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Abstract. In this paper, we introduce the notion of quasi-interior ideal as a generalization of quasi-ideal and interior ideal of semigroup and fuzzy quasi-interior ideal of semigroup. We characterize the regular semigroup in terms of fuzzy quasi-interior ideal of semigroup.

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Keywords: semigroup, regular semigroup, bi-quasi ideal, bi-interior ideal, fuzzy bi-quasi ideal, fuzzy bi-interior ideal, quasi-interior ideal, fuzzy quasi-interior ideal.

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1. Introduction

Semigroup, as the basic algebraic structure was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. The notion of ideals was introduced by Dedekind for the theory of algebraic numbers, was generalized by Noether for associative rings. The one and two sided ideals introduced by her, are still central concepts in ring theory and the notion of a one sided ideal of any algebraic structure is a generalization of notion of an ideal. The notion of the bi-ideal in semigroups is a special case of (m,n) ideals. In 1952, the concept of bi-ideals for semigroup was introduced by Good and Hughes [1] and the notion of bi-ideals in associative rings was introduced by Lazos and Szasz [6, 7]. Steinfeld [18] first introduced the notion of quasi ideals for semigroups and then for rings. The quasi ideals are generalization of left ideals and right ideals whereas the bi-ideals are generalization of quasi ideals. Iseki [2, 3] introduced the concept of quasi ideal for semirings. Rao [11, 12, 13, 14, 15] introduced the notion of bi-interior ideals for semigroups, bi-quasi-ideals and fuzzy bi-quasi-ideals in Γ- semigroups, tri-ideals and fuzzy tri-ideals, quasi-interior ideals and fuzzy quasi-interior ideals, weak-interior ideals and fuzzy weak-interior ideals of Γ- semirings. Quasi ideals in Γ-semirings were studied by Jagtap and Pawar [4].

The fuzzy set theory was developed by Zadeh [20] in 1965. Many papers on fuzzy sets appeared, showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. The fuzzification of algebraic structure was introduced by Rosenfeld [17] and he introduced the notion of fuzzy subgroups in 1971. In 1982, Liu [8] defined and studied fuzzy subrings as well as fuzzy ideals in rings. Swamy and Swamy [19]studied fuzzy prime ideals in rings in 1988. Mandal [9] studied fuzzy ideals and fuzzy interior ideals in ordered semirings. Rao [10] studied fuzzy soft Γ-semirings and fuzzy soft k-ideal over Γ-semirings. Kuroki [5] studied fuzzy interior ideals in semigroups. Rao [16] studied T-fuzzy ideals of ordered Γ-semirings. In this paper, we introduce the notion of quasi-interior ideal and fuzzy quasi-interior ideal of semigroup and we characterize the regular semigroup in terms of fuzzy quasi-interior ideal of semigroup.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. A semigroup is an algebraic system $(S, .)$ consisting of a non-empty set S together with an associative binary operation "$\cdot$".

Definition 2.2. A non-empty subset A of a semigroup M is called:

(i) a subsemigroup of M, if $AA \subseteq A$,
(ii) a quasi ideal of M, if A is a subsemigroup of M and $AM \cap MA \subseteq A$,
(iii) a bi-ideal of M, if A is a subsemigroup of M and $AMA \subseteq A$,
(iv) an interior ideal of M, if A is a subsemigroup of M and $MAM \subseteq A$,
(v) a left (right) ideal of M, if A is a subsemigroup of M and $MA \subseteq A(AM \subseteq A)$,
(vi) an ideal, if $AM \subseteq A$ and $MA \subseteq A$,
(vii) a bi-quasi-interior ideal of M, if A is a subsemigroup of M and $AMAMA \subseteq A$,
(viii) a bi-interior ideal of M, if A is a subsemigroup of M and $MAM \cap AMA \subseteq A$,
(ix) a left (right) bi-quasi ideal of M, if A is a subsemigroup of M and $MA \cap AMA \subseteq A$ $(AM \cap AMA \subseteq A)$,
(x) a bi-quasi ideal of M, if A is a subsemigroup of M, $MA \cap AMA \subseteq A$ *and* $AM \cap AMA \subseteq A$,
(xi) a left (right) tri- ideal of M if A is a subsemigroup of M and $AMAA \subseteq A$ $(AAMA \subseteq A)$.
(xii) a tri- ideal of M if A is a subsemigroup of M, $AMAA \subseteq A$ and $AAMA \subseteq A$.
(xiii) a left(right) weak-interior ideal of M if A is a subsemigroup of M and $MAA \subseteq A(AAM \subseteq A)$.
(xiv) a weak-interior ideal of M if A is a subsemigroup of M and A is a left weak-interior ideal and right weak-interior ideal of M.

