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Cubic bipolar structures of BCC-ideal on BCC-algebras

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Abstract. In this paper, we introduce the concept of cubic bipolar BCC-ideal and investigate several properties. Also, we give relations between cubic bipolar BCC-ideal and cubic bipolar BCC-ideal. The image and the pre-image of cubic bipolar BCC-ideal in BCC-algebras are defined and how the image and the pre-image of cubic bipolar BCC-ideal in BCC-algebras become cubic bipolar BCC-ideal are studied. Moreover, the Cartesian product of cubic bipolar BCC-ideal in Cartesian product BCC-algebras is given.

2010 AMS Classification: 06F35, 03G25, 08A72

Keywords: BCC-algebra, Fuzzy BCC-ideal, (Cubic bipolar) BCC-ideal, The pre-image of cubic bipolar BCC-ideal in BCC-algebras, Cartesian product of cubic bipolar BCC-ideal.

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1. Introduction

In 1966, Iséki and Tanaka [9, 10] introduced the notion of BCK-algebras. Iséki [8] introduced the notion of a BCI-algebra which is a generalization of BCK-algebra . Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other structures including lattices and Boolean algebras. There is a great deal of literature which has been produced on the theory of BCK/BCI-algebras, in particular, the emphasis seems to have been put on the ideal theory of BCK/BCI-algebras. Iséki posed an interesting problem (solved by Wroński [25] whether the class of BCK-algebras is a variety. In connection with this problem, Komori [15] introduced a notion of BCC-lgebras, and Dudek [2, 3] redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Komori. Dudek and Zhang [6] introduced a notion of BCC-ideals in BCC-algebras and described connections between such ideals and congruences. In 1965, Zadeh [26] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches. The idea of "intuitionistic fuzzy set" was first published by Atanassov [1] as a generalization of the notion of fuzzy sets. In 1991, Ougen [20] applied the concept of fuzzy sets to BCI, BCK, MV-algebras. Dudek and Jun [5] considered the fuzzification of BCC-ideals in BCC-algebras. Lee [17] introduced the notion of bipolar-valued fuzzy sets which is an extension of fuzzy sets. Lee [16] introduced the notion of bipolar fuzzy subalgebras/ideals of a BCK-algebra, and investigated several properties. Jun et al. [11, 12, 13, 14] introduced the concept of a cubic set and applied it to the cubic set to BCK/BCI-algebras. Senapati et al. [23, 24] make the combination of cubic set and intuionistic fuzzy set of several ideals. Riaz and Tehrim [21, 22] presented the concept of internal cubic bipolar fuzzy sets and external cubic bipolar fuzzy sets and discussed some properties. Now, in this paper, we use the idea of Riaz and Tehrim [21, 22] to establish the notion of cubic bipolar BCC-ideal of $BCC-$-algebras and investigate several properties as have been mentioned in the abstract.

2. Preliminaries

We review some definitions and one property that will be useful in our results.

Definition 2.1 (See [2, 3])**.** An algebraic system $(X, *, 0)$ of type $(2, 0)$ is called a BCC-algebra, if it satisfying the following conditions: for any $x,\ y,\ z \in X$,
(BCC-1) $((x * y) * (z * y)) * (x * z) = 0$,
(BCC-2) $x * 0 = x$,
(BCC-3) $x * x = 0$,
(BCC-4) $0 * x = 0$
(BCC-5) $x * y = 0$ and $y * x = 0$ imply $x = y$.

In a BCC-algebra X, we can define a partial ordering "$\leq$" by $x \leq y$ if and only if $x * y = 0$. Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebras (See [2]). Note that a BCC-algebra is a BCK-algebra if and only if for all $x,\ y,\ z \in X$, it satisfies the equality: $(x * y) * z = (x * z) * y$.

Proposition 2.2 (See [2, 3])**.** *Let $(X, *, 0)$ be a BCC-algebra. Then the followings hold: for any $x,\ y,\ z \in X$,*
(a_1) $(x * y) * x = 0$,
(a_2) $x \leq y$ *implies* $x * z \leq y * z$,
(a_3) $x \leq y$ *implies* $z * x \leq z * y$,
(a_4) $(x * y) * (z * y) \leq x * z = 0$.

Definition 2.3 (See [6])**.** A non-empty subset I of a BCC-algebra X is said to be a BCK-ideal of X if, it satisfies the following conditions: for any $x,\ y \in X$,
(I1) $0 \in I$,
(I2) $x * y \in I$ and $y \in I$ implies $x \in I$.

Definition 2.4 (See [6])**.** A non-empty subset I of a BCC-algebra X is said to be a BCC-ideal of X if, it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(bcc1) $0 \in I$,
(bcc2) $(x * y) * z \in I$ and $y \in I$ implies $x * z \in I$.

Definition 2.5 ([4])**.** Let $(X, *, 0)$ be a BCC-algebra. Then a fuzzy set A in X is called a fuzzy sub-algebra of X, if for any $x,\ y \in X$, $A(x * y) \geq A(x) \wedge A(y)$.

Definition 2.6 ([4])**.** Let $(X, *, 0)$ be a BCC-algebra. Then a fuzzy set A in X is called a fuzzy BCC-ideal of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(F1) $A(0) \geq A(x)$,
(F2) $A(x * z) \geq A((x * y) * z) \wedge A(y)$.

Definition 2.7 ([16])**.** Let $(X, *, 0)$ be an algebraic system of type $(2, 0)$ and let $A = (A^N, A^P)$ be a bipolar fuzzy set in X. Then A is called a bipolar fuzzy ideal of X, if it satisfies the following conditions: for any $x,\ y \in X$,
(B_1) $A^N(0) \leq A^N(x)$ and $A^P(0) \geq A^P(x)$,
(B_2) $A^N(x) \leq A^N(x * y) \vee A^N(y)$ and $A^P(x) \geq A^P(x * y) \wedge A^P(y)$.

Definition 2.8. Let $(X, *, 0)$ be a BCC-algebra and let $A = (A^N, A^P)$ be a bipolar fuzzy set in X. Then A is called a bipolar fuzzy BCC-ideal of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(BI_1) $A^N(0) \leq A^N(x)$ and $A^P(0) \geq A^P(x)$,
(BI_2) $A^N(x*z) \leq A^N((x*y)*z) \vee A^N(y)$ and $A^P(x*z) \geq A^P((x*y)*z) \wedge A^P(y)$.

Now, we begin with the concepts of interval-valued fuzzy sets. Let $D[0, 1]$ denote the set of all closed subintervals of $[0, 1]$, i.e.,

$$D[0, 1] = \{\widetilde{a} = [a^-, a^+] : \widetilde{a} \subset [0, 1] \text{ and } 0 \leq a^- \leq a^+ \leq 1\}.$$

Each member of $D[0, 1]$ is called an interval-valued number. We define the operations $\leq, \geq, =$, min (denoted by $\wedge$) and max (denoted by $\vee$) between two elements in $D[0, 1]$ as follows: for any $\widetilde{a},\ \widetilde{b} \in D[0, 1]$,
(i) $\widetilde{a} \leq \widetilde{b}$ iff $a^- \leq b^-,\ a^+ \leq b^+$,
(ii) $\widetilde{a} \geq \widetilde{b}$ iff $a^- \geq b^-,\ a^+ \geq b^+$,
(iii) $\widetilde{a} = \widetilde{b}$ iff $a^- = b^-,\ a^+ = b^+$,
(iv) $\widetilde{a} \wedge \widetilde{b} = [a^- \wedge b^-, a^+ \wedge b^+]$,
(v) $\widetilde{a} \vee \widetilde{b} = [a^- \vee b^-, a^+ \vee b^+]$.

