Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 19, No. 1, (February 2020) pp. 95–107
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2020.19.1.95

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 19, No. 1, February 2020

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Received 4 September 2019; Revised 10 October 2019; Accepted 29 October 2019

Abstract. Rough set theory is a powerful mathematical tool for dealing with inexact, uncertain or vague information. The core concept of rough set theory are information systems and approximation operators of approximation spaces. In this paper, we study the relationships between mset relations and mset topology. Moreover, this paper concerns generalized mset approximation spaces via topological methods and studies topological properties of rough msets. Classical compactness and connectedness for M-topological spaces are extended to generalized mset approximation spaces. Also, some properties of M-topological spaces induced by reflexive mset relation and some properties of M-topological spaces induced by tolerance mset relation are investigated.

2010 AMS Classification: 54D05, 54E45, 00A05, 03E70, 03E99

Keywords: Generalized rough msets, Reflexive mset relation, Tolerance mset relation, M-topology, GMA-space, Compactness.

Corresponding Author: E. A. Abo-Tabl (abotabl@yahoo.com)

1. Introduction

In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object is allowed in a set, then a mathematical structure, that is known as multiset (mset or bag, for short). Thus, a multiset differs from a set in the sense that each element has a multiplicity – a natural number not necessarily one – that indicates how many times it is a member of the multiset. One of the most natural and simplest examples is the multiset of prime factors of a positive integer n. The number 504 has the factorization $504 = 2^3 3^2 7^1$ which gives the multiset $\{2, 2, 2, 3, 3, 7\}$.

In any information system, some situations may occur, where the respective counts objects in the universe of discourse are not single. In such situations we replace its universe of discourse by multisets called rough multisets. The motivation to use rough multisets has come from the need to represent sub multisets of a multiset in terms of m-equivalence classes of a partition of that multiset (universe). The mset equivalence relation and mset partitions are explained in [9]. The mset partition characterizes an M-topological space, called an approximation mset space (M, E) where M is an mset called the universe and E is an equivalence mset relation. The m- equivalence classes of E are also known as granules with repetition or elementary msets or blocks. $[m/x] \subseteq M$ is used to denote the m-equivalence class containing m/x in M. In the approximation mset, there are two operators, the upper mset approximation and lower mset approximation of submsets. The concept of rough multisets and related properties with the help of lower mset approximation and upper mset approximations are important frameworks for certain types of information multisystems [6, 7].

Topology is an important branch of mathematics, which has the independent theoretic framework, background and broad applications. We can introduce topological methods to rough set theory and study the relationship between topological theory and rough set theory. This have deep theoretical and practical significance beyond doubt. Some researchers carried out this exploration. For example, Kondo [16] proved that every reflexive relation in a set can induce a topology, proposed a kind of compactness condition and got that a topology which satisfies the compactness condition can determine the lower and upper approximation operators induced by a similarity relation. Another kind of compactness condition was proposed in [18]. As a further result, a one-to-one correspondence between the set of all similarity relations and the set of all topologies which satisfy the proposed compactness condition was proved. The fact that the topology satisfying the compactness condition in [18] is exactly an Alexandrov topology was pointed out in [25]. Topological properties of different rough operators were discussed in [27].

We may relax equivalence relations so that rough set theory is able to solve more complicated problems in practice. The classical rough set theory based on equivalence relations has been extended to tolerance relations [5, 19], similarity relations [2, 20, 21], dominance relations [14], general binary relations [16, 23, 24, 28, 29], and coverings [4, 8, 17, 26, 30]. We call a pair (M, R) a generalized multiset approximation space (GMA-space for short), where M is a non-empty mset (maybe infinite) and R is a binary mset relation on M. Topological properties of GMA-space may have some application in information sciences. The purpose of this paper is to investigate further properties of the M-topological spaces induced by different binary mset relations.

We first introduce and study in Section 2 some properties of msets, mset relations and rough mset theory. Section 3 is concerned with the compactness and connectedness for GMA-spaces. In section 4, several basic concepts and results are introduced, Moreover, some properties of (M, τ_R) induced by a reflexive mset relation R, and some properties of (M, τ_R) induced by a tolerance mset relation R on M are investigated. At last, some conclusion is presented in section 5.