Definition 2.3. Let M be a semigroup. An element $1 \in M$ is said to be unity, if $x1 = 1x = x$ for all $x \in M$.

Definition 2.4. A semigroup M is a left (right) simple semigroup, if M has no proper left (right) ideals.

Definition 2.5. A semigroup M is a bi-quasi simple semigroup, if M has no proper bi-quasi ideals.

Definition 2.6. A semigroup M is said to be simple semigroup, if M has no proper ideals.

Definition 2.7. Let M be a non-empty set. A mapping $f : M \to [0, 1]$ is called a fuzzy subset of a semigroup M. If f is not a constant function, then f is called a non-empty fuzzy subset.

Definition 2.8. Let f be a fuzzy subset of a non-empty set M. Then for $t \in [0, 1]$, the set $f_t = \{x \in M \mid f(x) \geq t\}$ is called a level subset of M with respect to f.

Definition 2.9. Let M be a semigroup. A fuzzy subset μ of M is said to be fuzzy subsemigroup of M, if it satisfies the following condition: for all $x,\ y \in M$,

$$\mu(xy) \geq min\ \{\mu(x), \mu(y)\}.$$

Definition 2.10. A fuzzy subset μ of a semigroup M is called a fuzzy left (right) ideal of M, if it satisfies the following condition: for all $x,\ y \in M$,

$$\mu(xy) \geq \mu(y)\ (\mu(x)).$$

Definition 2.11. A fuzzy subset μ of a semigroup M is called a fuzzy ideal of M, if it satisfies the following condition: for all $x,\ y \in M$,

$$\mu(xy) \geq max\ \{\mu(x), \mu(y)\}.$$

Definition 2.12. For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$ for all $x \in M$.

Definition 2.13. Let f and g be fuzzy subsets of a semigroup M.
Then $f \circ g, f \cup g, f \cap g$, are defined by

$$f \circ g(z) = \begin{cases} \sup\limits_{z=xy} \{min\{f(x), g(y)\}\}, \\ 0, \quad \text{otherwise}, \end{cases}$$

$$f \cup g(z) = max\{f(z), g(z)\},\ f \cap g(z) = min\{f(z), g(z)\},$$

for all $x, y, z \in M$.

Definition 2.14. A function $f : R \to M$ is said to be semigroup homomorphism, if $f(ab) = f(a)f(b)$ for all $a,\ b \in R$, where R and M are semigroups.

Definition 2.15. Let A be a non-empty subset of M. The characteristic function of A is a fuzzy subset of M,it is denoted by $\chi_{_A}$ and defined as

$$\chi_{_A}(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$$

3. Quasi-interior ideals of semigroups

In this section, we introduce the notion of quasi-interior ideal as a generalization of quasi-ideal and interior ideal of semigroup and study the properties of quasi-interior ideal of semigroup.Throughout this paper M is a semigroup with unity element.

Definition 3.1. A non-empty subset B of a semigroup M is said to be left quasi-interior ideal of M, if B is a subsemigroup of M and $MBMB \subseteq B$.

Definition 3.2. A non-empty subset B of a semigroup M is said to be right quasi-interior ideal of M, if B is a subsemigroup of M and $BMBM \subseteq B$.