Here we consider that $\widetilde{0} = [0, 0]$ as the least element and $\widetilde{1} = [1, 1]$ as the greatest element. Now let $\{\widetilde{a}_j : j \in J\} \subset D[0, 1]$. Then its inf and sup are defined as follows:

$$\inf_{j \in J} \widetilde{a}_j = [\bigwedge_{j \in J} a_j^-, \bigwedge_{j \in J} a_j^+],\ \sup_{j \in J} \widetilde{a}_j = [\bigvee_{j \in J} a_j^-, \bigvee_{j \in J} a_j^+].$$

Let $D[-1, 0]$ denote the set of all closed subintervals of $[-1, 0]$. Then min, max, inf and sup of members of $D[-1, 0]$ are defined similarly to the above.

Definition 2.9 ([7, 19])**.** For a nonempty set X, a mapping $A : X \to [I]$ is called an interval-valued fuzzy set (briefly, an IVF set) in X. Let $[I]^X$ denote the set of all IVF sets in X. For each $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the degree of membership of an element x to A, where $A^-, A^+ \in I^X$ are called a lower fuzzy set and an upper fuzzy set in X, respectively. For each $A \in [I]^X$, we write $A = [A^-, A^+]$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy empty whole set in X.

Also, refer to [7, 19] for the inclusion, intersection, union of two IVF sets and the complement of an IVF set

Definition 2.10 ([14]). For a nonempty set X, a pair $\mathcal{A} = \langle A, \lambda \rangle$ is called a cubic set in X, where A is an IVF set in X and λ is a fuzzy set in X.

Definition 2.11 ([21, 22]). For a nonempty set X, a pair $\mathbf{A} = (\mathbf{A}^N, \mathbf{A}^P)$ is called an interval-valued bipolar fuzzy set (briefly, IVBFS) in X, if $\mathbf{A}^N : X \to D[-1, 0]$ and $\mathbf{A}^P : X \to D[0, 1]$. In this case for each $x \in X$, $\mathbf{A}^N(x) = [A^{N,-}(x), A^{N,+}(x)]$ and $\mathbf{A}^P(x) = [A^{P,-}(x), A^{P,+}(x)]$ are called the interval-valued positive and negative membership degree of x. In fact, for each $x \in X$,

$$\mathbf{A}(x) = ([A^{N,-}(x), A^{N,+}(x)], [A^{P,-}(x), A^{P,+}(x)]).$$

3. Cubic bipolar BCC-ideals in BCC-algebras

In this section, we introduce a new notion called cubic bipolar BCC-ideal in BCC-algebras and study its several properties.

Definition 3.1. Let X be a nonempty set. Then a pair $\mathcal{A} = \langle \mathbf{A}, A \rangle$ is called a cubic bipolar set in X, if $\mathbf{A} = (\mathbf{A}^N, \mathbf{A}^P)$ is an interval-valued bipolar fuzzy set and $A = (A^N, A^P)$ is an bipolar fuzzy set in X, where

$\mathbf{A}^N : X \to D[-1, 0],\ \mathbf{A}^P : X \to D[0, 1]$

and

$A^N : X \to [-1, 0],\ A^P : X \to [0, 1].$

We will denote the set of all cubic bipolar sets in X as $CB(X)$.

Definition 3.2. Let X be a BCC-algebra and let $\mathcal{A} = \langle \mathbf{A}, A \rangle \in CB(X)$. Then $\mathcal{A}$ is called a cubic bipolar subalgebra of X, if it satisfies the following conditions: for any $x,\ y \in X$,

(SC_1) $\mathbf{A}^P(x * y) \geq \mathbf{A}^P(x) \wedge \mathbf{A}^P(y)$, $\mathbf{A}^N(x * y) \leq \mathbf{A}^N(x) \vee \mathbf{A}^N(y)$,

(SC_2) $A^P(x * y) \geq A^P(x) \wedge A^P(y)$, $A^N(x * y) \leq A^N(x) \vee A^N(y)$.

Example 3.3. Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation $*$ defined by the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

Table 3.1

Then we can easily see that $X, *, 0)$ is a BCC-algebra. Consider the cubic bipolar fuzzy set $\mathcal{A} = \langle \mathbf{A}, A \rangle$ defined as follows:

$$\mathbf{A}(x) = \begin{cases} ([-0.9, -0.5], [0.3, 0.9]) & \text{if } x \in \{0, 1\} \\ ([-0.8, -0.2], [0.1, 0.6]) & \text{otherwise,} \end{cases}$$

$$A(0) = (-0.5, 0.7)\,,\ A(1) = (-0.3, 0.6)\,,$$
$$A(2) = (-0.2, 0.5)\,,\ A(3) = (-0.1, 0.4)\,.$$

Then we can easily check that $\mathcal{A}$ is a cubic bipolar subalgebra of X.

The following is the immediate result of Definitions 2.1 (BCC-3) and 3.2.

Lemma 3.4. *Let X be a BCC-algebra and let $\mathcal{A} \in CB(X)$. If $\mathcal{A}$ is a cubic bipolar subalgebra of X, then it satisfies the following conditions: for each $x \in X$,*
(1) $\mathbf{A}^P(0) \geq \mathbf{A}^P(x)$, $\mathbf{A}^N(0) \leq \mathbf{A}^N(x)$,
(2) $A^P(0) \geq A^P(x)$, $A^N(0) \leq A^N(x)$.

Definition 3.5. Let X be a BCC-algebra and let $\mathcal{A} = \langle \mathbf{A}, A \rangle \in CB(X)$. Then $\mathcal{A}$ is called a cubic bipolar BCK-ideal of X, if it satisfies the following conditions: for any $x,\ y \in X$,
(BC_1) $\mathbf{A}^P(x) \geq \mathbf{A}^P(x * y) \wedge \mathbf{A}^P(y)$,
(BC_2) $\mathbf{A}^N(x) \leq \mathbf{A}^N(x * y) \vee \mathbf{A}^N(y)$,
(BC_3) $A^P(x) \geq A^P(x * y) \wedge A^P(y)$,
(BC_4) $A^N(x) \leq A^N(x * y) \vee A^N(y)$.

Example 3.6. Let $X = \{0, 1, 2, 3\}$ be a BCC-algebra with Table 3.1. Consider the cubic bipolar fuzzy set $\mathcal{A} = \langle \mathbf{A}, A \rangle$ defined as follows:

$$\mathbf{A}(x) = \begin{cases} ([-0.9, -0.5], [0.5, 0.9]) & \text{if } x \in \{0, 2\} \\ ([-0.6, -0.2], [0.2, 0.6]) & \text{otherwise,} \end{cases}$$

$$A(0) = A(2) = (-04, 0.6),\ A(1) = A(3) = (-0.1, 0.3)\,.$$

Then we can easily check that $\mathcal{A}$ is a cubic bipolar BCK-ideal of X.

Definition 3.7. Let X be a BCC-algebra and let $\mathcal{A} = \langle \mathbf{A}, A \rangle \in CB(X)$. Then $\mathcal{A}$ is called a cubic bipolar BCC-ideal of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(CC_1) $\mathbf{A}^P(x * z) \geq \mathbf{A}^P((x * y) * z) \wedge \mathbf{A}^P(y)$
(CC_2) $\mathbf{A}^N(x * z) \leq \mathbf{A}^N((x * y) * z) \vee \mathbf{A}^N(y)$,
(CC_3) $A^P(x * z) \geq A^P((x * y) * z) \wedge A^P(y)$,
(CC_4) $A^N(x * z) \leq A^N((x * y) * z) \vee A^N(y)$.

Example 3.8. Let $X = \{0, 1, 2, 3\}$ be a BCC-algebra with Table 3.1. Consider the cubic bipolar fuzzy set $\mathcal{A} = \langle \mathbf{A}, A \rangle$ defined as follows:

$$\mathbf{A}(x) = \begin{cases} ([-0.9, -0.5], [0.2, 0.8]) & \text{if } x \in \{0, 1\} \\ ([-0.8, -0.2], [0.1, 0.5]) & \text{otherwise,} \end{cases}$$

$$A(0) = A(1) = (-0.3, 0.5),\ A(2) = A(3) = (-0.2, 0.3).$$

Then we can easily check that $\mathcal{A}$ is a cubic bipolar BCC-ideal of X. Moreover, we can check that $\mathcal{A}$ is a cubic bipolar BCK-ideal of X.

