oplus I$ with $\bar{a} \neq \bar{0}$, denoted by $A = x_{\bar{a}}$, if for each $y \in X$,

$$x_{\bar{a}}(y) = \begin{cases} \bar{a} & \text{if } y = x \\ \bar{0} & \text{otherwise.} \end{cases}$$

The set of all IF points in X is denoted by $IF_P(X)$.

For each $x_{\bar{a}} \in IF_P(X)$ and $A \in (I \oplus I)^X$, $x_{\bar{a}}$ is said to belong to A, denoted by $x_{\bar{a}} \in A$, if $a^{\in} \leq A^{\in}(x)$ and $a^{\notin} \geq A^{\notin}(x)$.

It is well-known (Theorem 2.4 in [17]) that $A = \bigcup_{x_{\bar{a}} \in A} x_{\bar{a}}$, for each $A \in (I \oplus I)^X$.

Definition 4.2 ([21]). $A \in ([I] \oplus [I])^X$ is called an interval-valued intuitionistic fuzzy point (briefly, an IVI point) with the support $x \in X$ and the value $\widetilde{\bar{a}} \in [I] \oplus [I]$ with $\widetilde{\bar{a}} \neq \widetilde{\bar{0}}$, denoted by $A = x_{\widetilde{\bar{a}}}$, if for each $y \in X$,

$$x_{\widetilde{\bar{a}}}(y) = \begin{cases} \widetilde{\bar{a}} & \text{if } y = x \\ \widetilde{\bar{0}} & \text{otherwise.} \end{cases}$$

The set of all IVI points in X is denoted by $IVIF_P(X)$.

For each $x_{\widetilde{\bar{a}}} \in IVF_P(X)$ and $A \in ([I]\oplus[I])^X$, $x_{\widetilde{\bar{a}}}$ is said to belong to A, denoted by $x_{\widetilde{\bar{a}}} \in A$, if $a^{\in,-} \leq A^{\in,-}(x)$, $a^{\in,+} \leq A^{\in,+}(x)$, $a^{\notin,-} \geq A^{\notin,-}(x)$ and $a^{\notin,-} \geq A^{\notin,-}(x)$.

It is well-known (See Theorem 1.1 in [21]) that $A = \bigcup_{x_{\widetilde{\bar{a}}} \in A} x_{\widetilde{\bar{a}}}$, for each $A \in ([I] \oplus [I])^X$.

Definition 4.3. Let $\mathcal{A} =< \mathbf{A}, A, \lambda >\in IVIO(X)$ and let $\widetilde{\widetilde{a}} = \left\langle \widetilde{\bar{a}}, \bar{a}, a \right\rangle$ be an IVI-octahedron number such that $\widetilde{\bar{a}} \neq \widetilde{\bar{0}}$, $\bar{b} \neq \bar{0}$ and $a \neq 0$. Then A is called an interval-valued intuitionistic octahedron point (briefly, IVI-octahedron point) with the support $x \in X$ and the value $\widetilde{\widetilde{a}}$, denoted by $A = x_{\widetilde{\widetilde{a}}}$, if for each $y \in X$,

$$x_{\widetilde{\widetilde{a}}}(y) = \begin{cases} \widetilde{\widetilde{a}} & \text{if } y = x \\ < \widetilde{\bar{0}}, \bar{0}, 0 > & \text{otherwise.} \end{cases}$$

The set of IVI-octahedron points in X is denoted by $IVIO_P(X)$.

Definition 4.4. Let $\mathcal{A} =< \mathbf{A}, A, \lambda >\in IVIO(X)$ and let $x_{\widetilde{\widetilde{a}}} \in IVIO_P(X)$. Then $x_{\widetilde{\widetilde{a}}}$ is said to:

(i) belong to $\mathcal{A}$ with respect to Type 1-order, denoted by $x_{\widetilde{\widetilde{a}}} \in_1 \mathcal{A}$, if

$\widetilde{\bar{a}} \leq \mathbf{A}(x)$, $\bar{a} \leq A(x)$ and $a \leq \lambda(x)$, i.e., $x_{\widetilde{\bar{a}}} \in \mathbf{A}$, $x_{\bar{a}} \in A$ and $x_a \in \lambda$,

(ii) belong to $\mathcal{A}$ with respect to Type 2-order, denoted by $x_{\widetilde{\widetilde{a}}} \in_2 \mathcal{A}$, if

$\widetilde{\bar{a}} \leq \mathbf{A}(x)$, $\bar{a} \leq A(x)$ and $a \geq \lambda(x)$,

(iii) belong to $\mathcal{A}$ with respect to Type 3-order, denoted by $x_{\widetilde{\widetilde{a}}} \in_3 \mathcal{A}$, if

$\widetilde{\bar{a}} \leq \mathbf{A}(x)$, $\bar{a} \geq A(x)$ and $a \leq \lambda(x)$,

(iv) belong to $\mathcal{A}$ with respect to Type 4-order, denoted by $x_{\widetilde{\widetilde{a}}} \in_4 \mathcal{A}$, if

$\widetilde{\bar{a}} \leq \mathbf{A}(x)$, $\bar{a} \geq A(x)$ and $a \geq \lambda(x)$.

It is clear that $A = \bigcup^i_{x_{\widetilde{\widetilde{a}}} \in_i \mathcal{A}} x_{\widetilde{\widetilde{a}}}$ $(i = 1,\ 2,\ 3,\ 4)$, for each $\mathcal{A} \in IVIO(X)$.

Theorem 4.5. *Let $x_{\widetilde{\widetilde{a}}} \in IVIO_P(X)$, $\mathcal{A} =< \mathbf{A}, A, \lambda >$ and $\mathcal{B} =< \mathbf{B}, B, \mu >\in IVIO(X)$. Then for any $i = 1,\ 2,\ 3,\ 4$,*

$$\mathcal{A} \subset_i \mathcal{B} \text{ if and only if } x_{\widetilde{\widetilde{a}}} \in_i \mathcal{B}, \text{ for each } x_{\widetilde{\widetilde{a}}} \in_i \mathcal{A}.$$

Proof. Suppose $\mathcal{A} \subset_1 \mathcal{B}$ and let $x_{\widetilde{\widetilde{a}}} \in_1 \mathcal{A}$. Then

$$\widetilde{\bar{a}} = ([a^{\in,-}, a^{\in,+}], [a^{\notin,-}, a^{\notin,+}]) \leq ([A^{\in,-}(x), A^{\in,+}(x)], [A^{\notin,-}(x), A^{\notin,+}(x)]) = \mathbf{A}(x),$$

$$\bar{a} = (a^{\in}, a^{\notin}) \leq (A^{\in}(x), A^{\notin}(x)) = A(x),\ a \leq \lambda(x).$$

Since $\mathcal{A} \subset_1 \mathcal{B}$, $\mathbf{A}(x) \leq \mathbf{B}(x)$, $A(x) \leq B(x)$, $\lambda(x) \leq \mu(x)$. Thus

$$\widetilde{\bar{a}} \leq \mathbf{B}(x),\ \bar{a} \leq B(x),\ a \leq \mu(x).$$

So $x_{\widetilde{\widetilde{a}}} \in_1 \mathcal{B}$.

Conversely, suppose the necessary condition holds and Assume that $\mathcal{A} \not\subset_1 \mathcal{B}$. Then there is $x_{\widetilde{\widetilde{a}}} \in IVIO_P(X)$ such that $x_{\widetilde{\widetilde{a}}} \in_1 \mathcal{A}$ but $x_{\widetilde{\widetilde{a}}} \notin_1 \mathcal{B}$. Thus $x_{\widetilde{\widetilde{a}}} \in \mathbf{A}$, $x_{\bar{a}} \in A$, $x_a \in \lambda$ but $x_{\widetilde{\widetilde{a}}} \notin \mathbf{B}$ or $x_{\bar{a}} \notin B$ or $x_a \notin \mu$. This is a contradiction. So $\mathcal{A} \subset_1 \mathcal{B}$.

The remainders can be proved similarly. □

Proposition 4.6. *Let $\mathcal{A} =< \mathbf{A}, A, \lambda >$, $\mathcal{B} =< \mathbf{B}, B, \mu >\in IVIO(X)$, let $(\mathcal{A}_j)_{j\in J} = (< \mathbf{A}_j, A_j, \lambda_j >)_{j\in J} \subset IVIO(X)$, $x_{\widetilde{\widetilde{a}}} \in IVIO_P(X)$ and let $i = 1,\ 2,\ 3,\ 4$.*

(1) *If $x_{\widetilde{\widetilde{a}}} \in_i \mathcal{A}$ or $x_{\widetilde{\widetilde{a}}} \in_i \mathcal{B}$, then $x_{\widetilde{\widetilde{a}}} \in_i \mathcal{A} \cup^i \mathcal{B}$.*

(2) *If there is $j \in J$ such that $x_{\widetilde{\widetilde{a}}} \in_i \mathcal{A}_j$, then $x_{\widetilde{\widetilde{a}}} \in_i \bigcup^i_{j\in J} \mathcal{A}_j$.*

Proof. The proofs are straightforward. □

The converse of Proposition 4.6 need not to be true in general as shown in the following example.

Example 4.7. Let $\mathcal{A} =< \mathbf{A}, A, \lambda >$ and $\mathcal{B} =< \mathbf{B}, B, \mu >$ be two IVI-octahedron sets in I given as follows: for each $x \in I$,

$$\mathcal{A}(x) =< ([\frac{1+x}{6}, \frac{1+x}{5}], [\frac{1+x}{8}, \frac{1+x}{4}]), (\frac{1+x}{3}, \frac{1-x}{3}), \frac{2+x}{5} >$$

and

$$\mathcal{B} =< ([\frac{x}{3}, \frac{1+x}{4}], [\frac{1+x}{7}, \frac{1+x}{6}]), (\frac{x}{3}, \frac{2-x}{5}), \frac{x}{4} > .$$

Then clearly, $\mathcal{A} \cup^1 \mathcal{B} =< ([\frac{1+x}{6}, \frac{1+x}{4}], [\frac{1+x}{8}, \frac{1+x}{4}], (\frac{1+x}{3}, \frac{2-x}{5}), \frac{2+x}{5} >$. Let $\widetilde{\widetilde{a}} = ([\frac{1}{5}, \frac{1}{4}], [\frac{1}{4}, \frac{1}{2}]$, $\bar{a} = (\frac{1}{3}, \frac{1}{5})$, $a = \frac{1}{3}$ and Consider octahedron point $0.5_{\widetilde{\widetilde{a}}}$. Then

$$(\mathcal{A} \cup^1 \mathcal{B})(0.5) =< ([\frac{1}{4}, \frac{3}{10}], [\frac{3}{16}, \frac{3}{8}], (\frac{1}{2}, \frac{1}{6}), \frac{1}{2} >,$$

$$\mathcal{A}(0.5) =< ([\frac{1}{4}, \frac{3}{10}], [\frac{3}{14}, \frac{1}{4}]), (\frac{1}{6}, \frac{3}{8}), \frac{1}{8} >,$$

$$\mathcal{B}(0.5) =< ([\frac{1}{5}, \frac{1}{4}], [\frac{1}{4}, \frac{1}{2}]), (\frac{1}{3}, \frac{1}{5}), \frac{1}{3} > .$$

Thus $\widetilde{\widetilde{a}} \leq_1 (\mathcal{A} \cup^1 \mathcal{B})(0.5)$ but $\bar{a} \not\leq B(0.5)$. So $0.5_{\widetilde{\widetilde{a}}} \in_1 \mathcal{A} \cup^1 \mathcal{B}$ but $0.5_{\bar{a}} \notin B$. Hence $0.5_{\widetilde{\widetilde{a}}} \in_1 \mathcal{A} \cup^1 \mathcal{B}$ but $0.5_{\widetilde{\widetilde{a}}} \notin_1 \mathcal{B}$.

Similarly, we can calculate that for $i = 2,\ 3,\ 4$, $0.5_{\widetilde{\widetilde{a}}} \in_i \mathcal{A} \cup^i \mathcal{B}$ but $0.5_{\widetilde{\widetilde{a}}} \notin_i \mathcal{A}$ or $0.5_{\widetilde{\widetilde{a}}} \notin_i \mathcal{B}$.

Theorem 4.8. *Let $\mathcal{A} =< \mathbf{A}, A, \lambda >$, $\mathcal{B} =< \mathbf{B}, B, \mu >\in IVIO(X)$, let $(\mathcal{A}_j)_{j\in J} = (< \mathbf{A}_j, A_j, \lambda_j >)_{j\in J} \subset IVIO(X)$, $x_{\widetilde{\widetilde{a}}} \in IVIO_P(X)$and let $i = 1,\ 2,\ ,\ 3,\ 4$. Then*

(1) *$x_{\widetilde{\widetilde{a}}} \in_i \mathcal{A} \cap^i \mathcal{B}$ if and only if $x_{\widetilde{\widetilde{a}}} \in_i \mathcal{A}$ and $x_{\widetilde{\widetilde{a}}} \in_i \mathcal{B}$,*

(2) *$x_{\widetilde{\widetilde{a}}} \in_i \cap^i_{j\in J}\mathcal{A}_j$ if and only if $x_{\widetilde{\widetilde{a}}} \in_i \mathcal{A}_j$, for each $j \in J$.*

Proof. (1) Suppose $x_{\widetilde{\widetilde{a}}} \in_1 \mathcal{A} \cap^1 \mathcal{B}$. Then $x_{\widetilde{\widetilde{a}}} \in \mathbf{A} \cap \mathbf{B}$, $x_{\bar{a}} \in A \cap B$ and $x_a \in \lambda \wedge \mu$. Thus

$$\begin{aligned}\widetilde{\widetilde{a}} &= ([a^{\in,-}, a^{\in,+}], [a^{\notin,-}, a^{\notin,+}]) \\ &\leq (\mathbf{A} \cap \mathbf{B})(x) \\ &= ([A^{\in,-}(x) \wedge B^{\in,-}(x), A^{\in,+}(x) \wedge B^{\in,+}(x)],\end{aligned}$$

$$[A^{\not\in,-}(x) \vee B^{\not\in,-}(x), A^{\not\in,+}(x) \vee B^{\not\in,+}(x)]),$$
$$\bar{a} = (a^{\in}, a^{\not\in}) \leq (A \cap B)(x) = (A^{\in}(x) \wedge B^{\in}(x), A^{\not\in}(x) \vee B^{\not\in}(x),$$
$$a \leq (\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x).$$

So $x_{\tilde{\tilde{a}}} \in \mathbf{A},\ x_{\bar{a}} \in A,\ x_a \in \lambda$ and $x_{\tilde{\tilde{a}}} \in \mathbf{B},\ x_{\bar{a}} \in B,\ x_a \in \mu$. Hence $x_{\tilde{\tilde{a}}} \in_1 \mathcal{A}$ and $x_{\tilde{\tilde{a}}} \in_1 \mathcal{B}$.

Conversely, suppose $x_{\tilde{\tilde{a}}} \in_1 \mathcal{A}$ and $x_{\tilde{\tilde{a}}} \in_1 \mathcal{B}$. Then

$$[a^{\in,-}(x), a^{\in,+}(x)] \leq \mathbf{A}^{\in}(x),\ [a^{\not\in,-}(x), a^{\not\in,+}(x)] \geq \mathbf{A}^{\not\in}(x),$$
$$a^{\in} \leq A^{\in}(x),\ a^{\not\in} \geq A^{\not\in}(x),\ a \leq \lambda(x),$$
$$[a^{\in,-}(x), a^{\in,+}(x)] \leq \mathbf{B}^{\in}(x),\ [a^{\not\in,-}(x), a^{\not\in,+}(x)] \geq \mathbf{B}^{\not\in}(x),$$
$$a^{\in} \leq B^{\in}(x),\ a^{\not\in} \geq B^{\not\in}(x),\ a \leq \mu(x).$$

Thus

$$[a^{\in,-}(x), a^{\in,+}(x)] \leq (\mathbf{A} \cap \mathbf{B})^{\in}(x),\ [a^{\not\in,-}(x), a^{\not\in,+}(x)] \geq (\mathbf{A} \cap \mathbf{B})^{\not\in}(x),$$

$$a^{\in} \leq A^{\in}(x) \wedge B^{\in}(x) = (A \cap B)^{\in}(x),\ a^{\not\in} \geq A^{\not\in}(x) \vee B^{\not\in}(x) = (A \cap B)^{\not\in}(x),$$
$$a \leq \lambda(x) \wedge \mu(x) = (\lambda \wedge \mu)(x).$$

So $x_{\tilde{\tilde{a}}} \in \mathbf{A} \cap \mathbf{B}$, $x_{\bar{a}} \in A \cap B$ and $x_a \in \lambda \wedge \mu$. Hence $x_{\tilde{\tilde{a}}} \in_1 \mathcal{A} \cap^1 \mathcal{B}$.

