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Abstract. Attribute reduction plays an important role in formal concept analysis. The existing work on attribute reduction of fuzzy formal context focuses mainly on one-sided fuzzy formal concepts. This paper is devoted to the study of attribute reduction for general L-fuzzy formal context. Some judgement theorems for a set of attributes to be consistent are presented. By using the discernibility function composed of discernibility attributes between related L-fuzzy formal concepts, an approach to compute attribute reductions is proposed. An illustrative example is presented which justify the method presented in this study.

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## 1. Introduction

The theory of formal concept analysis (FCA), proposed by Wille [20] in 1982, is one of the effective mathematical tools in conceptual data analysis and knowledge processing [14, 18, 19]. The main research object of FCA is formal context and formal concept. A formal context is a triple $(G, M, I)$, where $G$ is a set of objects, $M$ is a set of attributes, and $I$ is a crisp binary relation on $G \times M$. The formal concept, generated by using a pair of antitone Galois connection operators, is a pair $(X, B)$ of a set $X$ of objects and a set $B$ of attributes determined by a pair of derivation operators. The set of all formal concepts forms a complete lattice, called concept lattice.

In real life questions, fuzzy information is more ordinary than certain information. In this case, it is difficult to describe knowledge more accurately with classical formal context. Therefore, some scholars have applied fuzzy set theory to the formal concept analysis and proposed the fuzzy formal concept analysis. The binary relation in formal context is replaced by a fuzzy relation between objects and attributes, which induced the fuzzy formal context. For example, Bělohlávek [1] proposed the fuzzy formal concept in fuzzy formal contexts based on some implication operators of residuated lattice, and proved that the pair of operators forms a Galois connection. Popescu [15] presented a general method for constructing fuzzy concepts of a fuzzy formal context. Medina and Ojeda-Aciego [23] developed so-called t-concept lattice as a set of triples, which are related to graded tabular information interpreted in a non-commutative fuzzy logic. Jaoua and Elloumi [7] generalized the binary fuzzy relation to a real-set binary relation and studied the corresponding concept lattice theory. Krajči [8], Yahia and Jaoua [22] put forward the 'one-sided fuzzy concept' independently so that the number of fuzzy formal concepts generated by fuzzy formal context is reduced. Zhang et al. [24] further defined variable threshold concept lattice in fuzzy formal context.

Attribute reduction has always been a hot topic in the study of concept lattice theory, and its purpose is to find a minimal subset of attributes that keep some properties of formal context unchanged. Research on attribute reduction of classical formal context has been relatively mature. The criterion attribute reduction in classical formal contexts can be roughly divided into two categories: Firstly, attribute reduction to keep the concept lattice structure unchanged. In [25], Zhang et al. put forward the theory of attribute reduction based on concept lattice structure, and used the method of discernibility matrix and discernibility function(DM method) to calculate all reductions. Secondly, attribute reduction to keep the set of all object concepts. This kind of attribute reduction is also called granular reduction [**?**]. Additionally, Cao et al. in [9] proposed a method of concept reduction to keep the binary relation of formal context unchanged.

In recent years, many scholars have studied the approaches of attribute reduction for generalization model of concept lattice. For instance, Li. et al. [10] proposed the method of knowledge reduction in decision formal contexts. And Li et al. [11] presented a rule acquisition oriented framework of knowledge reduction for real decision formal contexts and formulated a reduction method. Li et al. [21] also proposed an approach for extracting non-redundant approximate decision rules from incomplete decision contexts and presented a knowledge reduction framework. There are also many studies on knowledge reduction in fuzzy formal contexts. Bělohlavek et al. [4] proposed a method to reduce the number of formal fuzzy concepts by only keeping the so-called crisply generated fuzzy concepts which are generated from some crisp subset of attributes, leaving out non-crisply generated fuzzy concepts. In some cases, this method will lead to a loss of formal contexts knowledge. Based on the Lukasiewicz implication, Elloumi et al. [6] gave a multi-level conceptual data reduction approach via the reduction of the object sets by keeping the minimal rows in a formal context. Li and Zhang [12] reconsidered the reduction issue in the formal fuzzy contexts by replacing the Lukasiewicz implication in [8] with T-implication. Based on the idea of variable threshold concept lattices [24], Shao et al. [16] proposed a method to reduce the attributes or objects of one-sided fuzzy formal concept. Li et al. [13] proposed a method of attribute reduction for formal fuzzy context by using the cut set of extent of formal fuzzy concepts.