The following is the immediate result of Definitions 2.1 (BCC-2), 3.5 and 3.7.

Proposition 3.9. *Every cubic bipolar BCC-ideal of X is a cubic bipolar BCK-ideal of X. But the converse is not true in general.*

Example 3.10. Let $\mathcal{A}$ be the cubic bipolar BCK-ideal of X given in Example 3.6. Then

$$\mathbf{A}^P(1 * 3) = [0.2, 0.6] \not\geq [0.5, 0.9] = \mathbf{A}^P((1 * 2) * 3) \wedge \mathbf{A}^P(2).$$

Thus $\mathcal{A}$ is not a cubic bipolar BCC-ideal of X.

Lemma 3.11. *Let X be a BCC-algebra and let $\mathcal{A} \in CB(X)$. If $\mathcal{A}$ is a cubic bipolar BCC-ideal of X, then it satisfies the following conditions: for each $x \in X$,*
(1) $\mathbf{A}^P(0) \geq \mathbf{A}^P(x)$, $\mathbf{A}^N(0) \leq \mathbf{A}^N(x)$,
(2) $A^P(0) \geq A^P(x)$, $A^N(0) \leq A^N(x)$.

Proof. It is straightforward. □

Lemma 3.12. *Let X be a BCC-algebra and let $\mathcal{A}$ be a BCC-ideal of X. Then for any $x,\ y \in X$ such that $x \leq y$, it satisfies the following conditions:*
(1) $\mathbf{A}^P(y) \geq \mathbf{A}^P(x)$, $\mathbf{A}^N(y) \leq \mathbf{A}^N(x)$,
(2) $A^P(y) \geq A^P(x)$, $A^N(y) \leq A^N(x)$.

Proof. Let $x,\ y \in X$ such that $x \leq y$. Then clearly, $x * y = 0$. Let $z = 0$. Then

$$\begin{aligned} \mathbf{A}^P(x) &= \mathbf{A}^P(x * 0) \text{ [By (BCC-2)]} \\ &\geq \mathbf{A}^P((x * y) * 0) \wedge \mathbf{A}^P(y) \text{ [By Definition 3.7]} \\ &= \mathbf{A}^P(0 * 0) \wedge \mathbf{A}^P(y) \\ &= \mathbf{A}^P(0) \wedge \mathbf{A}^P(y) \text{ [By (BCC-2)]} \\ &= \mathbf{A}^P(y), \text{ [By Lemma 3.11 (1)]} \\ A^N(x) &= A^N(x * 0) \text{ [By (BCC-2)]} \\ &\leq A^N((x * y) * 0) \vee A^N(y) \text{ [By Definition 3.7]} \\ &= A^N(0 * 0) \vee A^N(y) \\ &= A^N(0) \vee A^N(y) \text{ [By (BCC-2)]} \\ &= A^P(y). \text{ [By Lemma 3.11 (2)]} \end{aligned}$$

Similarly, we have $\mathbf{A}^N(x) \leq \mathbf{A}^N(y)$ and $A^P(x) \geq A^P(y)$. □

Lemma 3.13. *Let X be a BCC-algebra and let $\mathcal{A}$ be a BCC-ideal of X. Then for any $x,\ y,\ z \in X$ such that $x * y \leq z$, it satisfies the following conditions:*
(1) $\mathbf{A}^P(x) \geq \mathbf{A}^P(y) \wedge \mathbf{A}^P(z)$, $\mathbf{A}^N(x) \leq \mathbf{A}^P(y) \vee \mathbf{A}^P(z)$,
(2) $A^P(x) \leq A^P(y) \vee A^P(z)$, $A^N(x) \geq A^N(y) \wedge A^N(z)$.

Proof. Let $x,\ y,\ z \in X$ such that $x * y \leq z$. Then clearly, $(x * y) * z = 0$. Thus

$$\begin{aligned} \mathbf{A}^P(x) &= \mathbf{A}^P(x * 0) \text{ [By (BCC-2)]} \\ &\geq \mathbf{A}^P((x * y) * 0) \wedge \mathbf{A}^P(y) \text{ [By Definition 3.7]} \\ &= \mathbf{A}^P(x * y) \wedge \mathbf{A}^P(y) \text{ [By (BCC-2)]} \\ &\geq \mathbf{A}^P(z) \wedge \mathbf{A}^P(y) \text{ [Since } \mathbf{A}^P(x * y) \geq \mathbf{A}^P(z)\text{, by Lemma 3.12 (1)]} \\ &= \mathbf{A}^P(y) \wedge \mathbf{A}^P(z), \\ A^N(x) &= A^N(x * 0) \text{ [By (BCC-2)]} \\ &\leq A^N((x * y) * 0) \vee A^N(y) \text{ [By Definition 3.7]} \\ &= A^N(x * y) \vee A^N(y) \text{ [By (BCC-2)]} \\ &\leq A^N(z) \vee A^N(y) \text{ [Since } A^N(x * y) \leq A^N(z)\text{, by Lemma 3.12 (2)]} \\ &= A^N(y) \vee A^N(z). \end{aligned}$$

Similarly, we have $\mathbf{A}^N(x) \leq \mathbf{A}^N(y) \vee \mathbf{A}^N(z)$ and $A^P(x) \geq A^P(y) \wedge A^P(z)$. □

Proposition 3.14. *Let X be a BCC-algebra. Then every cubic bipolar BCK-ideal of X is a cubic bipolar subalgebra of X. But the converse is not true in general (See Example 3.16).*

Proof. Let $\mathcal{A}$ be a BCK-ideal of X and let $x,\ y \in X$. Since X is a BCC-algebra, $x * y \leq x$ by Proposition 2.2 (a_1). Then

$$\mathbf{A}^P(x * y) \geq \mathbf{A}^P((x * y) * x) \wedge \mathbf{A}^P(x) \text{ [By Definition 3.7 (CC}_1\text{)]}$$

$$
\begin{aligned}
&= \mathbf{A}^P(0) \wedge \mathbf{A}^P(x) \text{ [Since } x * y \leq x]\\
&\geq \mathbf{A}^P(y) \wedge \mathbf{A}^P(x) \text{ [By Lemma 3.13 (1)]}\\
&= \mathbf{A}^P(x) \wedge \mathbf{A}^P(y),\\
A^N(x*y) &\leq A^N((x*y)*x) \vee A^N(x) \text{ [By Definition 3.7 (CC}_4)]\\
&= A^N(0) \vee A^N(x) \text{ [Since } x * y \leq x]\\
&\leq A^N(y) \vee A^N(x) \text{ [By Lemma 3.13 (2)]}\\
&= A^N(x) \vee A^N(y).
\end{aligned}
$$

Similarly, we can show that the followings hold:

$$\mathbf{A}^N(x*y) \leq \mathbf{A}^N(x) \vee \mathbf{A}^N(y),\ A^P(x*y) \geq A^P(x) \wedge A^P(y).$$

Thus $\mathcal{A}$ is a cubic bipolar subalgebra of X. □

From Propositions 3.9 and 3.14, we can easily obtain the following result.

Corollary 3.15. *Let X be a BCC-algebra. Then every cubic bipolar BCC-ideal of X is a cubic bipolar subalgebra of X. But the converse is not true (See Example 3.16).*

Example 3.16. Let X be the BCC-algebra and $\mathcal{A}$ be the subalgebra in Example 3.3. Then

$$A^P(3*2) \wedge A^P(2) = 0.5 \not\leq 0.4 = A^P(3),$$
$$A^P((3*2)*1) \wedge A^P(2) = 05 \not\leq 0.4 = A^P(3*1).$$

Thus $\mathcal{A}$ is neither a BCK-ideal nor a BCC-ideal of X.