For $i = 2,\ 3,\ 4$, the proofs is similar. $\square$

From the definition of orders of IVI-octahedron numbers, we have the following definition.

Definition 4.9. Let X be a nonempty set, let $\tilde{\tilde{a}} \in ([I] \oplus [I]) \times (I \oplus I) \times I$ and let $\mathcal{A} =< \mathbf{A}, A, \lambda >\in IVIO(X)$. Then two subsets $[\mathcal{A}]_{\tilde{\tilde{a}}}$ and $[\mathcal{A}]^*_{\tilde{\tilde{a}}}$ of X are defined as follows:

$$[\mathcal{A}]_{\tilde{\tilde{a}}} = \{x \in X : \mathbf{A}(x) \geq \tilde{\tilde{a}},\ A(x) \geq \bar{a},\ \lambda(x) \geq a\},$$

$$[\mathcal{A}]^*_{\tilde{\tilde{a}}} = \{x \in X : \mathbf{A}(x) > \tilde{\tilde{a}},\ A(x) > \bar{a},\ \lambda(x) > a\}.$$

In this case, $[\mathcal{A}]_{\tilde{\tilde{a}}}$ is called an $\tilde{\tilde{a}}$-level set of $\mathcal{A}$ and $[\mathcal{A}]^*_{\tilde{\tilde{a}}}$ is called a strong $\tilde{\tilde{a}}$-level set of $\mathcal{A}$.

Example 4.10. Consider the IVI-octahedron set in I given by: for each $x \in I$,

$$\mathcal{A} =< ([\frac{1+x}{5}, \frac{1+x}{3}], [\frac{1+x}{7}, \frac{1+x}{6}]), (\frac{1+x}{2}, \frac{x}{5}), \frac{1+x}{4} > .$$

Let $\tilde{\tilde{a}} = ([\frac{1}{5}, \frac{1}{3}], [\frac{1}{6}, \frac{1}{5}],\ \bar{a} = (\frac{2}{7}, \frac{5}{7}),\ a = \frac{1}{4}$. Then

$$[\mathcal{A}]_{\tilde{\tilde{0}}} = \{x \in I : [\tfrac{1+x}{5}, \tfrac{1+x}{3}] \geq \mathbf{0},\ [\tfrac{1+x}{7}, \tfrac{1+x}{6}] \leq \mathbf{1},\ (\tfrac{1+x}{2}, \tfrac{x}{5}) \geq \bar{0},\ \tfrac{1+x}{4} \geq 0\}$$
$$= I,$$
$$[\mathcal{A}]^*_{\tilde{\tilde{0}}} = \{x \in I : [\tfrac{1+x}{5}, \tfrac{1+x}{3}] > \mathbf{0},\ [\tfrac{1+x}{7}, \tfrac{1+x}{6}] < \mathbf{1},\ (\tfrac{1+x}{2}, \tfrac{x}{5}) > \bar{0},\ \tfrac{1+x}{4} > 0\}$$
$$= I \setminus \{1\},$$
$$[\mathcal{A}]_{\tilde{\tilde{1}}} = \{x \in I : [\tfrac{1+x}{5}, \tfrac{1+x}{3}] \geq \mathbf{1},\ [\tfrac{1+x}{7}, \tfrac{1+x}{6}] \leq \mathbf{0},\ (\tfrac{1+x}{2}, \tfrac{x}{5}) \geq \bar{1},\ \tfrac{1+x}{4} \geq 1\}$$
$$= \phi = [\mathcal{A}]^*_{\tilde{\tilde{1}}},$$
$$[\mathcal{A}]_{\tilde{\tilde{a}}} = \{x \in I : ([\tfrac{1+x}{5}, \tfrac{1+x}{3}], [\tfrac{1+x}{7}, \tfrac{1+x}{6}]) \geq \bar{\tilde{a}},\ (\tfrac{1+x}{2}, \tfrac{x}{5}) \geq \bar{b},\ \tfrac{1+x}{4} \geq \alpha\}$$

$$= [0, \tfrac{1}{7}],$$
$$[\mathcal{A}]^*_{\tilde{\tilde{a}}} = \{x \in I : ([\tfrac{1+x}{5}, \tfrac{1+x}{3}], [\tfrac{1+x}{7}, \tfrac{1+x}{6}]) > \tilde{\bar{a}},\ (\tfrac{1+x}{2}, \tfrac{x}{5}) > \bar{b},\ \tfrac{1+x}{4} > \alpha\}$$
$$= (0, \tfrac{1}{7}).$$

It is obvious that that for each $\tilde{\tilde{a}} \in ([I] \oplus [I]) \times (I \oplus I) \times I$ and each $\mathcal{A} \in IVIO(X)$, $[\mathcal{A}]^*_{\tilde{\tilde{a}}} \subset [\mathcal{A}]_{\tilde{\tilde{a}}}$.

Proposition 4.11. *Let $\mathcal{A} \in IVIO(X)$ and let $\tilde{\tilde{a}}$, $\tilde{\tilde{b}} \in ([I] \oplus [I]) \times (I \oplus I) \times I$. Then have the following properties:*

(1) *if $\tilde{\tilde{a}} \leq_1 \tilde{\tilde{b}}$, then $[\mathcal{A}]_{\tilde{\tilde{b}}} \subset [\mathcal{A}]_{\tilde{\tilde{a}}}$,*

(2) $[\mathcal{A}]_{\tilde{\tilde{a}}} = \bigcap_{\tilde{\tilde{b}} <_1 \tilde{\tilde{a}}} [\mathcal{A}]_{\tilde{\tilde{b}}}$, *where $\tilde{\tilde{a}} \neq \tilde{\tilde{0}}$, $\bar{b} \neq \bar{0}$, $a \neq 0$,*

$(1)'$ *if $\tilde{\tilde{a}} \leq_1 \tilde{\tilde{b}}$, then $[\mathcal{A}]^*_{\tilde{\tilde{b}}} \subset [\mathcal{A}]^*_{\tilde{\tilde{a}}}$,*

$(2)'$ $[\mathcal{A}]^*_{\tilde{\tilde{a}}} = \bigcup_{\tilde{\tilde{b}} >_1 \tilde{\tilde{a}}} [\mathcal{A}]^*_{\tilde{\tilde{b}}}$, *where $\tilde{\tilde{a}} \neq \tilde{\tilde{1}}$, $\bar{b} \neq \bar{0}$, $\alpha \neq 0$.*

Proof. (1) The proof is obvious from Definition 4.9.

(2) From (1), it is obvious that $([\mathcal{A}]_{\tilde{\tilde{a}}})_{\tilde{\tilde{a}} \in (([I] \oplus [I]) \times (I \oplus I) \times I) \setminus \{\tilde{\tilde{0}}\}}$ is a descending family of subsets of X. Then clearly, for each $\tilde{\tilde{a}} >\in (([I] \oplus [I]) \times (I \oplus I) \times I) \setminus \{\tilde{\tilde{0}}\}$,

$$[\mathcal{A}]_{\tilde{\tilde{a}}} \subset \bigcap_{\tilde{\tilde{b}}h <_1 \tilde{\tilde{a}}} [\mathcal{A}]_{\tilde{\tilde{b}}}.$$

Assume that $x \notin [\mathcal{A}]_{\tilde{\tilde{a}}}$. Then $\mathcal{A}(x) <_1 \tilde{\tilde{a}}$. Thus

$$\exists\ \tilde{\tilde{b}} \in (([I] \oplus [I]) \times (I \oplus I) \times I) \setminus \{\tilde{\tilde{0}}\}$$

such that $\mathcal{A}(x) <_1 \tilde{\tilde{b}} <_1 \tilde{\tilde{a}}$. So for some $\tilde{\tilde{b}} \in (([I] \oplus [I]) \times (I \oplus I) \times I) \setminus \{\tilde{\tilde{0}}\}$ such that $\tilde{\tilde{b}} <_1< \tilde{\tilde{a}}$, $x \notin [\mathcal{A}]_{\tilde{\tilde{b}}}$, i.e., $x \notin \bigcap_{\tilde{\tilde{b}} <_1 \tilde{\tilde{a}}} [\mathcal{A}]_{\tilde{\tilde{b}}}$. Hence $\bigcap_{\tilde{\tilde{b}} <_1 \tilde{\tilde{a}}} [\mathcal{A}]_{\tilde{\tilde{b}}} \subset [\mathcal{A}]_{\tilde{\tilde{a}}}$. Therefore $[\mathcal{A}]_{\tilde{\tilde{a}}} = \bigcap_{\tilde{\tilde{b}} <_1 \tilde{\tilde{a}}} [\mathcal{A}]_{\tilde{\tilde{b}}}$.

$(2)'$ The proof is similar to (2). □

5. Mappings of IVI-octahedron set

Definition 5.1 ([5]). Let X, Y be two sets, let $f : X \to Y$ be a mapping and let $A \in (I \oplus I)^X$, $B \in (I \oplus I)^Y$.

(i) The preimage of B under f, denoted by $f^{-1}(B)$, is the IF set in X defined as follows: for each $x \in X$,

$$f^{-1}(B)(x) = (B^{\in}(f(x)), B^{\notin}(f(x))) = ((B^{\in} \circ f)(x), (B^{\notin} \circ f)(x)).$$

(ii) The image of A under f, denoted by $f(A) = (f(A^{\in}), f_{-}(A^{\notin}))$, is the IF set in Y defined as follows: for each $y \in Y$,

$$f(A^{\in})(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} A^{\in}(x) & \text{if } f^{-1}(y) \neq \phi \\ \bar{0} & \text{otherwise,} \end{cases}$$

$$f_{-}(A^{\notin})(y) = (1 - f(1 - A^{\notin}))(y) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} A^{\notin,-}(x) & \text{if } f^{-1}(y) \neq \phi \\ \bar{1} & \text{otherwise.} \end{cases}$$

Definition 5.2 ([21])**.** Let X, Y be two sets, let $f : X \to Y$ be a mapping and let $A \in ([I] \oplus [I])^X$, $B \in ([I] \oplus [I])^Y$.

(i) The preimage of B under f, denoted by $f^{-1}(B)$, is the IVI set in X defined as follows: for each $x \in X$,

$$f^{-1}(B)(x) = (B^{\in}(f(x)), B^{\notin}(f(x))) = ((B^{\in} \circ f)(x)), (B^{\notin} \circ f)(x)), \text{ where}$$

$$B^{\in}(f(x)) = [B^{\in,-}(f(x)), B^{\in,+}(f(x))] \text{ and } B^{\notin}(f(x)) = [B^{\notin,-}(f(x)), B^{\notin,+}(f(x))].$$

(ii) The image of A under f, denoted by $f(A) = (f(A^{\in}), f(A^{\notin}))$, is the IVI set in Y defined as follows: for each $y \in Y$,

$$f(A^{\in})(y) = \begin{cases} [\bigvee_{x \in f^{-1}(y)} A^{\in,-}(x), \bigvee_{x \in f^{-1}(y)} A^{\in,+}(x)] & \text{if } f^{-1}(y) \neq \phi \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

$$f(A^{\notin})(y) = \begin{cases} [\bigwedge_{x \in f^{-1}(y)} A^{\notin,-}(x), \bigwedge_{x \in f^{-1}(y)} A^{\notin,+}(x)] & \text{if } f^{-1}(y) \neq \phi \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Definition 5.3. Let X, Y be two sets, let $f : X \to Y$ be a mapping and let $\mathcal{A} =< \mathbf{A}, A, \lambda >\in IVIO(X)$, $\mathcal{B} =< \mathbf{B}, B, \mu >\in IVIO(Y)$.

(i) The preimage of $\mathcal{B}$ under f, denoted by $f^{-1}(\mathcal{B}) =< f^{-1}(\mathbf{B}), f^{-1}(B), f^{-1}(\mu) >$, is an IVI-octahedron set in X defined as follows: for each $x \in X$,

$$f^{-1}(\mathcal{B})(x) =< ([(B^{\in,-} \circ f)(x), (B^{\in,+} \circ f)(x))], [(B^{\notin,-} \circ f)(x), (B^{\notin,+} \circ f)(x))]) \\ , ((B^{\in} \circ f)(x), (B^{\notin} \circ f)(x)), (\mu \circ f)(x) > .$$

(ii) The image of $\mathcal{A}$ under f, denoted by $f(\mathcal{A}) =< f(\mathbf{A}), f(A), f(\lambda) >$, is an IVI-octahedron set in Y defined as follows: for each $y \in Y$,

$$f(\mathbf{A}^{\in})(y) = \begin{cases} [\bigvee_{x \in f^{-1}(y)} A^{\in,-}(x), \bigvee_{x \in f^{-1}(y)} A^{\in,+}(x)] & \text{if } f^{-1}(y) \neq \phi \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

$$f(\mathbf{A}^{\notin})(y) = \begin{cases} [\bigwedge_{x \in f^{-1}(y)} A^{\notin,-}(x), \bigwedge_{x \in f^{-1}(y)} A^{\notin,+}(x)] & \text{if } f^{-1}(y) \neq \phi \\ \mathbf{1} & \text{otherwise,} \end{cases}$$

$$f(A)(y) = \begin{cases} (\bigvee_{x \in f^{-1}(y)} A^{\in}(x), \bigwedge_{x \in f^{-1}(y)} A^{\notin}(x)) & \text{if } f^{-1}(y) \neq \phi \\ \bar{0} & \text{otherwise,} \end{cases}$$

$$f(\lambda)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \lambda(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that $f(x_{<\widetilde{a}, \bar{b}, \alpha>}) = [f(x)]_{<\widetilde{a}, \bar{b}, \alpha>}$, for each $x_{<\widetilde{a}, \bar{b}, \alpha>} \in O_P(X)$.

Example 5.4. Let $X = \{x, y, z\}$, $Y = \{a, b, c, d\}$ and let $f : X \to Y$ be the mapping defined by: $f(x) = f(y) = a,\ f(z) = c$.
Let $\mathcal{A} =< \mathbf{A}, A, \lambda >$ be the IVI-octahedron set in X and let $\mathcal{B} =< \mathbf{B}, B, \mu >$ be the octahedron set in Y defined by the following Table:
Then $f(\mathbf{A}^{\in})(a) = [\bigvee_{t \in f^{-1}(a)} A^{\in,-}(t), \bigwedge_{t \in f^{-1}(a)} A^{\in,+}(t)]$

X	$\mathbf{A}(t)$	$A(t)$	$\lambda(t)$
x	$([0.2, 0.6], [0.1, 0.3])$	$(0.6, 0.3)$	0.7
y	$([0.3, 0.5], [0.3, 0.4])$	$(0.5, 0.2)$	0.6
z	$([0.4, 0.7], [0.2, 0.2])$	$(0.7, 0.2)$	0.8

Table 5.1

$$
\begin{aligned}
&= [A^{\in,-}(x) \vee A^{\in,-}(y), A^{\in,+}(x) \vee A^{\in,+}(y)] \\
&= [0.3, 0.6], \\
f(\mathbf{A}^{\notin})(a) &= [\bigwedge_{t \in f^{-1}(a)} A^{\notin,-}(t), \bigvee_{t \in f^{-1}(a)} A^{\notin,+}(t)] \\
&= [A^{\notin,-}(x) \wedge A^{\notin,-}(y), A^{\notin,+}(x) \wedge A^{\notin,+}(y)]) \\
&= [0.1, 0.3], \\
f(\mathbf{A})(b) &= f(\mathbf{A})(d) = \tilde{\tilde{0}},\ f(\mathbf{A})(c) = ([0.4, 0.7], [0.2, 0.2]), \\
f(A)(a) &= (\bigvee_{t \in f^{-1}(a)} A^{\in}(t), \bigwedge_{t \in f^{-1}(a)} A^{\notin}(t)) \\
&= (A^{\in}(x) \vee A^{\in}(y), A^{\notin}(x) \wedge A^{\notin}(y)) \\
&= (0.6, 0.2), \\
f(A)(b) &= f(A)(d) = \bar{0},\ f(A)(c) = (0.7, 0.2), \\
f(\lambda)(a) &= \bigvee_{t \in f^{-1}(a)} \lambda(t) \\
&= \lambda(x) \vee \lambda(y) = 0.7, \\
f(\lambda)(b) &= f(\lambda)(d) = 0,\ f(\lambda)(c) = 0.8.
\end{aligned}
$$

Thus we have Table 5.2 for $f(\mathcal{A})$:

Y	$f(\mathbf{A})(x)$	$f(A)(x)$	$f(\lambda)(x)$
a	$([0.3, 0.6], [0.1, 0.3])$	$(0.6, 0.2)$	0.7
b	$\tilde{\tilde{0}}$	$\tilde{0}$	0
c	$([0.4, 0.7], [0.2, 0.2])$	$(0.7, 0.2)$	0.8
d	$\tilde{\tilde{0}}$	$\tilde{0}$	0

Table 5.2

Now let $\mathcal{B} =< \mathbf{B}, B, \mu >$ be the IVI-octahedron set in Y defined by Table 5.3:

Y	$\mathbf{B}(x)$	$B(x)$	$\mu(x)$
a	$([0.3, 0.5], [0.2, 0.4])$	$(0.5, 0.4)$	0.6
b	$([0.2, 0.6], [0.3, 0.3])$	$(0.7, 0.2)$	0.8
c	$([0.4, 0.7], [0.1, 0.2])$	$(0.6, 0.3)$	0.7
d	$([0.2, 0.5], [0.3, 0.4])$	$(0.4, 0.5)$	0.5

Table 5.3

Then
$$
\begin{aligned}
f^{-1}(\mathbf{B}^{\in})(x) &= [B^{\in,-}(f(x)), B^{\in,+}(f(x))] \\
&= [B^{\in,-}(a), B^{\in,+}(a)] \\
&= [0.3, 0.5] \\
&= f^{-1}(\mathbf{B}^{\in})(y), \\
f^{-1}(\mathbf{B}^{\notin})(x) &= [B^{\notin,-}(f(x)), B^{\notin,+}(f(x))] \\
&= [B^{\notin,-}(a), B^{\notin,+}(a)] \\
&= [0.2, 0.4]
\end{aligned}
$$

$= f^{-1}(\mathbf{B}^{\not\in})(y)$.
Similarly, we can calculate the followings:
$f^{-1}(\mathbf{B})(z) = ([0.4, 0.7], [0.1, 0.2])$,
$f^{-1}(B)(x) = f^{-1}(B)(y) = (0.5, 0.4),\ f^{-1}(B)(z) = (0.6, 0.3)$,
$f^{-1}(\mu)(x) = f^{-1}(\mu)(y) = 0.6,\ f^{-1}(\mu)(z) = 0.7$.
So we have Table 5.4 for $f^{-1}(\mathcal{B})$:

X	$f^{-1}(\mathbf{B})(t)$	$f^{-1}(B)(t)$	$f^{-1}(\mu)(t)$
x	$([0.3, 0.5], [0.2, 0.4])$	$(0.5, 0.4)$	0.6
y	$([0.3, 0.5], [0.2, 0.4])$	$(0.5, 0.4)$	0.6
z	$([0.4, 0.7], [0.1, 0.2])$	$(0.6, 0.3)$	0.7

Table 5.4

Proposition 5.5. *Let $\mathcal{A} =< \mathbf{A}, A, \lambda >$, $\mathcal{A}_1 =< \mathbf{A}_1, A_1, \lambda_1 >$, $\mathcal{A}_2 =< \mathbf{A}_2, A_2, \lambda_2 >\in IVIO(X)$, $(\mathcal{A}_j)_{j\in J} = (< \mathbf{A}_j, A_j, \lambda_j >)_{j\in J} \subset IVIO(X)$, let $\mathcal{B} =< \mathbf{B}, B, \mu >$, $\mathcal{B}_1 =< \mathbf{B}_1, B_1, \mu_1 >$, $\mathcal{B}_2 =< \mathbf{B}_2, B_2, \mu_2 >\in IVIO(Y)$, $(\mathcal{B}_j)_{j\in J} = (< \mathbf{B}_j, B_j, \mu_j >)_{j\in J} \subset IVIO(Y)$ and let $f : X \to Y$ be a mapping. Then for each $i = 1,\ 2,\ 3,\ 4$,*
(1) *if $\mathcal{A}_1 \subset_i \mathcal{A}_2$, then $f(\mathcal{A}_1) \subset_i f(\mathcal{A}_2)$,*
(2) *if $\mathcal{B}_1 \subset_i \mathcal{B}_2$, then $f^{-1}(\mathcal{B}_1) \subset_i f^{-1}(\mathcal{B}_2)$,*
(3) *$\mathcal{A} \subset_1 f^{-1}(f(\mathcal{A}))$ and if f is injective, then $\mathcal{A} = f^{-1}(f(\mathcal{A}))$,*
(4) *$f(f^{-1}(\mathcal{B})) \subset_1 \mathcal{B}$ and if f is surjective, $f(f^{-1}(\mathcal{B})) = \mathcal{B}$,*
(5) *$f^{-1}(\bigcup^i_{j\in J} \mathcal{B}_j) = \bigcup^i_{j\in J} f^{-1}(\mathcal{B}_j)$,*
(6) *$f^{-1}(\bigcap^i_{j\in J} \mathcal{B}_j) = \bigcap^i_{j\in J} f^{-1}(\mathcal{B}_j)$,*
(7) *$f(\bigcup^1_{j\in J} \mathcal{A}_j) = \bigcup^1_{j\in J} f(\mathcal{A}_j)$,*
(8) *$f(\bigcap^i_{j\in J} \mathcal{A}_j) \subset_i \bigcap^i_{j\in J} f(\mathcal{A}_j)$ and if f is injective, then $f(\bigcap^i_{j\in J} \mathcal{A}_j) = \bigcap^i_{j\in J} f(\mathcal{A}_j)$,*
(9) *if f is surjective, then $f(\mathcal{A})^c \subset_1 f(\mathcal{A}^c)$.*
(10) *$f^{-1}(\mathcal{B}^c) = f^{-1}(\mathcal{B})^c$.*
(11) *$f^{-1}(\ddot{\mathbf{0}}) = \ddot{\mathbf{0}}$, $f^{-1}(\ddot{\mathbf{1}}) = \ddot{\mathbf{1}}$, $f^{-1}(< \widetilde{\widetilde{0}}, \bar{\mathbf{0}}, 1 >) =< \widetilde{\widetilde{0}}, \bar{\mathbf{0}}, 1 >$,*
$f^{-1}(< \widetilde{\widetilde{0}}, \bar{\mathbf{1}}, 0 >) =< \widetilde{\widetilde{0}}, \bar{\mathbf{1}}, 0 >$, $f^{-1}(< \widetilde{\widetilde{1}}, \bar{\mathbf{0}}, 0 >) =< \widetilde{\widetilde{1}}, \bar{\mathbf{0}}, 0 >$,
$f^{-1}(< \widetilde{\widetilde{0}}, \bar{\mathbf{1}}, 1 >) =< \widetilde{\widetilde{0}}, \bar{\mathbf{1}}, 1 >$, $f^{-1}(< \widetilde{\widetilde{1}}, \bar{\mathbf{0}}, 1 >) =< \widetilde{\widetilde{1}}, \bar{\mathbf{0}}, 1 >$,
$f^{-1}(< \widetilde{\widetilde{1}}, \bar{\mathbf{1}}, 0 >) =< \widetilde{\widetilde{1}}, \bar{\mathbf{1}}, 0 >$.
(12) *$f(\ddot{\mathbf{0}}) = \ddot{\mathbf{0}}$ and if f is surjective, then the following hold:*

$$f(< \widetilde{\widetilde{0}}, \bar{\mathbf{0}}, 1 >) =< \widetilde{\widetilde{0}}, \bar{\mathbf{0}}, 1 >,\ f(< \widetilde{\widetilde{0}}, \bar{\mathbf{1}}, 0 >) =< \widetilde{\widetilde{0}}, \bar{\mathbf{1}}, 0 >,$$

$$f(< \widetilde{\widetilde{1}}, \bar{\mathbf{0}}, 0 >) =< \widetilde{\widetilde{1}}, \bar{\mathbf{0}}, 0 >,\ f(< \widetilde{\widetilde{0}}, \bar{\mathbf{1}}, 1 >) =< \widetilde{\widetilde{0}}, \bar{\mathbf{1}}, 1 >,$$

$$f(< \widetilde{\widetilde{1}}, \bar{\mathbf{0}}, 1 >) =< \widetilde{\widetilde{1}}, \bar{\mathbf{0}}, 1 >,\ f(< \widetilde{\widetilde{1}}, \bar{\mathbf{1}}, 0 >) =< \widetilde{\widetilde{1}}, \bar{\mathbf{1}}, 0 >,\ f(\ddot{\mathbf{1}}) = \ddot{\mathbf{1}}.$$

Proof. The proofs are straightforward. □

Example 5.6. Let f be the mapping and let $\mathcal{A}$ be the IVI-octahedron set in X given in Example 5.4. Then $f^{-1}(f(\mathcal{A}))(x) =< ([0.3, 0.6], [0.1, 0.3]), (0.6, 0.2), 0.7 >\geq \mathcal{A}(x)$, $f^{-1}(f(\mathcal{A}))(y) =< ([0.3, 0.6], [0.1, 0.3]), (0.6, 0.2), 0.7 >\geq \mathcal{A}(y)$ and $f^{-1}(f(\mathcal{A}))(z) = \mathcal{A}(z)$. Then $\mathcal{A} \subset_1 f^{-1}(f(\mathcal{A}))$. Moreover, $\mathcal{A} \neq f^{-1}(f(\mathcal{A}))$. On the other hand, we can easily calculate that $f(f^{-1}(\mathcal{B}))=\mathcal{B}$. Thus we can confirm that Proposition 5.5 (3) and (4) hold. Note that f is surjective but not injective.

Remark 5.7. $f(\bigcup^i_{j\in J}\mathcal{A}_j)\neq\bigcup^i_{j\in J}f(\mathcal{A}_j)$ for $i=2,\ 4,\ 4$, in general.

Example 5.8. Let $X=\{x,y,z\}$, $Y=\{a,b,c\}$, let $\mathcal{A}_1=<\mathbf{A}_1,A_1,\lambda_1>$ and $\mathcal{A}_2=<\mathbf{A}_2,A_2,\lambda_2>$ be two IVI-octahedron set in X given by:

$\mathcal{A}_1(x)=<([0.3,0.6],[0.2,0.3]),(0.6,0.3),0.6>$,
$\mathcal{A}_1(y)=<([0.2,0.7],[0.1,0.2]),(0.7,0.2),0.7>$,
$\mathcal{A}_1(z)=<([0.5,0.6],[0.3,0.3]),(0.7,0.1),0.5>$,
$\mathcal{A}_2(x)=<([0.4,0.5],[0.3,0.4]),(0.7,0.2),0.7>$,
$\mathcal{A}_2(y)=<([0.3,0.4],[0.4,0.5])),(0.6,0.3),0.6>$,
$\mathcal{A}_2(z)=<([0.3,0.8],[0.1,0.1]),(0.5,0.3),0.7>$.

Let $f:X\to Y$ be the mapping defined by $f(x)=f(y)=a,\ f(z)=c$. Then

$$f(\lambda_1\wedge\lambda_2)(a)=0.6\neq 0.7=(f(\lambda_1)\wedge f(\lambda_2))(a)$$

and

$$f(A_1\cap A_2)(a)=(0.6,0.3)\neq(0.7,0.2)=(f(A_1)\cap f(A_2).$$

Thus $f(\mathcal{A}_1\cup^2\mathcal{A}_2)(a)\neq(f(\mathcal{A}_1)\cup^2 f(\mathcal{A}_1))(a)$, $f(\mathcal{A}_1\cup^3\mathcal{A}_2)(a)\neq(f(\mathcal{A}_1)^4 f(\mathcal{A}_1))(a)$ and $f(\mathcal{A}_1\cup^4\mathcal{A}_2)(a)\neq(f(\mathcal{A}_1)\cup^4 f(\mathcal{A}_1))(a)$. So $f(\bigcup^i_{j\in J}\mathcal{A}_j)\neq\bigcup^i_{j\in J}f(\mathcal{A}_j)$ for $i=2,\ 3,\ 4$.

The following is an immediate result of Definition 5.3 (i).

Proposition 5.9. *If $g:Y\to Z$ is a mapping, then $(g\circ f)^{-1}(C)=f^{-1}(g^{-1}(C))$, for each $\mathcal{C}\in IVIO(X)$, where $g\circ f$ is the composition of f and g.*

6. IVI-octahedron groupoids

In this section, we introduce the concept of IVI-octahedron groupoids and study some of its properties.

Throughout this section and next sections, for an octahedron set $\mathcal{A}=\langle\mathbf{A},A,\lambda\rangle$ in a set X, $\mathcal{A}\neq\ddot{0}$ [resp. $\left\langle\tilde{\tilde{0}},\bar{\mathbf{0}},1\right\rangle$, $\left\langle\tilde{\tilde{0}},\bar{\mathbf{1}},0\right\rangle$ and $\left\langle\tilde{\tilde{0}},\bar{\mathbf{1}},1\right\rangle$] means that

$$\mathbf{A}\neq\tilde{\tilde{0}},\ A\neq\bar{\mathbf{0}},\ \lambda\neq 0$$

[resp. $\mathbf{A}\neq\tilde{\tilde{0}},\ A\neq\bar{\mathbf{0}},\ \lambda\neq 1$, $\mathbf{A}\neq\tilde{\tilde{0}},\ A\neq\bar{\mathbf{1}},\ \lambda\neq 0$ and $\mathbf{A}\neq\tilde{\tilde{0}},\ A\neq\bar{\mathbf{1}},\ \lambda\neq 1$].

Definition 6.1 ([18])**.** Let $(X,\cdot)$ be a groupoid and let $\lambda,\ \mu\in I^X$. Then the product of λ and μ, denoted by $\lambda\circ_F\mu$, is a fuzzy set in X defined as follows: for each $x\in X$,

$$(\lambda\circ_F\mu)(x)=\begin{cases}\bigvee_{yz=x,\ y,\ z\in X}[\lambda(y)\wedge\mu(z)] & \text{if } yz=x\\ 0 & \text{otherwise.}\end{cases}$$

Definition 6.2 ([9])**.** Let $(X,\cdot)$ be a groupoid and let $\mathbf{A},\ \mathbf{B}\in(I\oplus I)^X$. Then the product of A and B, denoted by $A\circ_{IF}B$, is an IF set in X defined as follows: for each $x\in X$,

$$(A\circ_{IF}B)(x)$$
$$=\begin{cases}(\bigvee_{yz=x,\ y,\ z\in X}[A^{\in}(y)\wedge B^{\in}(z)],\bigwedge_{yz=x,\ y,\ z\in X}[A^{\notin}(y)\wedge B^{\notin}(z)] & \text{if } yz=x\\ (0,1) & \text{otherwise.}\end{cases}$$

Definition 6.3. Let $(X, \cdot)$ be a groupoid and let $\mathbf{A},\ \mathbf{B} \in ([I] \oplus [I])^X$. Then the product of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \circ_{IVI} \mathbf{B}$, is an IVI set in X defined as follows: for each $x \in X$,

$$(\mathbf{A} \circ_{IVI} \mathbf{B})(x) = \begin{cases} [\bigvee_{yz=x,\ y,\ z\in X}[\mathbf{A}^{\in}(y) \wedge \mathbf{B}^{\in}(z)], \bigwedge_{yz=x,\ y,\ z\in X}[\mathbf{A}^{\notin}(y) \wedge \mathbf{B}^{\notin}(z)] & \text{if } yz = x \\ \widetilde{\widetilde{0}} & \text{otherwise,} \end{cases}$$

where $\mathbf{A}^{\in}(y) = [A^{\in,-}(y), A^{\in,+}(y)]$ and $\mathbf{A}^{\notin}(y) = [A^{\notin,-}(y), A^{\notin,+}(y)]$.

Proposition 6.4. *Let $(X, \cdot)$ be a groupoid and let $\mathbf{A},\ \mathbf{B} \in (I \oplus I)^X$. Then*

$$\mathbf{A} \circ_{IVI} \mathbf{B} = \bigcup_{x_{\widetilde{\widetilde{a}}} \in \mathbf{A},\ x_{\widetilde{\widetilde{b}}} \in \mathbf{B}} x_{\widetilde{\widetilde{a}}} \circ x_{\widetilde{\widetilde{b}}}.$$

Proof. The proof follows from Proposition 2.2 (2) in [9]. □

By using the definitions the inf and the sup of IVI-octahedron numbers, we can find the product of two IVI-octahedron sets as follows.