We note that most of the existing work on attribute reduction of fuzzy formal context focused on one-sided fuzzy formal concepts or on the fuzzy formal concepts containing classical information. There are few researches on attribute reduction based on general fuzzy formal concepts in fuzzy formal contexts. In view of the conceptual induction operator defined by Bělohlávek [1], this paper proposed a new method to reduced attributes of fuzzy formal contexts similarly to the research on reduction for classical formal contexts [13]. In addition, this method is also applicable when objects sets and attributes sets are fuzzy sets on general complete residuated lattices.

## 2. Preliminaries

In this section, we briefly review some notions of fuzzy formal contexts and fuzzy concept lattice which are used in this paper.

**Definition 2.1** ([1, 3])**.** A residuated lattice is a structure $\mathbf{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ such that:

(i) $(L, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,

(ii) $(L, \otimes, 1)$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$)

(iii) $\otimes, \rightarrow$ form an adjoint pair, i.e., $x \leq y \rightarrow z$ iff $x \otimes y \leq z$ hold for all $x, y, z \in L$.

If the lattices $(L, \wedge, \vee, 0, 1)$ is complete, the residuated lattice $\mathbf{L}$ is called complete. In a fuzzy setting, the lattices $\mathbf{L}$ defines the scale of degrees of truth. A common choice of $L$ is a structure with $L = [0, 1]$, there are typically three kinds of companion pairs $(\otimes, \rightarrow)$: If $x \otimes y = max(x + y - 1, 0)$ and $x \rightarrow y = min(1 - x + y, 1)$, we obtain the Lukasiewicz residuated lattices. If $(x \otimes y = min(x, y)$ and $x \rightarrow y = 1$ if $x \leq y$, $x \rightarrow y = y$ otherwise, we get the Godel residuated lattices. If $x \otimes y = x \cdot y$, $x \rightarrow y = 1$ if $x \leq y$ and $x \rightarrow y = y/x$ otherwise, we obtain the product residuated lattices.

**Proposition 2.2.** *Let $\boldsymbol{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a complete residuated lattice. The following properties are established* ([17])*:*

(1) *the multiplication operation $\otimes$ is isotone in every argument, i.e., for all $x_1, x_2, y_1, y_2 \in L$, if $x_1 \leq x_2$ and $y_1 \leq y_2$, then $x_1 \otimes x_2 \leq y_1 \otimes y_2$,*

(2) *the operation $\rightarrow$ is antitone in its first argument, and isotone in its second argument, i.e., for all $x_1,\ x_2,\ y \in L$, if $x_1 \leq x_2$, then $x_2 \rightarrow y \leq x_1 \rightarrow y$, and for all $x,\ y_1,\ y_2 \in L$, if $y_1 \leq y_2$, then $x \rightarrow y_1 \leq x \rightarrow y_2$,*

(3) *$\otimes$ distributes over arbitrary joins, i.e., for all $x \in L$, $\{y_i\ ;\ i \in I\} \subseteq L$,*

$$x \otimes (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \otimes y_i),$$

(4) *for all $x \in L$, $\{y_i\ ;\ i \in I\} \subseteq L$, $x \rightarrow \bigwedge_{i \in I} y_i = \bigvee_{i \in I} (x \rightarrow y_i)$, $\bigvee_{i \in I} y_i \rightarrow x = \bigwedge_{i \in I} (y_i \rightarrow x)$,*

(5) *for all $x,\ y \in L$, $x \rightarrow y = \bigvee \{z \in L | x \otimes z \leq y\}$,*

(6) *$1 \rightarrow x = x$, $x \rightarrow 1 = 1$,*

(7) *$x \rightarrow y = 1$ iff $x \leq y$,*

(8) *$0 \otimes x = x \otimes 0 = 0$,*

(9) *$x \otimes y \leq x$, $x \otimes y \leq y$,*

(10) *$(x \rightarrow y) \otimes (y \rightarrow z) \leq x \rightarrow z$,*

(11) *$x \leq \neg(\neg x)$,*

(12) $x \to y \leq \neg y \to \neg x$,
*where* $\neg y = y \to 0$.

**Definition 2.3** ([24]). An L-fuzzy formal context is a triple $(G,\ M,\ I)$, where $G = \{g_1, g_2, \cdots, g_n\}$ is a nonempty finite set of objects, $M = \{m_1, m_2, \cdots, m_t\}$ is a nonempty finite set of attributes, $I$ is a L-fuzzy relation between $G$ and $M$, i.e., $I\colon G \times M \to L$, and $L$ is a finite residuated lattice.