Proposition 3.17. *Let X be a BCC-algebra and let $\mathcal{A}$ be a cubic bipolar subalgebra of X. Suppose the following conditions hold: for any $x,\ y,\ z \in X$ such that $x*y \leq z$,*
(i) $\mathbf{A}^P(x) \geq \mathbf{A}^P(y) \wedge \mathbf{A}^P(z),\ \mathbf{A}^N(x) \leq \mathbf{A}^N(y) \vee \mathbf{A}^N(z)$,
(ii) $A^P(x) \geq A^P(y) \wedge A^P(z),\ A^N(x) \leq A^N(y) \vee A^N(z)$.
Then $\mathcal{A}$ is a cubic bipolar BCC-ideal of X.

Proof. Let $\mathcal{A}$ be a cubic bipolar subalgebra of X and let $x. \in X$. Then from Lemma 3.4, we have

$$\mathbf{A}^P(0) \geq \mathbf{A}^P(x),\ \mathbf{A}^N(0) \leq \mathbf{A}^N(x),\ A^P(0) \geq A^P(x),\ A^N(0) \leq A^N(x).$$

Let $x,\ y,\ z \in X$ such that $x*y \leq z$. Then clearly, $(x*z)*((x*y)*z) \leq y$. Thus by Lemma 3.13, we have

$$\mathbf{A}^P(x*z) \geq \mathbf{A}^P((x*z)*y) \wedge \mathbf{A}^P(y),\ \mathbf{A}^N(x*z) \leq \mathbf{A}^N((x*z)*y) \vee \mathbf{A}^N(y),$$

$$A^P(x*z) \geq A^P((x*z)*y) \wedge A^P(y),\ A^N(x*z) \leq A^N((x*z)*y) \vee A^N(y).$$

So $\mathcal{A}$ is a cubic bipolar BCC-ideal of X. □

The members of $(D[-1,0] \times D[0,1]) \times ([-1,0] \times [0,1])$ will be called cubic bipolar numbers and denoted by $\widetilde{\bar{a}},\ \widetilde{\bar{b}},\ \widetilde{\bar{c}}$, etc., where

$$\widetilde{\bar{a}} = \langle \widetilde{a}, \bar{a} \rangle = \left\langle ([a^{N,-}, a^{N,+}], [a^{P,-}, a^{P,+}]), (a^N, a^P) \right\rangle.$$

Now we define the operations $\leq$, $=$, min (denoted by $\wedge$) and max (denoted by $\vee$) between two cubic bipolar numbers, and inf and sup of arbitrary cubic bipolar numbers as follows: for any $\widetilde{\bar{a}},\ \widetilde{\bar{b}} \in (D[-1,0] \times D[0,1]) \times ([-1,0] \times [0,1])$,
(i) $\widetilde{\bar{a}} \leq \widetilde{\bar{b}}$ iff $a^{N,-} \geq b^{N,-},\ a^{P,-} \leq b^{P,+},\ a^N \geq b^N,\ a^p \leq b^p$,

(ii) $\widetilde{\widetilde{a}} = \widetilde{\widetilde{b}}$ iff $\widetilde{a} = \widetilde{b},\ \bar{a} = \bar{b}$,

(iii) $\widetilde{\widetilde{a}} \wedge \widetilde{\widetilde{b}}$
$= \langle([a^{N,-} \vee b^{N,-}, a^{N,+} \vee b^{N,+}], [a^{P,-} \wedge b^{P,-}, a^{P,+} \wedge b^{P,+}]), (a^N \vee b^N, a^P \wedge b^P)\rangle$,

(iv) $\widetilde{\widetilde{a}} \vee \widetilde{\widetilde{b}}$
$= \langle([a^{N,-} \wedge b^{N,-}, a^{N,+} \wedge b^{N,+}], [a^{P,-} \vee b^{P,-}, a^{P,+} \vee b^{P,+}]), (a^N \wedge b^N, a^P \vee b^P)\rangle$.

Definition 3.18. Let X be a nonempty set, let $\mathcal{A} \in CB(X)$ and let $\widetilde{\widetilde{a}}$ be any cubic bipolar number. Then the subset $[\mathcal{A}]_{\widetilde{\widetilde{a}}}$ of X

$$[\mathcal{A}]_{\widetilde{\widetilde{a}}} = \{x \in X : \mathbf{A}^P(x) \geq \widetilde{a}^P,\ \mathbf{A}^N(x) \leq \widetilde{a}^N,\ A^P(x) \geq a^P,\ A^N(x) \leq a^N\}$$

is called the $\widetilde{\widetilde{a}}$-level set of $\mathcal{A}$,
where $\widetilde{a}^P = [a^{P,-}, a^{P,+}] \in D[0,1]$ and $\widetilde{a}^N = [a^{N,-}, a^{N,+}] \in D[-1,0]$.

Theorem 3.19. *Let X be a BCC-algebra, let $\widetilde{\widetilde{a}}$ be any cubic bipolar number and let $\mathcal{A} \in CB(X)$. Then $\mathcal{A}$ is a cubic bipolar BCC-ideal of X if and only if $[\mathcal{A}]_{\widetilde{\widetilde{a}}}$ is a BCC-ideal of X, where $[\mathcal{A}]_{\widetilde{\widetilde{a}}} \neq \phi$.*

Proof. Suppose $\mathcal{A}$ is a cubic bipolar BCC-ideal of X and $[\mathcal{A}]_{\widetilde{\widetilde{a}}} \neq \phi$. For any $x,\ y,\ z \in X$, let $(x * y) * z,\ y \in [\mathcal{A}]_{\widetilde{\widetilde{a}}}$ and let $\widetilde{a}^P = [a^{P,-}, a^{P,+}],\ \widetilde{a}^N = [a^{N,-}, a^{N,+}]$. Then

$$\mathbf{A}^P((x * y) * z) \geq \widetilde{a}^P,\ \mathbf{A}^N((x * y) * z) \leq \widetilde{a}^N, \tag{3.1}$$

$$A^P((x * y) * z) \geq a^P,\ A^N((x * y) * z) \leq a^N, \tag{3.2}$$

$$\mathbf{A}^P(y) \geq \widetilde{a}^P,\ \mathbf{A}^N(y) \leq \widetilde{a}^N, \tag{3.3}$$

$$A^P(y) \geq a^P,\ A^N(y) \leq a^N. \tag{3.4}$$

Thus

$$\begin{aligned}\mathbf{A}^P(x * z) &\geq \mathbf{A}^P((x * y) * z) \wedge \mathbf{A}^P(y) \text{ [By the hypothesis]}\\ &\geq \widetilde{a}^P \wedge \widetilde{a}^P, \text{ [By (3.1) and (3.3)]}\\ &= \widetilde{a}^P,\\ A^N(x * z) &\leq A^N((x * y) * z) \vee A^N(y) \text{ [By the hypothesis]}\\ &\leq a^N \vee a^N, \text{ [By (3.2) and (3.4)]}\\ &= a^N.\end{aligned}$$

Similarly, we have $\mathbf{A}^N(x * z) \leq \widetilde{a}^N$ and $A^P(x * z) \geq a^P$. So $x * z \in [\mathcal{A}]_{\widetilde{\widetilde{a}}}$. Hence $[\mathcal{A}]_{\widetilde{\widetilde{a}}}$ is a BCC-ideal of X.

Conversely, suppose $[\mathcal{A}]_{\widetilde{\widetilde{a}}} (\neq \phi)$ is a BCC-ideal of X. We prove that $\mathcal{A}$ satisfies the conditions of Definition 3.7.