Definition 3.3. A non-empty subset B of a semigroup M is said to be quasi-interior ideal of M, if B is a subsemigroup of M, B is a left quasi-interior ideal and a right quasi-interior ideal of M.

Remark 3.4. A quasi-interior ideal of a semigroup M need not be quasi-ideal, interior ideal, bi-interior ideal. and bi-quasi ideal of semigroup M.

Example 3.5. Let Q be the set of all rational numbers and $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\}$. Then M is a semigroup with respect to usual addition of matrices and the usual matrix multiplication. If $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, 0 \neq b \in Q \right\}$, then A is not a bi-ideal of semigroup M.

Example 3.6. Let $M = \{0, 1, 2, 3, 4\}$ .Binary operation is defined $(x, y) \rightarrow x + y$ as $x + y$ if$x + y \leq 4, 4$ if$x + y > 4$, where $+$ is the usual addition of integers. Then M is a semigroup.Let $I = \{0, 2, 4\}$ .Then I is a quasi- interior ideal of M.

In the following theorem, we mention some important properties and we omit the proofs since they are straight forward.

Theorem 3.7. *Let M be a semigroup. Then the following are hold.*

(1) *Every left ideal is a left quasi-interior ideal of M.*
(2) *Every right ideal is a right quasi-interior ideal of M.*
(3) *Every quasi ideal is a quasi-interior ideal of M.*
(4) *Every ideal is a quasi-interior ideal of M.*
(5) *The intersection of a right ideal and a left ideal of M is a quasi-interior ideal of M.*
(6) *If L is a left ideal and R is a right ideal of a semigroup M then $B = RL$ is a quasi-interior ideal of M.*
(7) *If B is a quasi-interior ideal and T is a subsemigroup of M then $B \cap T$ is a quasi-interior ideal of M.*
(8) *If B be a subsemigroup of M and $MMMB \subseteq B$ then B is a left quasi-interior ideal of M.*
(9) *If B is a subsemigroup of M, $MMMB \subseteq B$ and $BMMM \subseteq B$ then B is a quasi-interior ideal of M.*
(10) *If L is a left ideal and R is a right ideal of M then $B = R \cap L$ is a quasi-interior ideal of M.*

Theorem 3.8. *If B is a left quasi-interior ideal of a semigroup M, then B is a left bi-quasi ideal of M.*

Proof. Suppose B is a left quasi-interior ideal of the semigroup M. Then $MBMB \subseteq B$. Thus $BMB \subseteq MBMB$. So $MB \cap BMB \subseteq BMB \subseteq MBMB \subseteq B$. Hence B is a left bi-quasi ideal of M. □

Corollary 3.9. *If B is a right quasi-interior ideal of a semigroup M, then B is a right bi-quasi ideal of M.*

Corollary 3.10. *If B is a quasi-interior ideal of a semigroup M, then B is a bi-quasi ideal of M.*

Theorem 3.11. *If B is a left quasi-interior ideal of a semigroup M, then B is a bi–interior ideal of M.*

Proof. Suppose B is a left quasi-interior ideal of the semigroup M. Then $MBMB \subseteq B$. Thus $MBM \cap BMB \subseteq BMB \subseteq MBMB \subseteq B$. So B is a bi–interior ideal of M. □

Corollary 3.12. *If B is a right quasi-interior ideal of a semigroup M, then B is a bi–interior ideal of M.*

Corollary 3.13. *If B is a quasi-interior ideal of a semigroup M, then B is a bi–interior ideal of M.*

Theorem 3.14. *Every left quasi-interior ideal of a semigroup M is a bi-ideal of M.*

Proof. Let B be a left quasi-interior ideal of the semigroup M. Then $BMB \subseteq MBMB \subseteq B$. Thus $BMB \subseteq B$. So every left quasi-interior ideal of M is a bi-ideal of the semigroup M. □

Corollary 3.15. *Every right quasi-interior ideal of a semigroup M is a bi-ideal of M.*

Corollary 3.16. *Every quasi-interior ideal of a semigroup M is a bi-ideal of M.*

4. Fuzzy quasi interior ideals of semigroups

In this section, we introduce the notion of fuzzy right(left) bi-quasi ideal as a generalization of fuzzy bi-ideal of a semigroup and study the properties of fuzzy right(left) bi-quasi ideals.