Definition 6.5. Let $(X, \cdot)$ be a groupoid and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle,\ \mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in IVIO(X)$. Then the i-product of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \circ_i \mathcal{B}$ ($i = 1,\ 2,\ ,3,\ 4$), is an IVI-octahedron set in X defined as follows: for each $x \in X$,

$$(\mathcal{A} \circ_1 \mathcal{B})(x) = \begin{cases} \bigvee^1_{yz=x,\ y,\ z\in X}[\mathcal{A}(y) \wedge^1 \mathcal{B}(z)] & \text{if } yz = x \\ \widetilde{\widetilde{\bar{0}}} & \text{otherwise,} \end{cases}$$

$$(\mathcal{A} \circ_2 \mathcal{B})(x) = \begin{cases} \bigvee^2_{yz=x,\ y,\ z\in X}[\mathcal{A}(y) \wedge^2 \mathcal{B}(z)] & \text{if } yz = x \\ \left\langle \widetilde{\widetilde{0}}, \bar{\mathbf{0}}, 1 \right\rangle & \text{otherwise,} \end{cases}$$

$$(\mathcal{A} \circ_3 \mathcal{B})(x) = \begin{cases} \bigvee^3_{yz=x,\ y,\ z\in X}[\mathcal{A}(y) \wedge^3 \mathcal{B}(z)] & \text{if } yz = x \\ \left\langle \widetilde{\widetilde{0}}, \bar{\mathbf{1}}, 0 \right\rangle & \text{otherwise,} \end{cases}$$

$$(\mathcal{A} \circ_4 \mathcal{B})(x) = \begin{cases} \bigvee^4_{yz=x,\ y,\ z\in X}[\mathcal{A}(y) \wedge^4 \mathcal{B}(z)] & \text{if } yz = x \\ \left\langle \widetilde{\widetilde{0}}, \bar{\mathbf{1}}, 1 \right\rangle & \text{otherwise.} \end{cases}$$

Remark 6.6. From Definitions 6.1, 6.2 and 6.3, we can easily see that followings hold:

(1) $\mathcal{A} \circ_1 \mathcal{B} = \langle \mathbf{A} \circ_{IVI} \mathbf{B}, A \circ_{IF} B, \lambda \circ_F \mu \rangle$,

(2) $\mathcal{A} \circ_1 \mathcal{B} = \langle \mathbf{A} \circ_{IVI} \mathbf{B}, A \circ_{IF} B, \lambda \circ_2 \mu \rangle$, where

$$(\lambda \circ_2 \mu)(x) = \begin{cases} \bigwedge_{yz=x,\ y,\ z\in X}[\lambda(y) \vee \mu(z)] & \text{if } yz = x \\ 1 & \text{otherwise,} \end{cases}$$

(3) $\mathcal{A} \circ_3 \mathcal{B} = \langle \mathbf{A} \circ_{IVI} \mathbf{B}, A \circ_3 B, \lambda \circ_F \mu \rangle$, where

$$(A \circ_3 B)(x) = \begin{cases} (\bigwedge_{yz=x,\ y,\ z\in X}[A^{\in}(y) \vee B^{\in}(z)], \bigvee_{yz=x,\ y,\ z\in X}[A^{\notin}(y) \wedge B^{\notin}(z)] & \text{if } yz = x \\ (1, 0) & \text{otherwise,} \end{cases}$$

(4) $\mathcal{A} \circ_4 \mathcal{B} = \langle \mathbf{A} \circ_{IVI} \mathbf{B}, A \circ_3 B, \lambda \circ_2 \mu \rangle$.

$\cdot$	a	b	c
a	a	a	a
b	b	a	b
c	c	c	a

Table 6.1

Example 6.7. Let $X = \{a, b, c\}$ be the groupoid with the following Cayley table: Consider two octahedron sets $\mathcal{A}$ and $\mathcal{B}$ in X, respectively given by:

$$\mathcal{A}(a) = \langle([0.3, 0.6], [0.2, 0.3]), (0.7, 0.2), 0.5\rangle\,,$$

$$\mathcal{A}(b) = \langle([0.2, 0.4], [0.4, 0.5]), (0.6, 0.3), 0.7\rangle\,,$$

$$\mathcal{A}(c) = \langle([0.4, 0.7], [0.1, 0.2]), (0.5, 0.4), 0.3\rangle\,,$$

$$\mathcal{B}(a) = \langle([0.2, 0.6], [0.2, 0.3]), (0.6, 0.3), 0.7\rangle\,,$$

$$\mathcal{B}(b) = \langle([0.3, 0.5], [0.2, 0.4]), (0.5, 0.2), 0.6\rangle\,,$$

$$\mathcal{B}(c) = \langle([0.4, 0.7], [0.1, 0.3]), (0.7, 0.2), 0.8\rangle\,.$$

Then we can easily calculate $\mathcal{A} \circ_i \mathcal{B}$ having Tables 6.2 and 6.3:

	$(\mathcal{A} \circ_1 \mathcal{B})(t)$	$(\mathcal{A} \circ_2 \mathcal{B})(t)$
a	$\langle([0.4, 0.7], [0.1, 0.3]), (0.5, 0.2), 0.6\rangle$	$\langle([0.4, 0.7], [0.1, 0.3]), (0.5, 0.2), 0.6\rangle$
b	$\langle([0.2, 0.4], [0.4, 0.5]), (0.6, 0.3), 0.7\rangle$	$\langle([0.2, 0.4], [0.4, 0.5]), (0.6, 0.3), 0.7\rangle$
c	$\langle([0.3, 0.6], [0.2, 0.3]), (0.5, 0.4), 0.3\rangle$	$\langle([0.3, 0.6], [0.2, 0.3]), (0.5, 0.4), 0.7\rangle$

Table 6.2

	$(\mathcal{A} \circ_3 \mathcal{B})(t)$	$(\mathcal{A} \circ_4 \mathcal{B})(t)$
a	$\langle([0.4, 0.7], [0.1, 0.3]), (0.6, 0.2), 0.6\rangle$	$\langle([0.4, 0.7], [0.1, 0.3]), (0.6, 0.2), 0.6\rangle$
b	$\langle([0.2, 0.4], [0.4, 0.5]), (0.6, 0.2), 0.7\rangle$	$\langle([0.2, 0.4], [0.4, 0.5]), (0.6, 0.2), 0.7\rangle$
c	$\langle([0.3, 0.6], [0.2, 0.3]), (0.6, 0.3), 0.3\rangle$	$\langle([0.3, 0.6], [0.2, 0.3]), (0.6, 0.3), 0.7\rangle$

Table 6.3

Proposition 6.8. *Let $(X, \cdot)$ be a groupoid, let $\mathcal{A} = \langle \mathbf{A}, A, \lambda\rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu\rangle \in IVIO(X)$ and let $x_{\tilde{\tilde{a}}},\ y_{\tilde{\tilde{b}}} \in IVIO_P(X)$. Then we have*

(1) $x_{\tilde{\tilde{a}}} \circ_i y_{\tilde{\tilde{b}}} = (xy)_{\tilde{\tilde{a}} \wedge^i \tilde{\tilde{b}}}$, *for* $i = 1,\ 2,\ 3,\ 4$, *i.e.,*

$$x_{\tilde{\tilde{a}}} \circ_1 y_{\tilde{\tilde{b}}} = \left\langle (xy)_{\tilde{\tilde{a}} \wedge \tilde{\tilde{b}}}, (xy)_{\bar{a} \wedge \bar{b}}, (xy)_{a \wedge b} \right\rangle,\ x_{\tilde{\tilde{a}}} \circ_2 y_{\tilde{\tilde{b}}} = \left\langle (xy)_{\tilde{\tilde{a}} \wedge \tilde{\tilde{b}}}, (xy)_{\bar{a} \wedge \bar{b}}, (xy)_{a \vee b} \right\rangle,$$

$$x_{\tilde{\tilde{a}}} \circ_3 y_{\tilde{\tilde{b}}} = \left\langle (xy)_{\tilde{\tilde{a}} \wedge \tilde{\tilde{b}}}, (xy)_{\bar{a} \vee \bar{b}}, (xy)_{a \wedge b} \right\rangle,\ x_{\tilde{\tilde{a}}} \circ_4 y_{\tilde{\tilde{b}}} = \left\langle (xy)_{\tilde{\tilde{a}} \wedge \tilde{\tilde{b}}}, (xy)_{\bar{a} \vee \bar{b}}, (xy)_{a \vee b} \right\rangle,$$

(2) $\mathcal{A} \circ_i \mathcal{B} = \bigcup^i_{x_{\tilde{\tilde{a}}} \in_i \mathcal{A},\ y_{\tilde{\tilde{b}}} \in_i \mathcal{B}} x_{\tilde{\tilde{a}}} \circ_i y_{\tilde{\tilde{b}}}$, *for* $i = 1,\ 2,\ 3,\ 4$.

Proof. (1) The proofs are obvious from Definition 6.5.

(2) Case 1: Suppose $i = 1$. Then the proof follows from Proposition 1.1 [18], Proposition 2.2 [9] and Proposition 6.4.

Case 2: Suppose $i = 2$. From Remark 6.6 (2), it is sufficient to prove that $\lambda \circ_2 \mu = \bigcap_{x_a \in_2 \lambda,\ y_b \in_2 \mu} x_a \circ_2 y_b$. Let $C = \bigcap_{x_a \in_2 \lambda, y_b \in_2 \mu} x_a \circ_2 y_b$. For each $z \in X$, we may suppose that there are $u,\ v \in X$ such that $uv = z$, $x_a \neq 1$ and $y_b \neq 1$ without loss of generality. Then

$$\begin{aligned}(\lambda \circ_2 \mu)(z) &= \bigwedge_{z=uv}[\lambda(u) \vee \mu(v)]\\ &\leq \bigwedge_{z=uv}(\bigwedge_{x_a \in_2 \lambda,\ y_b \in_2 \mu}[x_a(u) \vee y_b(v)])\\ &= \bigwedge_{x_a \in_2 \lambda,\ y_b \in_2 \mu}(\bigwedge_{z=uv}[x_a(u) \vee y_b(v)])\\ &= (\bigcap_{x_a \in_2 \lambda,\ y_b \in_2 \mu} x_a \circ_2 y_b)(z)\\ &= C.\end{aligned}$$

Since $u_{\lambda(u)} \in_2 \lambda$ and $v_{\mu(v)} \in_2 \mu$,

$$\begin{aligned}(\bigcap_{x_a \in_2 \lambda,\ y_b \in_2 \mu} x_a \circ_2 y_b)(z) &= \bigwedge_{x_a \in_2 \lambda,\ y_b \in_2 \mu} \bigwedge_{z=uv}[x_a(u) \vee y_b(v)]\\ &\leq \bigwedge_{z=uv}[u_{\lambda(u)}(u) \vee v_{\mu(v)}(v)]\\ &= \bigwedge_{z=uv}[\lambda(u) \vee \mu(v)]\\ &= (\lambda \circ_2 \mu)(z).\end{aligned}$$

Thus $(\lambda \circ_2 \mu)(z) = C(z)$. So $\mathcal{A} \circ_2 \mathcal{B} = \bigcup^2_{x_{\tilde{\tilde{a}}} \in_2 \mathcal{A},\ y_{\tilde{\tilde{b}}} \in_2 \mathcal{B}} x_{\tilde{\tilde{a}}} \circ_2 y_{\tilde{\tilde{b}}}$.

Case 3: Suppose $i = 3$. From Remark 6.6 (3), it is sufficient to prove that

$$A \circ_3 B = (\bigcap_{x_a \in_3 A,\ y_b \in_3 B} x_a \circ_3 y_b, \bigcup_{x_a \in_3 A,\ y_b \in_3 B} x_a \circ_3 y_b),$$

where $(A \circ_3 B)^{\in} = \bigcap_{x_a \in_3 A,\ y_b \in_3 B} x_a \circ_3 y_b$ and $(A \circ_3 B)^{\notin} = \bigcup_{x_a \in_3 A,\ y_b \in_3 B} x_a \circ_3 y_b$. Let $z \in X$. Then from the proof of Case 2 and Proposition 1.1 [18] (ii), we have

$$(A \circ_3 B)^{\in}(z) = (\bigcap_{x_{\bar{a}} \in_3 A,\ y_{\bar{b}} \in_3 B} x_{\bar{a}} \circ_3 y_{\bar{b}})(z),\ (A \circ_3 B)^{\notin}(z) = (\bigcup_{x_{\bar{a}} \in_3 A,\ y_{\bar{b}} \in_3 B} x_{\bar{a}} \circ_3 y_{\bar{b}})(z).$$

Thus $\mathcal{A} \circ_3 \mathcal{B} = \bigcup^3_{x_{\tilde{\tilde{a}}} \in_3 \mathcal{A},\ y_{\tilde{\tilde{b}}} \in_3 \mathcal{B}} x_{\tilde{\tilde{a}}} \circ_3 y_{\tilde{\tilde{b}}}$.

Case 4: Suppose $i = 4$. Then from Cases 2 and 3, the proof is obvious. □

The followings are immediate results of Definition 6.5.

Proposition 6.9. *Let $(X, \cdot)$ be a groupoid and let $i = 1,\ 2,\ 3,\ 4$.*

(1) *If "$\cdot$" is associative [resp. commutative] in X, then so is "$\circ_i$" in $IVIO(X)$.*

(2) *If "$\cdot$" has an identity $e \in X$, then we have*

(2_a) $e_{\ddot{1}} \in IVIO_P(X)$ *is an identity of "$\circ_1$" in $IVIO(X)$, i.e.,*
$\mathcal{A} \circ e_{\ddot{1}} = e_{\ddot{1}} \circ \mathcal{A} = \mathcal{A}$, *for each* $\mathcal{A} \in IVIO(X)$,

(2_b) $e_{\left\langle \tilde{1}, \bar{\mathbf{1}}, 0 \right\rangle} \in IVIO_P(X)$ *is an identity of "$\circ_2$" in $IVIO(X)$, i.e.,*
$\mathcal{A} \circ_2 e_{\left\langle \tilde{1}, \bar{\mathbf{1}}, 0 \right\rangle} = e_{\left\langle \tilde{1}, \bar{\mathbf{1}}, 0 \right\rangle} \circ_2 \mathcal{A} = \mathcal{A}$, *for each* $\mathcal{A} \in IVIO(X)$,

(2_c) $e_{\left\langle \tilde{1}, \bar{\mathbf{0}}, 1 \right\rangle} \in \mathcal{O}_P(X)$ *is an identity of "$\circ_3$" in $IVIO(X)$, i.e.,*
$\mathcal{A} \circ_3 e_{\left\langle \tilde{1}, \bar{\mathbf{0}}, 1 \right\rangle} = e_{\left\langle \tilde{1}, \bar{\mathbf{0}}, 1 \right\rangle} \circ_3 \mathcal{A} = \mathcal{A}$, *for each* $\mathcal{A} \in IVIO(X)$,

(2_d) $e_{\left\langle \tilde{1}, \bar{\mathbf{0}}, 0 \right\rangle} \in IVIO_P(X)$ *is an identity of "$\circ_4$" in $IVIO(X)$, i.e.,*
$\mathcal{A} \circ_4 e_{\left\langle \tilde{1}, \bar{\mathbf{0}}, 0 \right\rangle} = e_{\left\langle \tilde{1}, \bar{\mathbf{0}}, 0 \right\rangle} \circ_4 \mathcal{A} = \mathcal{A}$, *for each* $\mathcal{A} \in IVIO(X)$.

Definition 6.10. Let $(X, \cdot)$ be a groupoid and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in IVIO(X)$. Then

(i) $\ddot{0} \neq \mathcal{A}$ is called a 1-octahedron subgroupoid in X, if $\mathcal{A} \circ_1 \mathcal{A} \subset_1 \mathcal{A}$, i.e.,

$$\mathbf{A} \circ_{IVI} \mathbf{A} \subset \mathbf{A},\ A \circ_{IF} A \subset A,\ \lambda \circ_F \lambda \subset \lambda,$$

(ii) $\left\langle \widetilde{\widetilde{0}}, \bar{\mathbf{0}}, 1 \right\rangle \neq \mathcal{A}$ is called a 2-octahedron subgroupoid in X, if $\mathcal{A} \circ_2 \mathcal{A} \subset_2 \mathcal{A}$, i.e.,

$$\mathbf{A} \circ_{IVI} \mathbf{A} \subset \mathbf{A},\ A \circ_{IF} A \subset A,\ \lambda \circ_2 \lambda \supset \lambda,$$

(iii) $\left\langle \widetilde{\widetilde{0}}, \bar{\mathbf{1}}, 0 \right\rangle \neq \mathcal{A}$ is called a 3-octahedron subgroupoid in X, if $\mathcal{A} \circ_3 \mathcal{A} \subset_3 \mathcal{A}$, i.e.,

$$\mathbf{A} \circ_{IVI} \mathbf{A} \subset \mathbf{A},\ A \circ_3 A \supset A,\ \lambda \circ_F \lambda \subset \lambda,$$

(iv) $\left\langle \widetilde{\widetilde{0}}, \bar{\mathbf{1}}, 1 \right\rangle \neq \mathcal{A}$ is called a 4-octahedron subgroupoid in X, if $\mathcal{A} \circ_4 \mathcal{A} \subset_4 \mathcal{A}$, i.e.,

$$\mathbf{A} \circ_{IVI} \mathbf{A} \subset \mathbf{A},\ A \circ_3 A \supset A,\ \lambda \circ_2 \lambda \supset \lambda.$$

We will denote the set of all i-IVIGPs in X as $IVIOGP_i(X)$ $(i = 1,\ 2,\ 3,\ 4)$.

In (i), if $\mathbf{A} \circ_{IVI} \mathbf{A} \subset \mathbf{A}$, then $\mathbf{A}$ will be called an interval-valued intuionistic fuzzy subgroupoid (briefly, IVIGP) in X. We will denote the set of all IVIGPs in X as $IVIGP(X)$.