An L-fuzzy formal context can be represented by a table in which the rows are headed by the object names and the columns are headed by the attribute names. Table 1 depicts an example of L-fuzzy formal context, where the set of objects $G = \{x_1, x_2, x_3\}$ and the set of attributes $M = \{a_1, a_2, a_3, a_4, a_5\}$, $L = \{0, 0.5, 1\}$ defines the scale of degree of truth. Example $(x_1, a_2) = 0.5$ indicated that the degree of the object $x_1$ with the attribute $a_2$ is 0.5.

Table 1. The fuzzy formal concepts

| $I$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|
| $x_1$ | 1 | 0.5 | 0 | 1 | 1 |
| $x_2$ | 0.5 | 1 | 0.5 | 0 | 0 |
| $x_3$ | 0 | 1 | 1 | 0.5 | 0.5 |

The knowledge of a formal context is the formal concept, Bělohlávek [1] define the operators as follows:

**Definition 2.4** ([1]). Let $(G,\ M,\ I)$ be an L-fuzzy formal context. For $A \in L^G$ and $B \in L^M$, two operators are defined as below:

$$A^{\uparrow}(m) = \bigwedge_{x \in G} (A(x) \to I(x, m)), m \in M,$$

$$B^{\downarrow}(g) = \bigwedge_{m \in M} (B(m) \to I(g, m)), g \in G.$$

By Definition 2.4, it is straightforward to conclude the following properties:

**Proposition 2.5.** *Let $(G,\ M,\ I)$ be an L-fuzzy formal context, $X,\ X_1,\ X_2,\ X_i \in L^G$, and $B,\ B_1,\ B_2,\ B_i \in L^M$, $i \in I$, $I$ is an index set, then*
(1) $X_1 \subseteq X_2 \Rightarrow X_2^{\uparrow} \subseteq X_1^{\uparrow}$, $B_1 \subseteq B_2 \Rightarrow B_2^{\downarrow} \subseteq B_1^{\downarrow}$,
(2) $X \subseteq X^{\uparrow\downarrow}$, $B \subseteq B^{\downarrow\uparrow}$,
(3) $X^{\uparrow} = X^{\uparrow\downarrow\uparrow}$, $B^{\downarrow} = B^{\downarrow\uparrow\downarrow}$,
(4) $X \subseteq A^{\downarrow} \iff A \subseteq X^{\uparrow}$,
(5) $(\bigcup_{i \in I} X_i)^{\uparrow} = \bigcap_{i \in I} X_i^{\uparrow}$, $(\bigcup_{i \in I} B_i)^{\downarrow} = \bigcap_{i \in I} B_i^{\downarrow}$.

It is shown that the pair $(\uparrow, \downarrow)$ forms a Galois connection between $L^G$ and $L^M$. A pair $(A, B)$ is called an L-fuzzy formal concept, if $A^{\uparrow} = B$ and $B^{\downarrow} = A$, where $A$ is referred to as the extent of the L-formal fuzzy concept $(A, B)$ and $B$ is called its intent. The set of all L-fuzzy formal concepts forms a complete lattice called an

L-fuzzy concept lattice and is denoted by $L(G,M,I)$. The meet $\wedge$ and join $\vee$ of the fuzzy lattice are given by: for any $(X_1,B_1),\ (X_2,B_2)\in L(G,M,I)$,

$$(X_1,\ B_1)\wedge(X_2,\ B_2)=(X_1\cap X_2,\ (B_1\cup B_2)^{\downarrow\uparrow}),$$

$$(X_1,\ B_1)\vee(X_2,\ B_2)=((X_1\cup X_2)^{\downarrow\uparrow},\ B_1\cap B_2).$$

The corresponding partial order relation $\leq$ on the fuzzy formal concept lattice $L(G,M,I)$ is given by:

$$(X_1,B_1)\leq(X_2,B_2)\iff X_1\subseteq X_2 \text{ (or equally } B_2\subseteq B_1)$$

If $(X_1,B_1)\leq(X_2,B_2)$, $(X_1,B_1)$ is called a sub-concept of $(X_2,B_2)$ and $(X_2,B_2)$ is called a super-concept of $(X_1,B_1)$. The notation $(X_1,B_1)<(X_2,B_2)$ denotes that $(X_1,B_1)\leq(X_2,B_2)$ and $(X_1,B_1)\neq(X_2,B_2)$. If $(X_1,B_1)<(X_2,B_2)$ and there does not exist a concept $(X,B)$ such that $(X_1,B_1)<(X,B)<(X_2,B_2)$, then $(X_1,B_1)$ is called an immediate sub-concept of $(X_2,B_2)$ and $(X_2,B_2)$ is called an immediate super-concept of $(X_1,B_1)$, which is denoted by $(X_1,B_1)\prec(X_2,B_2)$.

