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Abstract. In this paper, we introduce the concept of interval-valued intuitionistic cubic medial-ideal and investigate several properties .Also, we give relations between interval-valued intuitionistic cubic medial-ideal and interval-valued intuitionistic cubic BCI-ideal.The image and the pre-image of interval-valued intuitionistic cubic medial-ideal under homomorphism of BCI-algebras are defined and how the image and the pre-image of interval-valued intuitionistic cubic medial-ideal under homomorphism of BCI-algebras become cubic intuitionistic medial-ideal are studied. Moreover, the Cartesian product of interval-valued intuitionistic cubic medial-ideal in Cartesian product BCI-algebras is given.

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Keywords: Medial BCI-algebra, Fuzzy medial-ideal, Interval-valued intuitionistic cubic) medial-ideal, The pre-image of interval-valued intuitionistic cubic medial-ideal under homomorphism of BCI-algebras, Cartesian product of interval-valued intuitionistic cubic medial-ideal.

Corresponding Author: J. G. Lee (jukolee@wku.ac.kr)

1. Introduction

In 1966, K. Iséki [6] introduced the notion of BCK-algebras (See [18]). K. Iséki and Tanaka [7, 8] introduced the notion of a BCI-algebra which is a generalization of BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other structures including lattices and Boolean algebras. There is a great deal of literature which has been produced on the theory of BCK/BCI-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras. In 1956, Zadeh [22] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches. There are several kinds of fuzzy sets extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets [1], interval valued fuzzy sets [5, 23], interval valued intuitionistic fuzzy sets [2] etc. In 1991, Xi [20] applied the concept of fuzzy sets to BCI, BCK, MV-algebras. Meng and Jun [17] studied medial BCI-algebras. Mostafa et al. [19] introduced the notion of medial ideals in BCI-algebras, they state the fuzzification of medial ideals and investigate its properties. Jun et al. [10, 11, 12] introduced the notion of cubic subalgebras/ideals in BCK/BCI-algebras, and then they investigated several properties. They discussed the relationship between a cubic subalgebra and a cubic ideal. Also, they provided characterizations of a cubic subalgebra/ideal and considered a method to make a new cubic subalgebra from an old one, and then this notion is applied to several algebraic structures (Also see [13, 15, 21]. Jun [9] introduce the notion of cubic intuitionistic set which is an extended concept of a cubic set (See [14]) and applied it to the cubic intuitionistic set (which will be called an interval-valued intuitionistic cubic set) BCK/BCI-algebras. In this paper, we introduce the notion of interval-valued intuitionistic cubic medial-ideals of BCI-algebras and then we study the homomorphic image and inverse image of interval-valued intuitionistic cubic medial ideals under homomorphism of BCI-algebras.

2. Preliminaries

Now we review some definitions and properties that will be useful in our results.

Definition 2.1 ([6])**.** An algebraic system $(X, *, 0)$ of type $(2, 0)$ is called a BCI-algebra, if it satisfying the following conditions: for any $x,\ y,\ z \in X$,

(BCI-1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCI-2) $(x * (x * y)) * y = 0$,
(BCI-3) $x * x = 0$,
(BCI-4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

In a BCI-algebra X, we can define a partial ordering "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

In what follows, X will denote a BCI-algebra unless otherwise specified.

Definition 2.2 ([17])**.** A BCI-algebra $(X, *, 0)$ of type $(2, 0)$ is called a medial BCI-algebra, if it satisfying the following condition: for any $x,\ y,\ z,\ u \in X$,

$$(x * y) * (z * u) = (x * z) * (y * u).$$

Lemma 2.3 ([17])**.** *An algebra $(X, *, 0)$ of type $(2, 0)$ is a medial BCI-algebra if and only if it satisfying the following conditions: for any $x,\ y,\ z \in X$,*

(i) $x * (y * z) = z * (y * x)$,
(ii) $x * 0 = x$,
(iii) $x * x = 0$.

Lemma 2.4 ([17])**.** *In a medial BCI-algebra X, the following holds:*

$$x * (x * y) = y, \text{ for any } x,\ y \in X.$$

Lemma 2.5 ([18])**.** *Let X be a medial BCI-algebra X. Then*

$$0 * (y * x) = x * y, \text{ for any } x,\ y \in X.$$

Definition 2.6 ([19])**.** A nonempty subset S of a medial BCI-algebra X is said to be medial sub-algebra of X, if $x * y \in S$, for any $x,\ y \in S$.

Definition 2.7 ([6])**.** A nonempty subset I of a BCI-algebra X is said to be a BCI-ideal of X, if it satisfies the following conditions:
(I1) $0 \in I$,
(I2) $x * y \in I$ and $y \in I$ imply $x \in I$, for any $x,\ y \in X$.

Definition 2.8 ([19])**.** A nonempty subset M of a medial BCI-algebra X is said to be a medial ideal of X, if it satisfies the following conditions:
(M1) $0 \in M$,
(M2) $z * (y * x) \in M$ and $y * z \in M$ imply $x \in M$, for any $x,\ y,\ z \in X$.

Proposition 2.9 ([19])**.** *Any medial ideal of BCI-algebra must be a BCI-ideal but the converse is not true.*

Proposition 2.10 ([19])**.** *Any BCI-ideal of a medial BCI-algebra is a medial ideal.*

Let $I \oplus I = \{\bar{a} = (a^{\in}, a^{\notin}) \in I \times I : a^{\in} + a^{\notin} \leq 1\}$. Then each member $\bar{a}$ of $I \oplus I$ is called an intuitionistic point or intuitionistic number. In particuar, we denote $(0, 1)$ and $(1, 0)$ as $\bar{0}$ and $\bar{1}$, respectively. We define relations $\leq$ and $=$ on $I \oplus I$ as follows (See [4]):

$$(\forall \bar{a},\ \bar{b} \in I \oplus I)(\bar{a} \leq \bar{b} \Longleftrightarrow a^{\in} \leq b^{\in} \text{ and } a^{\notin} \geq b^{\notin}),$$
$$(\forall \bar{a},\ \bar{b} \in I \oplus I)(\bar{a} = \bar{b} \Longleftrightarrow a^{\in} = b^{\in} \text{ and } a^{\notin} = b^{\notin}).$$

For each $\bar{a} \in I \oplus I)$, the complement of $\bar{a}$, denoted by $\bar{a}^c$, is defined as follows:

$$\bar{a}^c = (a^{\notin}, a^{\in}).$$

For any $(\bar{a}_j)_{j \in J} \subset I \oplus I)$, its inf $\bigwedge_{j \in J} \bar{a}_j$ and sup $\bigvee_{j \in J} \bar{a}_j$ are defined as follows:

$$\bigwedge_{j \in J} \bar{a}_j = (\bigwedge_{j \in J} a_j^{\in}, \bigvee_{j \in J} a_j^{\notin}),$$
$$\bigvee_{j \in J} \bar{a}_j = (\bigvee_{j \in J} a_j^{\in}, \bigwedge_{j \in J} a_j^{\notin}).$$

From Theorem 2.1 in [4], we can see that $(I \oplus I, \leq)$ is a complete distributive lattice with the greatest element $\bar{1}$ and the least element $\bar{0}$ satisfying DeMorgan's laws.