Let us denote the set of all fuzzy [resp. intuitionistic fuzzy] subgroupoids in a groupoid X in the sense of Liu [18] [resp. Hur et al. [9]] as $FGP(X)$ [resp. $IFGP(X)$].

Remark 6.11. Let $(X, \cdot)$ be a groupoid and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in IVIO(X)$. Then

(1) $\mathcal{A} \in IVIOGP_1(X)$ if and only if $\mathbf{A} \in IVIGP(X),\ A \in IFGP(X),\ \lambda \in FGP(X)$,

(2) $\mathcal{A} \in IVIOGP_2(X)$ if and only if $\mathbf{A} \in IVIGP(X),\ A \in IFGP(X),\ \lambda \circ_2 \lambda \supset \lambda$,

(3) $\mathcal{A} \in IVIOGP_3(X)$ if and only if $\mathbf{A} \in IVIGP(X),\ A \circ_3 A \supset A,\ \lambda \in FGP(X)$,

(4) $\mathcal{A} \in IVIOGP_3(X)$ if and only if $\mathbf{A} \in IVIGP(X),\ A \circ_3 A \supset A,\ \lambda \circ_2 \lambda \supset \lambda$.

Example 6.12. (1) Let $(x, \cdot)$ be the subgroupoid and let $\mathcal{A}$ be the IVI-octahedron set in X given in Example 6.7. Then we can easily calculate that

$$(\mathbf{A} \circ_{IVI} \mathbf{A})(a) = ([0.4, 0.7], [0.1, 0.3]) \not\leq ([0.3, 0.6], [0.2, 0.3]) = \mathbf{A}(a),$$

$$(\lambda \circ_2 \lambda)(a) = 0.3 \not\geq 0.5 = \lambda(a),$$

$$(A \circ_3 A)(a) = (0.5, 0.4) \not\geq (0.7, 0.2) = A(a).$$

Thus $\mathcal{A} \notin IVIOGP_i(X)$, for $i = 1,\ 2,\ 3,\ 4$.

(2) Let $X = \{a, b, c\}$ be the groupoid with the following Cayley table:

$\cdot$	a	b	c
a	a	a	a
b	b	a	b
c	c	c	c

Table 6.4

Consider the IVI-octahedron sets $\mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \mathcal{D}$ in X defined as follows:

$$\mathcal{A}(a) = \langle ([0.3, 0.6], [0.2, 0.3]), (0.7, 0.2), 0.8 \rangle\,,$$

$$\mathcal{A}(b) = \langle([0.2, 0.4], [0.4, 0.5]), (0.6, 0.3), 0.7\rangle\,,$$
$$\mathcal{A}(c) = \langle([0.4, 0.7], [0.1, 0.2]), (0.5, 0.4), 0.6\rangle\,,$$
$$\mathcal{B}(a) = \langle([0.3, 0.6], [0.2, 0.3]), (0.7, 0.2), 0.6\rangle\,,$$
$$\mathcal{B}(b) = \langle([0.2, 0.4], [0.4, 0.5]), (0.6, 0.3), 0.7\rangle\,,$$
$$\mathcal{B}(c) = \langle([0.4, 0.7], [0.1, 0.2]), (0.5, 0.4), 0.8\rangle\,,$$
$$\mathcal{C}(a) = \langle([0.3, 0.6], [0.2, 0.3]), (0.5, 0.4), 0.8\rangle\,,$$
$$\mathcal{C}(b) = \langle([0.2, 0.4], [0.4, 0.5]), (0.6, 0.3), 0.7\rangle\,,$$
$$\mathcal{C}(c) = \langle([0.4, 0.7], [0.1, 0.2]), (0.7, 0.2), 0.6\rangle\,,$$
$$\mathcal{D}(a) = \langle([0.3, 0.6], [0.2, 0.3]), (0.5, 0.4), 0.6\rangle\,,$$
$$\mathcal{D}(b) = \langle([0.2, 0.4], [0.4, 0.5]), (0.6, 0.3), 0.7\rangle\,,$$
$$\mathcal{D}(c) = \langle([0.4, 0.7], [0.1, 0.2]), (0.7, 0.2), 0.8\rangle\,.$$

Then we can easily see that $\mathcal{A} \in IVIOGP_1(X)$, $\mathcal{B} \in IVIOGP_2(X)$, $\mathcal{C} \in IVIOGP_3(X)$ and $\mathcal{D} \in IVIOGP_4(X)$.

(3) Let $(X, \cdot)$ be a groupoid and let $\mathbf{A} \in IVIGP(X)$. Then clearly, $\mathcal{IO}_{\mathbf{A}} \in IVIOGP_1(X)$, where $\mathcal{IO}_{\mathbf{A}}$ is the IVI-octahedron set in X induced by $\mathbf{A}$ (See Example 3.2 (3)).

(4) Let $(X, \cdot)$ be a groupoid and let $A \in IFGP(X)$. Then clearly, $\mathcal{IO}_A \in IVIOGP_1(X)$, where $\mathcal{IO}_A$ is the IVI-octahedron set in X induced by A (See Example 3.2 (4)).

The followings are immediate results of Definitions 6.5, 6.10, Proposition 6.9 (1) and Remark 6.11 (1).

Proposition 6.13. *Let $(X, \cdot)$ be a groupoid and let $\ddot{0} \neq \mathcal{A} = \langle \mathbf{A}, A, \lambda\rangle \in IVIO(X)$. Then the followings are equivalent:*

(1) $\mathcal{A} \in IVIOGP_1(X)$,

(2) *for any* $x_{\tilde{\tilde{a}}},\ y_{\tilde{\tilde{b}}} \in_1 \mathcal{A}$, $x_{\tilde{\tilde{a}}} \circ_1 y_{\tilde{\tilde{b}}} \in_1 \mathcal{A}$, *i.e.,* $(\mathcal{A}, \circ_1)$ *is a groupoid,*

(3) *for any* $x,\ y \in X$, $\mathcal{A}(xy) \geq_1 \mathcal{A}(x) \wedge^1 \mathcal{A}(y)$, *i.e.,*

 (i) $A^{\in,-}(xy) \geq A^{\in,-}(x) \wedge A^{\in,-}(y)$, $A^{\in,+}(xy) \geq A^{\in,+}(x) \wedge A^{\in,+}(y)$,
 $A^{\notin,-}(xy) \leq A^{\notin,-}(x) \vee A^{\notin,-}(y)$, $A^{\notin,+}(xy) \leq A^{\notin,+}(x) \vee A^{\notin,+}(y)$,

 (ii) $A^{\in}(xy) \geq A^{\in}(x) \wedge A^{\in}(y)$, $A^{\notin}(xy) \leq A^{\notin}(x) \vee A^{\notin}(y)$,

 (iii) $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$.

In fact, from the above Proposition, we can easily see that $\mathcal{A} \in IVIOGP_1(X)$ if and only if $\mathbf{A} \in IVIGP(X)$, $A \in IFGP(X)$ and $\lambda \in FGP(X)$.

From Remark 6.11 (1) and the above Proposition, it is obvious that $(\mathcal{A}, \circ_1)$ is a groupoid if and only if $(\mathbf{A}, \circ_{IVI})$, $(A, \circ_{IF})$ and $(\lambda, \circ_F)$ are groupoids.

Proposition 6.14. *Let $(X, \cdot)$ be a groupoid and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda\rangle \in IVIOGP_1(X)$.*

(1) *If "$\cdot$" is associative in X, then so is "$\circ_1$" in $\mathcal{A}$, i.e., for any* $x_{\tilde{\tilde{a}}},\ y_{\tilde{\tilde{b}}},\ z_{\tilde{\tilde{c}}} \in_1 \mathcal{A}$,

$$(x_{\tilde{\tilde{a}}} \circ_1 y_{\tilde{\tilde{b}}}) \circ_1 z_{\tilde{\tilde{c}}} = x_{\tilde{\tilde{a}}} \circ_1 (y_{\tilde{\tilde{b}}} \circ_1 z_{\tilde{\tilde{c}}}),$$

(2) *If "$\cdot$" is commutative in X, then so is "$\circ_1$" in $\mathcal{A}$, i.e., for any* $x_{\tilde{\tilde{a}}},\ y_{\tilde{\tilde{b}}} \in_1 \mathcal{A}$,

$$x_{\tilde{\tilde{a}}} \circ_1 y_{\tilde{\tilde{b}}} = y_{\tilde{\tilde{b}}} \circ_1 x_{\tilde{\tilde{a}}},$$

(3) *If "$\cdot$" has an identity $e \in X$, then for each $x_{\tilde{\tilde{a}}} \in_1 \mathcal{A}$,*

$$e_{\ddot{1}} \circ_1 x_{\tilde{\tilde{a}}} = x_{\tilde{\tilde{a}}} = x_{\tilde{\tilde{a}}} \circ_1 e_{\ddot{1}}.$$

The followings are immediate results of Definitions 6.5, 6.10, Proposition 6.9 (2) and Remark 6.11 (2).

Proposition 6.15. *Let $(X, \cdot)$ be a groupoid and let $\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle \neq \mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in IVIO(X)$. Then the followings are equivalent:*

(1) $\mathcal{A} \in IVIOGP_2(X)$,

(2) *for any $x_{\tilde{\tilde{a}}},\ y_{\tilde{\tilde{b}}} \in_2 \mathcal{A}$, $x_{\tilde{\tilde{a}}} \circ_2 y_{\tilde{\tilde{b}}} \in_2 \mathcal{A}$, i.e., $(\mathcal{A}, \circ_2)$ is a groupoid,*

(3) *for any $x,\ y \in X$, $\mathcal{A}(xy) \geq_2 \mathcal{A}(x) \wedge^2 \mathcal{A}(y)$, i.e.,*

(i) $A^{\in,-}(xy) \geq A^{\in,-}(x) \wedge A^{\in,-}(y),\ A^{\in,+}(xy) \geq A^{\in,+}(x) \wedge A^{\in,+}(y),$
$A^{\notin,-}(xy) \leq A^{\notin,-}(x) \vee A^{\notin,-}(y),\ A^{\notin,+}(xy) \leq A^{\notin,+}(x) \vee A^{\notin,+}(y),$

(ii) $A^{\in}(xy) \geq A^{\in}(x) \wedge A^{\in}(y),\ A^{\notin}(xy) \leq A^{\notin}(x) \vee A^{\notin}(y),$

(iii) $\lambda(xy) \leq \lambda(x) \vee \lambda(y)$.

In fact, from the above Proposition, it is obvious that $\mathcal{A} \in IVIOGP_2(X)$ if and only if $\mathcal{A} \in IVIOGP_1(X)$ if and only if $\mathbf{A} \in IVIGP(X)$, $A \in IFGP(X)$ and λ satisfies the condition (iii).

Proposition 6.16. *Let $(X, \cdot)$ be a groupoid and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in IVIOGP_2(X)$.*

(1) *If "$\cdot$" is associative in X, then so is "$\circ_2$" in $\mathcal{A}$, i.e., for any $x_{\tilde{\tilde{a}}},\ y_{\tilde{\tilde{b}}},\ z_{\tilde{\tilde{c}}} \in_2 \mathcal{A}$,*

$$(x_{\tilde{\tilde{a}}} \circ_2 y_{\tilde{\tilde{b}}}) \circ_2 z_{\tilde{\tilde{c}}} = x_{\tilde{\tilde{a}}} \circ_2 (y_{\tilde{\tilde{b}}} \circ_2 z_{\tilde{\tilde{c}}}),$$

(2) *If "$\cdot$" is commutative in X, then so is "$\circ_2$" in $\mathcal{A}$, i.e., for any $x_{\tilde{\tilde{a}}},\ y_{\tilde{\tilde{b}}} \in_2 \mathcal{A}$,*

$$x_{\tilde{\tilde{a}}} \circ_2 y_{\tilde{\tilde{b}}} = y_{\tilde{\tilde{b}}} \circ_2 x_{\tilde{\tilde{a}}},$$

(3) *If "$\cdot$" has an identity $e \in X$, then for each $x_{\tilde{\tilde{a}}} \in_2 \mathcal{A}$,*

$$e_{\langle \tilde{1}, \bar{\mathbf{1}}, 0 \rangle} \circ_2 x_{\tilde{\tilde{a}}} = x_{\tilde{\tilde{a}}} = x_{\tilde{\tilde{a}}} \circ_2 e_{\langle \tilde{1}, \bar{\mathbf{1}}, 0 \rangle}.$$

The followings are immediate results of Definitions 6.5, 6.10, Proposition 6.9 (3) and Remark 6.11 (3).

Proposition 6.17. *Let $(X, \cdot)$ be a groupoid and let $\left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle \neq \mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in IVIO(X)$. Then the followings are equivalent:*

(1) $\mathcal{A} \in IVIOGP_3(X)$,

(2) *for any $x_{\tilde{\tilde{a}}},\ y_{\tilde{\tilde{b}}} \in_3 \mathcal{A}$, $x_{\tilde{\tilde{a}}} \circ_3 y_{\tilde{\tilde{b}}} \in_3 \mathcal{A}$, i.e., $(\mathcal{A}, \circ_3)$ is a groupoid,*

(3) *for any $x,\ y \in X$, $\mathcal{A}(xy) \geq_3 \mathcal{A}(x) \wedge^3 \mathcal{A}(y)$, i.e.,*

(i) $A^{\in,-}(xy) \geq A^{\in,-}(x) \wedge A^{\in,-}(y),\ A^{\in,+}(xy) \geq A^{\in,+}(x) \wedge A^{\in,+}(y),$
$A^{\notin,-}(xy) \leq A^{\notin,-}(x) \vee A^{\notin,-}(y),\ A^{\notin,+}(xy) \leq A^{\notin,+}(x) \vee A^{\notin,+}(y),$

(ii) $A^{\in}(xy) \leq A^{\in}(x) \vee A^{\in}(y),\ A^{\notin}(xy) \geq A^{\notin}(x) \wedge A^{\notin}(y),$

(iii) $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$.

In fact, from the above Proposition, it is obvious that $\mathcal{A} \in IVIOGP_3(X)$ if and only if $\mathcal{A} \in IVIOGP_1(X)$ if and only if $\mathbf{A} \in IVIGP(X)$, $\lambda \in IGP(X)$ and A satisfies the condition (ii).

Proposition 6.18. *Let $(X, \cdot)$ be a groupoid and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in IVIOGP_3(X)$.*
(1) *If "$\cdot$" is associative in X, then so is "$\circ_3$" in $\mathcal{A}$, i.e., for any $x_{\widetilde{\bar{a}}}$, $y_{\widetilde{\bar{b}}}$, $z_{\widetilde{\bar{c}}} \in_3 \mathcal{A}$,*

$$(x_{\widetilde{\bar{a}}} \circ_3 y_{\widetilde{\bar{b}}}) \circ_3 z_{\widetilde{\bar{c}}} = x_{\widetilde{\bar{a}}} \circ_3 (y_{\widetilde{\bar{b}}} \circ_3 z_{\widetilde{\bar{c}}}),$$

(2) *If "$\cdot$" is commutative in X, then so is "$\circ_3$" in $\mathcal{A}$, i.e., for any $x_{\widetilde{\bar{a}}}$, $y_{\widetilde{\bar{b}}} \in_3 \mathcal{A}$,*

$$x_{\widetilde{\bar{a}}} \circ_3 y_{\widetilde{\bar{b}}} = y_{\widetilde{\bar{b}}} \circ_3 x_{\widetilde{\bar{a}}},$$

(3) *If "$\cdot$" has an identity $e \in X$, then for each $x_{\widetilde{\bar{a}}} \in_3 \mathcal{A}$,*

$$e_{\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \rangle} \circ_3 x_{\widetilde{\bar{a}}} = x_{\widetilde{\bar{a}}} = x_{\widetilde{\bar{a}}} \circ_3 e_{\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \rangle}.$$

The followings are immediate results of Definitions 6.5, 6.10, Proposition 6.9 (4) and Remark 6.11 (4).

Proposition 6.19. *Let $(X, \cdot)$ be a groupoid and let $\left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle \neq \mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in IVIO(X)$. Then the followings are equivalent:*
(1) $\mathcal{A} \in IVIOGP_4(X)$,
(2) *for any $x_{\widetilde{\bar{a}}}$, $y_{\widetilde{\bar{b}}} \in_4 \mathcal{A}$, $x_{\widetilde{\bar{a}}} \circ_4 y_{\widetilde{\bar{b}}} \in_4 \mathcal{A}$, i.e., $(\mathcal{A}, \circ_4)$ is a groupoid,*
(3) *for any x, $y \in X$, $\mathcal{A}(xy) \geq_4 \mathcal{A}(x) \wedge^4 \mathcal{A}(y)$, i.e.,*
  (i) $A^{\in,-}(xy) \geq A^{\in,-}(x) \wedge A^{\in,-}(y)$, $A^{\in,+}(xy) \geq A^{\in,+}(x) \wedge A^{\in,+}(y)$,
  $A^{\notin,-}(xy) \leq A^{\notin,-}(x) \vee A^{\notin,-}(y)$, $A^{\notin,+}(xy) \leq A^{\notin,+}(x) \vee A^{\notin,+}(y)$,
  (ii) $A^{\in}(xy) \leq A^{\in}(x) \vee A^{\in}(y)$, $A^{\notin}(xy) \geq A^{\notin}(x) \wedge A^{\notin}(y)$,
  (iii) $\lambda(xy) \leq \lambda(x) \vee \lambda(y)$.