## 3. Attribute reduction in L-fuzzy formal contexts

Let $(G,\ M,\ I)$ be an L-fuzzy formal context, $D\subseteq M$. $D$ induces an L-fuzzy formal context $(G,\ D,\ I_D)$ which is called a subcontext of $(G,\ M,\ I)$, and $I_D$ is given by $I_D=I\big|_{G\times D}$. To avoid confusion, the operators defined for $(G,\ D,\ I_D)$ will be denoted by $\uparrow_D$ and $\downarrow_D$ respectively. One can conclude the following property:

(1) for any $A\in L^G, A^{\uparrow_D}=A^{\uparrow}\big|_D$,

(2) for any $B\in L^D,\ B^{\downarrow_D}=\overline{B}^{\downarrow}$, where $\overline{B}\in L^M$ is given by:

$$\overline{B}(m)=\begin{cases}B(m), & m\in D\\ 0, & m\in M-D\end{cases}$$

**Definition 3.1.** Let $L(G,\ M_1,\ I_1)$ and $L(G,\ M_2,\ I_2)$ be two L-fuzzy formal concept lattices. $L(G,\ M_1,\ I_1)$ is called finer than $L(G,\ M_2,\ I_2)$, denoted by $L(G,\ M_1,\ I_1)\leq L(G,\ M_2,\ I_2)$, if for each $(A,B)\in L(G,\ M_2,\ I_2)$, there always exists $C\in L^{M_1}$ such that $(A,C)\in L(G,\ M_1,\ I_1)$.

Clearly, $L(G,\ M_1,\ I_1)\leq L(G,\ M_2,\ I_2)$ if and only if $ExtL(G,\ M_2,\ I_2)\subseteq ExtL(G,\ M_1,\ I_1)$, where

$$ExtL(G,\ M_1,\ I_1)=\{X\in L^G;\ \ \exists A\in L^{M_1}\ ((X,A)\in L(G,\ M_1,\ I_1))\}$$

is the set of all extents of concepts in $L(G,\ M_1,\ I_1)$. If $L(G,\ M_1,\ I_1)\leq L(G,\ M_2,\ I_2)$ and $L(G,\ M_2,\ I_2)\leq L(G,\ M_1,\ I_1)$, then the lattices $L(G,\ M_1,\ I_1)$ and $L(G,\ M_2,\ I_2)$ are isomorphic, and denoted by $L(G,\ M_1,\ I_1)\cong L(G,\ M_2,\ I_2)$.

**Theorem 3.2.** *Let $(G,\ M,\ I)$ be an L-fuzzy formal context, $D\subseteq M$, $(G,\ D,\ I_D)$ is a subcontext of $(G,\ M,\ I)$. Then $L(G,\ M,\ I)\leq L(G,\ D,\ I_D)$.*

*Proof.* It suffices to show that for any $(A,B)\in L(G,\ D,\ I_D)$ , we have $(A,A^{\uparrow})\in L(G,\ M,\ I)$. In fact, since $A^{\uparrow}\supseteq\overline{A^{\uparrow}\big|_D}$, from properties of the operator, we have

$$A\subseteq A^{\uparrow\downarrow}\subseteq\overline{(A^{\uparrow}\big|_D)}^{\downarrow}=(A^{\uparrow}\big|_D)^{\downarrow_D}=A^{\uparrow_D\downarrow_D}=A.$$

Thus $A=A^{\uparrow\downarrow}$. So $(A,A^{\uparrow})\in L(G,\ M,\ I)$. □

**Definition 3.3.** Let $(G, M, I)$ be an L-fuzzy formal context, $D \subseteq M$, $D$ is called a consistent set of $(G,\ M,\ I)$, if $L(G,\ D,\ I_D) \cong L(G,\ M,\ I)$. Furthermore, if $D$ is a consistent set of $(G,\ M,\ I)$, and no proper subset of $D$ is a consistent set of $(G,\ M,\ I)$, i.e., for any $d \in D$, $L(G,\ D-\{d\},\ I_{D-\{d\}}) \ncong L(G,\ M,\ I)$, then we say $D$ is an attribute reduction of $(G,\ M,\ I)$.