Definition 4.1. A fuzzy subset μ of a semigroup M is called a fuzzy left (right) bi-quasi ideal, if $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu\big(\mu \circ \chi_M \circ \mu \circ \chi_M \subseteq \mu\big)$.

Definition 4.2. A fuzzy subset μ of a semigroup M is called a fuzzy quasi-interior ideal, if it is both a fuzzy left quasi-interior ideal and a fuzzy right quasi-interior ideal of M.

Example 4.3. Let M be a set of all natural numbers. Binary operation is defined as $(x, y) \to x+y$, where $+$ is the usual addition of integers. Then M is a semigroup. Let I be a set of all even natural numbers. Then I is a quasi-interior ideal of M but not quasi-ideal, interior ideal, bi-interior ideal and bi-quasi ideal of M.

Define $\mu : M \to [0,1]$ such that $\mu(x) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{otherwise.} \end{cases}$
Then μ is a fuzzy quasi-interior ideal of M.

Theorem 4.4. *Every fuzzy right ideal of a semigroup M is a fuzzy right quasi-interior ideal of M.*

Proof. Let μ be a fuzzy right ideal of the semigroup M and $x \in M$. Then

$$\begin{aligned} \mu \circ \chi_M(x) &= \sup_{x=ab} \min\{\mu(a), \chi_M(b)\}, a, b \in M \\ &= \sup_{x=ab} \mu(a) \\ &\leq \sup_{x=ab} \mu(ab) \\ &= \mu(x). \end{aligned}$$

Thus $\mu \circ \chi_M(x) \leq \mu(x)$.

On the other hand

$$\begin{aligned} \mu \circ \chi_M \circ \mu \circ \chi_M(x) &= \sup_{x=uvs} \min\{\mu \circ \chi_M(uv), \mu \circ \chi_M(s)\} \\ &\leq \sup_{x=uvs} \min\{\mu(uv), \mu(s)\} \\ &= \mu(x). \end{aligned}$$

So μ is a fuzzy right quasi-interior ideal of the semigroup M. □

Corollary 4.5. *Every fuzzy left ideal of a semigroup M is a fuzzy left quasi-interior ideal of M.*

Corollary 4.6. *Every fuzzy ideal of a semigroup M is a fuzzy quasi-interior ideal of M.*

Theorem 4.7. *Let M be a semigroup and μ be a non-empty fuzzy subset of M. A fuzzy subset μ is a fuzzy left quasi-interior ideal of a semigroup M if and only if the level subset μ_t of μ is a left quasi-interior ideal of M for every $t \in [0,1]$, where $\mu_t \neq \phi$.*

Proof. Let M be a semigroup and μ be a non-empty fuzzy subset of M. Suppose μ is a fuzzy left quasi-interior ideal of a semigroup M, $\mu_t \neq \phi, t \in [0,1]$ and $a, b \in \mu_t$. Then $\mu(a) \geq t,\ \mu(b) \geq t$. Thus $\mu(ab) \geq \min\{\mu(a), \mu(b)\} \geq t$. So $ab \in \mu_t$.

Let $x \in M\mu_t M\mu_t$. Then $x = badc$, where $b, d \in M, a, c \in \mu_t$. Thus

$$\chi_M \circ \mu \circ \chi_M \circ \mu(x) \geq t.$$

So $\mu(x) \geq \chi_M \circ \mu \circ \chi_M \circ \mu(x) \geq t$. Hence $x \in \mu_t$. Therefore μ_t is a left quasi-interior ideal of the semigroup M.