Case 1: Assume that (CC_1) does not true. Then there are $x_0,\ y_0,\ z_0 \in X$ such that $\mathbf{A}^P(x_0 * z_0) < \mathbf{A}^P((x_0 * y_0) * z_0) \wedge \mathbf{A}^P(y_0)$. Let

$\mathbf{A}^P((x_0 * y_0) * z_0) = [b^{P,-}, b^{P,+}],\ \mathbf{A}^P(y_0) = [c^{P,-}, c^{P,+}],\ \mathbf{A}^P(x_0 * z_0) = [a^{P,-}, a^{P,+}]$,

where $[b^{P,-}, b^{P,+}],\ [c^{P,-}, c^{P,+}],\ [a^{P,-}, a^{P,+}] \in D[0,1]$. Then clearly,

$$[a^{P,-}, a^{P,+}] < [b^{P,-}, b^{P,+}] \wedge [c^{P,-}, c^{P,+}] = [b^{P,-} \wedge c^{P,-}, b^{P,+} \wedge c^{P,+}].$$

Let $[d^{P,-}, d^{P,+}] = [\frac{1}{2}(a^{P,-} + b^{P,-} \wedge c^{P,-}), \frac{1}{2}(a^{P,+} + b^{P,+} \wedge c^{P,+})]$. Then we can easily obtain the following inequalities:

$$b^{P,-} \wedge c^{P,-} > d^{P,-} = \frac{1}{2}(a^{P,-} + b^{P,-} \wedge c^{P,-}) > a^{P,-},$$

$$b^{P,+} \wedge c^{P,+} > d^{P,+} = \frac{1}{2}(a^{P,+} + b^{P,+} \wedge c^{P,+}) > a^{P,+}.$$

Thus $[b^{P,-} \wedge c^{P,-}, b^{P,+} \wedge c^{P,+}] > [d^{P,-}, d^{P,+}] > [a^{P,-}, a^{P,+}] = \mathbf{A}^P(x_0 * z_0)$. So $(x_0 * y_0) * z_0,\ y_0 \in [\mathcal{A}]_{\widetilde{a}}$ but $(x_0 * z_0) \notin [\mathcal{A}]_{\widetilde{a}}$.

Case 2: Assume that (CC_2) does not true. Then there are $x_0,\ y_0,\ z_0 \in X$ such that $\mathbf{A}^N(x_0 * z_0) > \mathbf{A}^N((x_0 * y_0) * z_0) \vee \mathbf{A}^N(y_0)$. Let

$$\mathbf{A}^N((x_0*y_0)*z_0) = [b^{N,-}, b^{N,+}],\ \mathbf{A}^N(y_0) = [c^{N,-}, c^{N,+}],\ \mathbf{A}^N(x_0*z_0) = [a^{N,-}, a^{N,+}],$$

where $[b^{N,-}, b^{N,+}],\ [c^{N,-}, c^{N,+}],\ [a^{N,-}, a^{N,+}] \in D[-1, 0]$. Then clearly,

$$[a^{N,-}, a^{N,+}] > [b^{N,-}, b^{N,+}] \vee [c^{N,-}, c^{N,+}] = [b^{N,-} \vee c^{N,-}, b^{N,+} \vee c^{N,+}].$$

Let $[d^{N,-}, d^{N,+}] = [\frac{1}{2}(a^{N,-} + b^{N,-} \vee c^{N,-}), \frac{1}{2}(a^{N,+} + b^{N,+} \vee c^{N,+})]$. Then we can easily obtain the following inequalities:

$$b^{N,-} \vee c^{N,-} < d^{N,-} = \frac{1}{2}(a^{N,-} + b^{N,-} \vee c^{N,-}) < a^{N,-},$$

$$b^{N,+} \vee c^{N,+} < d^{N,+} = \frac{1}{2}(a^{N,+} + b^{N,+} \vee c^{N,+}) < a^{N,+}.$$

Thus $[b^{N,-} \vee c^{N,-}, b^{N,+} \vee c^{N,+}] < [d^{N,-}, d^{N,+}] < [a^{N,-}, a^{N,+}] = \mathbf{A}^N(x_0 * z_0)$. So $(x_0 * y_0) * z_0,\ y_0 \in [\mathcal{A}]_{\widetilde{a}}$ but $(x_0 * z_0) \notin [\mathcal{A}]_{\widetilde{a}}$.

Case 3: Assume that (CC_3) does not true. Then there are $x_0,\ y_0,\ z_0 \in X$ such that $A^P(x_0 * z_0) < A^P((x_0 * y_0) * z_0) \wedge A^P(y_0)$. Let

$$A^P((x_0 * y_0) * z_0) = b^P,\ A^P(y_0) = c^P,\ A^P(x_0 * z_0) = a^P,$$

where $b^P,\ c^P,\ a^P \in [0, 1]$. Then clearly, $a^P < b^P \wedge c^P$. Let $d^P = \frac{1}{2}(a^P + b^P \vee c^P)$. Then we can easily obtain the following inequalities:

$$b^P \wedge c^P > d^P = \frac{1}{2}(a^P + b^P \wedge c^P) > a^P.$$

Thus $b^P \wedge c^P > d^P > a^P = A^P(x_0 * z_0)$. So $(x_0 * y_0) * z_0,\ y_0 \in [\mathcal{A}]_{\widetilde{a}}$ but $(x_0 * z_0) \notin [\mathcal{A}]_{\widetilde{a}}$.

Case 4: Assume that (CC_4) does not true. Then there are $x_0,\ y_0,\ z_0 \in X$ such that $A^N(x_0 * z_0) > A^N((x_0 * y_0) * z_0) \vee A^N(y_0)$. Let

$$A^N((x_0 * y_0) * z_0) = b^N,\ A^N(y_0) = c^N,\ A^N(x_0 * z_0) = a^N,$$

where $b^N,\ c^N,\ a^N \in [-1, 0]$. Then clearly, $a^N > b^N \vee c^N$. Let $d^N = \frac{1}{2}(a^N + b^N \vee c^N)$. Then we can easily obtain the following inequalities:

$$b^N \vee c^N < d^N = \frac{1}{2}(a^N + b^N \vee c^N) > a^N.$$

Thus $b^N \vee c^N < d^N < a^N = A^N(x_0 * z_0)$. So $(x_0 * y_0) * z_0,\ y_0 \in [\mathcal{A}]_{\widetilde{a}}$ but $(x_0 * z_0) \notin [\mathcal{A}]_{\widetilde{a}}$. This completes the proof. □

Theorem 3.20. *Let X be a BCC-algebra and let $\mathcal{A} \in CB(X)$. Then $\mathcal{A}$ is a cubic bipolar BCC-ideal of X if and only if $\mathbf{A}$ is a bipolar interval-valued fuzzy BCC-ideal and A is a bipolar fuzzy BCC-ideal of X.*

Proof. Suppose $\mathcal{A}$ is a cubic bipolar BCC-ideal of X. Then by Lemma 3.11, we have for each $x \in X$,

(3.5)
$$A^{P,-}(0) \geq A^{P,-}(x),\ A^{P,+}(0) \geq A^{P,+}(x),\ A^{N,-}(0) \leq A^{N,-}(x),\ A^{N,+}(0) \leq A^{N,+}(x),$$

$$A^P(0) \geq A^P(x),\ A^N(0) \leq A^N(x). \quad (3.6)$$

Now let $x,\ y,\ z \in X$. Then

$$\begin{aligned} &[A^{P,-}(x*z), A^{P,+}(x*z)] \\ &= \mathbf{A}^P(x*z) \\ &\geq \mathbf{A}^P((x*y)*z) \wedge \mathbf{A}^P(y) \\ &= [A^{P,-}((x*y)*z), A^{P,+}((x*y)*z)] \wedge [A^{P,-}(y), A^{P,+}(y)] \\ &= [A^{P,-}((x*y)*z) \wedge A^{P,-}(y), A^{P,+}((x*y)*z) \wedge A^{P,+}(y)], \end{aligned}$$

$$\begin{aligned} &[A^{N,-}(x*z), A^{N,+}(x*z)] \\ &= \mathbf{A}^N(x*z) \\ &\leq \mathbf{A}^N((x*y)*z) \vee \mathbf{A}^N(y) \\ &= [A^{N,-}((x*y)*z), A^{N,+}((x*y)*z)] \vee [A^{N,-}(y), A^{N,+}(y)] \\ &= [A^{,-}((x*y)*z) \vee A^{N,-}(y), A^{N,+}((x*y)*z) \vee A^{N,+}(y)]. \end{aligned}$$

Thus we have

(3.7)
$$A^{P,-}(x*z)) \geq A^{P,-}((x*y)*z) \wedge A^{P,-}(y),\ A^{P,+}(x*z)) \geq A^{P,+}((x*y)*z) \wedge A^{P,+}(y),$$

(3.8)
$$A^{N,-}(x*z)) \leq A^{N,-}((x*y)*z) \vee A^{N,-}(y),\ A^{N,+}(x*z)) \leq A^{N,+}((x*y)*z) \vee A^{N,+}(y).$$

Also we can easily see that the followings hold:

$$A^P(x*z) \geq A^P((x*y)*z) \wedge A^P(y),\ A^N(x*z) \leq A^N((x*y)*z) \vee A^N(y). \quad (3.9)$$

So by (3.5), (3.7), (3.8) and (3.6), (3.9), $\mathbf{A}$ is a bipolar interval-valued fuzzy BCC-ideal of X and A is a bipolar fuzzy BCC-ideal of X.