In [16], Shao et al. solved the problem of attribute reduction in fuzzy formal contexts, but his work is different from that in this paper. He proposed the object consistent set and the attribute consistent set for the one-sided variable threshold fuzzy concepts, and calculated all reductions of fuzzy formal context by discernibility matrices and discernibility functions. In [5], Shao et al. also discussed the granular reduction of fuzzy formal contexts. The lattice structure of concepts involved in their paper is based on the 'crisp-fuzzy concept' proposed independently by Yahia [8] and Krajča [22]. Its granular reduction method is to obtain a minimal attribute set preserving all the object granules of a concept lattice. But in this paper the extension and intension of fuzzy formal concepts are both fuzzy set on a complete residuated lattice L, which is generated by the concept derivation operators proposed by Bělohlávek. Under certain conditions, this model is closer to people's minds.

**Theorem 3.4.** *Let $(G,\ M,\ I)$ be an L-fuzzy formal context, $D \subseteq M$. Then $D$ is a consistent set of $(G,\ M,\ I)$ if and only if $L(G,\ D,\ I_D) \leq L(G,\ M,\ I)$.*

*Proof.* It follows directly from Definition 3.3 and Theorem 3.2. □

**Theorem 3.5.** *Let $(G,\ M,\ I)$ be an L-fuzzy formal context, $D \subseteq M$. Then $D$ is a consistent set of $(G,\ M,\ I)$ if and only if for any $(A_1,\ B_1),\ (A_2,\ B_2) \in L(G,\ M,\ I)$, there exist $d \in D$ such that $B_1(d) \neq B_2(d)$ if $(A_1,\ B_1) \neq (A_2,\ B_2)$.*

*Proof.* ($\Longrightarrow$) Assume that $D$ is a consistent set of $(G,\ M,\ I)$ and $(A_1,\ B_1),\ (A_2,\ B_2) \in L(G,\ M,\ I)$ such that $(A_1,\ B_1) \neq (A_2,\ B_2)$. Then $(A_1,\ A_1^{\uparrow_D})$ , $(A_2,\ A_2^{\uparrow_D}) \in L(G,\ D,\ I_D)$ and $(A_1,\ A_1^{\uparrow_D}) \neq (A_2,\ A_2^{\uparrow_D})$ by $A_1 \neq A_2$. Thus we have $A_1^{\uparrow_D} \neq A_2^{\uparrow_D}$, i.e., $A_1^{\uparrow}\big|_D \neq A_2^{\uparrow}\big|_D$. So there exist $d \in D$ such that $B_1(d) \neq B_2(d)$.

($\Longleftarrow$) By Theorem 3.4, we have $L(G,\ M,\ I) \leq L(G,\ D,\ I_D)$. Then it suffices to prove that $L(G,\ D,\ I_D) \leq L(G,\ M,\ I)$, i.e., $(A,\ A^{\uparrow_D}) \in L(G,\ D,\ I_D)$ for any $(A,\ B) \in L(G,\ M,\ I)$.

That is need to prove $A = A^{\uparrow_D\downarrow_D}$. If not, by $A \subseteq A^{\uparrow_D\downarrow_D}$, it follows that $A \subset A^{\uparrow_D\downarrow_D}$. Notice that $A^{\uparrow_D\downarrow_D} = \overline{A^{\uparrow_D}}^{\downarrow}$. Thus

$$(A^{\uparrow_D\downarrow_D},\ A^{\uparrow_D\downarrow_D\uparrow}) = (\overline{A^{\uparrow_D}}^{\downarrow},\ \overline{A^{\uparrow_D}}^{\downarrow\uparrow}) \in L(G,\ M,\ I).$$

By $(A,\ B) \in L(G,\ M,\ I)$ and $A^{\uparrow_D\downarrow_D} \neq A$, we have

$$(A,\ B) \neq (A^{\uparrow_D\downarrow_D}, A^{\uparrow_D\downarrow_D\uparrow}).$$

By assumption, we conclude that there exists $d \in D$ such that $B(d) \neq \overline{A^{\uparrow_D}}^{\downarrow\uparrow}(d)$, i.e., $B(d) \neq A^{\uparrow_D\downarrow_D\uparrow}(d)$. Moreover, by $A \subseteq A^{\uparrow_D\downarrow_D}$, we know that$A^{\uparrow_D\downarrow_D\uparrow} \subseteq A^{\uparrow}$. So

$$A^{\uparrow_D\downarrow_D\uparrow}(d) \leq A^{\uparrow}(d).$$

Besides, from the properties of the operator, we have

$$A^{\uparrow_D\downarrow_D\uparrow}(d) = (\overline{A^{\uparrow_D}})^{\downarrow\uparrow}(d) \geq \overline{A^{\uparrow_D}}(d).$$