Definition 2.11 ([1])**.** Let X be a nonempty set X. Then a mapping $A : X \to I \oplus I$ is called an intuitionistic fuzzy set (briefly, IF set) in X, where for each $x \in X$, $A(x) = (A^{\in}(x), A^{\notin}(x))$, and $A^{\in}(x)$ and $A^{\notin}(x)$ represent the degree of membership and the degree of nonmembership of an element x to A, respectively. Let $(I \oplus I)^X$ denote the set of all IF sets in X and for each $A \in (I \oplus I)^X$, we write $A = (A^{\in}, A^{\notin})$. In particular, $\bar{\mathbf{0}}$ and $\bar{\mathbf{1}}$ denote the IF empty set and the IF whole set in X defined by, respectively: for each $x \in X$,

$$\bar{\mathbf{0}}(x) = \bar{0} \text{ and } \bar{\mathbf{1}}(x) = \bar{1}.$$

We define relations $\subset$ and $=$ on $(I \oplus I)^X$ as follows:

$$(\forall\ A,\ B \in (I \oplus I)^X)(A \subset B \Longleftrightarrow (x \in X)(A(x) \leq B(x)),$$

$$(\forall\ A,\ B \in (I \oplus I)^X)(A = B \Longleftrightarrow (x \in X)(A(x) = B(x)).$$

For each $A \in (I \oplus I)^X$, the complement of A, denoted by A^c, is defined as follows: for each $x \in X$,

$$A^c(x) = (A^{\notin}(x), A^{\in}(x)).$$

The set of all closed subintervals of I is denoted by $[I]$, and members of $[I]$ are called interval numbers and are denoted by $\widetilde{a},\ \widetilde{b},\ \widetilde{c}$, etc., where $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$. In particular, if $a^- = a^+$, then we write as $\widetilde{a} = \mathbf{a}$.

We define relations $\leq$ and $=$ on $[I]$ as follows:

$$(\forall\ \widetilde{a},\ \widetilde{b} \in [I])(\widetilde{a} \leq \widetilde{b} \Longleftrightarrow a^- \leq b^- \text{ and } a^+ \leq b^+),$$

$$(\forall\ \widetilde{a},\ \widetilde{b} \in [I])(\widetilde{a} = \widetilde{b} \Longleftrightarrow a^- = b^- \text{ and } a^+ = b^+).$$

For any $\widetilde{a},\ \widetilde{b} \in [I]$, their minimum and maximum, denoted by $\widetilde{a} \wedge \widetilde{b}$ and $\widetilde{a} \vee \widetilde{b}$, are defined as follows:

$$\widetilde{a} \wedge \widetilde{b} = [a^- \wedge b^-, a^+ \wedge b^+],$$

$$\widetilde{a} \vee \widetilde{b} = [a^- \vee b^-, a^+ \vee b^+].$$

Let $(\widetilde{a_j})_{j \in J} \subset [I]$. Then its inf and sup, denoted by $\bigwedge_{j \in J} \widetilde{a_j}$ and $\bigvee_{j \in J} \widetilde{a_j}$ are defined as follows:

$$\bigwedge_{j \in J} \widetilde{a_j} = [\bigwedge_{j \in J} a_j^-, \bigwedge_{j \in J} a_j^+],$$

$$\bigvee_{j \in J} \widetilde{a_j} = [\bigvee_{j \in J} a_j^-, \bigvee_{j \in J} a_j^+].$$

For each $\widetilde{a} \in [I]$, its complement, denoted by $\widetilde{a}^c$, is defined as follows:

$$\widetilde{a}^c = [1 - a^+, 1 - a^-].$$

Definition 2.12 ([5, 23])**.** Let X be a nonempty set X. Then a mapping $A : X \to [I]$ is called an interval-valued fuzzy set (briefly, an IVF set) in X. Let $[I]^X$ denote the set of all IVF sets in X. For each $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the degree of membership of an element x to A, where $A^-, A^+ \in I^X$ are called a lower fuzzy set and an upper fuzzy set in X, respectively. For each $A \in [I]^X$, we write $A = [A^-, A^+]$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy empty whole set in X, respectively. We define relations $\subset$ and $=$ on $[I]^X$ as follows:

$$(\forall\ A,\ B \in [I]^X)(A \subset B \Longleftrightarrow (x \in X)(A(x) \leq B(x)),$$

$$(\forall\ A,\ B \in [I]^X)(A = B \Longleftrightarrow (x \in X)(A(x) = B(x)).$$

For each $A \in [I]^X$, the complement of A, denoted by A^c, is defined as follows: for each $x \in X$,

$$A^c(x) = [1 - A^+(x), 1 - A^-(x)].$$

For any $(A_j)_{j \in J} \subset [I]^X$, its intersection $\bigcap_{j \in J} A_j$ and union $\bigcup_{j \in J} A_j$ are defined, respectively as follows: for each $x \in X$,

$$(\bigcap_{j \in J} A_j)(x) = \bigwedge_{j \in J} A_j(x),$$

$$(\bigcup_{j\in J} A_j)(x) = \bigvee_{j\in J} A_j(x).$$

Let $[I]\oplus[I] = \{\bar{\tilde{a}} = (\tilde{a}^{\in}, \tilde{a}^{\notin}) \in [I]\times[I] : a^{\in,+} + a^{\notin,-} \leq 1\}$, where

$$\tilde{a}^{\in} = [a^{\in,-}, a^{\in,+}],\ \tilde{a}^{\notin} = [a^{\notin,-}, a^{\notin,+}] \in [I].$$

Each member of $[I]\oplus[I]$ is called an interval-valued intuitionistic number. In particular, we write as $\bar{\tilde{0}} = (\mathbf{0}, \mathbf{1})$ and $\bar{\tilde{1}} = (\mathbf{1}, \mathbf{0})$.

We define relations $\leq$ and $=$ on $[I]\oplus[I]$ as follows: for any $\forall\ \bar{\tilde{a}},\ \bar{\tilde{b}} \in [I]\oplus[I]$,

$$\bar{\tilde{a}} \leq \bar{\tilde{b}} \Longleftrightarrow a^{\in,-} \leq b^{\in,-},\ a^{\in,+} \leq b^{\in,+} \text{ and } a^{\notin,-} \geq b^{\notin,-},\ a^{\notin,+} \geq b^{\notin,+},$$

$$\bar{\tilde{a}} = \bar{\tilde{b}} \Longleftrightarrow \bar{\tilde{a}} \leq \bar{\tilde{b}} \text{ and } \bar{\tilde{a}} \geq \bar{\tilde{b}}.$$

For any $\bar{\tilde{a}},\ \bar{\tilde{b}} \in [I]\oplus[I]$, their minimum and maximum, denoted by $\bar{\tilde{a}}\wedge\bar{\tilde{b}}$ and $\bar{\tilde{a}}\vee\bar{\tilde{b}}$, are defined as follows:

$$\bar{\tilde{a}}\wedge\bar{\tilde{b}} = ([a^{\in,-}\wedge b^{\in,-}, a^{\in,+}\wedge b^{\in,+}], [a^{\notin,-}\vee b^{\notin,-}, a^{\notin,+}\vee b^{\notin,+}]),$$

$$\bar{\tilde{a}}\vee\bar{\tilde{b}} = ([a^{\in,-}\vee b^{\in,-}, a^{\in,+}\vee b^{\in,+}], [a^{\notin,-}\wedge b^{\notin,-}, a^{\notin,+}\wedge b^{\notin,+}]).$$

Let $(\bar{\tilde{a}}_j)_{j\in J} \subset [I]\oplus[I]$. Then its inf and sup, denoted by $\bigwedge_{j\in J}\bar{\tilde{a}}_j$ and $\bigvee_{j\in J}\bar{\tilde{a}}_j$, are defined as follows:

$$\bigwedge_{j\in J}\bar{\tilde{a}}_j = ([\bigwedge_{j\in J} a_j^{\in,-}, \bigwedge_{j\in J} a_j^{\in,+}], [\bigvee_{j\in J} a_j^{\notin,-}, \bigvee_{j\in J} a_j^{\notin,+}]),$$

$$\bigvee_{j\in J}\bar{\tilde{a}}_j = ([\bigvee j\in J a_j^{\in,-}, \bigvee_{j\in J} a_j^{\in,+}], [\bigwedge_{j\in J} a_j^{\notin,-}, \bigwedge_{j\in J} a_j^{\notin,+}]).$$

For each $\bar{\tilde{a}} \in [I]\oplus[I]$, its complement, denoted by $\bar{\tilde{a}}^c$, is defined as follows:

$$\bar{\tilde{a}}^c = (\tilde{a}^{\notin}, \tilde{a}^{\in}).$$

From Proposition 2.1 in [16], we can see that $([I]\oplus[I], \vee, \wedge, ^c)$ is a complete distributive lattice with the greatest element $\bar{\tilde{1}}$ and the least element $\bar{\tilde{0}}$ satisfying DeMorgan's laws.