Conversely, suppose that μ_t is a left quasi-interior ideal of the semigroup M, for all $t \in Im(\mu)$. Let $x, y \in M, \mu(x) = t_1, \mu(y) = t_2$ and $t_1 \geq t_2$. Then $x, y \in \mu_{t_2}$. Thus $xy \in \mu_{t_2}$. So $\mu(xy) \geq t_2 = \min\{t_1, t_2\} = \min\{\mu(x), \mu(y)\}$. Hence

$$\mu(xy) \geq t_2 = \min\{\mu(x), \mu(y)\}.$$

We have $M\mu_l M\mu_l \subseteq \mu_l$, for all $l \in Im(\mu)$.
Suppose $t = \min\{Im(\mu)\}$. Then $M\mu_t M\mu_t \subseteq \mu_t$. Thus $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$. So μ is a fuzzy left quasi-interior ideal of the semigroup M. □

Corollary 4.8. *Let M be a semigroup and μ be a non-empty fuzzy subset of M. A fuzzy subset μ is a fuzzy right quasi-interior ideal of a semigroup if and only if the level subset μ_t of μ is a right quasi-interior ideal of M for every $t \in [0,1]$, where $\mu_t \neq \phi$.*

Theorem 4.9. *Let I be a non-empty subset of a semigroup M and χ_I be the characteristic function of I. Then I is a right quasi-interior ideal of a semigroup M if and only if χ_I is a fuzzy right quasi-interior ideal of a semigroup M.*

Proof. Let I be a non-empty subset of the semigroup M and χ_I be the characteristic function of I. Suppose I is a right quasi-interior ideal of the semigroup M. Obviously χ_I is a fuzzy subsemigroup of M. Thus we have $IMIM \subseteq I$. So

$$\begin{aligned}\chi_I \circ \chi_M \circ \chi_I \circ \chi_M &= \chi_{IMIM}\\ &= \chi_{IMIM}\\ &\subseteq \chi_I.\end{aligned}$$

Hence χ_I is a fuzzy right quasi-interior ideal of the semigroup M.

Conversely, suppose that χ_I is a fuzzy right quasi-interior ideal of M. Then I is a subsemigroup of M. Thus we have $\chi_I \circ \chi_M \circ \chi_I \circ \chi_M \subseteq \chi_I$. So $\chi_{IMIM} \subseteq \chi_I$. Hence $IMIM \subseteq I$. Therefore I is a right quasi-interior ideal of the semigroup M. □

Theorem 4.10. *If μ and λ are fuzzy left quasi-interior ideals of a semigroup M, then $\mu \cap \lambda$ is a fuzzy left quasi-interior ideal of M.*

Proof. Let μ and λ be fuzzy left bi- quasi ideals of the semigroup M and $x, y \in M$. Then

$$\begin{aligned}\mu \cap \lambda(xy) &= \min\{\mu(xy), \lambda(xy)\}\\ &\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\}\\ &= \min\{\min\{\mu(x), \lambda(x)\}, \min\{\mu(y), \lambda(y)\}\}\\ &= \min\{\mu \cap \lambda(x), \mu \cap \lambda(y)\},\\ \chi_M \circ \mu \cap \lambda(x) &= \sup_{x=ab} \min\{\chi_M(a), \mu \cap \lambda(b)\}\\ &= \sup_{x=ab} \min\{\chi_M(a), \min\{\mu(b), \lambda(b)\}\}\\ &= \sup_{x=ab} \min\{\min\{\chi_M(a), \mu(b)\}, \min\{\chi_M(a), \lambda(b)\}\}\\ &= \min\{\sup_{x=ab} \min\{\chi_M(a), \mu(b)\}, \sup_{x=ab} \min\{\chi_M(a), \lambda(b)\}\}\\ &= \min\{\chi_M \circ \mu(x).\chi_M \circ \lambda(x)\}\\ &= \chi_M \circ \mu \cap \chi_M \circ \lambda(x).\end{aligned}$$