By the properties of operators, we know that $\overline{A^{\uparrow_D}} = A^{\uparrow_D}$, when $d \in D$. Hence $\overline{A^{\uparrow_D}}(d) = A^{\uparrow_D}(d) = A^{\uparrow}|_D(d) = A^{\uparrow}(d) = B(d)$. It follows that $B(d) = A^{\uparrow_D \downarrow_D \uparrow}(d)$. This is a contradiction. Therefore $A^{\uparrow_D \downarrow_D} = A$ and $(A,\ A^{\uparrow_D}) \in L(G,\ D,\ I_D)$. □

**Definition 3.6.** Let $(G,\ M,\ I)$ be an L-fuzzy formal context. For $(A_1,\ B_1),\ (A_2,\ B_2) \in L(G,\ M,\ I)$ with $(A_1,\ B_1) \neq (A_2,\ B_2)$, the set of discernibility attributes of $(A_1,\ B_1)$, $(A_2,\ B_2)$ is $K((A_1,\ B_1),\ (A_2,\ B_2))$, where $K((A_1,\ B_1), (A_2,\ B_2))$ is given by:

$$K((A_1,\ B_1),\ (A_2,\ B_2)) = \{a \in M;\ B_1(a) \neq B_2(a)\}$$

From Theorem 3.5 and the definition of discernibility attributes, it is obvious that the following theorem holds.

**Theorem 3.7.** *Let $(G,\ M,\ I)$ be an L-fuzzy formal context, $D \subseteq M$. Then $D$ is a consistent set of $(G,\ M,\ I)$ if and only if for any $(X_1,\ A_1),\ (X_2,\ A_2) \in L(G,\ M,\ I)$, $(X_1,\ A_1) \neq (X_2,\ A_2)$ implies that $D \cap K((X_1,\ A_1),\ (X_2,\ A_2)) \neq \varnothing$.*

In general, there are a large number of L-fuzzy formal concepts generated by L-fuzzy formal context. For simplicity, we also prove the following method for calculation the consistent set.

**Theorem 3.8.** *Let $(G,\ M,\ I)$ be an L-fuzzy formal context, $D \subseteq M$. Then $D$ is a consistent set of $(G,\ M,\ I)$ if and only if for any $(A_1,\ B_1),\ (A_2,\ B_2) \in L(G,\ M,\ I)$, $(A_1,\ B_1) \prec (A_2,\ B_2)$ implies that $D \cap K((A_1,\ B_1),\ (A_2,\ B_2)) \neq \varnothing$.*

*Proof.* $(\Longrightarrow)$ Obviously.

$(\Longleftarrow)$ Let $(X_1,\ A_1),\ (X_2,\ A_2) \in L(G,\ M,\ I)$. If $(X_1,\ A_1) \neq (X_2,\ A_2)$, we have $(X_1,\ A_1) < (X_1,\ A_1) \vee (X_2,\ A_2)$ or $(X_2,\ A_2) < (X_1,\ A_1) \vee (X_2,\ A_2)$. Might as well assume $(X_1, A_1) < (X_1, A_1) \vee (X_2, A_2)$, and notice that

$$(X_1,\ A_1) \vee (X_2,\ A_2) = ((X_1 \cup X_2)^{\uparrow\downarrow}, A_1 \cap A_2)$$

There must exists $(X_3,\ A_3)$ such that

$$(X_1,\ A_1) \prec (X_3,\ A_3) \leq ((X_1 \cup X_2)^{\uparrow\downarrow}, A_1 \cap A_2)$$

By assumption, it follows that $D \cap K((X_1,\ A_1),\ (X_3,\ A_3)) \neq \varnothing$. By the property of the discernibility attribute set, we know that $K((X_1,\ A_1),\ (X_3,\ A_3)) \subseteq K(((X_1 \cup X_2)^{\uparrow\downarrow},\ A_1 \cap A_2)) \subseteq K((X_1,\ A_1),\ (X_2,\ A_2))$. Then

$$D \cap K((X_1, A_1),\ (X_2, A_2)) \neq \varnothing.$$

Thus by Theorem 3.5, $D$ is a consistent of $(G,\ M,\ I)$. □

Given an L-fuzzy formal context $(G,\ M,\ I)$. Denote

$$f = \bigwedge_{(A_1,\ B_1) \prec (A_2,\ B_2)} \ \bigvee_{a \in K((A_1,\ B_1),(A_2,\ B_2))} a$$

$f$ is called the discernibility function of $(G,\ M,\ I)$. By using the discernibility function, we can compute all the reductions of L-fuzzy formal context.