Definition 2.13 ([2])**.** Let X be a nonempty set. Then a mapping $\mathbf{A} = (\mathbf{A}^{\in}, \mathbf{A}^{\notin}) : X \to [I]\oplus[I]$ is called an interval-valued intuitionistic fuzzy set (briefly, IVIF set) in X, where for each $x \in X$, $\mathbf{A}^{\in} = [\mathbf{A}^{\in,-}(x), \mathbf{A}^{\in,+}(x)]$, $\mathbf{A}^{\notin} = [\mathbf{A}^{\notin,-}(x), \mathbf{A}^{\notin,+}(x)]$ and $\mathbf{A}^{\in,+}(x) + \mathbf{A}^{\notin,+}(x) \leq 1$.

In particular, $\bar{\tilde{\mathbf{0}}}$ (resp. $\bar{\tilde{\mathbf{1}}}$) will be called an IVIF empty set (resp. IVIF whole set) in X. We will denote the set of all IVIF sets as $([I]\oplus[I])^X$.

The relations $(\subset, =)$, operations $(\cup, \cap, ^c)$ and operators $([\ ], \diamond)$ on $([I]\oplus[I])^X$ are defined as follows.

Definition 2.14 ([2])**.** Let $\mathbf{A} = (\mathbf{A}^{\in}, \mathbf{A}^{\notin})$, $\mathbf{B} = (\mathbf{B}^{\in}, \mathbf{B}^{\notin}) \in ([I]\oplus[I])^X$ and let $(\mathbf{A}_j)_{j\in J} = ((\mathbf{A}_j^{\in}, \mathbf{A}^{\notin})_j)_{j\in J} \subset ([I]\oplus[I])^X$. Then

(i) $\mathbf{A} \subset \mathbf{B} \Longleftrightarrow (\forall x \in X)(\mathbf{A}^{\in,-}(x) \leq \mathbf{B}^{\in,-}(x),\ \mathbf{A}^{\in,+}(x) \leq \mathbf{B}^{\in,+}(x)$
and $\mathbf{A}^{\notin,-}(x) \geq \mathbf{B}^{\notin,-}(x),\ \mathbf{A}^{\notin,+}(x) \geq \mathbf{B}^{\notin,+}(x))$,

(ii) $\mathbf{A} = \mathbf{B} \Longleftrightarrow \mathbf{A} \subset \mathbf{B}$ and $\mathbf{B} \subset \mathbf{A}$,

(iii) $\mathbf{A}^c(x) = (\mathbf{A}^{\notin}(x), \mathbf{A}^{\in}(x))$ for each $x \in X$,

(iv) $(\mathbf{A}\cup\mathbf{B})(x)=([\mathbf{A}^{\in,-}(x)\vee\mathbf{B}^{\in,-}(x),\mathbf{A}^{\in,+}(x)\vee\mathbf{B}^{\in,+}(x)],$
$[\mathbf{A}^{\notin,-}(x)\wedge\mathbf{B}^{\notin,-}(x),\mathbf{A}^{\notin,+}(x)\wedge\mathbf{B}^{\notin,+}(x)])$ for each $x\in X$,
(v) $(\mathbf{A}\cap\mathbf{B})(x)=([\mathbf{A}^{\in,-}(x)\wedge\mathbf{B}^{\in,-}(x),\mathbf{A}^{\in,+}(x)\wedge\mathbf{B}^{\in,+}(x)],$
$[\mathbf{A}^{\notin,-}(x)\vee\mathbf{B}^{\notin,-}(x),\mathbf{A}^{\notin,+}(x)\vee\mathbf{B}^{\notin,+}(x)])$ for each $x\in X$,
(vi) $(\bigcup_{j\in J}\mathbf{A}_j)(x)=([\bigvee_{j\in J}\mathbf{A}_j^{\in,-}(x),\bigvee_{j\in J}\mathbf{A}_j^{\in,+}(x)],$
$[\bigwedge_{j\in J}\mathbf{A}_j^{\notin,-}(x),\bigwedge_{j\in J}\mathbf{A}_j^{\notin,+}(x)])$ for each $x\in X$,
(vii) $(\bigcap_{j\in J}\mathbf{A}_j)(x)=([\bigwedge_{j\in J}\mathbf{A}_j^{\in,-}(x),\bigwedge_{j\in J}\mathbf{A}_j^{\in,+}(x)],$
$[\bigvee_{j\in J}\mathbf{A}_j^{\notin,-}(x),\bigvee_{j\in J}\mathbf{A}_j^{\notin,+}(x)])$ for each $x\in X$,
(viii) $[\ ]\mathbf{A}(x)=(\mathbf{A}^{\in}(x),[(\mathbf{A}^{\notin,-}(x),1-\mathbf{A}^{\in,+}(x)])$ for each $x\in X$,
(ix) $\diamond\mathbf{A}(x)=([\mathbf{A}^{\in,-}(x),1-\mathbf{A}^{\notin,+}(x)],\mathbf{A}^{\notin}(x))$ for each $x\in X$.

3. IVIC medial-ideal in BCI-algebras

Definition 3.1 ([16])**.** Let X be a nonempty set. Then a mapping

$$\mathcal{A}=<\mathbf{A},A>: X\to([I]\oplus[I])\times(I\oplus I)$$

is called a cubic intuitionistic fuzzy set (briefly, CIFS) in X (See [9]) or an interval-valued intuitionisti cubic set (briefly, IVIC set) in X, where $\mathbf{A}$ is an IVF set in X and A is an IF set in X.

An IVIC set $\mathcal{A}=<\mathbf{A},A>$ in which $\mathbf{A}(x)=\tilde{\bar{0}}$ and $A(x)=\bar{1}$ (resp. $\mathbf{A}(x)=\tilde{\bar{1}}$ and $A(x)=\bar{0}$) for each $x\in X$ is denoted by $\ddot{\mathbf{0}}$ (resp. $\ddot{\mathbf{1}}$).

An IVIC set $\mathcal{B}=<\mathbf{B},B>$ in which $\mathbf{B}(x)=\tilde{\bar{0}}$ and $A(x)=\bar{0}$ (resp. $\mathbf{B}(x)=\tilde{\bar{1}}$ and $B(x)=\bar{1}$) for each $x\in X$ is denoted by $\hat{\mathbf{0}}$ (resp. $\hat{\mathbf{1}}$).

In particular, $\hat{\mathbf{0}}$ (resp. $\hat{\mathbf{1}}$) will be called the IVIC empty set (resp. the IVIC whole set). We will denote the set of all IVIC sets as $[([I]\oplus[I])\times(I\oplus I)]^X$.

Definition 3.2 ([16])**.** Let X be a nonempty set, let $\mathcal{A}=<\mathbf{A},A>$ and let $\mathcal{B}=<\mathbf{B},B>$ be IVIC sets in X. Then we define the following relations:
(i) (Equality) $\mathcal{A}=\mathcal{B}\Leftrightarrow\mathbf{A}=\mathbf{B}$ and $A=B$,
(ii) (P-order) $\mathcal{A}\sqsubset\mathcal{B}\Leftrightarrow\mathbf{A}\subset\mathbf{B}$ and $A\subset B$,
(iii) (R-order) $\mathcal{A}\Subset\mathcal{B}\Leftrightarrow\mathbf{A}\subset\mathbf{B}$ and $A\supset B$.