Conversely, suppose $\mathbf{A}$ is a bipolar interval-valued fuzzy BCC-ideal of X and A is a bipolar fuzzy BCC-ideal of X. Then clearly, we have for each $x \in X$,

$$\mathbf{A}^P(0) \geq \mathbf{A}^P(x),\ \mathbf{A}^N(0) \leq \mathbf{A}^N(x), \quad (3.10)$$

$$A^P(0) \geq A^P(x),\ A^N(0) \leq A^N(x). \quad (3.11)$$

Let $x,\ y,\ z \in X$. Then

$$\begin{aligned} \mathbf{A}^P(x*z) &= [A^{P,-}(x*z), A^{P,+}(x*z)] \\ &\geq [A^{P,-}((x*y)*z) \wedge A^{P,-}(y). A^{P,+}((x*y)*z) \wedge A^{P,+}(y)] \\ &= [A^{P,-}((x*y)*z), A^{P,+}((x*y)*z)] \wedge [A^{P,-}(y), A^{P,+}(y)] \\ &= \mathbf{A}^P((x*y)*z) \wedge \mathbf{A}^P(y). \end{aligned}$$

Thus we have

$$\mathbf{A}^P(x*z) \geq \mathbf{A}^P((x*y)*z) \wedge \mathbf{A}^P(y). \quad (3.12)$$

$A^P(x*z) \leq A^P((x*y)*z) \vee A^P(y)$.

Similarly, we have

$$\mathbf{A}^N(x*z) \leq \mathbf{A}^{/n}((x*y)*z) \vee \mathbf{A}^P(y), \quad (3.13)$$

(3.14) $A^P(x*z) \geq A^P((x*y)*z) \wedge A^P(y),\ A^N(x*z) \leq A^N((x*y)*z) \vee A^N(y).$

Thus $\mathcal{A}$ is a cubic bipolar BCC-ideal of X. □

Proposition 3.21. *Let X be a BCC-algebra and let $(\mathcal{A}_j)_{j\in J}$ be a family of cubic bipolar BCC-ideals of X. Then $\bigcap_{j\in J}\mathcal{A}_j$ is a cubic bipolar BCC-ideal of X.*

Proof. The proof is straightforward. □

4. The image and the preimage of a cubic bipolar BCC-ideal under a homomorphism of BCC-algebras

Definition 4.1. Let $(X,*,0)$ and $(Y,*',0')$ be two BCC-algebras. Then a mapping $f: X \to Y$ is called a homomorphism, if $f(x*y) = f(x)*' f(y)$ for any $x,\ y \in X$

It is clear that $f(0) = 0'$.

Definition 4.2. Let $(X,*,0)$ and $(Y,*',0')$ be two BCC-algebras, and let $\mathcal{A} \in CB(X),\ \mathcal{B} \in CB(Y)$.

(i) The image of $\mathcal{A}$, denoted by $f(\mathcal{A})$, is a cubic bipolar set in Y defined as follows: for each $y \in Y$,

$$f(\mathcal{A})(y) = \begin{cases} \bigwedge_{x\in f^{-1}(y)} \mathcal{A}(x) & \text{if } f^{-1}(y) \neq \phi \\ \langle([0,0],[0,0]),(0,0)\rangle & \text{otherwise,} \end{cases}$$

where $\bigwedge_{x\in f^{-1}(y)} \mathcal{A}(x) = \left\langle \bigwedge_{x\in f^{-1}(y)} \mathbf{A}(x), \bigwedge_{x\in f^{-1}(y)} A(x) \right\rangle$,

$\bigwedge_{x\in f^{-1}(y)} \mathbf{A}(x) = ([\bigvee_{x\in f^{-1}(y)} A^{N,-}(x), \bigvee_{x\in f^{-1}(y)} A^{N,+}(x)],$
$[\bigwedge_{x\in f^{-1}(y)} A^{P,-}(x), \bigwedge_{x\in f^{-1}(y)} A^{P,+}(x)]),$

$\bigwedge_{x\in f^{-1}(y)} A(x) = (\bigvee_{x\in f^{-1}(y)} A^N(x), \bigwedge_{x\in f^{-1}(y)} A^P(x)]).$

(ii) The preimage of $\mathcal{B}$ under f, denoted by $f^{-1}(\mathcal{B})$, is a cubic bipolar set in X defined as follows: for each $x \in X$,

$$[f^{-1}(\mathcal{B})](x) = \mathcal{B}(f(x)) = \langle \mathbf{B}(f(x)), B(f(x))\rangle = \left\langle [f^{-1}(\mathbf{B})](x), [f^{-1}(B)](x)\right\rangle.$$

Proposition 4.3. *Let $f: X \to Y$ be a homomorphism of BCC-algebras and let $\mathcal{B} \in CB(Y)$. If $\mathcal{B}$ is a cubic bipolar BCC-ideal of Y, then $f^{-1}(\mathcal{B})$ is a cubic bipolar BCC-ideal of X.*

Proof. Suppose $\mathcal{B}$ is a cubic bipolar BCC-ideal of Y. Then clearly, by Definition 3.7, $\mathbf{B}$ is a bipolar interval-valued fuzzy BCC-ideal of X and B is a bipolar fuzzy BCC-ideal of X. Let $x \in X$. Then

$[f^{-1}(B^P)](x) = B^P(f(x))$
$\leq B^P(0')$ [Since B is a bipolar fuzzy BCC-ideal of Y]
$= B^P(f(0))$ [Since f is a homomorphism]
$= [f^{-1}(B^P)](0)$,

$[f^{-1}(\mathbf{B}^P)](x) = \mathbf{B}^P(f(x))$
$\leq \mathbf{B}^P(0')$
[Since $\mathbf{B}$ is a bipolar interval-valued fuzzy BCC-ideal of Y]
$= \mathbf{B}^P(f(0))$ [Since f is a homomorphism]
$= [f^{-1}(\mathbf{B}^P)](0)$.

Similarly, we have $[f^{-1}(B^N)](x) \geq [f^{-1}(B^N)](0)$ and $[f^{-1}(\mathbf{B}^N)](x) \geq [f^{-1}(\mathbf{B}^N)](0)$.

Now let $x,\ y,\ z \in X$. Then

$$\begin{aligned}
[f^{-1}(B^P)](x*z) &= B^P(f(x*z)) \\
&= B^P(f(x)*' f(z)) \text{ [Since } f \text{ is a homomorphism]} \\
&\geq B^P((f(x)*' f(y))*' f(z)) \wedge B^P(f(y)) \\
&\quad \text{[Since } B \text{ is a bipolar fuzzy } BCC\text{-ideal of } Y] \\
&= B^P((f(x*y)*z)) \wedge B^P(f(y)) \\
&\quad \text{[Since } f \text{ is a homomorphism]} \\
&= [f^{-1}(B^P)]((x*y)*z) \wedge [f^{-1}(B^P)](y), \\
[f^{-1}(\mathbf{B}^P)](x*z) &= \mathbf{B}^P(f(x*z)) \\
&= \mathbf{B}^P(f(x)*' f(z)) \text{ [Since } f \text{ is a homomorphism]} \\
&\geq \mathbf{B}^P((f(x)*' f(y))*' f(z)) \wedge \mathbf{B}^P(f(y)) \\
&\quad \text{[Since } \mathbf{B} \text{ is a bipolar interval-valued fuzzy } BCC\text{-ideal of } Y] \\
&= \mathbf{B}^P((f(x*y)*z)) \wedge \mathbf{B}^P(f(y)) \\
&\quad \text{[Since } f \text{ is a homomorphism]} \\
&= [f^{-1}(\mathbf{B}^P)]((x*y)*z) \wedge [f^{-1}(\mathbf{B}^P)](y).
\end{aligned}$$