In fact, from the above Proposition, it is obvious that $\mathcal{A} \in IVIOGP_4(X)$ if and only if $\mathcal{A} \in IVIOGP_1(X)$ if and only if $\mathbf{A} \in IVIGP(X)$, A satisfies the condition (ii) and λ satisfies the condition (iii).

From Propositions 6.13, 6.15, 6.17 and 6.19, Note that for any $\mathcal{A} \in IVIOGP_i(X)$ $(i = 1,\ 2,\ 3,\ 4)$, we have: for each $x \in X$,

$$\mathcal{A}(x^n) \geq^i \mathcal{A}(x), \text{ i.e.,}$$

where x^n is any composite of x's.

Proposition 6.20. *Let $(X, \cdot)$ be a groupoid and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in IVIOGP_4(X)$.*
(1) *If "$\cdot$" is associative in X, then so is "$\circ_4$" in $\mathcal{A}$, i.e., for any $x_{\widetilde{\bar{a}}}$, $y_{\widetilde{\bar{b}}}$, $z_{\widetilde{\bar{c}}} \in_4 \mathcal{A}$,*

$$(x_{\widetilde{\bar{a}}} \circ_4 y_{\widetilde{\bar{b}}}) \circ_4 z_{\widetilde{\bar{c}}} = x_{\widetilde{\bar{a}}} \circ_4 (y_{\widetilde{\bar{b}}} \circ_4 z_{\widetilde{\bar{c}}}),$$

(2) *If "$\cdot$" is commutative in X, then so is "$\circ_4$" in $\mathcal{A}$, i.e., for any $x_{\widetilde{\bar{a}}}$, $y_{\widetilde{\bar{b}}} \in_4 \mathcal{A}$,*

$$x_{\widetilde{\bar{a}}} \circ_4 y_{\widetilde{\bar{b}}} = y_{\widetilde{\bar{b}}} \circ_4 x_{\widetilde{\bar{a}}},$$

(3) *If "$\cdot$" has an identity $e \in X$, then for each $x_{\widetilde{\bar{a}}} \in_4 \mathcal{A}$,*

$$e_{\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \rangle} \circ_3 x_{\widetilde{\bar{a}}} = x_{\widetilde{\bar{a}}} = x_{\widetilde{\bar{a}}} \circ_4 e_{\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \rangle}.$$

Remark 6.21. Let $(X, \cdot)$ be a groupoid and let $A \in 2^X$. Then we have

$$\chi_{\mathcal{A}} \in IVIOGP_1(X) \iff A \text{ is a subgroupoid of } X.$$

Definition 6.22. Let $(X, \cdot)$ be a groupoid, $\mathcal{A} \in IVIO(X)$ and let $i = 1,\ 2,\ 3,\ 4.$ Then $\mathcal{A}$ is called a:

(i) i-IVI octahedron left ideal (simply, i-IVIOLI) of X, if for any $x,\ y \in X$,

$$\mathcal{A}(xy) \geq_i \mathcal{A}(y),$$

(ii) i-IVI octahedron right ideal (simply, i-IVIORI) of X, if for any $x,\ y \in X$,

$$\mathcal{A}(xy) \geq_i \mathcal{A}(x),$$

(iii) i-IVI octahedron ideal (simply, i-IVIOI) of X, if it is both a i-IVIOLI and a i-IVIORI of X.

In this case, we will denote the set of all i-IVIOIs [resp. i-IVIOLIs and i-IVIORIs] of X as $IVIOI_i(X)$ [resp. $IVIOLI_i(X)$ and $IVIORI_i(X)$].

For a groupoid $(X, \cdot)$, let us denote the set of all fuzzy ideals [resp. left ideals and right ideals] (See [24]) and the set of all IFIs [resp. IFLIs, IFRIs] (See [9]) of X as $FI(X)$ [resp. $FLI(X)$ and $FRL(X)$].

Remark 6.23. From Definition 6.22, we have the followings:

(1) $\mathcal{A} \in IVIOLI_1(X) \Longleftrightarrow$ for any $x,\ y \in X$,

$$\mathbf{A}(xy) \geq \mathbf{A}(y),\ A(xy) \geq A(y),\ \lambda(xy) \geq \lambda(y),\ \text{i.e.,}$$

$$A^{\in,-}(xy) \geq A^{\in,-}(y),\ A^{\in,+}(xy) \geq A^{\in,+}(y), \tag{6.1}$$

$$A^{\notin,-}(xy) \leq A^{\notin,-}(y),\ A^{\notin,+}(xy) \leq A^{\notin,+}(y), \tag{6.2}$$

$$A^{\in}(xy) \geq A^{\in}(y),\ A^{\notin}(xy) \leq A^{\notin}(y), \tag{6.3}$$

$$\lambda(xy) \geq \lambda(y), \tag{6.4}$$

consequently, $\mathcal{A} \in IVIOLI_1(X) \Longleftrightarrow \mathbf{A} \in IVILI(X),\ A \in IFLI(x),\ \lambda \in FLI(X)$ (See 6.24 (2)),

$\mathcal{A} \in IVIOLI_2(X) \Longleftrightarrow \mathbf{A} \in IVILI(X),\ A \in IFLI(x)$ and it satisfies the condition 6.5,

$$\lambda(xy) \leq \lambda(y) \text{ for any } x,\ y \in X, \tag{6.5}$$

$\mathcal{A} \in IVIOLI_3(X) \Longleftrightarrow \mathbf{A} \in IVILI(X),\ \lambda \in FLI(x)$ and it satisfies the condition 6.6,

$$A^{\in}(xy) \leq A^{\in}(y),\ A^{\notin}(xy) \geq A^{\notin}(y) \text{ for any } x,\ y \in X, \tag{6.6}$$

$\mathcal{A} \in IVIOLI_4(X) \Longleftrightarrow \mathbf{A} \in IVILI(X)$, and it satisfies the conditions 6.5 and 6.6,

(2) $\mathcal{A} \in IVIORI_1(X) \Longleftrightarrow$ for any $x,\ y \in X$,

$$\mathbf{A}(xy) \geq \mathbf{A}(x),\ A(xy) \geq A(x),\ \lambda(xy) \geq \lambda(x),\ \text{i.e.,}$$

$$A^{\in,-}(xy) \geq A^{\in,-}(x),\ A^{\in,+}(xy) \geq A^{\in,+}(x), \tag{6.7}$$

$$A^{\notin,-}(xy) \leq A^{\notin,-}(x),\ A^{\notin,+}(xy) \leq A^{\notin,+}(x), \tag{6.8}$$

$$A^{\in}(xy) \geq A^{\in}(x),\ A^{\notin}(xy) \leq A^{\notin}(x), \tag{6.9}$$

$$\lambda(xy) \geq \lambda(x), \tag{6.10}$$

consequently, $\mathcal{A} \in IVIORI_1(X) \Longleftrightarrow \mathbf{A} \in IVIRI(X),\ A \in IFRI(X),\ \lambda \in FRI(X)$ (See 6.24 (2)),

$\mathcal{A} \in IVIORI_2(X) \Longleftrightarrow \mathbf{A} \in IVIRI(X),\ A \in IFRI(X)$ and it satisfies the condition 6.11,

$$\lambda(xy) \leq \lambda(x) \text{ for any } x,\ y \in X, \tag{6.11}$$

$\mathcal{A} \in IVIORI_3(X) \Longleftrightarrow \mathbf{A} \in IVIRI(X),\ A \in FRI(X)$ and it satisfies the condition 6.12,

$$A^{\in}(xy) \leq A^{\in}(x),\ A^{\notin}(xy) \geq A^{\notin}(x), \tag{6.12}$$

$\mathcal{A} \in IVIOLI_4(X) \Longleftrightarrow \mathbf{A} \in IVIRI(X),\ A \in FRI(X)$ and it satisfies the condition 6.12, it satisfies the conditions 6.11 and 6.12,

(3) $\mathcal{A} \in IVIOI_1(X) \Longleftrightarrow$ for any $x,\ y \in X$,

$$\mathbf{A}(xy) \geq \mathbf{A}(x) \vee \mathbf{A}(y),\ A(xy) \geq A(x) \vee A(y),\ \lambda(xy) \geq \lambda(x) \vee \lambda(y), \text{ i.e.,}$$

$$A^{\in,-}(xy) \geq A^{\in,-}(x) \vee A^{\in,-}(y),\ A^{\in,+}(xy) \geq A^{\in,+}(x) \vee A^{\in,+}(y), \tag{6.13}$$

$$A^{\notin,-}(xy) \leq A^{\notin,-}(x) \wedge A^{\notin,-}(y),\ A^{\notin,+}(xy) \leq A^{\notin,+}(x) \wedge A^{\notin,+}(y), \tag{6.14}$$

$$A^{\in}(xy) \geq A^{\in}(x) \vee A^{\in}(y),\ A^{\notin}(xy) \leq A^{\notin}(x) \wedge A^{\notin}(x), \tag{6.15}$$

$$\lambda(xy) \geq \lambda(x) \vee \lambda(y), \tag{6.16}$$

consequently, $\mathcal{A} \in IVIORI_1(X) \Longleftrightarrow \mathbf{A} \in IVII(X),\ A \in IFI(X),\ \lambda \in FI(X)$ (See 6.24 (2)),

$\mathcal{A} \in IVIOI_2(X) \Longleftrightarrow \mathbf{A} \in IVII(X),\ A \in IFI(X)$ and it satisfies the condition 6.17,

$$\lambda(xy) \leq \lambda(x) \wedge \lambda(y), \tag{6.17}$$

$\mathcal{A} \in IVIOI_3(X) \Longleftrightarrow \mathbf{A} \in IVII(X),\ \lambda \in IFI(X)$ and it satisfies the condition 6.18,

$$A^{\in}(xy) \leq A^{\in}(x) \wedge A^{\in}(y),\ A^{\notin}(xy) \geq A^{\notin}(x) \vee A^{\notin}(x), \tag{6.18}$$

$\mathcal{A} \in IVIOLI_4(X) \Longleftrightarrow \mathbf{A} \in IVII(X)$, and it satisfies the conditions 6.17 and 6.18,

Remark 6.24. (1) A i-IVIOLI [resp. IVIORI and IVIOI] in a semigroup S, a group G and a ring G is defined as Definition 6.22.

(2) An interval-valued intuitionistic fuzzy set $\mathbf{A}$ in a groupoid $(X, \cdot)$ satisfying the conditions 6.1, 6.2 [resp. 6.7, 6.8 and 6.13, 6.14] will be called an interval-valued intuitionistic fuzzy left ideal (briefly, IVILI) [resp. right ideal (briefly, IVIRI) and ideal (briefly, IVII)] in X and the set of all IVILIs [resp. IVIRIs and IVIIs] in X as $IVILI(X)$ [resp. $IVIRI(X)$ and $IVII(X)$].

(3) It is obvious that $\mathcal{A} \in IVIOGP_i(X)$, for each $\mathcal{A} \in IVIOI_i(X)$ [resp. $IVIOLI_i(X)$ and $IVIORI_i(X)$] $(i = 1,\ 2,\ 3,\ 4)$ but the converse is not true in general (See Example 6.25 (1)).

Example 6.25. (1) Let $(X, \cdot)$ be the groupoid and $\mathcal{A} \in IVIOGP_1(X)$ given in Example 6.12 (2). Then clearly, $\lambda(ab) = 0.5 \ngeq 0.7 = \lambda(b)$. Thus $\lambda \notin FLI(X)$. So $\mathcal{A} \notin IVIOLI_1(X)$.

(2) Let $X = \{a, b, c\}$ be the groupoid with the following Cayley table:

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	b	c

Table 6.5

Consider two IVI-octahedron sets $\mathcal{A}$ and $\mathcal{B}$ in X given by:

$$\mathcal{A}(a) = \langle ([0.4, 0.8], [0.1, 0.2]), (0.5, 0.2), 0.8 \rangle\,,$$
$$\mathcal{A}(b) = \langle ([0.3, 0.7], [0.2, 0.2]), (0.6, 0.3), 0.7 \rangle\,,$$
$$\mathcal{A}(c) = \langle ([0.2, 0.6], [0.2, 0.3]), (0.7, 0.4), 0.5 \rangle\,,$$
$$\mathcal{B}(a) = \langle ([0.4, 0.8], [0.1, 0.2]), (0.5, 0.2), 0.5 \rangle\,,$$
$$\mathcal{B}(b) = \langle ([0.3, 0.7], [0.2, 0.2]), (0.6, 0.3), 0.7 \rangle\,,$$
$$\mathcal{B}(c) = \langle ([0.2, 0.6], [0.2, 0.3]), (0.7, 0.4), 0.8 \rangle\,.$$

Then we can easily calculate that $\mathcal{A} \in IVIOLI_1(X)$ and $\mathcal{B} \in IVIOLI_2(X)$. But $A^{\in,-}(bc) = B^{\in,-}(bc) = 0.2 \ngeq 0.3 = A^{\in,-}(b) = B^{\in,-}(b)$. Thus $\mathbf{A},\ \mathbf{B} \notin IVIRI(X)$. So $\mathcal{A} \notin IVIORI_1(X)$ and $\mathcal{B} \notin IVIORI_2(X)$.

(3) Let $X = \{a, b, c\}$ be the groupoid with the following Cayley table:

$\cdot$	a	b	c
a	a	a	a
b	b	b	a
c	c	a	c

Table 6.6

Consider two IVI-octahedron sets $\mathcal{A}$ and $\mathcal{B}$ in X given by:

$$\mathcal{A}(a) = \langle ([0.4, 0.8], [0.1, 0.2]), (0.7, 0.2), 0.9 \rangle\,,$$

$$\mathcal{A}(b) = \langle([0.3, 0.7], [0.2, 0.2]), (0.6, 0.3), 0.7\rangle,$$
$$\mathcal{A}(c) = \langle([0.2, 0.6], [0.2, 0.3]), (0.5, 0.4), 0.8\rangle,$$
$$\mathcal{B}(a) = \langle([0.4, 0.8], [0.1, 0.2]), (0.7, 0.2), 0.9\rangle,$$
$$\mathcal{B}(b) = \langle([0.3, 0.7], [0.2, 0.2]), (0.6, 0.3), 0.7\rangle,$$
$$\mathcal{B}(c) = \langle([0.2, 0.6], [0.2, 0.3]), (0.5, 0.4), 0.8\rangle.$$

Then we can easily calculate that $\mathcal{A} \in IVIORI_1(X)$ and $\mathcal{B} \in IVIORI_2(X)$. But $A^{\in,-}(ba) = B^{\in,-}(ba) = 0.3 \not\geq 0.4 = A^{\in,-}(a) = B^{\in,-}(a)$. Thus $\mathbf{A}$, $\mathbf{B} \notin IVIOLI(X)$. So $\mathcal{A} \notin IVIOLI_1(X)$ and $\mathcal{B} \notin IVIOLI_2(X)$.

From Proposition 3.2 in [24], we have the following result.

Theorem 6.26. *Let $(X, \cdot)$ be a groupoid and let $A \in 2^X$. Then $\chi_{\mathcal{A}} \in IVIOLI_1(X)$ [resp. $IVIORI_1(X)$ and $IVIOI_1(X)$] if and only if A is a left ideal [resp. a right ideal and an ideal] of X.*

The following is an immediate result of Proposition 6.13 and Definitions 6.10 and 6.22.