**Theorem 3.9.** *For an L-fuzzy formal context $(G,\ M,\ I)$, let*

$$f = \bigvee_{t=1}^{k} (\bigwedge_{s=1}^{r_t} a_{t,s})$$

*be the minimal disjunctive normal form of the discernibility function of* $(G,\ M,\ I)$. *Then* $\{a_{t,s}|1 \le s \le r_t\}(t \le k)$ *are all the reductions of* $(G,\ M,\ I)$.

**Example 3.10.** Take the fuzzy formal context $(G,\ M,\ I)$ in Table 1 as an example. Assume that $x \to y = min(1 - x + y, 1)$. We can conclude that there are 19 fuzzy formal concepts in the formal context determined by the operators in this paper, as is shown below:

FC1. $(\varnothing, \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5})$,
FC2. $(\frac{1}{x_1} + \frac{0}{x_2} + \frac{0}{x_3}, \frac{1}{a_1} + \frac{0.5}{a_2} + \frac{0}{a_3} + \frac{1}{a_4} + \frac{1}{a_5})$,
FC3. $(\frac{0.5}{x_1} + \frac{0}{x_2} + \frac{0}{x_3}, \frac{1}{a_1} + \frac{1}{a_2} + \frac{0.5}{a_3} + \frac{1}{a_4} + \frac{1}{a_5})$,
FC4. $(\frac{0}{x_1} + \frac{0.5}{x_2} + \frac{0}{x_3}, \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{0.5}{a_4} + \frac{0.5}{a_5})$,
FC5. $(\frac{0.5}{x_1} + \frac{0}{x_2} + \frac{0}{x_3}, \frac{1}{a_1} + \frac{1}{a_2} + \frac{0.5}{a_3} + \frac{1}{a_4} + \frac{1}{a_5})$,
FC6. $(\frac{0}{x_1} + \frac{0.5}{x_2} + \frac{0}{x_3}, \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{0.5}{a_4} + \frac{0.5}{a_5})$,
FC7. $(\frac{0}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3}, \frac{0}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{0.5}{a_4} + \frac{0.5}{a_5})$,
FC8. $(\frac{0.5}{x_1} + \frac{0.5}{x_2} + \frac{0}{x_3}, \frac{1}{a_1} + \frac{1}{a_2} + \frac{0.5}{a_3} + \frac{0.5}{a_4} + \frac{0.5}{a_5})$,
FC9. $(\frac{1}{x_1} + \frac{0.5}{x_2} + \frac{0}{x_3}, \frac{1}{a_1} + \frac{0.5}{a_2} + \frac{0}{a_3} + \frac{0.5}{a_4} + \frac{0.5}{a_5})$,
FC10. $(\frac{1}{x_1} + \frac{1}{x_2} + \frac{0.5}{x_3}, \frac{0.5}{a_1} + \frac{0.5}{a_2} + \frac{0}{a_3} + \frac{0}{a_4} + \frac{0}{a_5})$,
FC11. $(\frac{0.5}{x_1} + \frac{0}{x_2} + \frac{0.5}{x_3}, \frac{0.5}{a_1} + \frac{1}{a_2} + \frac{0.5}{a_3} + \frac{1}{a_4} + \frac{1}{a_5})$,
FC12. $(\frac{0.5}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3}, \frac{0}{a_1} + \frac{1}{a_2} + \frac{0.5}{a_3} + \frac{0.5}{a_4} + \frac{0.5}{a_5})$,
FC13. $(\frac{1}{x_1} + \frac{0}{x_2} + \frac{0.5}{x_3}, \frac{0.5}{a_1} + \frac{0.5}{a_2} + \frac{0}{a_3} + \frac{1}{a_4} + \frac{1}{a_5})$,
FC14. $(\frac{1}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3}, \frac{0}{a_1} + \frac{0.5}{a_2} + \frac{0}{a_3} + \frac{0.5}{a_4} + \frac{0.5}{a_5})$,
FC15. $(\frac{0}{x_1} + \frac{0.5}{x_2} + \frac{0.5}{x_3}, \frac{0.5}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{0.5}{a_4} + \frac{0.5}{a_5})$,
FC16. $(\frac{0.5}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, \frac{0}{a_1} + \frac{1}{a_2} + \frac{0.5}{a_3} + \frac{0}{a_4} + \frac{0}{a_5})$,
FC17. $(\frac{0.5}{x_1} + \frac{0.5}{x_2} + \frac{0.5}{x_3}, \frac{0.5}{a_1} + \frac{1}{a_2} + \frac{0.5}{a_3} + \frac{0.5}{a_4} + \frac{0.5}{a_5})$,
FC18. $(\frac{1}{x_1} + \frac{0.5}{x_2} + \frac{0.5}{x_3}, \frac{0.5}{a_1} + \frac{0.5}{a_2} + \frac{0}{a_3} + \frac{0.5}{a_4} + \frac{0.5}{a_5})$,
FC19. $(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, \frac{0}{a_1} + \frac{0.5}{a_2} + \frac{0}{a_3} + \frac{0}{a_4} + \frac{0}{a_5})$.