2. M-relations and M-topology

In this subsection, a brief survey of the notion of msets introduced by Yager [22], the different types of collections of msets and the basic definitions and notions of relations in mset context introduced by Girish and John [9, 10, 11, 12, 13] are presented.

Definition 2.1 ([15])**.** An mset M drawn from the set X is represented by a function Count M or C_M defined as $C_M : X \to N$, where N represents the set of non negative integers.

In Definition 2.1, $C_M(x)$ is the number of occurrences of the element x in the mset M. However those elements which are not included in the mset M have zero count.

Let M_1 and M_2 be two msets drawn from a set X. Then the following are defined [13]:

(i) $M_1 = M_2$, if $C_{M_1}(x) = C_{M_2}(x)$ for all $x \in X$,
(ii) $M_1 \subseteq M_2$, if $C_{M_1}(x) \leq C_{M_2}(x)$ for all $x \in X$,
(iii) $P = M_1 \cup M_2$, if $C_P(X) = Max\{C_{M_1}(x), C_{M_2}(x)\}$ for all $x \in X$,
(iv) $P = M_1 \cap M_2$, if $C_P(x) = Min\{C_{M_1}(x), C_{M_2}(x)\}$ for all $x \in X$.

Definition 2.2 ([15])**.** A domain X, is defined as a set of elements from which msets are the mset space $[X]^m$ is the set of all msets whose elements are X such that no element in the mset occurs more than m times.

If $X = \{x_1, x_2, ..., x_k\}$, then

$$[X]^m = \{\{m_1/x_1, m_2/x_2, ..., m_k/x_k\} : \text{for} i = 1, 2, ..., k; m_i \in \{0, 1, 2, ..., m\}\}.$$

Henceforth M stands for a multiset drawn from the multiset space $[X]^m$.

Definition 2.3 ([15])**.** Let M be an mset drawn from a set X. The support set of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$, i.e., M^* is an ordinary set and it is also called root set.

Definition 2.4 ([15])**.** Let X be a support set and $[X]^m$ be the mset space defined over X. Then for any mset $M \in [X]^m$, the complement M^c of M in $[X]^m$ is an element of $[X]^m$ such that $C^c_M(x) = m - C_M(x)$ for all $x \in X$.

Let M be an mset from $X = \{x_1, x_2, ..., x_n\}$ with x appearing n times in M. It is denoted by $x \in^n M$. The mset $M = \{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$ drawn from X means that M is an mset with x_1 appearing k_1 times, x_2 appearing k_2 times and so on. A new notation can be introduced for the purpose of defining Cartesian product $M_1 \times M_2$ of two multisets M_1 and M_2, relation on multisets and its domain and co-domain. The entry of the form $(m/x, n/y)/mn$ denotes that x is repeated m-times in M_1, y is repeated n-times in M_2 and the pair (x, y) is repeated mn-times in $M_1 \times M_2$. The counts of the members of the domain and co-domain vary in relation to the counts of the x co-ordinate and y co-ordinate in $(m/x, n/y)/k$. The notation $C_1(x, y)$ and $C_2(x, y)$ is therefore introduced. $C_1(x, y)$ denotes the count of the first co-ordinate in the ordered pair (x, y) and $C_2(x, y)$ denotes the count of the second co-ordinate in the ordered pair (x, y).

Definition 2.5 ([3])**.** Let $M \in [X]^m$ be an mset. The power mset $P(M)$ of M is the set of all the sub msets of M, i.e., $A \in P(M)$ if and only if $A \subseteq M$.

If $A = \varnothing$, then $A \in^1 P(M)$; and if $A \neq \varnothing$, then $A \in^k P(M)$ where $k = \prod_z \binom{|[M]_z|}{|[A]_z|}$, the product $\prod_z$ is taken over by distinct elements of z of the mset A and $|[M]_z| = m$ iff $z \in^m M$, $|[A]_z| = n$ iff $z \in^n A$, then $\prod_z \binom{|[M]_z|}{|[A]_z|} = \binom{m}{n} = m!/n!(m-n)!$.

The power set of an mset is the support set of the power mset and is denoted by $P^*(M)$. Power mset is an mset but its support set is an ordinary set whose elements are msets.

Definition 2.6 ([9])**.** A sub mset R of $M \times M$ is said to be an mset relation on M, if every member $(m/x, n/y)$ of R has a count, the product of $C_1(x, y)$ and $C_2(x, y)$. m/x related to n/y is denoted by $(m/x)R(n/y)$.