Definition 3.3. Let $(X,*,0)$ be a BCI-algebra and let $\mathcal{A}=<\mathbf{A},A>$ be an IVIC set in X. Then $\mathcal{A}=<\mathbf{A},A>$ is called an interval-valued intuitionistic cubic subalgebra (briefly, IVICSA) of X, if it satisfies the following conditions: for any $x,\ y\in X$,
(IVICSA-1) $\mathbf{A}^{\in}(x*y)\geq\mathbf{A}^{\in}(x)\wedge\mathbf{A}^{\in}(y)$, i.e.,

$$\mathbf{A}^{\in,-}(x*y)\geq\mathbf{A}^{\in,-}(x)\wedge\mathbf{A}^{\in,-}(y),\ \mathbf{A}^{\in,+}(x*y)\geq\mathbf{A}^{\in,+}(x)\wedge\mathbf{A}^{\in,+}(y),$$

(IVICSA-2) $\mathbf{A}^{\notin}(x*y)\leq\mathbf{A}^{\notin}(x)\vee\mathbf{A}^{\notin}(y)$, i.e.,

$$\mathbf{A}^{\notin,-}(x*y)\leq\mathbf{A}^{\notin,-}(x)\vee\mathbf{A}^{\notin,-}(y),\ \mathbf{A}^{\notin,+}(x*y)\leq\mathbf{A}^{\notin,+}(x)\vee\mathbf{A}^{\notin,+}(y),$$

(IVICSA-3) $A^{\in}(x*y)\leq A^{\in}(x)\vee A^{\in}(y)$,
(IVICSA-4) $A^{\notin}(x*y)\geq A^{\notin}(x)\wedge A^{\notin}(y)$.

Definition 3.4. Let $(X,*,0)$ be a BCI-algebra and let $\mathcal{A}=<\mathbf{A},A>$ be an IVIC set in X. Then $\mathcal{A}=<\mathbf{A},A>$ is called an interval-valued intuitionistic cubic ideal (briefly, IVICI) of X, if it satisfies the following conditions: for any $x,\ y\in X$,
(IVICI-1) $\mathbf{A}^{\in}(0)\geq\mathbf{A}^{\in}(x)$ and $\mathbf{A}^{\notin}(0)\leq\mathbf{A}^{\notin}(x)$,

(IVICI-2) $\mathbf{A}^{\in}(x) \geq \mathbf{A}^{\in}(x * y) \wedge \mathbf{A}^{\in}(y)$,
(IVICI-3) $\mathbf{A}^{\notin}(x) \leq \mathbf{A}^{\notin}(x * y) \vee \mathbf{A}^{\notin}(y)$,
(IVICI-4) $A^{\in}(0) \leq A^{\in}(x)$ and $A^{\notin}(0) \geq A^{\notin}(x)$,
(IVICI-5) $A^{\in}(x) \leq A^{\in}(x * y) \vee A^{\in}(y)$,
(IVICI-6) $A^{\notin}(x) \geq A^{\notin}(x * y) \wedge A^{\notin}(y)$.

Definition 3.5. Let $(X, *, 0)$ be a BCI-algebra and let $\mathcal{A} =< \mathbf{A}, A >$ be an IVIC set in X. Then $\mathcal{A} =< \mathbf{A}, A >$ is called an interval-valued intuitionistic cubic medial ideal (briefly, IVICMI) of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(IVICMI-1) $\mathbf{A}^{\in}(0) \geq \mathbf{A}^{\in}(x)$ and $\mathbf{A}^{\notin}(0) \leq \mathbf{A}^{\notin}(x)$,
(IVICMI-2) $\mathbf{A}^{\in}(x) \geq \mathbf{A}^{\in}(z * (y * x)) \wedge \mathbf{A}^{\in}(y * z)$,
(IVICMI-3) $\mathbf{A}^{\notin}(x) \leq \mathbf{A}^{\notin}(z * (y * x)) \vee \mathbf{A}^{\notin}(y * z)$,
(IVICMI-4) $A^{\in}(0) \leq A^{\in}(x)$ and $A^{\notin}(0) \geq A^{\notin}(x)$,
(IVICMI-5) $A^{\in}(x) \leq A^{\in}(z * (y * x)) \vee A^{\in}(y * z)$,
(IVICMI-6) $A^{\notin}(x) \geq A^{\notin}(z * (y * x)) \wedge A^{\notin}(y * z)$.

Example 3.6. Let $X = \{0, 1, 2, 3\}$ be the set with the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Table 4.1

Then we can prove that $(X, *)$ is a BCI-algebra.

Let $\mathbf{A} : X \to [I] \oplus [I]$ and $A : X \to I \oplus I$ be the mappings defined as follows, respectively:

$$\mathbf{A}(0) = \mathbf{A}(1) = ([0.2, 0.6], [0.1, 0.3]),\ \ \mathbf{A}(2) = \mathbf{A}(3) = ([0.1, 0.4], [0.4, 0.5]),$$

$$A(0) = (0.2, 0.6),\ \ A(1) = (0.2, 0.5),\ \ A(2) = (0.6, 0.3),\ \ A(3) = (0.7, 0.2).$$

Then it is easy to check that $\mathcal{A} =< \mathbf{A}, A >$ is an IVIC sub-BCI-algebra of X.

Lemma 3.7. *Let X be a BCI-algebra. If $\mathcal{A} =< \mathbf{A}, A >$ is an IVIC sub-BCI-algebra of X, then $\mathbf{A}_{\in}(0) \geq \mathbf{A}_{\in}(x)$, $\mathbf{A}_{\notin}(0) \leq \mathbf{A}_{\in}(x)$, $A^{\in}(0) \leq A^{\in}(x)$ and $A^{\notin}(0) \geq A^{\in}(x)$, for each $x \in X$.*

Proof. Let $x \in X$. Then by Lemma 2.3 (iii) and Definition 3.3,

$$\mathbf{A}^{\in}(0) = \mathbf{A}^{\in}(x * x) \geq \mathbf{A}^{\in}(x) \wedge \mathbf{A}^{\in}(x) = \mathbf{A}^{\in}(x),$$

$$\mathbf{A}^{\notin}(0) = \mathbf{A}^{\in}(x * x) \leq \mathbf{A}^{\notin}(x) \vee \mathbf{A}^{\notin}(x) = \mathbf{A}^{\notin}(x),$$

$$A^{\in}(0) = A^{\in}(x * x) \geq A^{\in}(x) \wedge A^{\in}(x) = A^{\in}(x),$$

$$A^{\notin}(0) = A^{\in}(x * x) \leq A^{\notin}(x) \vee A^{\notin}(x) = A^{\notin}(x).$$

Thus the results hold. □

Lemma 3.8. *Let X be a BCI-algebra, let $\mathcal{A} =< \mathbf{A}, A >$ be an IVICMI of X and let $x,\ y \in X$. If $x \leq y$, then $\mathbf{A}_{\in}(x) \geq \mathbf{A}_{\in}(y)$, $\mathbf{A}_{\notin}(x) \leq \mathbf{A}_{\in}(y)$, $A^{\in}(x) \leq A^{\in}(y)$ and $A^{\notin}(x) \geq A^{\in}(y)$.*

Proof. Let $x,\ y \in X$ such that $x \leq y$. Then clearly, $x * y = 0$. Thus

$$\begin{aligned}\mathbf{A}^{\in}(x) &\geq \mathbf{A}^{\in}(0 * (y * x)) \wedge \mathbf{A}^{\in}(y * 0) \text{ [By the condition (IVICMI-2)]}\\ &= \mathbf{A}^{\in}(y * x) \wedge \mathbf{A}^{\in}(y) \text{ [By Lemmas 2.3 and 2.5]}\\ &= \mathbf{A}^{\in}(0) \wedge \mathbf{A}^{\in}(y) \text{ [Since } x * y = 0\text{]}\\ &= \mathbf{A}^{\in}(y). \text{ [By Lemma 3.7]}\end{aligned}$$

Similarly, we can prove the remainder's inequalities. □

Lemma 3.9. *Let X be a BCI-algebra and let $\mathcal{A} =< \mathbf{A}, A >$ be an IVICMI of X. Suppose the inequality $x * y \leq z$, for any $x,\ y,\ z \in X$. Then*
(1) $\mathbf{A}^{\in}(x) \geq \mathbf{A}^{\in}(y) \wedge \mathbf{A}^{\in}(z)$,
(2) $\mathbf{A}^{\notin}(x) \leq \mathbf{A}^{\notin}(y) \vee \mathbf{A}^{\notin}(z)$,
(3) $A^{\in}(x) \leq A^{\in}(y) \vee A^{\in}(z)$,
(4) $A^{\notin}(x) \geq A^{\notin}(y) \wedge A^{\notin}(z)$.