Proposition 6.27. *Let $(X, \cdot)$ be a groupoid and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda\rangle \in IVIO(X)$. If $\mathcal{A} \in IVIOGP_1(X)$ or $\mathcal{A} \in IVIOLI_1(X)$ [resp. $IVIORI_1(X)$ and $IVIOI_1(X)$], then $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}}$ is a subgroupoid or a left ideal [resp. a right ideal and an ideal] of X, for each $\tilde{\tilde{\tilde{a}}} \in ([I] \oplus [I]) \times (I \oplus I) \times I$.*

Proposition 6.28. *Let $(X, \cdot)$ be a groupoid. If $(\mathcal{A}_j)_{j\in J} = (\langle \mathbf{A}_j, A_j, \lambda_j\rangle)_{j\in J} \subset IVIOGP_i(X)$, then $\bigcap^i_{j\in J} \mathcal{A}_j \in IVIOGP_i(X)$, where J denotes an index set and $j - 1,\ 2,\ 3,\ 4$.*

Proof. Case 1: Let $i = 1$. Then it is obvious that $\bigcap_{j\in J} A_j \in IFGP(X)$ by Proposition 3.9 in [9] and $\bigcap_{j\in J} \lambda_j \in FGP(X)$ by Proposition 3.1 in [24]. Thus it is sufficient to prove that $\bigcap_{j\in J} \mathbf{A}_j \in IVGP(X)$. Let $x,\ y \in X$. Since $\mathbf{A}_j \in IVGP(X)$ for each $j \in J$, we have

$$A^{\in,-}_j(xy) \geq A^{\in,-}_j(x) \wedge A^{\in,-}_j(y),\ \ A^{\in,+}_j(xy) \geq A^{\in,+}_j(x) \wedge A^{\in,+}_j(y),$$
$$A^{\notin,-}_j(xy) \leq A^{\notin,-}_j(x) \vee A^{\notin,-}_j(y),\ \ A^{\notin,+}_j(xy) \leq A^{\notin,+}_j(x) \vee A^{\notin,+}_j(y),$$

for each $j \in J$. Thus

$$\bigwedge_{j\in J} A^{\in,-}_j(xy) \geq \bigwedge_{j\in J} (A^{\in,-}_j(x) \wedge A^{\in,-}_j(y)) = \bigwedge_{j\in J} A^{\notin,-}_j(x) \wedge \bigwedge_{j\in J} A^{\notin,-}_j(y).$$

Similarly, we have

$$\bigwedge_{j\in J} A^{\in,+}_j(xy) \geq \bigwedge_{j\in J} A^{\in,+}_j(x) \wedge \bigwedge_{j\in J} A^{\in,+}_j(y),$$
$$\bigvee_{j\in J} A^{\notin,-}_j(xy) \leq \bigvee_{j\in J} A^{\notin,-}_j(x) \vee \bigvee_{j\in J} A^{\notin,+}_j(y),$$
$$\bigvee_{j\in J} A^{\notin,+}_j(xy) \leq \bigvee_{j\in J} A^{\notin,+}_j(x) \vee \bigvee_{j\in J} A^{\notin,+}_j(y).$$

So $\bigcap_{j\in J} \mathbf{A}_j \in IVGP(X)$. Hence by Proposition 6.13, $\bigcap^1_{j\in J} \mathcal{A}_j \in IVIOGP_1(X)$.

Case 2: Since $\bigcap_{j\in J}\mathbf{A}_j \in IVGP(X)$ and $\bigcap_{j\in J}A_j \in IFGP(X)$ from Case 1, it is sufficient to show that $\bigvee_{j\in J}\lambda_j(xy) \leq \bigvee_{j\in J}\lambda_j(x)\vee\bigvee_{j\in J}\lambda_j(y)$, for any $x,\ y\in X$. Since $\mathcal{A}_j \in IVIOGP_2(X)$ for each $j\in J$, by Proposition 6.15 (iii),

$$\lambda_j(xy) \leq \lambda_j(x)\vee\lambda_j(y) \text{ for each } j\in J.$$

Then $\bigvee_{j\in J}\lambda_j(xy) \leq \bigvee_{j\in J}(\lambda_j(x)\vee\lambda_j(y)) = \bigvee_{j\in J}\lambda_j(x)\vee\bigvee_{j\in J}\lambda_j(y)$. Thus by Proposition 6.15, $\bigcap^2_{j\in J}\mathcal{A}_j \in IVIOGP_2(X)$.

Case 3: Since $\bigcap_{j\in J}\mathbf{A}_j \in IVGP(X)$ and $\bigcap_{j\in J}\lambda_j \in FGP(X)$ from Case 1, it is sufficient to show that for any $x,\ y\in X$,

$$\bigvee_{j\in J}A^{\in}_j(xy) \leq \bigvee_{j\in J}A^{\in}_j(x)\vee\bigvee_{j\in J}A^{\in}_j(y),\ \bigwedge_{j\in J}A^{\notin}_j(xy) \geq \bigwedge_{j\in J}A^{\notin}_j(x)\wedge\bigwedge_{j\in J}A^{\notin}_j(y). \tag{6.19}$$

Since $\mathcal{A}_j \in IVIOGP_3(X)$ for each $j\in J$, by Proposition 6.17 (iii),

$$A^{\in}_j(xy) \leq A^{\in}_j(x)\vee A^{\in}_j(y),\ A^{\notin}_j(xy) \geq A^{\notin}_j(x)\wedge A^{\notin}_j(y) \text{ for each } j\in J.$$

Then we can easily see that (6.19) holds. Thus by Proposition 6.17, $\bigcap^3_{j\in J}\mathcal{A}_j \in IVIOGP_3(X)$.

Case 4: The proof follows from Cases 1, 2 and 3. □

Remark 6.29. For any $\mathcal{A},\ \mathcal{B}\in IVIOGP_1(X)$, $\mathcal{A}\cup^1\mathcal{B}\notin IVIOGP_1(X)$ in general.

Example 6.30. Let $(X,\cdot)$ be the groupoid and $\mathcal{A}\in OGP(X)$ given in Example 6.12 (2). Consider the octahedron subgroupoid in X given by:

$$\mathcal{B}(a)=\mathcal{B}(b)=\mathcal{B}(c)=\langle([0.1,0.7],[0.1,0.2]),(0.5,0.4),0.6\rangle\,.$$

Then $(A\cup B)(ab)=(0.7,0.4)\ngeq(0.6,0.3)=(A\cup B)(a)\wedge(A\cup B)(b)$. Thus $A\cup B\notin IFGP(X)$. So $\mathcal{A}\cup\mathcal{B}\notin IVIOGP_1(X)$.

Remark 6.31. Let $(X,\cdot)$ be a groupoid and let $(\mathcal{A}_j)_{j\in J}=(\langle\mathbf{A}_j,A_j,\lambda_j\rangle)_{j\in J}\subset IVIOGP_1(X)$. Then from Proposition 6.28, we can easily see that

$$\bigcap^1\{\mathcal{A}\in IVIOGP_1(X):\bigcup^1_{j\in J}\mathcal{A}_j\subset_1\mathcal{A}\}\in IVIOGP_1(X).$$

In this case, we will denote $\bigcap^1\{\mathcal{A}\in IVIOGP_1(X):\bigcup^1_{j\in J}\mathcal{A}_j\subset_1\mathcal{A}\}$ as $\bigvee_{j\in J}\mathcal{A}_j$.

It is obvious that $(IVIOGP_1(X),\subset_1)$ is a complete lattice with the least element $\ddot{0}$ and the greatest element $\ddot{1}$, where for each $(\mathcal{A}_j)_{j\in J}\subset IVIOGP_1(X)$, the inf and the sup of $(\mathcal{A}_j)_{j\in J}$ are $inf_{j\in J}\mathcal{A}_j=\bigcap^1_{j\in J}\mathcal{A}_j$ and $sup_{j\in J}\mathcal{A}_j=\bigvee_{j\in J}\mathcal{A}_j$.

The following is an immediate result of Proposition 6.28.

Corollary 6.32. *Let $(X,\cdot)$ be the groupoid, $\mathcal{A}\in IVIO(X)$ and let*

$$(\mathcal{A})=\bigcap^1\{\mathcal{B}\in IVIOGP_1(X):\mathcal{A}\subset_1\mathcal{B}\}.$$

Then $(\mathcal{A})\in IVIOGP_1(X)$.

In this case, $(\mathcal{A})$ is called the interval-valued intuitionistic octahedron subgroupoid in X generated by $\mathcal{A}$.

Proposition 6.33. *Let $(X, \cdot)$ be a groupoid, and let (A) be the subgroupoid generated by A and $\chi_{(A)} = \left\langle [\chi_{(A)}, \chi_{(A)}], (\chi_{(A)}, \chi_{(A^c)}), \chi_{(A)} \right\rangle$ for each $A \in 2^X$. Then*

$$(\chi_{\mathcal{A}}) = \chi_{(A)}.$$

Proof. From Remark 6.21 and Corollary 6.32, it is obvious that $\chi_{(A)} \in IVIOGP_1(X)$. Let $\mathcal{B} \in IVIOGP_1(X)$ such that $\mathcal{B} \supset_1 \chi_{\mathcal{A}}$. Then clearly,

$$\mathcal{B}(x) = \langle [1, 1], (1, 0), 1 \rangle \text{, for each } x \in A.$$

Since $\mathcal{B} \in IVIOGP_1(X)$, $\mathcal{B}(xy) = \langle [1, 1], (1, 0), 1 \rangle$ for any $x,\ y \in A$. Thus $\mathcal{B} \supset_1 \chi_{(\mathcal{A})}$. So

$$\chi_{(\mathcal{A})} \subset_1 \bigcap \{\mathcal{B} \in IVIOGP_1(X) : \mathcal{B} \supset_1 \chi_{\mathcal{A}}\} = (\chi_{\mathcal{A}}).$$

We can easily prove that $(\chi_{\mathcal{A}}) \subset_1 \chi_{(\mathcal{A})}$. Hence $(\chi_{\mathcal{A}}) = \chi_{(A)}$. □

From the above Proposition, the subgoupoid lattice of X can be regarded as a sublattice of the interval-valued intuitionistic octahedron subgroupoid lattice of X.

Proposition 6.34. *Let $(X, \cdot)$ be a groupoid. Then the i-intersection and the i-union of any i-IVIOLIs [resp. i-IVIORIs and i-IVIOIs] is an i-IVIOLI [resp. i-IVIORI and i-IVIOI], for $i = 1,\ 2,\ 3,\ 4$.*

Proof. Let $(\mathcal{A}_j)_{j \in J} \subset IVIOLI(X)$ [resp. $IVIORI(X)$ and $IVIOI(X)$], where $\mathcal{A}_j = \langle \mathbf{A}_j, A_j, \lambda_j \rangle$. Let $\mathcal{A} = \bigcap^1_{j \in J} \mathcal{A}_j$, let $\mathcal{B} = \bigcup^1_{j \in J} \mathcal{A}_j$ and let us prove only that $\mathcal{A},\ \mathcal{B} \in IVIOLI_1(X)$. The remainder's proofs are omitted. Clearly,

$$\mathcal{A} = \bigcap^1_{j \in J} \mathcal{A}_j = \left\langle \bigcap_{j \in J} \mathbf{A}_j, \bigcap_{j \in J} A_j, \bigcap_{j \in J} \lambda_j \right\rangle$$

and

$$\mathcal{B} = \bigcup^1_{j \in J} \mathcal{A}_j = \left\langle \bigcup_{j \in J} \mathbf{A}_j, \bigcup_{j \in J} A_j, \bigcup_{j \in J} \lambda_j \right\rangle.$$

Then from Proposition 3.3 in [24] and 3.10 in [9],

$$\bigcap_{j \in J} A_j \in FLI(X),\ \bigcap_{j \in J} A_j \in IFLI(X)$$

and

$$\bigcup_{j \in J} A_j \in FLI(X),\ \bigcup_{j \in J} A_j \in IFLI(X).$$

Now we prove that $\bigcap_{j \in J} \mathbf{A}_j \in IVLI(X)$ and $\bigcup_{j \in J} \mathbf{A}_j \in IVLI(X)$. Let $x,\ y \in X$. Then

$$\begin{aligned}
(\textstyle\bigcap_{j \in J} \mathbf{A}_j)^{\in}(xy) &= [\textstyle\bigwedge_{j \in J} A_j^{\in,-}(xy), \bigwedge_{j \in J} A_j^{\in,+}(xy)] \\
&\geq [\textstyle\bigwedge_{j \in J} A_j^{\in,-}(y), \bigwedge_{j \in J} A_j^{\in,+}(y)] \text{ [Since } \mathbf{A}_j \in IVILI(X)] \\
&= (\textstyle\bigcap_{j \in J} \mathbf{A}_j)^{\in}(y), \\
(\textstyle\bigcap_{j \in J} \mathbf{A}_j)^{\notin}(xy) &= [\textstyle\bigvee_{j \in J} A_j^{\notin,-}(xy), \bigvee_{j \in J} A_j^{\notin,+}(xy)] \\
&\leq [\textstyle\bigvee_{j \in J} A_j^{\notin,-}(y), \bigvee_{j \in J} A_j^{\notin,+}(y)] \text{ [Since } \mathbf{A}_j \in IVILI(X)] \\
&= (\textstyle\bigcap_{j \in J} \mathbf{A}_j)^{\notin}(y), \\
(\textstyle\bigcup_{j \in J} \mathbf{A}_j)^{\in}(xy) &= [\textstyle\bigvee_{j \in J} A_j^{\in,-}(xy), \bigvee_{j \in J} A_j^{\in,+}(xy)]
\end{aligned}$$

$$\begin{aligned}
&\geq [\bigvee_{j\in J} A_j^{\in,-}(y), \bigvee_{j\in J} A_j^{\in,+}(y)] \text{ [Since } \mathbf{A}_j \in IVILI(X)] \\
&= (\bigcup_{j\in J} \mathbf{A}_j)^{\in}(y), \\
(\bigcup_{j\in J} \mathbf{A}_j)^{\notin}(xy) &= [\bigwedge_{j\in J} A_j^{\notin,-}(xy), \bigwedge_{j\in J} A_j^{\notin,+}(xy)] \\
&\leq [\bigwedge_{j\in J} A_j^{\notin,-}(y), \bigwedge_{j\in J} A_j^{\notin,+}(y)] \text{ [Since } \mathbf{A}_j \in IVILI(X)] \\
&= (\bigcup_{j\in J} \mathbf{A}_j)^{\notin}(y).
\end{aligned}$$

Thus $\mathbf{A} \in IViLI(X)$ and $\mathbf{A} \in IVILI(X)$. So by Remark 6.23 (1), $\mathcal{A},\ \mathcal{B} \in IVIOLI(X)$. □

Proposition 6.35. *Let $f : X \to Y$ be a groupoid homomorphism, let $\mathcal{B} = \langle \mathbf{B}, B, \mu\rangle \in IVIO(Y)$ and let $i = 1,\ 2,\ 3,\ 4$.*

(1) *If $\mathcal{B} \in IVIOGP_i(Y)$, then $f^{-1}(\mathcal{B}) \in IVIOGP_i(X)$.*

(2) *If $\mathcal{B} \in IVIOLI_i(Y)$ [resp. $IVIORI_i(Y)$ and $IVIOI_i(Y)$], then $f^{-1}(\mathcal{B}) \in IVIOLI_i(X)$ [resp. $IVIORI_i(X)$ and $IVIOI_i(X)$].*

Proof. (1) Case 1: Let $i = 1$. From Propositions 4.1 in [24] and 4.1 (1) in [9], $f^{-1}(\mu) \in FGP(X)$ and $f^{-1}(B) \in IFGP(X)$. It is sufficient to prove that $f^{-1}(\mathbf{B}) \in IVIGP(X)$. Let $x,\ y \in X$. Then

$$\begin{aligned}
f^{-1}(\mathbf{B}^{\in})(xy) &= [(B^{\in,-}\circ f)(xy), (B^{\in,+}\circ f)(xy)] = [B^{\in,-}(f(xy)), B^{\in,+}(f(xy))] \\
&= [B^{\in,-}(f(x)f(y)), B^{\in,+}(f(x)f(y))] \\
&\quad \text{[Since } f \text{ is a groupoid homomorphism]} \\
&\geq [B^{\in,-}(f(x)) \wedge B^{\in,-}(f(y)), B^{\in,+}(f(x)) \wedge B^{\in,+}(f(y))] \\
&\quad \text{[Since } \mathbf{B} \in IVIGP(Y)] \\
&= [B^{\in,-}(f(x)), B^{\in,+}(f(x))] \wedge [B^{\in,-}(f(y)), B^{\in,+}(f(y))] \\
&= f^{-1}(\mathbf{B}^{\in})(x) \wedge f^{-1}(\mathbf{B}^{\in})(y).
\end{aligned}$$

Similarly, we can see that $f^{-1}(\mathbf{B}^{\notin})(xy) \leq f^{-1}(\mathbf{B}^{\notin})(x) \vee f^{-1}(\mathbf{B}^{\notin})(y)$. Thus $f^{-1}(B) \in IVIGP(X)$. So by Proposition 6.13, $f^{-1}(\mathcal{B}) \in IVIOGP_1(X)$.

Case 2: Let $i = 2$. By Case 1, it is clear that $f^{-1}(\mathbf{A} \in IVIGP(X)$ and $f^{-1}(A) \in IFGP(X)$. It is sufficient to show that for any $x,\ y \in X$,

$$f^{-1}(\mu)(xy) \leq f^{-1}(\mu)(x) \vee f^{-1}(\mu)(y).$$

Let $x,\ y \in X$. Then

$$\begin{aligned}
f^{-1}(\mu)(xy) &= \mu(f(xy)) \\
&= \mu(f(x)f(y)) \text{ [Since } f \text{ is a groupoid homomorphism]} \\
&\leq \mu(f(x)) \vee \mu(f(y)) \text{ [Since } \mathcal{B} \in IVIOGP_2(X)] \\
&= f^{-1}(\mu)(x) \vee f^{-1}(\mu)(y).
\end{aligned}$$

Thus by Proposition 6.15, $f^{-1}(\mathcal{B}) \in IVIOGP_2(X)$.