Definition 2.7 ([9])**.** Let M be an mset in $[X]^m$ and let R be an mset relation on M. Then R is said to be:
(i) reflexive, if $(m/x)R(m/x)$ for each m/x in M,
(ii) irreflexive, if $(m/x)R(m/x)$ never holds for each m/x in M,
(iii) symmetric, if $(m/x)R(n/y)$ implies $(n/y)R(m/x)$ for any $x,\ y \in X$,
(iv) antisymmetric, if $(m/x)R(n/y)$ and $(n/y)R(m/x)$ imply $m/x = n/y$,
(v) transitive, if $(m/x)R(n/y)$ and $(n/y)R(k/z)$ imply $(m/x)R(k/z)$.

A mset relation R on a mset M is called an equivalence mset relation, if it is reflexive, symmetric and transitive. A mset relation R on a mset M is called a partial ordered mset relation, if it is reflexive, antisymmetric and transitive. A mset relation R on a mset M is called a preorder relation, if it is reflexive and transitive.

Definition 2.8 ([13])**.** Let $M \in [X]^m$ and $P^*(M)$. Then τ is called a multiset topology, if τ satisfies the following properties:
(i) $\varnothing,\ M \in \tau$,
(ii) The union of the elements of any sub collection of τ is in τ,
(iii) The intersection of the elements of any finite sub collection of τ is in τ.

Mathematically, a multiset topological space is an ordered pair (M, τ) consisting of an mset $M \in [X]^m$ and a multiset topology $\tau \subseteq P^*(M)$ on M. Note that τ is an ordinary set whose elements are msets and the multiset topology is abbreviated as an M-topology. A submset U of an M-topological space M is an open mset of M, if U belongs to the M-topology. Also, a submset U of an M-topological space M is called closed, if U^c is open (See[13]).

Definition 2.9 ([13])**.** Let R be an mset relation on M.
(i) The successor-mset of $x \in^m M$ is defined as:

$$(m/x)R = \{n/y : \exists\ k \text{ with } (k/x)R(n/y)\}.$$

(ii) The successor-mset of $A \subseteq M$ is defined as:

$$(A)R = \{n/y : \exists\ k \text{ with } (k/x)R(n/y) \text{ for } \text{ some } k/x \in A\}.$$

(iii) The predecessor-mset of $x \in^m M$ is defined as:

$$R(m/x) = \{n/y : \exists\ k \text{ with } (n/y)R(k/x)\}.$$

(iv) The predecessor-mset of $A \subseteq M$ is defined as:

$$R(A) = \{n/y : \exists\ k \text{ with } (n/y)R(k/x) \text{ for some } k/x \in A\}.$$

Definition 2.10 ([13]). Let R be any binary mset relation on a nonempty multiset M. For any submset $A \subseteq M$, the lower and the upper mset approximations of A according to R are defined as:

$$\underline{R}(A) = \{m/x : (m/x)R \subseteq A\},$$
$$\overline{R}(A) = \{m/x : (m/x)R \cap A \neq \phi\}.$$

Theorem 2.11 ([13]). *If R is an mset relation on M, then the successor class $= \{(m/x)R : x \in^m M\}$ form a sub M-base for an M-topology τ on M and the predecessor class $= \{R(m/x) : x \in^m M\}$ form a sub M-base for a dual M-topology of τ on M.*

Definition 2.12 ([13]). If M is an mset, an M-basis for an M-topology on M is a collection $\mathbf{B}$ of partial whole submsets of M (called M-basis element) such that

(i) for each $x \in^m M$, for some $m > 0$, there is at least one M-basis element $B \in \mathbf{B}$ containing m/x, i.e., for each mset in $\mathbf{B}$ there is at least one element with full multiplicity as in M,

(ii) if m/x belongs to the intersection of two M-basis elements M_1 and M_2, then there is an M-basis element M_3 containing m/x such that $M_3 \subseteq M_1 \cap M_2$, i.e., there is an M-basis element M_3 containing an element with full multiplicity as in M and that element must be in M_1 and M_2 also.

3. Compactness and connectedness of GMA-spaces

In this section we consider some global properties of GMA-spaces such as compactness and connectedness. We recall first compactness of M-topological spaces.