Proof. (1) Let $x,\ y,\ z \in X$ such that $x * y \leq z$. Then

$$\begin{aligned}\mathbf{A}^{\in}(x) &\geq \mathbf{A}^{\in}(0 * (y * x)) \wedge \mathbf{A}^{\in}(y * 0) \text{ [By the condition (IVICMI-2)]}\\ &= \mathbf{A}^{\in}(y * x) \wedge \mathbf{A}^{\in}(y) \text{ [By Lemmas 2.3 and 2.5]}\\ &\geq \mathbf{A}^{\in}(z) \wedge \mathbf{A}^{\in}(y). \text{ [Since } \mathbf{A}^{\in}(y * x) \geq \mathbf{A}^{\in}(z)\text{, by Lemma 3.8]}\end{aligned}$$

Similarly, we can prove that (2), (3) and (4) hold. □

Lemma 3.10. *Let X be a BCk-algebra and let $\mathcal{A} =< \mathbf{A}, A >$ be any IVICMI of X. Then $\mathcal{A}$ is an IVICSA of X.*

Proof. From Definition 2.2, it is obvious that $x * y \leq x$, for any $x,\ y \in X$. Let $x,\ y \in X$. Then

$$\begin{aligned}\mathbf{A}^{\in}(x * y) &\geq \mathbf{A}^{\in}(x) \text{ [By Lemma 3.8]}\\ &\geq \mathbf{A}^{\in}((0 * (y * x)) \wedge \mathbf{A}^{\in}(y * 0) \text{ [By the condition (IVICMI-2)]}\\ &= \mathbf{A}^{\in}(x * y) \wedge \mathbf{A}^{\in}(y) \text{ [Lemmas 2.3 and 2.5]}\\ &\geq \mathbf{A}^{\in}(x) \wedge \mathbf{A}^{\in}(y). \text{ [Since } \mathbf{A}^{\in}(x * y) \geq \mathbf{A}^{\in}(x)\text{, by Lemma 3.8]}\end{aligned}$$

Thus

$$\mathbf{A}^{\in}(x * y) \geq \mathbf{A}^{\in}(x) \wedge \mathbf{A}^{\in}(y). \tag{3.1}$$

Similarly, we have

$$\mathbf{A}^{\notin}(x * y) \leq \mathbf{A}^{\notin}(x) \vee \mathbf{A}^{\notin}(y), \tag{3.2}$$

$$A^{\in}(x * y) \leq A^{\in}(x) \vee A^{\in}(y), \tag{3.3}$$

$$A^{\notin}(x * y) \geq A^{\notin}(x) \wedge A^{\notin}(y). \tag{3.4}$$

So from (3.1), (3.2), (3.3) and (3.4), $\mathcal{A}$ is an IVICSA of X. □

Proposition 3.11. *Let X be a BCI-algebra and let $\mathcal{A} =< \mathbf{A}, A >$ be an IVICSA of X. Suppose for any $x,\ y,\ z \in X$ such that $x * y \leq z$, the following inequalities hold:*

$$\mathbf{A}^{\in}(x) \geq \mathbf{A}^{\in}(y) \wedge \mathbf{A}^{\in}(z),\ \ \mathbf{A}^{\notin}(x) \leq \mathbf{A}^{\notin}(y) \vee \mathbf{A}^{\notin}(z),$$

$$A^{\in}(x) \leq A^{\in}(y) \vee A^{\in}(z),\ \ A^{\notin}(x) \geq A^{\notin}(y) \wedge A^{\notin}(z).$$

Then $\mathcal{A}$ is an IVICMI of X.

Proof. By Lemma 3.7, it is obvious that the conditions (IVICMI-1) and (IVICMI-4) hold. Moreover, for any $x,\ y,\ z \in X$, we have $x*(z*(y*x)) = (y*x)*(z*x) \leq y*z$. Substituting $z*(y*x)$ for y and $y*z$ for z in (1), (2), (3) and (4) of Lemma 3.9, then we have

$$\mathbf{A}^{\in}(x) \geq \mathbf{A}^{\in}(z*(y*x)) \wedge \mathbf{A}^{\in}(y*z),\ \ \mathbf{A}^{\notin}(x) \leq \mathbf{A}^{\notin}(z*(y*x)) \vee \mathbf{A}^{\notin}(y*z),$$

$$A^{\in}(x) \leq A^{\in}(z*(y*x)) \vee A^{\in}(y*z),\ \ A^{\notin}(x) \geq A^{\notin}(z*(y*x)) \wedge A^{\notin}(y*z).$$

Thus $\mathcal{A}$ is an IVICMI of X. □

Theorem 3.12. *Let X be a BCI-algebra. Then $\mathcal{A} =< \mathbf{A}, A >$ is an IVICMI of X if and only if $\mathbf{A}$ is an interval-valued intuitionistic fuzzy medial ideal and A is an intuitionistic anti fuzzy medial ideal of X.*

Proof. Suppose $\mathbf{A}$ is an interval-valued intuitionistic fuzzy medial ideal and A is an intuitionistic anti fuzzy medial ideal of X. Then clearly, for each $x \in X$,

$$\mathbf{A}^{\in}(0) \geq \mathbf{A}^{\in}(x),\ \ \mathbf{A}^{\notin}(0) \leq \mathbf{A}^{\notin}(x),\ \ A^{\in}(0) \leq A^{\in}(x),\ \ A^{\notin}(0) \geq A^{\notin}(x).$$

Thus $\mathcal{A}$ satisfies the conditions (IVICMI-1) and (IVICMI-4).

Let $x,\ y,\ z \in X$. Then

$$\begin{aligned}
\mathbf{A}^{\in}(x) &= [\mathbf{A}^{\in,-}(x), \mathbf{A}^{\in,+}(x)] \\
&\geq [\mathbf{A}^{\in,-}(z*(y*x)) \wedge \mathbf{A}^{\in,-}(y*z), \mathbf{A}^{\in,+}(z*(y*x)) \wedge \mathbf{A}^{\in,+}(y*z)] \\
&= [\mathbf{A}^{\in,-}(z*(y*x)), \mathbf{A}^{\in,+}(z*(y*x))] \wedge [\mathbf{A}^{\in,-}(y*z), \mathbf{A}^{\in,+}(y*z)] \\
&= \mathbf{A}^{\in}(z*(y*x)) \wedge \mathbf{A}^{\in}(y*z), \\
\mathbf{A}^{\notin}(x) &= [\mathbf{A}^{\notin,-}(x), \mathbf{A}^{\notin,+}(x)] \\
&\leq [\mathbf{A}^{\notin,-}(z*(y*x)) \vee \mathbf{A}^{\notin,-}(y*z), \mathbf{A}^{\notin,+}(z*(y*x)) \vee \mathbf{A}^{\notin,+}(y*z)] \\
&= [\mathbf{A}^{\notin,-}(z*(y*x)), \mathbf{A}^{\notin,+}(z*(y*x))] \vee [\mathbf{A}^{\notin,-}(y*z), \mathbf{A}^{\notin,+}(y*z)] \\
&= \mathbf{A}^{\notin}(z*(y*x)) \vee \mathbf{A}^{\notin}(y*z), \\
A^{\in}(x) &\leq A^{\in}(z*(y*x)) \vee A^{\in}(y*z), \\
A^{\notin}(x) &\geq A^{\notin}(z*(y*x)) \wedge A^{\notin}(y*z).
\end{aligned}$$

Thus $\mathcal{A}$ satisfies the conditions (IVICMI-2), (IVICMI-3), (IVICMI-5) and (IVICMI-6). So $\mathcal{A}$ is an IVICMI of X.