Thus

$$\chi_M \circ \mu \cap \lambda = \chi_M \circ \mu \cap \chi_M \circ \lambda$$

$$\begin{aligned}&\chi_M \circ \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda(x)\\ &= \sup_{x=abc} \min\{\chi_M \circ \mu \cap \chi_M \circ \lambda(a), \chi_M \circ \mu \cap \lambda(bc)\}\\ &= \sup_{x=abc} \min\{\chi_M \circ \mu \cap \chi_M \circ \lambda(a), \chi_M \circ \mu \cap \chi_M \circ \lambda(bc)\}\end{aligned}$$

$$= \sup_{x=abc} \min\{\chi_M \circ \mu(a), \chi_M \circ \lambda(a)\}\}, \min\{\chi_M \circ \mu(bc), \chi_M \circ \lambda(bc)\}\}$$
$$= \sup_{x=abc} \min\{\min\{\chi_M \circ \mu(a), \chi_M \circ \mu(bc)\}, \min\{\chi_M \circ \lambda(a), \chi_M \circ \lambda(bc)\}\}$$
$$\min\{\sup_{x=abc} \min\{\chi_M \circ \mu(a), \chi_M \circ \mu(bc)\}, \sup_{x=abc} \min\{\chi_M \circ \lambda(a), \chi_M \circ \lambda(bc)\}\}$$
$$= \min\{\chi_M \circ \mu \circ \chi_M \circ \mu(x), \chi_M \circ \lambda \circ \chi_M \circ \lambda(x)\}$$
$$= \chi_M \circ \mu \circ \chi_M \circ \mu \cap \chi_M \circ \lambda \circ \chi_M \circ \lambda(x).$$

So $\chi_M \circ \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = \chi_M \circ \mu \circ \chi_M \circ \mu \cap \chi_M \circ \lambda \circ \chi_M \circ \lambda$. Hence

$$\chi_M \circ \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = \chi_M \circ \mu \circ \chi_M \circ \mu \cap \chi_M \circ \lambda \circ \chi_M \circ \lambda \subseteq \mu \cap \lambda.$$

Therefore $\mu \cap \lambda$ is a left fuzzy quasi-interior ideal of M. □

Corollary 4.11. *If μ and λ are fuzzy right quasi-interior ideals of a semigroup M then $\mu \cap \lambda$ is a fuzzy right quasi-interior ideal of M.*

Corollary 4.12. *Let μ and λ be fuzzy quasi-interior ideals of a semigroup M . Then $\mu \cap \lambda$ is a fuzzy quasi-interior ideal of M.*

The proofs of the following theorems are similar to theorems in [15]. So we omit the proofs.

Theorem 4.13. *M is a regular semigroup if and only if $AB = A \cap B$ for any right ideal A and left ideal B of M.*

Theorem 4.14. *If μ is a fuzzy quasi-ideal of a regular semigroup M then μ is a fuzzy ideal of M.*

Theorem 4.15. *A semigroup M is a regular if and only if $\lambda \circ \mu = \lambda \cap \mu$, for any fuzzy right ideal λ and fuzzy left ideal μ of M.*

Theorem 4.16. *Let M be a regular semigroup. Then μ is a fuzzy left quasi-interior ideal of M if and only if μ is a fuzzy quasi ideal of M.*

Proof. Let μ be a fuzzy left quasi-interior ideal of the semigroup M and $x \in M$. Then $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$. Suppose $\chi_M \circ \mu(x) > \mu(x)$ and $\mu \circ \chi_M(x) > \mu(x)$. Since M is a regular, there exists $y \in M$ such that $x = xyx$. Then

$$\begin{aligned} \mu \circ \chi_M(x) &= \sup_{x=xyx} \min\{\mu(x), \chi_M(yx)\} \\ &= \sup_{x=xyx} \min\{\mu(x), 1\} \\ &= \sup_{x=xyx} \mu(x) \\ &> \mu(x), \end{aligned}$$

$$\begin{aligned} \mu \circ \chi_M \circ \mu \circ \chi_M(x) &= \sup_{x=xyx} \min\{\mu \circ \chi_M(x), \mu \circ \chi_M(yx)\}) \\ &> \sup_{x=xyx} \min\{\mu(x), \mu(yx)\} \\ &= \mu(x). \end{aligned}$$

Which is a contradiction. Thus μ is a fuzzy quasi ideal of M.