Similarly, we have the following inequalities:

$$[f^{-1}(B^N)](x*z) \leq [f^{-1}(B^N)]((x*y)*z) \vee [f^{-1}(B^N)](y),$$

$$[f^{-1}(\mathbf{B}^N)](x*z) \leq [f^{-1}(\mathbf{B}^N)]((x*y)*z) \vee [f^{-1}(\mathbf{B}^N)](y).$$

Thus $f^{-1}(B)$ is a bipolar fuzzy BCC-ideal of X and $f^{-1}(\mathbf{B})$ is a bipolar interval-valued fuzzy BCC-ideal of X. So by Theorem 3.20, $f^{-1}(\mathcal{B})$ is a cubic bipolar BCC-ideal of X. □

Proposition 4.4. *Let $f : X \to Y$ be an epihomomorphism of BCC-algebras and let $\mathcal{B} \in CB(Y)$. If $f^{-1}(\mathcal{B})$ is a cubic bipolar BCC-ideal of X, $\mathcal{B}$ is a cubic bipolar BCC-ideal of Y.*

Proof. Suppose $f^{-1}(\mathcal{B})$ is a cubic bipolar BCC-ideal of X and let $a \in Y$. Since f is surjective, there is $x \in X$ such that $f(x) = a$. Then

$$\begin{aligned}
B^P(a) &= B^P(f(x)) = [f^{-1}(B^P)](x) \\
&\leq [f^{-1}(B^P)](0) \text{ [Since } f^{-1}(B) \text{ is a bipolar fuzzy } BCC\text{-ideal of } X] \\
&= B^P(f(0)) = B^P(0'), \text{ [Since } f \text{ is a homomorphism]} \\
\mathbf{B}^P(a) &= \mathbf{B}^P(f(x)) = [f^{-1}(\mathbf{B}^P)](x) \\
&\leq [f^{-1}(\mathbf{B}^P)](0) \\
&\quad \text{[Since } f^{-1}(\mathbf{B}) \text{ is a bipolar interval-valued fuzzy } BCC\text{-ideal of } X] \\
&= \mathbf{B}^P(f(0)) = \mathbf{B}^P(0').
\end{aligned}$$

Similarly, we have $B^N(b) \geq B^N(0')$, $\mathbf{B}^N(b) \geq \mathbf{B}^N(0')$.

Now let $a,\ b,\ c \in Y$. Then clearly, there are $x,\ y,\ z \in X$ such that

$$f(x) = a,\ f(y) = b,\ f(z) = c.$$

Thus

$$\begin{aligned}
B^P(a*' c) &= B^P(f(x*z)) \text{ [Since } f \text{ is a homomorphism]} \\
&= [f^{-1}(B^P)](x*z) \\
&\geq [f^{-1}(B^P)]((x*y)*z) \wedge [f^{-1}(B^P)](y) \\
&\quad \text{[Since } f^{-1}(B) \text{ is a bipolar fuzzy } BCC\text{-ideal of } X] \\
&= B^P(((f(x)*' f(y))*' f(z)) \wedge B^P(f(y)) \\
&\quad \text{[Since } f \text{ is a homomorphism]}
\end{aligned}$$

$$
\begin{aligned}
&= B^P((a*' b)*' c) \wedge B^P(b),\\
\mathbf{B}^P(a*' c) &= \mathbf{B}^P(f(x*z))\\
&= [f^{-1}(\mathbf{B}^P)](x*z)\\
&\geq [f^{-1}(\mathbf{B}^P)]((x*y)*z) \wedge [f^{-1}(\mathbf{B}^P)](y)\\
&\quad \text{[Since } f^{-1}(\mathbf{B}) \text{ is a bipolar interval-valued fuzzy } BCC\text{-ideal of } X]\\
&= \mathbf{B}^P(((f(x)*' f(y))*' f(z)) \wedge \mathbf{B}^P(f(y))\\
&\quad \text{[Since } f \text{ is a homomorphism]}\\
&= \mathbf{B}^P((a*' b)*' c) \wedge \mathbf{B}^P(b).
\end{aligned}
$$

Similarly, we have the following inequalities:

$$B^N(a*' c) \leq B^N((a*' b)*' c) \vee B^N(b),$$

$$\mathbf{B}^N(a*' c) \leq \mathbf{B}^N((a*' b)*' c) \vee \mathbf{B}^N(b).$$

So B is a a bipolar fuzzy BCC-ideal of Y and $\mathbf{B}$ is a bipolar interval-valued fuzzy BCC-ideal of Y. Hence by Theorem 3.20, $\mathcal{B}$ is a cubic bipolar BCC-ideal of Y. □

5. Product of cubic bipolar BCC-ideals

Definition 5.1. Let X, Y be nonempty sets and let $\mathcal{A} \in CB(X)$, $\mathcal{B} \in CB(Y)$. Then the Cartesian product of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \times \mathcal{B} = \langle \mathbf{A} \times \mathbf{B}, A \times B \rangle$, is a cubic bipolar set in $X \times X$ defined as follows: for each $(x, y) \in X \times Y$,

$$(\mathbf{A} \times \mathbf{B})^P(x, y) = \mathbf{A}^P(x) \wedge \mathbf{B}^P(y),\ (\mathbf{A} \times \mathbf{B})^N(x, y) = \mathbf{A}^N(x) \vee \mathbf{B}^N(y),$$

$$(A \times B)^P(x, y) = A^P(x) \wedge B^P(y),\ (A \times B)^N(x, y) = A^N(x) \vee B^P(y).$$

In fact, $(\mathbf{A} \times \mathbf{B})^P : X \times Y \to D[0, 1]$, $(\mathbf{A} \times \mathbf{B})^N : X \times Y \to D[-1, 0]$, $(A \times B)^P : X \times Y \to [0, 1]$, $(A \times B)^N : X \times Y \to [-1, 0]$.

Remark 5.2. Let X, Y be two BCC-algebras. We define the binary operation $*$ on $X \times Y$ as follows: for any $(x_1, y_1),\ (x_2, y_2) \in X \times Y$,

$$(x_1, y_1) * (x_2, y_2) = (x_1 * x_2, y_1 * y_2).$$

Then clearly, $((X \times Y, *, (0,))$ is a BCC-algebra.

Proposition 5.3. *Let X, Y be two BCC-algebras and let $\mathcal{A} \in CB(X)$, $\mathcal{B} \in CB(Y)$. If $\mathcal{A}$ is a cubic bipolar BCC-ideal of X and $\mathcal{B}$ is a cubic bipolar BCC-ideal of Y, then $\mathcal{A} \times \mathcal{B}$ is a cubic bipolar BCC-ideal of $X \times Y$.*

Proof. Suppose $\mathcal{A}$ is a cubic bipolar BCC-ideal of X and $\mathcal{B}$ is a cubic bipolar BCC-ideal of Y. Then clearly, $\mathbf{A}$ [resp. $\mathbf{B}$] is a bipolar interval-valued fuzzy BCC-ideal of X [resp. Y] and A [resp. B] is a bipolar fuzzy BCC-ideal of X [resp. Y]. Let $(x, y) \in X \times Y$. Then

$$(\mathbf{A} \times \mathbf{B})^P(x, y) = \mathbf{A}^P(x) \wedge \mathbf{B}^P(y) \leq \mathbf{A}^P(0) \wedge \mathbf{B}^P(0) = (\mathbf{A} \times \mathbf{B})^P(0, 0),$$

$$(A \times B)^P(x, y) = A^P(x) \wedge B^P(y) \leq A^P(0) \wedge B^P(0) = (A \times B)^P(0, 0).$$

Similarly, we have $(\mathbf{A}\times\mathbf{B})^N(x, y) \geq (\mathbf{A}\times\mathbf{B})^N(0, 0)$, $(A\times B)^N(x, y) \geq (A\times B)^N(0, 0)$.