Conversely, suppose $\mathcal{A}$ is an IVICMI of X. Then from Definition 3.5, we can easily prove that $\mathbf{A}$ is an interval-valued intuitionistic fuzzy medial ideal and A is an intuitionistic anti fuzzy medial ideal of X. □

Proposition 3.13. *Let X be a BCI-algebra and let $(\mathcal{A}_j)_{j\in J} = (< \mathbf{A}_j, A_j >)_{j\in J}$ be a family of IVICMIs of X. Then $\bigcap_{j\in J} \mathcal{A}_j$ is an IVICMI of X.*

Proof. Let $\mathcal{A} = \bigcap_{j\in J} \mathcal{A}_j$, where $\mathbf{A} = \bigcap_{j\in J} \mathbf{A}_j$ and $A = \bigcap_{j\in J} A_j$. Let $x,\ y,\ z \in X$. Then

$$\begin{aligned}
&\mathbf{A}(0) = (\textstyle\bigcap_{j\in J} \mathbf{A}_j)(0) = \textstyle\bigwedge_{j\in J} \mathbf{A}_j(0) \geq \textstyle\bigwedge_{j\in J} \mathbf{A}_j(x) = (\textstyle\bigcap_{j\in J} \mathbf{A}_j)(x) = \mathbf{A}(x), \\
&\mathbf{A}^{\in}(x) = (\textstyle\bigcap_{j\in J} \mathbf{A}_j^{\in})(x) = \textstyle\bigwedge_{j\in J} \mathbf{A}_j^{\in}(x) \\
&\qquad \geq \textstyle\bigwedge_{j\in J} \mathbf{A}_j^{\in}(z*(y*x)) \wedge \mathbf{A}_j^{\in}(y*z) \\
&\qquad = (\textstyle\bigwedge_{j\in J} \mathbf{A}_j^{\in}(z*(y*x))) \wedge (\textstyle\bigwedge_{j\in J} \mathbf{A}_j^{\in}(y*z)) \\
&\qquad = (\textstyle\bigcap_{j\in J} \mathbf{A}_j^{\in})(z*(y*x))) \wedge (\textstyle\bigcap_{j\in J} \mathbf{A}_j^{\in})(y*z)) \\
&\qquad = \mathbf{A}^{\in}(z*(y*x))) \wedge \mathbf{A}^{\in}(y*z)), \\
&\mathbf{A}^{\notin}(x) = (\textstyle\bigcap_{j\in J} \mathbf{A}_j^{\notin})(x) = \textstyle\bigvee_{j\in J} \mathbf{A}_j^{\notin}(x)
\end{aligned}$$

$$\begin{aligned}
&\leq \bigvee_{j\in J} \mathbf{A}_j^{\notin}(z*(y*x)) \vee \mathbf{A}_j^{\notin}(y*z)\\
&= (\bigvee_{j\in J} \mathbf{A}_j^{\notin}(z*(y*x))) \vee (\bigvee_{j\in J} \mathbf{A}_j^{\notin}(y*z))\\
&= (\bigcap_{j\in J} \mathbf{A}_j^{\notin})(z*(y*x))) \vee (\bigcap_{j\in J} \mathbf{A}_j^{\notin})(y*z))\\
&= \mathbf{A}^{\notin}(z*(y*x))) \vee \mathbf{A}^{\notin}(y*z)),
\end{aligned}$$

$$A(0) = (\bigcap_{j\in J} A_j)(0) = \bigvee_{j\in J} A_j(0) \leq \bigvee_{j\in J} A_j(x) = (\bigcap_{j\in J} A_j)(x) = A(x),$$

$$\begin{aligned}
A^{\in}(x) &= (\bigcap_{j\in J} A_j^{\in})(x) = \bigvee_{j\in J} A_j^{\in}(x)\\
&\leq \bigvee_{j\in J} A_j^{\in}(z*(y*x)) \vee A_j^{\in}(y*z)\\
&= (\bigvee_{j\in J} A_j^{\in}(z*(y*x))) \vee (\bigvee_{j\in J} A_j^{\in}(y*z))\\
&= (\bigcap_{j\in J} A_j^{\in})(z*(y*x))) \vee (\bigcap_{j\in J} A_j^{\in})(y*z))\\
&= A^{\in}(z*(y*x))) \vee A^{in}(y*z)),
\end{aligned}$$

$$\begin{aligned}
A^{\notin}(x) &= (\bigcap_{j\in J} A_j^{\notin})(x) = \bigwedge_{j\in J} A_j^{\notin}(x)\\
&\geq \bigwedge_{j\in J} A_j^{\notin}(z*(y*x)) \wedge A_j^{\notin}(y*z)\\
&= (\bigwedge_{j\in J} A_j^{\notin}(z*(y*x))) \wedge (\bigwedge_{j\in J} A_j^{\notin}(y*z))\\
&= (\bigcap_{j\in J} A_j^{\notin})(z*(y*x))) \wedge (\bigcap_{j\in J} A_j^{\notin})(y*z))\\
&= A^{\notin}(z*(y*x))) \wedge A^{\notin}(y*z)).
\end{aligned}$$

Thus $\bigcap_{j\in J} \mathcal{A}_j$ is an interval-valued intuitionistic cubic medial ideal of X. □

4. Image and preimage of IVICMI under homomorphisms of BCI-algebras

Definition 4.1 ([3]). Let X, Y be two sets, let $f : X \to Y$ be a mapping and let $A \in (I \oplus I)^X$, $B \in (I \oplus I)^Y$.

(i) The preimage of B under f, denoted by $f^{-1}(B)$, is the IF set in X defined as follows: for each $x \in X$,

$$f^{-1}(B)(x) = (f^{-1}(B^{\in}(x), f^{-1}(B^{\notin}(x)) = ((B^{\in} \circ f)(x), (B^{\notin} \circ f)(x)).$$

(ii) The image of A under f, denoted by $f(A) = (f(A^{\in}), f_{-}(A^{\notin}))$, is the IF set in Y defined as follows: for each $y \in Y$,

$$f(A^{\in})(y) = \begin{cases} \bigvee_{x\in f^{-1}(y)} A^{\in}(x) & \text{if } f^{-1}(y) \neq \phi\\ \bar{0} & \text{otherwise,}\end{cases}$$

$$f_{-}(A^{\notin})(y) = (1 - f(1 - A^{\notin}))(y) = \begin{cases} \bigwedge_{x\in f^{-1}(y)} A^{\notin,-}(x) & \text{if } f^{-1}(y) \neq \phi\\ \bar{1} & \text{otherwise.}\end{cases}$$

Definition 4.2 ([16]). Let X, Y be two sets, let $f : X \to Y$ be a mapping and let $\mathcal{A} =< \mathbf{A}, A >\in [([I] \oplus [I]) \times (I \oplus I)]^X$, $\mathcal{B} =< \mathbf{B}, B >\in [([I] \oplus [I]) \times (I \oplus I)]^X$.

(i) The preimage of $\mathcal{B}$ under f, denoted by $f^{-1}(\mathcal{B}) =< f^{-1}(\mathbf{B}), f^{-1}(B) >$, is the IVIC set in X defined as follows: for each $x \in X$,

$$\begin{aligned}
&f^{-1}(\mathcal{B})(x)\\
&=< ([f^{-1}(\mathbf{B}^{\in,-})(x), f^{-1}(\mathbf{B}^{\in,+})(x)], [f^{-1}(\mathbf{B}^{\notin,-})(x), f^{-1}(\mathbf{B}^{\notin,+})(x)]),\\
&\qquad (f^{-1}(B^{\in})(x), f^{-1}(B^{\notin})(x)) >\\
&=< ([(B^{\in,-} \circ f)(x), (B^{\in,+} \circ f)(x)], [(B^{\notin,-} \circ f)(x), (B^{\notin,+} \circ f)(x)]),\\
&\qquad ((B^{\in} \circ f^{-1})(x), (B^{\notin} \circ f^{-1})(x)) >.
\end{aligned}$$

(ii) The image of $\mathcal{A}$ under f, denoted by $f(\mathcal{A}) =< f(\mathbf{A}), f(A) >$, is the IVIC set in Y defined as follows: for each $y \in Y$,

$$f(\mathbf{A}^{\in})(y) = \begin{cases} [\bigvee_{x\in f^{-1}(y)} \mathbf{A}^{\in,-}(x), \bigvee_{x\in f^{-1}(y)} \mathbf{A}^{\in,+}(x)] & \text{if } f^{-1}(y) \neq \phi \\ \tilde{\bar{0}} & \text{otherwise,} \end{cases}$$

$$f(\mathbf{A}^{\notin})(y) = \begin{cases} [\bigwedge_{x\in f^{-1}(y)} \mathbf{A}^{\notin,-}(x), \bigwedge_{x\in f^{-1}(y)} \mathbf{A}^{\notin,+}(x)] & \text{if } f^{-1}(y) \neq \phi \\ \tilde{\bar{1}} & \text{otherwise,} \end{cases}$$

where $f(\mathbf{A}) = (f(\mathbf{A}^{\in}), f(\mathbf{A}^{\notin}))$ and $f(A)$ denotes the image of an IF set A under f.