Case 3: Let $i = 3$. Then from Case 1, it is obvious that $f^{-1}(\mathbf{A} \in IVIGP(X)$ and $f^{-1}(\mu) \in FGP(X)$. Thus it is sufficient to prove that $f^{-1}(B)$ satisfies Proposition 6.19 (3) (ii). Let $x,\ y \in X$. Then

$$\begin{aligned}
f^{-1}(A^{\in})(xy) &= A^{\in}(f(xy)) \\
&= A^{\in}(f(x)f(y)) \text{ [Since } f \text{ is a groupoid homomorphism]} \\
&\leq A^{\in}(f(x)) \vee A^{\in}(f(y)) \text{ [Since } \mathcal{B} \in IVIOGP_3(X)] \\
&= f^{-1}(A^{\in})(x) \vee f^{-1}(A^{\in})(y).
\end{aligned}$$

Similarly, we have $f^{-1}(A^{\notin})(xy) \geq f^{-1}(A^{\notin})(x) \vee f^{-1}(A^{\notin})(y)$. Thus $f^{-1}(\mathcal{B}) \in IVIOGP_3(X)$.

Case 4: Let $i = 4$. Then the proof is obvious.

(2) Case 1: Let $i = 1$. Then from Propositions 4.1 in [24] and 4.1 (2) in [9], $f^{-1}(\mu) \in FLI(X)$ [resp. $FRI(X)$ and $FI(X)$] and $f^{-1}(B) \in IFLI(X)$ [resp.

$IFRI(X)$ and $IFI(X)$]. Thus we prove only that $f^{-1}(\mathbf{B}) \in IVILI(X)$ and the remainder's proofs are omitted. Let $x,\ y \in X$. Then

$$\begin{aligned} f^{-1}(\mathbf{B}^{\in})(xy) &= [B^{\in,-}(f(x)f(y)), B^{\in,+}(f(x)f(y))] \\ &\geq [B^{\in,-}(f(y)), B^{\in,+}(f(y))] \text{ [Since } \mathbf{B} \in IVILI(Y)] \\ &= f^{-1}(\mathbf{B}^{\in})(y). \end{aligned}$$

Similarly, we have $f^{-1}(\mathbf{B}^{\notin})(xy) \leq f^{-1}(\mathbf{B}^{\notin})(y)$. Thus $f^{-1}(\mathbf{B}) \in IVILI(X)$. So $f^{-1}(\mathcal{B}) \in IVIOLI(X)$.

Case 2: Let $i = 2$. Then by Case 1, $f^{-1}(\mathbf{B}) \in IVILI(X)$ and $f^{-1}(B) \in IFLI(X)$. Thus it is sufficient to show that λ satisfies the condition (6.11). Let $x,\ y \in X$. Then

$$\begin{aligned} f^{-1}(\mu)(xy) &= \mu(f(x)f(y)) \\ &\leq \mu(f(y)) \text{ [Since } \mathcal{B} \in IVIOLI_2(Y)] \\ &= f^{-1}(\mu)(y). \end{aligned}$$

Thus $f^{-1}(\mathcal{B}) \in IVIOLI_2(X)$.

Case 3: Let $i = 3$. Then from Case 1, it is obvious that $f^{-1}(\mathbf{B}) \in IVILI(X)$ and $f^{-1}(\mu) \in FLI(X)$. Thus it is sufficient to show that λ satisfies the condition (6.12). Let $x,\ y \in X$. Then

$$\begin{aligned} f^{-1}(B^{\in})(xy) &= B^{\in}(f(x)f(y)) \\ &\leq B^{\in}(f(y)) \text{ [Since } \mathcal{B} \in IVIOLI_3(Y)] \\ &= f^{-1}(B^{\in})(y). \end{aligned}$$

Similarly, we have $f^{-1}(B^{\notin})(xy) \geq f^{-1}(B^{\notin})(y)$. Thus $f^{-1}(\mathcal{B}) \in IVIOLI_3(X)$.

Case 4: Let $i = 4$. Then the proof is clear. □

Definition 6.36. Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in IVIO(X)$. Then we say that $\mathcal{A}$ has the i-sup-property ($i = 1,\ 2,\ 3,\ 4$), if for each $T \in 2^X$, there is $t_0 \in T$ such that

$$\mathcal{A}(t_0) = \bigvee_{t \in T}^{i} \mathcal{A}(t), \text{ i.e.,}$$

$$\mathcal{A}(t_0) = \bigvee_{t \in T}^{1} \mathcal{A}(t) = \left\langle \bigvee_{t \in T} \mathbf{A}(t), \bigvee_{t \in T} A(t), \bigvee_{t \in T} \lambda(t) \right\rangle,$$

$$\mathcal{A}(t_0) = \bigvee_{t \in T}^{2} \mathcal{A}(t) = \left\langle \bigvee_{t \in T} \mathbf{A}(t), \bigvee_{t \in T} A(t), \bigwedge_{t \in T} \lambda(t) \right\rangle,$$

$$\mathcal{A}(t_0) = \bigvee_{t \in T}^{3} \mathcal{A}(t) = \left\langle \bigvee_{t \in T} \mathbf{A}(t), \bigwedge_{t \in T} A(t), \bigvee_{t \in T} \lambda(t) \right\rangle,$$

$$\mathcal{A}(t_0) = \bigvee_{t \in T}^{4} \mathcal{A}(t) = \left\langle \bigvee_{t \in T} \mathbf{A}(t), \bigwedge_{t \in T} A(t), \bigwedge_{t \in T} \lambda(t) \right\rangle.$$

It is obvious that $\mathcal{A} \in IVIO(X)$ has the 1-sup-property if and only if $\mathbf{A}$, A and λ have the sup-property. Furthermore, if $\mathcal{A}$ takes on only finitely many values, then it has the i-sup-property ($i = 1,\ 2,\ 3,\ 4$).

Proposition 6.37. *Let $f : X \to Y$ be a groupoid homomorphism, let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in IVIO(X)$ has the i-sup-property, for $i = 1,\ 2, 3,\ 4$.*

(1) *If $\mathcal{A} \in IVIOGP_i(X)$, then $f(\mathcal{A}) \in IVIOGP_i(Y)$.*

(2) *If* $\mathcal{A} \in IVIOLI_i(X)$ *[resp.* $IVIORI_i(X)$ *and* $IVIOI_i(X)$*], then* $f(\mathcal{A}) \in IVIOLI_i(Y)$ *[resp.* $IVIORI_i(Y)$ *and* $IVIOI_i(Y)$*].*

Proof. (1) Case 1: Let $i = 1$. Then from Propositions 4.2 in [24] and 4.4 (1) in [9], $f(\mathcal{A}) \in FGP(X)$ and $f(\mathcal{A}) \in IFGP(X)$. Thus it is sufficient to prove that $f(\mathcal{A}) \in IVIGP(X)$. Let $y,\ y^{'} \in Y$. Then we can consider the followings:

(a) $f^{-1}(y) \neq \varnothing$ and $f^{-1}(y^{'}) \neq \varnothing$, (b) $f^{-1}(y) \neq \varnothing$ and $f^{-1}(y^{'})t = \varnothing$,

(c) $f^{-1}(y) = \varnothing$ and $f^{-1}(y^{'}) \neq \varnothing$, (d) $f^{-1}(y) = \varnothing$ and $f^{-1}(y^{'}) = \varnothing$.

We prove only the case (a) and omit the remainders. Since $\mathcal{A}$ has the 1-sup-property, there are $x_0 \in f^{-1}(y)$ and $x_0^{'} \in f^{-1}(y^{'})$ such that

$$\mathcal{A}(x_0) = \bigvee^{1}_{t\in f^{-1}(y)} \mathcal{A}(x) \text{ and } \mathcal{A}(x_0^{'}) = \bigvee^{1}_{t^{'}\in f^{-1}(y^{'})} \mathcal{A}(t^{'}).$$

Then

$$\begin{aligned} f(\mathbf{A}^{\in})(yy^{'}) &= [\textstyle\bigvee_{z\in f^{-1}(yy^{'})} A^{\in,-}(z), \bigvee_{z\in f^{-1}(yy^{'})} A^{\in,+}(z)] \\ &\geq [A^{\in,-}(x_0x_0^{'}), A^{\in,+}(x_0x_0^{'})] \text{ [Since } f(x_0x_0^{'}) = f(x_0)f(x_0^{'}) = yy^{'}] \\ &\geq [A^{\in,-}(x_0) \wedge A^{\in,-}(x_0^{'}), A^{\in,+}(x_0) \wedge A^{\in,+}(x_0^{'})] \\ &\quad \text{[Since } A \in IVIGP(X)] \\ &= \textstyle\bigvee_{t\in f^{-1}(y)} A^{\in}(t) \wedge \bigvee_{t^{'}\in f^{-1}(y^{'})} A^{\in}(t^{'}) \\ &= f(A^{\in})(y) \wedge f(A^{\in})(y^{'}). \end{aligned}$$

Similarly, we have $f(\mathbf{A}^{\notin})(yy^{'}) \leq f(\mathbf{A}^{\notin})(y) \vee f(\mathbf{A}^{\notin})(y^{'})$. Thus $f(\mathbf{A}) \in IVIGP(Y)$. So $f(\mathcal{A}) \in IVIOGP_1(Y)$.

Case 2: Let $i = 2$. Then by Case 1, $f(\mathbf{A}) \in IVIGP(Y)$ and $f(A) \in IFGP(Y)$. Thus is is sufficient to show that $f(\lambda)$ satisfies the condition (iii) of Proposition 6.15. Let $x,\ y \in X$. Since $\mathcal{A}$ has the 1-sup-property, there are $x_0 \in f^{-1}(y)$ and $x_0^{'} \in f^{-1}(y^{'})$ such that

$$\mathcal{A}(x_0) = \bigvee^{2}_{t\in f^{-1}(y)} \mathcal{A}(x) \text{ and } \mathcal{A}(x_0^{'}) = \bigvee^{2}_{t^{'}\in f^{-1}(y^{'})} \mathcal{A}(t^{'}).$$

Then

$$\begin{aligned} f(\lambda)(yy^{'}) &= \textstyle\bigwedge_{z\in f^{-1}(yy^{'})} \lambda(z) \text{ [Since } \mathcal{A} \in IVIOGP_2(X)] \\ &\leq \lambda(x_0x_0^{'}) \text{ [Since } f(x_0x_0^{'}) = f(x_0)f(x_0^{'}) = yy^{'}] \\ &\leq \lambda(x_0) \vee \lambda(x_0^{'}) \text{ [By Proposition 6.15 (iii)]} \\ &= \textstyle\bigwedge_{t\in f^{-1}(y)} \lambda(t) \vee \bigwedge_{t^{'}\in f^{-1}(y^{'})} \lambda(t^{'}) \\ &= f(\lambda)(y) \vee f(\lambda)(y^{'}). \end{aligned}$$

Thus $f(\mathcal{A}) \in IVIOGP_2(Y)$.

Case 3: Let $i = 2$. Then by Case 1, $f(\mathbf{A}) \in IVIGP(Y)$ and $f(\lambda) \in FGP(Y)$. Thus is is sufficient to show that $f(A)$ satisfies the condition (ii) of Proposition 6.17. Let $x,\ y \in X$. Since $\mathcal{A}$ has the 1-sup-property, there are $x_0 \in f^{-1}(y)$ and $x_0^{'} \in f^{-1}(y^{'})$ such that

$$\mathcal{A}(x_0) = \bigvee^{3}_{t\in f^{-1}(y)} \mathcal{A}(x) \text{ and } \mathcal{A}(x_0^{'}) = \bigvee^{3}_{t^{'}\in f^{-1}(y^{'})} \mathcal{A}(t^{'}).$$

Then

$$\begin{aligned} f(A^{\in})(yy^{'}) &= \bigwedge_{z\in f^{-1}(yy^{'})} A^{\in}(z) \text{ [Since } \mathcal{A} \in IVIOGP_3(X)] \\ &\leq A^{\in}(x_0x_0^{'}) \text{ [Since } f(x_0x_0^{'}) = f(x_0)f(x_0^{'}) = yy^{'}] \\ &\leq A^{\in}(x_0) \vee A^{\in}(x_0^{'}) \text{ [By Proposition 6.17 (ii)]} \\ &= \bigwedge_{t\in f^{-1}(y)} A^{\in}(t) \vee \bigwedge_{t'\in f^{-1}(y')} A^{\in}(t^{'}) \\ &= f(A^{\in})(y) \vee f(A^{\in})(y^{'}). \end{aligned}$$

Similarly, we have $f(A^{\notin})(yy^{'}) \geq f(A^{\notin})(y) \wedge f(A^{\notin})(y^{'})$. Thus $f(\mathcal{A}) \in IVIOGP_3(Y)$.

Case 4: Let $i = 4$. The the proof is obvious.

(2) Since the proof is similar to (1)l it is omitted. □

Definition 6.38. Let X, Y be sets, $f : X \to Y$ be a mapping and let $\mathcal{A} \in IVIO(X)$. Then $\mathcal{A}$ is said to be f-invariant, if for any $x,\ y \in X$, $f(x) = f(y)$ implies $\mathcal{A}(x) = \mathcal{A}(y)$.

It is obvious that $\mathcal{A}$ is f-invariant if and only if $\mathbf{A}$, A and λ are f-invariant. Moreover, we can easily see that if $\mathcal{A}$ is f-invariant, then $f^{-1}(f(\mathcal{A})) = \mathcal{A}$.

Example 6.39. Let $X = \{a, b, c\}$, $Y = \{x, y\}$ be sets and $f : X \to Y$ be the mapping defined by $f(a) = f(b) = x$ and $f(c) = y$. Consider two IVI-octahedron sets $\mathcal{A}$, $\mathcal{A}$ in X given by:

$$\mathcal{A}(a) = \langle([0.4, 0.8], [0.1, 0.2]), (0.5, 0.2), 0.8\rangle\,,$$

$$\mathcal{A}(b) = \langle([0.4, 0.8], [0.1, 0.2]), (0.5, 0.2), 0.8\rangle\,,$$

$$\mathcal{A}(c) = \langle([0.2, 0.6], [0.2, 0.3]), (0.7, 0.4), 0.5\rangle\,,$$

$$\mathcal{B}(a) = \langle([0.4, 0.8], [0.1, 0.2]), (0.5, 0.2), 0.5\rangle\,,$$

$$\mathcal{B}(b) = \langle([0.3, 0.7], [0.2, 0.2]), (0.6, 0.3), 0.7\rangle\,,$$

$$\mathcal{B}(c) = \langle([0.2, 0.6], [0.2, 0.3]), (0.7, 0.4), 0.8\rangle\,.$$

Then we can easily check that $\mathcal{A}$ is invariant but $\mathcal{B}$ is not invariant. Moreover, we can easily confirm that $f^{-1}(f(\mathcal{A})) = \mathcal{A}$.

We have the similar result to Theorem 18.4 in [23].

Proposition 6.40. *Let X, Y be sets, let $f : X \to Y$ be a mapping and let*

$$\Omega = \{\mathcal{A} \in IVIO(X) : \mathcal{A} \text{ is f-invariant}\}.$$

Then there is a one-to-one correspondence between Ω and $IVIO(Imf)$, where Imf denotes the image of f.

Also, we obtain the similar result to Theorem 26.5 in [23].

Proposition 6.41. *Let $f : X \to Y$ be a groupoid homomorphism and let*

$$\Phi_i = \{\mathcal{A} \in IVIOGP_i(X) : \mathcal{A} \text{ is f-invariant and has the i-sup-property}\}.$$

Then there is a one-to-one correspondence between Φ_i and $IVIOGP_i(Imf)$, for $i = 1,\ 2,\ 3,\ 3$.

7. Conclusions

We defined an (internal, external) IVI-octahedron set, and discussed some related properties. Also, we defined an IVI-octahedron point and the level set, and obtained some their properties including the characterizations of Type i-union (Type i-intersection). Moreover, we defined the image and preimage of an IVI-octahedron set under a mapping and studied some of their properties. Finally, we introduced the concepts of an IVIOGP and an IVIOLI [resp. IVIORI and IVIOI] of type i ($i = 1,\ 2,\ 3, 4$), and dealt with their some properties. In the future, we expect that one applies IVI-octahedron sets to group and ring theories, BCI/BCK-algebras, topologies, category theory and decision-making, etc. Also, we will introduce a new concept in which the neutrosophic set of each of the first, second and third components of the IVI-octahedron set is changed, and research for its properties.

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