Definition 3.1 ([1]). Let (M, R) be a GMA-space. If for all $m/x, n/y \in M$, $m/x \neq n/y$ implies $(n/y)R \cap (m/x)R = \varnothing$, then (M, R) is called a T_2^a GMA-space.

Proposition 3.2 ([1]). *If (M, R) is a normal GMA-space, then for all $m/x, n/y \in M$, $(m/x)R(n/y)$ implies that there is $v \in^t M$ such that $(t/v)R(m/x)$ and $(t/v)R(n/y)$.*

Definition 3.3. An M-topological space M is called compact, if every open cover of M has a finite subcover.

Corollary 3.4. *Let (M, R) be a M-topological GMA-space. Then (M, τ_R) is a compact space iff there is a finite mset $A \subseteq M$ such that $(A)R = M$.*

By Corollary 3.4, for general GMA-spaces, we give the following definition.

Definition 3.5. Let (M, R) be a GMA-space. If there is a finite $A \subseteq M$ such that for any $m/x \in M$, there exists $l/a \in A$ with $(l/a)R(m/x)$, then we call (M, R) a compact GMA-space.

Remark 3.6. It is easy to see by Definition 2.3 that the mset relation R of a compact GMA-space (M, R) is inverse serial, that is, for each $m/x \in M$, there is $n/y \in M$ such that $(n/y)R(m/x)$. Conversely, if M is finite and R is inverse serial, then (M, R) is compact.

For an M-topological space (M, τ) and its induced GMA-space (M, R_τ), we have

Theorem 3.7. *Let (M, τ) be an M-topological space. If (M, R_τ) is a compact GMA-space, then (M, τ) is a compact M-topological space.*

Proof. Since (M, R_τ) is an M-topological GMA-space, (M, R_τ) is compact implies that $(M, (\tau_R)_\tau)$ is compact. It follows from $\tau \subseteq (\tau_R)_\tau$ that (M, τ) is compact too. □

It is well known that for M-topological spaces compactness and T_2 imply normality. But for GMA-spaces, we have the following counterexample.

Example 3.8. Let $M = \{2/a, 3/b, 2/c, 4/d\}$ and $R = \{(2/a, 2/a), (3/b, 3/b), (3/b, 2/c)$ $, (2/c, 4/d)\}$. Then $(2/a)R = \{2/a\}, (3/b)R = \{3/b, 2/c\}, (2/c)R = \{4/d\} and (4/dR) = \phi$. Clearly (M, R) is a T_2^a GMA-space. Since M is finite and R is inverse serial, (M, R) is compact by Remark 3.6. But for $2/c, 4/d \in M$ with $(2/c)R(4/d)$, there is no $2/x \in M$ satisfying $(2/x)R(2/c)$ and $(2/x)R(4/d)$. By Proposition 3.2, (M, R) is not a normal GMA-space.

Now we pass to connectedness.

Definition 3.9. An M-topological space M is said to be connected, if its only clopen submsets are ϕ and M; otherwise, M is said to be disconnected.

Let M be an M-topological space and $A, B \subseteq M$. If $A \cap cl(B) = B \cap cl(A) = \phi$, then we call A and B a pair of separated submsets of M. It is well known that M is disconnected iff there are two non-empty separated submsets $A, B \subseteq M$ such that $M = A \cup B$ iff there are two non-empty closed msets $A, B \subseteq M$ satisfying $A \cap B = \phi$ and $A \cup B = M$ iff there are two non-empty open msets $A, B \subseteq M$ satisfying $A \cap B = \phi$ and $A \cup B = M$.

Corollary 3.10. *Let (M, R) be an M-topological GMA-space. Then (M, τ_R) is a disconnected space iff there is $A \subseteq M$ with $\phi \neq A \neq M$ such that $(A)R = A = R(A)$.*

To consider connectedness of GMA-spaces, we introduce the concepts of R-open msets, R-closed msets, R-clopen msets and R-separated submsets first.

Definition 3.11. Let (M, R) be a GMA-space and $A, B \subseteq M$. If $\overline{R}(A) \subseteq A$, then A is called an R-closed mset of (M, R). If $B \subseteq \underline{R}(B)$, then B is called an R-open mset of (M, R). If A is both R-closed and R-open, then we call A an R-clopen mset. If $A \cap B = \phi$ and $\overline{R}(A) \cap B = \overline{R}(B) \cap A = \phi$, then we call A and B a pair of R-separated submsets of (M, R).