Definition 4.3 ([17])**.** Let $(X, *, 0)$ and $(Y, *^{'}, 0^{'})$ be BCI-algebras. Then a mapping $f : X \to Y$ is called a homomorphism, if $f(x * y) = f(x) *^{'} f(y)$, for each $x,\ y \in X$.

Proposition 4.4. *Let $f : X \to Y$ be a homomorphism of BCI-algebras. If $\mathcal{B} =< \mathbf{B}, B >$ is an IVICMI of Y, then $f^{-1}(\mathcal{B}) =< f^{-1}(\mathbf{B}), f^{-1}(B) >$ is an IVICMI of X.*

Proof. Let $x \in X$. Since f is a a homomorphism of BCI-algebras, by Definition 4.2,

$$f^{-1}(\mathbf{B}^{\in})(0) = \mathbf{B}^{\in}(f(0)) = \mathbf{B}^{\in}(0^{'}) \geq \mathbf{B}^{\in}(f(x)) = f^{-1}(\mathbf{B}^{\in})(x).$$

Similarly, we have the following inequalities:

$$f^{-1}(\mathbf{B}^{\notin})(0) \leq f^{-1}(\mathbf{B}^{\notin})(x),\ f^{-1}(B^{\in})(0) \leq f^{-1}(B^{\in})(x),\ f^{-1}(B^{\notin})(0) \geq f^{-1}(B^{\notin})(x).$$

Then $f^{-1}(\mathcal{B})$ satisfies the conditions (IVICMI-1) and (IVICMI-4).

Let $x,\ y,\ z \in X$. Then

$$\begin{aligned} f^{-1}(\mathbf{B}^{\in})(x) &= \mathbf{B}^{\in}(f(x)) \\ &\geq \mathbf{B}^{\in}(f(z) * (f(y) * f(x))) \wedge \mathbf{B}^{\in}(f(y) * f(z)) \text{ [By the hypothesis]} \\ &= \mathbf{B}^{\in}(f(z * (y * x))) \wedge \mathbf{B}^{\in}(f(y * z)) \text{ [Since } f \text{ is a homomorphism]} \\ &= f^{-1}(\mathbf{B}^{\in})(z * (y * x)) \wedge f^{-1}(\mathbf{B}^{\in})(y * z). \end{aligned}$$

Similarly, we have the following inequalities:

$$f^{-1}(\mathbf{B}^{\notin})(x) \leq f^{-1}(\mathbf{B}^{\notin})(z * (y * x)) \vee f^{-1}(\mathbf{B}^{\notin})(y * z),$$

$$f^{-1}(B^{\in})(x) \leq f^{-1}(B^{\in})(z * (y * x)) \vee f^{-1}(B^{\in})(y * z),$$

$$f^{-1}(B^{\notin})(x) \geq f^{-1}(B^{\notin})(z * (y * x)) \wedge f^{-1}(B^{\notin})(y * z).$$

Thus $f^{-1}(\mathcal{B})$ satisfies the conditions (IVICMI-2), (IVICMI-3), (IVICMI-5) and (IVICMI-6). So $f^{-1}(\mathcal{B})$ is an IVICMI of X. □

Proposition 4.5. *Let $f : X \to Y$ be an epimorphism of BCI-algebras and let $\mathcal{B} =< \mathbf{B}, B >$ be an IVIC set in Y. If $f^{-1}(\mathcal{B}) =< f^{-1}(\mathbf{B}), f^{-1}(B) >$ is an IVICMI of X, then $\mathcal{B}$ is an IVICMI of Y.*

Proof. Let $y \in Y$. Then clearly there is $x \in X$ such that $y = f(x)$. Thus

$$\begin{aligned} \mathbf{B}^{\in}(y) &= \mathbf{B}^{\in}(f(x)) = f^{-1}(\mathbf{B}^{\in})(x) \\ &\leq f^{-1}(\mathbf{B}^{\in})(0) \text{ [By the condition (IVICMI-1)]} \\ &= \mathbf{B}^{\in}(f(0)) \\ &= \mathbf{B}^{\in}(0^{'}). \text{ [Since } f \text{ is a homomorphism]} \end{aligned}$$

Similarly, we have the following inequalities:

$$\mathbf{B}^{\notin}(y) \leq \mathbf{B}^{\notin}(0^{'}),\ B^{\in}(y) \leq B^{\in}(0^{'}),\ B^{\notin}(y) \geq B^{\notin}(0^{'}).$$

So $\mathcal{B}$ satisfies the conditions (IVICMI-1) and (IVICMI-4).

Let $a,\ b,\ c \in Y$. Then there are $x,\ y,\ z \in X$ such that $f(x) = a,\ f(y) = b,\ f(z) = c$. Thus

$$\begin{aligned}\mathbf{B}^{\in}(a) &= \mathbf{B}^{\in}(f(x)) = f^{-1}(\mathbf{B}^{\in})(x)\\ &\geq f^{-1}(\mathbf{B}^{\in})(z*(y*x)) \wedge f^{-1}(\mathbf{B}^{\in})(y*z) \text{ [By the condition (IVICMI-2)]}\\ &= \mathbf{B}^{\in}(f(z*(y*x))) \wedge \mathbf{B}^{\in}(f(y*z))\\ &= \mathbf{B}^{\in}(f(z)*(f(y)*f(x))) \wedge \mathbf{B}^{\in}(f(y)*f(z)) \text{ [Since } f \text{ is a homomorphism]}\\ &= \mathbf{B}^{\in}(c*(b*a)) \wedge \mathbf{B}^{\in}(b*c).\end{aligned}$$

Similarly, we have the following inequalities:

$$\mathbf{B}^{\notin}(a) \leq \mathbf{B}^{\notin}(c*(b*a)) \vee \mathbf{B}^{\notin}(b*c),$$

$$B^{\in}(a) \leq B^{\in}(c*(b*a)) \vee B^{\in}(b*c),$$

$$B^{\notin}(a) \geq B^{\notin}(c*(b*a)) \wedge B^{\notin}(b*c).$$

So $\mathcal{B}$ satisfies the conditions (IVICMI-2), (IVICMI-3), (IVICMI-5) and (IVICMI-6). Hence $\mathcal{B}$ is an IVICMI of Y. □

5. Product of IVICMIs

Definition 5.1. Let $\mathcal{A} =< \mathbf{A}, A >$ and $\mathcal{B} =< \mathbf{B}, B >$ be two IVIC sets in a set X. Then the Cartesian product of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \times \mathcal{B} =< \mathbf{A} \times \mathbf{B}, A \times B >$, is an IVIC set in $X \times X$ defined as follows: for each $(x, y) \in X \times X$,

$$(\mathbf{A} \times \mathbf{B})^{\in}(x, y) = \mathbf{A}^{\in}(x) \wedge \mathbf{B}^{\in}(y),$$

$$(\mathbf{A} \times \mathbf{B})^{\notin}(x, y) = \mathbf{A}^{\notin}(x) \vee \mathbf{B}^{\notin}(y),$$

$$(A \times B)^{\in}(x, y) = A^{\in}(x) \vee B^{\in}(y),$$

$$(A \times B)^{\notin}(x, y) = A^{\notin}(x) \wedge B^{\notin}(y).$$

Remark 5.2. Let X and Y be medial BCI-algebras. We define an operation $*$ on $X \times Y$ as follows: for any $(x, y),\ (u, v) \in X \times Y$,

$$(x, y) * (u, v) = (x * u, y * v).$$

Then it is obvious that $(X \times Y, *)$ is a BCI-algebra.