By Theorem 4.14, converse is true. □

Corollary 4.17. *Let M be a regular semigroup.Then μ is a fuzzy right quasi-interior ideal of M if and only if μ is a fuzzy quasi ideal of M.*

Theorem 4.18. *Let M be a regular left simple semigroup Then B is a left quasi-interior ideal of a regular semigroup M if and only if $MBMB = B$ for all left quasi-interior ideals B of M.*

Proof. Suppose M is a regular left simple semigroup, B is a left quasi-interior ideal of M and $x \in B$. Then $MBMB \subseteq B$ and there exists $y \in M$, such that $x = xyxyx \in MBMB$. Thus $x \in MBMB$. So $MBMB = B$.

Conversely, suppose that $MBMB = B$ for all left quasi-interior ideals B of M. Let $B = R \cap L$ where R is a right ideal and L is an ideal of M. Then B is a quasi-interior ideal of M. Thus $(R \cap L)M(R \cap L)M = R \cap L$. On the other hand,

$$\begin{aligned} R \cap L &= (R \cap L)M(R \cap L)M \\ &\subseteq RMLM \\ &\subseteq RLM, \\ R \cap L &= (R \cap L)M(R \cap L)M \\ &\subseteq RLMRLM \\ &\subseteq RL \\ &\subseteq R \cap L \text{ (since } RL \subseteq L \text{ and } RL \subseteq R). \end{aligned}$$

So $R \cap L = RL$. Hence M is a regular$-$semigroup. □

Corollary 4.19. *Let M be a regular right simple semigroup. Then B is a right quasi-interior ideal of a regular semigroup M if and only if $BMBM = B$ for all right quasi-interior ideals B of M.*

Corollary 4.20. *Let M be a simple semigroup. Then B is a quasi-interior ideal of a regular semigroup M if and only if $BMBM = B$ and $MBMB = B$ for all quasi-interior ideals B of M.*

Theorem 4.21. *Let M be a left simple semigroup. Then M is a regular if and only if $\mu = \chi_M \circ \mu \circ \chi_M \circ \mu$, for any fuzzy left quasi-interior ideal μ of M.*

Proof. Let μ be a fuzzy left quasi-interior ideal of the regular semigroup M and $x \in M$. Then $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$. Thus

$$\begin{aligned} \chi_M \circ \mu \circ \chi_M \circ \mu(x) &= \sup_{x=xyx} \{\min\{\chi_M \circ \mu(x), \chi_M \circ \mu(yx)\}\} \\ &\geq \sup_{x=xyx} \{\min\{\mu(x), \mu(x)\}\} \\ &= \mu(x). \end{aligned}$$

So $\mu \subseteq \chi_M \circ \mu \circ \chi_M \circ \mu$. Hence $\chi_M \circ \mu \circ \chi_M \circ \mu = \mu$.

Conversely, suppose that $\mu = \chi_M \circ \mu \circ \chi_M \circ \mu$, for any fuzzy quasi-interior ideal μ of M. Let B be a left quasi-interior ideal of the semigroup M. Then by Theorem

4.8, χ_B be a fuzzy quasi-interior ideal of M. Thus

$$\begin{aligned} \chi_B &= \chi_M \circ \chi_B \circ \chi_M \circ \chi_B \\ &= \chi_{MBMB} \\ B &= MBMB. \end{aligned}$$

So by Theorem 4.18, M is a regular semigroup . □

5. Conclusion

In this paper, we introduced the notion of fuzzy right (left) quasi-interior ideal of a semigroup and characterized the regular semigroup in terms of fuzzy right(left)quasi-interior ideals of a semigroup and studied some of their algebraical properties.

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