It is easy to see that A is R-closed iff $(A)^C$ is R-open. If R is a preorder, then $\underline{R}(A) \subseteq A \subseteq \overline{R}(A)$ and the R-open (R-closed, R-clopen) msets of GMA-space (M, R) are exactly the open (closed, clopen) msets of M-topological space (M, τ_R).

Proposition 3.12. *Let (M, R) be a GMA-space and $A, B \subseteq M$. Then A and B are R-separated iff $A \cap B = \phi$, $(m/x)R^C(n/y)$ and $(n/y)R^C(m/x)$ for all $m/x \in A$ and $n/y \in B$.*

Proof. Assume that A and B are R-separated. Then trivially, $A \cap B = \phi$. Suppose that there are $m/x \in A$ and $n/y \in B$ such that $(m/x)R(n/y)$. Then $n/y \in (m/x)R \cap B \neq \phi$. Thus $m/x \in \overline{R}(B)$. This contradicts $\overline{R}(B) \cap A = \phi$. So $(m/x)R^C(n/y)$ for all $m/x \in A$ and $n/y \in B$. By the same argument, we have that $(n/y)R^C(m/x)$ for all $m/x \in A$ and $n/y \in B$.

Conversely, for all $m/x \in A$, since $(m/x)R^C(n/y)$ for all $n/y \in B$, we have $(m/x)R \cap B = \phi$ and $m/x \in \overline{R}(B)$. Then $\overline{R}(B) \cap A = \phi$. By the same argument, we have $\overline{R}(A) \cap B = \phi$. Thus A and B are R-separated. □

Definition 3.13. A GMA-space (M, R) is said to be connected, if its only R-clopen submsets are ϕ and M; otherwise, (M, R) is said to be disconnected.

Theorem 3.14. *Let (M, R) be a GMA-space. Then the following are equivalent:*
(1) *(M, R) is a disconnected GMA-space,*
(2) *there are two non-empty R-separated submsets $A,\ B \subseteq M$ such that$A \cup B = M$*
,
(3) *there are two non-empty R-closed msets $A,\ B \subseteq M$ satisfying $A \cap B = \phi$ and $A \cup B = M$,*
(4) *there are two non-empty R-open msets $A,\ B \subseteq M$ satisfying $A \cap B = \phi$ and A $A \cup B = M$.*

Proof. (1)⇒(2): Since (M, R) is a disconnected GMA-space, there is an R-clopen mset $A \subseteq M$ with $\phi \neq A \neq M$. Let $B = A^C$. Then $A,\ B \neq \phi$, $A \cap B = \phi$ and A $A \cup B = M$. Since A and B are both R-clopen, we have that $\overline{R}(A) \cap B = \phi$ and $\overline{R}(B) \cap A = \phi$. Thus A and B are R-separated.

(2)⇒(3): By (2), there are two non-empty R-separated submsets $A,\ B \subseteq M$ such that $A \cup B = M$. Then $A \cap B = \phi$ and $\overline{R}(A) \cap B = \overline{R}(B) \cap A = \phi$. Thus

$$\overline{R}(B) = \overline{R}(B) \cap M = \overline{R}(B) \cap (A \cup B) = (\overline{R}(B) \cap A) \cup (\overline{R}(B) \cap B) = \overline{R}(B) \cap B.$$

It follows that $\overline{R}(B) \subseteq B$. Similarly, $\overline{R}(A) \subseteq A$. So A and B are R-closed msets.

(3) ⇒ (4) : For the two R-closed msets A, B in (3), we have $A = B^C$ and $B = A^C$. Then $A,\ B$ are also R-open. Thus (4) holds.