Proposition 5.3. *Let X be a medial BCI-algebra and let $\mathcal{A} =< \mathbf{A}, A >$ and $\mathcal{B} =< \mathbf{B}, B >$ be two iIVIC MIs of X. Then $\mathcal{A} \times \mathcal{B}$ is an IVICMI of $X \times X$.*

Proof. Let $(x, y) \in X \times X$. Then

$$\begin{aligned}(\mathbf{A} \times \mathbf{B})^{\in}(0, 0) &= \mathbf{A}^{\in}(0) \wedge \mathbf{B}^{\in}(0)\\ &\geq \mathbf{A}^{\in}(x) \wedge \mathbf{B}^{\in}(y)\\ &= (\mathbf{A} \times \mathbf{B})^{\in}(x, y).\end{aligned}$$

Similarly, we have the following inequalities:

$$(\mathbf{A} \times \mathbf{B})^{\notin}(0, 0) \leq (\mathbf{A} \times \mathbf{B})^{\notin}(x, y),$$

$$(A \times B)^{\in}(0, 0) \leq (A \times B)^{\in}(x, y),$$

$$(A \times B)^{\notin}(0, 0) \geq (A \times B)^{\notin}(x, y).$$

Thus $\mathcal{A} \times \mathcal{B}$ satisfies the conditions (IVICMI-1) and (IVICMI-4).

Let $(x_1, x_2),\ (y_1, y_2),\ (z_1, z_2) \in X \times X$. Then

$$(\mathbf{A} \times \mathbf{B})^{\in}((z_1, z_2) * ((y_1, y_2) * (x_1, x_2))) \wedge (\mathbf{A} \times \mathbf{B})^{\in}((y_1, y_2) * (z_1, z_2))$$

$= (\mathbf{A}\times\mathbf{B})^{\in}(z_1 * (y_1 * z_1), z_2 * (y_2 * z_2)) \wedge (\mathbf{A}\times\mathbf{B})^{\in}(y_1 * z_1, y_2 * z_2)$
$= [\mathbf{A}^{\in}(z_1 * (y_1 * z_1)) \wedge \mathbf{B}^{\in}(z_2 * (y_2 * z_2))] \wedge [\mathbf{A}^{\in}(y_1 * z_1) \wedge \mathbf{B}^{\in}(y_2 * z_2)]$
$\leq \mathbf{A}^{\in}(x_1) \wedge \mathbf{B}^{\in}(x_2)$
$= (\mathbf{A}\times\mathbf{B})^{\in}(x_1, x_2)$.

Similarly, we have the following inequalities:

$$(\mathbf{A}\times\mathbf{B})^{\notin}(x_1,x_2) \leq (\mathbf{A}\times\mathbf{B})^{\notin}((z_1,z_2)*((y_1,y_2)*(x_1,x_2)))\vee(\mathbf{A}\times\mathbf{B})^{\notin}((y_1,y_2)*(z_1,z_2)),$$

$$(A\times B)^{\in}(x_1,x_2) \leq (A\times B)^{\in}((z_1,z_2)*((y_1,y_2)*(x_1,x_2)))\vee(A\times B)^{\in}((y_1,y_2)*(z_1,z_2)),$$

$$(A\times B)^{\notin}(x_1,x_2) \geq (A\times B)^{\notin}((z_1,z_2)*((y_1,y_2)*(x_1,x_2)))\wedge(A\times B)^{\notin}((y_1,y_2)*(z_1,z_2)).$$

Thus $\mathcal{A}\times\mathcal{B}$ satisfies the conditions (IVICMI-2), (IVICMI-3), (IVICMI-5) and (IVICMI-6). So $\mathcal{A}\times\mathcal{B}$ is an IVICMI of $X \times X$. □

6. Conclusions

We defined an interval-valued intuitionistic cubic medial ideal in BCI-algebras and we obtained some of its properties. Next, how the image and the preimage of interval-valued intuitionistic cubic medial ideal in BCI-algebras become interval-valued intuitionistic cubic medial ideal studied. Finally, the product of interval-valued intuitionistic cubic medial ideals was established. Furthermore, we constructed some algorithms applied to medial-ideal in BCI-algebras. The main purpose of our future work is to investigate the interval-valued intuitionistic cubic foldedness of medial ideal in BCI-algebras.

7. Algorithm for BCI-algebras

Input (X: set, $*$: binary operation)
Output ("X is a BCI-algebra or not")
Begin
If $X = \phi$, then go to (1.);
End If
If $0 \notin X$, then go to (1.);
End If
Stop: =false;
$i := 1$;
While $i \leq\mid X \mid$ and not (Stop) do
If $x_i * x_i \neq 0$, then
Stop: = true;
End If
$j := 1$;
While $j \leq\mid X \mid$ and not (Stop) do
If $x_i * (x_i * y_j)) * y_j \neq 0$, then
Stop: = true;
End If
End If
$k := 1$;
While $k \leq\mid X \mid$ and not (Stop) do
If $(x_i * y_j) * (x_i * z_k)) * (z_k * y_j \neq 0$, then

Stop: = true;
End If
End While
End While
End While
If Stop then
(1.)→ Output ("X is not a BCI-algebra")
Else
Output ("X is a BCI-algebra")
End If.

8. Algorithm for fuzzy subsets

Input (BCI-algebra, $\mu : X \to [0,1]$);
Output ("A is a fuzzy subset of X or not")
Begin
Stop: =false;
$i := 1$;
While $i \leq\mid X \mid$ and not (Stop) do
If ($\mu(x) < 0$) or ($\mu(x) > 1$), then
Stop: = true;
End If
End While
If Stop then
Output ("μ is a fuzzy subset of X ")
Else
Output ("μ is not a fuzzy subset of X ")
End If
End.

9. Algorithm for medial ideals

Input (BCI-algebra, I: subset of X);
Output ("I is an medial ideals of X or not");
Begin
If $I = \phi$, then go to (1.);
End If
If $0 \notin I$, then go to (1.);
End If
Stop: =false;
$i := 1$;
While $i \leq\mid X \mid$ and not (Stop) do
$j := 1$;
While $j \leq\mid X \mid$ and not (Stop) do
$k := 1$;
While $k \leq\mid X \mid$ and not (Stop) do
If $z_k * (y_j * x_i) \in I$ and $y_j * z_k \in I$, then
If $x_i \notin I$, then

Stop: = true;
End If
End If
End While
End While
End While
If Stop then
Output (“I is is an medial ideal of X”)
Else
(1.) Output (“I is not is an medial ideal of X”)
End If
End.

10. Algorithm for cubic medial ideal of X

Input (BCI-algebra, $*$: binary operation, λ anti fuzzy subsets and $\widetilde{\mu}$ interval value of);
Output (“$B = (x, \lambda, \widetilde{\mu})$ is cubic medial ideal of X or notÔÇØ)
Begin
Stop: =false;
$i := 1$;
While $i \leq\mid X \mid$ and not (Stop) do
If $\widetilde{\mu}(0) > \widetilde{\mu}(x)$ and $\widetilde{\mu}(0) < \widetilde{\mu}(x)$, then
Stop: = true;
End If
$j := 1$;
While $j \leq\mid X \mid$ and not (Stop) do
$k := 1$;
While $k \leq\mid X \mid$ and not (Stop) do
If $\lambda(x) > max\{\lambda(x * y), \widetilde{\mu}(x)\} < rmin\{\widetilde{\mu}(x * y), \widetilde{\mu}(x)\}$, then
Stop: = true;
End If
End While
End While
End While
If Stop then
Output (“$B = (x, \lambda, \widetilde{\mu})$ is not cubic medial ideal of X”)
Else
Output(“$B = (x, \lambda, \widetilde{\mu})$ is cubic medial ideal of X”)
End If.
End.

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J. Kim (junhikim@wku.ac.kr)
Department of Mathematics Education, Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea

Minseok Cheong (poset@korea.ac.kr)
Information Security Convergence, College of Informatics, Korea University, Seoul, Korea

J. G. Lee (jukolee@wku.ac.kr)

Department of Applied Mathematics, Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea

K. Hur (kulhur@wku.ac.kr)
Department of Applied Mathematics, Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea

S. M. Mostafa (samymostafa@yahoo.com)
Faculty of Education Department of Math., Ain Shams University, Roxy, Cairo, Egypt