The fuzzy concept lattice generated by all the fuzzy formal concepts is shown in Figure 1. For convenience, we use serial numbers 1 to 19 to represent concepts FC1 to FC19.

According to Theorem 3.8, two concepts need to be distinguished when they have an immediate sub-concept (super-concept) relationship. From the concept lattice shown in Figure 1, we can see that the discernibility attributes set are as follows:where $K_{1,3}$ denotes the set of discernibility attributes of concepts FC1 and FC3.

$K_{1,3} = \{a_3\}$, $K_{1,4} = \{a_4, a_5\}$, $K_{1,6} = \{a_1\}$, $K_{3,2} = \{a_2, a_3\}$,
$K_{3,8} = \{a_4, a_5\}$, $K_{4,8} = \{a_3\}$, $K_{4,11} = \{a_1, a_3, a_4, a_5\}$, $K_{4,15} = \{a_1\}$,
$K_{6,11} = \{a_3\}$, $K_{6,15} = \{a_4, a_5\}$, $K_{2,9} = \{a_4, a_5\}$, $K_{2,13} = \{a_1\}$,
$K_{8,9} = \{a_2, a_3\}$, $K_{8,17} = \{a_1\}$, $K_{11,17} = \{a_4, a_5\}$, $K_{11,13} = \{a_2, a_3\}$,
$K_{15,17} = \{a_3\}$, $K_{15,7} = \{a_1\}$, $K_{9,18} = \{a_1\}$, $K_{17,5} = \{a_4, a_5\}$,
$K_{17,18} = \{a_2, a_3\}$, $K_{13,18} = \{a_4, a_5\}$, $K_{7,12} = \{a_3\}$, $K_{5,10} = \{a_2, a_3\}$,
$K_{18,10} = \{a_4, a_5\}$, $K_{18,14} = \{a_1\}$, $K_{12,14} = \{a_2, a_3\}$, $K_{12,16} = \{a_4, a_5\}$,
$K_{10,19} = \{a_1\}$, $K_{14,19} = \{a_4, a_5\}$, $K_{16,19} = \{a_2, a_3\}$.

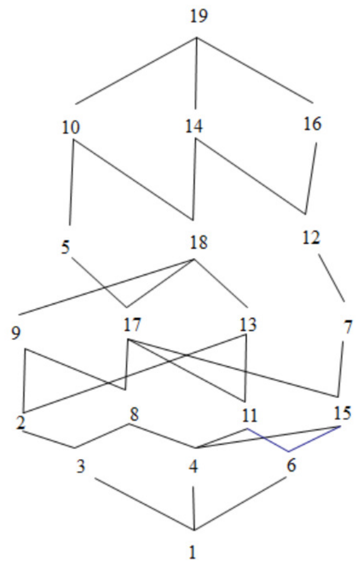


Figure 1. The fuzzy concept lattice

Its discernibility function is as follows:

$$f = a_3 \wedge (a_4 \vee a_5) \wedge a_1 \wedge (a_2 \vee a_3) \wedge (a_1 \vee a_3 \vee a_4 \vee a_5)$$

$$= a_3 \wedge a_1 \wedge (a_4 \vee a_5) = (a_1 \wedge a_3 \wedge a_4) \vee (a_1 \wedge a_3 \wedge a_5)$$

So the fuzzy formal context shown in Table1 has two attribute reductions $\{a_1, a_3, a_4\}$ and $\{a_1, a_3, a_5\}$ .

## 4. Conclusion

Fuzzy formal concept analysis is a theory of information processing and knowledge discovery. Attribute reduction of formal context is also a hot research topic. In this paper, we proposed a more general approach of attribute reduction of L-fuzzy formal context. Based on Bělohlávek fuzzy concept operator, we define the discernibility attributes set and discernibility function of fuzzy formal concepts, and propose a method to obtain all reductions of L-fuzzy formal context by distinguishing function. However, a sufficient and necessary condition for attribute characteristics similar to[13] has not been given, and further research is needed.

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