Let $x = (x_1, y_1),\ y = (x_2, y_2),\ z = (x_3, y_3) \in X \times Y$. Then

$$
\begin{aligned}
&(\mathbf{A} \times \mathbf{B})^P(x * z)\\
&= (\mathbf{A} \times \mathbf{B})^P((x_1, y_1) * (x_3, y_3))\\
&= \mathbf{A}^P(x_1 * x_3) \wedge \mathbf{A}^P(y_1 * y_3)
\end{aligned}
$$

$$\begin{aligned}
&\geq [\mathbf{A}^P((x_1 * x_2) * x_3) \wedge \mathbf{A}^P(x_2)] \wedge [\mathbf{B}^P((y_1 * y_2) * y_3) \wedge \mathbf{B}^P(y_2)]\\
&= [\mathbf{A}^P((x_1 * x_2) * x_3) \wedge \mathbf{B}^P((y_1 * y_2) * y_3)] \wedge [\mathbf{A}^P(x_2) \wedge \mathbf{B}^P(y_2)]\\
&= \mathbf{A}^P(((x_1, y_1) * (x_2, y_2)) * (x_3, y_3)) \wedge \mathbf{A}^P(x_2, y_2)\\
&= (\mathbf{A}^P \times \mathbf{B}^P)((x * y) * z) \wedge (\mathbf{A}^P \times \mathbf{B}^P)(y),\\
&= (\mathbf{A} \times \mathbf{B})^P((x * y) * z) \wedge (\mathbf{A} \times \mathbf{B})^P(y),
\end{aligned}$$

$$\begin{aligned}
&\ (A \times B)^P(x * z)\\
&= (A \times B)^P((x_1, y_1) * (x_3, y_3))\\
&= A^P(x_1 * x_3) \wedge A^P(y_1 * y_3)\\
&\geq [A^P((x_1 * x_2) * x_3) \wedge A^P(x_2)] \wedge [B^P((y_1 * y_2) * y_3) \wedge B^P(y_2)]\\
&= [A^P((x_1 * x_2) * x_3) \wedge A^P((y_1 * y_2) * y_3)] \wedge [A^P(x_2) \wedge B^P(y_2)]\\
&= A^P(((x_1, y_1) * (x_2, y_2)) * (x_3, y_3)) \wedge A^P(x_2, y_2)\\
&= (A^P \times B^P)((x * y) * z) \wedge (A^P \times B^P)(y)\\
&= (A \times B)^P((x * y) * z) \wedge (A \times B)^P(y).
\end{aligned}$$

Similarly, we have

$$(\mathbf{A} \times \mathbf{B})^N(x * z) \leq (\mathbf{A} \times \mathbf{B})^N((x * y) * z) \vee (\mathbf{A} \times \mathbf{B})^N(y),$$

$$(A \times B)^N(x * z) \leq (A \times B)^N((x * y) * z) \vee (A \times B)^N(y).$$

Thus $\mathbf{A} \times \mathbf{B}$ is a bipolar interval-valued fuzzy BCC-ideal of $X \times Y$ and $A \times B$ is a bipolar fuzzy BCC-ideal of $X \times Y$. So by Theorem 3.20, $\mathcal{A} \times \mathcal{B}$ is a cubic bipolar BCC-ideal of $X \times Y$. □

6. Conclusions

We have studied the cubic bipolar BCC-ideal in BCC-algebras. Also we discussed few results of cubic bipolar BCC-ideal in BCC-algebras, the image and the preimage of cubic bipolar BCC-ideal in BCC-algebras under a homomorphism were defined. How the image and the preimage of cubic bipolar BCC-ideal in BCC-algebras became cubic bipolar BCC-ideal were studied. Moreover, the product of cubic bipolar BCC-ideal to product cubic bipolar BCC-ideal is established. Furthermore, we construct some algorithms theory applied to BCC-ideal in BCC-algebras. The main purpose of our future work is to investigate the cubic bipolar ideal of algebraic structure such as a cubic bipolar hyper KU-ideal in hyper KU-algebras.

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References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.
[2] W. A. Dudek, On proper BCC-algebras, Bull. Inst. Math. Academia Sinica 20 (1992) 137–150.
[3] W. A. Dudek, A new characterization of ideals in BCC-algebras, Novi Sad J. Math. 29 (1999) 139–145.
[4] W. A. Dudek and Y. B. Jun, Fuzzy BCC-ideals in BCC-algebras, Math. Montisnigri 10 (1999) 21–30.
[5] W. A. Dudek and Y. B. Jun, Normalizations of fuzzy BCC-ideals in BCC-algebras, Math. Moravica 3 (1999) 17–24.
[6] W. A. Dudek and X. H. Zhang, On ideals and congruences in BCC-algebras, Czechoslovak Math. J. 48 (123) (1998) 21–29.

[7] M. B. Gorzalczany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, Fuzzy sets and Systems 21 (1987) 1–17.
[8] K. Iséki, On BCI-algebras, Mathematics Seminar Notes 8 (1) (1980) 125–130.
[9] K. Iséki and S. Tanaka, An introduction to the theory of BCK -algebras, Mathematica Japonica 23 (1) (1978) 1–26.
[10] K. Iséki and S. Tanaka, Ideal theory of BCK-algebras, Mathematica Japonica 21 (4) (1976) 351–366.
[11] Y. B. Jun, C. S. Kim, and M. S. Kang, Cubic subalgebras and ideals of BCK/BCI-algebras, Far East Journal of Mathematical Sciences 44 (2) (2010) 239–250.
[12] Y. B. Jun, C. S. Kim, and J. G. Kang, Cubic q-ideals of BCIalgebras, Ann. Fuzzy Math. Inform. 1 (1) (2011) 25–34.
[13] Y. B. Jun, K. J. Lee, and M. S. Kang, Cubic structures applied to ideals of BCI-algebras, Computers & Mathematics with Applications 62 (9) (2011) 3334–3342.
[14] Y. B. Jun, C. S.Kim, and K. O. Yang, Cubic sets, Ann. Fuzzy Math. Inform. 4 (1) (2012) 83–98.
[15] Y. Komori, The class of BCC-algebras is not a variety, Math. Japonica 29(1984) 391–394.
[16] K. J. Lee, Bipolar fuzzy subalgerbas and bipolar fuzzy ideals of BCK/BCI- algerbas, Bull. Malays. Math. Sci. Soc. (in press).
[17] K. M. Lee, Bipolar-valued fuzzy sets and their operations, Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand (2000) 307–312.
[18] K. M. Lee, Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets, J. Fuzzy Logic Intelligent Systems 14 (2) (2004) 125–129.
[19] T. K. Mondal and S. K. Samanta, Topology of interval-valued fuzzy sets, Indian J. Pure Appl. Math. 30 (1) (1999) 133–189.
[20] X. Ougen, Fuzzy BCK-algebra, Math. Japon. 36 (5) (1991) 935–942.
[21] M. Riaz and S. TehrimCubic, bipolar fuzzy ordered weighted geometric aggregation operators and their application using internal and external cubic bipolar fuzzy data, Computational and Applied Mathematics (2019) 38–87.
[22] M. Riaz and S. Tehrim, Cubic bipolar fuzzy set with application to multi criteria group decision making, Group Decis Negotiation (Submitted).
[23] T. Senapati, Y. B. Jun, G. Muhiuddin, Cubic intuitionistic structures applied to ideals of BCI-algebras, An St Univ Ovidius Constanta 25 (2) (2019) 213–232.
[24] T. Senapati, Y. B. Jun, K. P. Shum, Cubic Intuitionistic Implicative Ideals of BCK-Algebras Proceedings of the National Academy of Sciences, India Section A: Physical Sciences (2020).
[25] A. Wroński, BCK-algebras do not form a variety, Math. Japon. 28 (2) (1983) 211–213.
[26] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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