(4) ⇒ (1) : If $A,\ B \subseteq M$ are the two R-open msets satisfying the conditions in (4), then $A = B^C$ is also R-closed. Thus A is an R-clopen mset of (M, R) and $\phi \neq A \neq M$. So (M, R) is a disconnected GMA-space. □

For an M-topological space (M, τ) and its induced GMA-space (M, R_τ), we have

Theorem 3.15. *Let (M, τ) be an M-topological space. If (M, R_τ) is a connected GMA-space, then (M, τ) is a connected M-topological space.*

Proof. Since (M, R_τ) is an M-topological GMA-space, (M, R_τ) is connected implies that $(M, (\tau_R)_\tau)$ is connected. It follows from $\tau \subseteq (\tau_R)_\tau$ that (M, τ) is connected. □

4. On the structure of generalized rough msets

4.1. **Definitions and proposition.** In this section, we introduce some basic concepts and relational propositions. Let M be a nonempty mset and R a binary mset relation on M. For any $A \subseteq M$, we define $\tau_R = \{A \subseteq M : \underline{R}(A) = A\}$.

The author [1] proved that if R is a reflexive mset relation on M, then τ_R is a mset topology on M, which may be called the M-topology induced by R on M.

Definition 4.1. Let R be a reflexive mset relation on M. Then (M, τ_R) is called the M-topological space induced by R on M.

Definition 4.2. Let R be a binary mset relation on M. Then R is called a tolerance mset relation on M, if R is both reflexive and symmetric.

Definition 4.3. Let R and R_s be two binary mset relations on M. If for all $m/x, n/y \in M, (m/x)R_s(n/y)$ if and only if $(m/x)R(n/y)$ or there exists $\{m_1/v_1, m_2/v_2, ..., m_n/v_n\} \subseteq M$ such that $(m/x)R(m_1/v_1), (m_1/v_1)R(m_2/v_2), ..., (m_n/v_n)R(n/y)$, then R_s is called the transmitting expression of R.

Proposition 4.4. *Let R be a binary mset relation on M and R_s the transmitting expression of R. Then R_s is a transitive relation on M. Moreover,*
(1) *if R is reflexive, then R_s is also reflexive,*
(2) *if R is transitive, then $R_s = R$,*
(3) *if R is symmetric, then R_s is also symmetric.*

Proof. Suppose $(m/x)R_s(n/y)$ and $(n/y)R_s(k/z)$. Then there exists

$$\{m_1/v_1, m_2/v_2, ..., m_n/v_n\} \subseteq M \text{ such that}$$

$$(m/x)R(m_1/v_1), (m_1/v_1)R(m_2/v_2), ..., (m_n/v_n)R(n/y)$$

and there exists $\{n_1/u_1, n_2/u_2, ..., n_n/u_n\} \subseteq M$ such that

$$(n/y)R(n_1/u_1), (n_1/u_1)R(n_2/u_2), ..., (n_n/u_n)R(k/z).$$

Thus there exists $\{m_1/v_1, m_2/v_2, ..., m_n/v_n, n_1/u_1, n_2/u_2, ..., n_n/u_n\} \subseteq M$ such that
$(m/x)R(m1/v_1), (m_1/v_1)R(m_2/v_2), ...,$
$(m_n/v_n)R(n/y), (n/y)R(n_1/u_1), (n_1/u_1)R(n_2/u_2), ..., (n_n/u_n)R(k/z).$
So $(m/x)R_s(k/z)$. Hence R_s is a transitive mset relation.

(1) Suppose R is reflexive. Then $(m/x)R(m/x)$ for all $m/x \in M$. Thus $(m/x)R_s(m/x)$, for all $m/x \in M$. So R_s is reflexive.

(2) Suppose R is transitive and $R_s \neq R$. then there exists $m/x,\ n/y \in M$ such that $(m/x)R_s(n/y)$ and $(n/y)R_s(m/x)$. Thus there exists $\{m_1/v_1, m_2/v_2, ..., m_n/v_n\} \subseteq M$ such that $(m/x)R(m_1/v_1), (m_1/v_1)R(m_2/v_2), ..., (m_n/v_n)R(n/y)$. Since R is transitive, $(m/x)R(n/y)$, which it is a contradiction. So $R_s = R$.

(3) Suppose R is symmetric and $(m/x)R_s(n/y)$. Then by definition, $(m/x)R(n/y)$. Thus $(n/y)R(m/x)$. So $(n/y)R_s(m/x)$. Hence R_s is symmetric. □

Definition 4.5. (i) Let ϑ_R be a base of M-topological space (M, τ_R), which is induced by a reflexive mset relation R on M. For $P \in \vartheta_R$, if there does not exist $P' \in \vartheta_R - \{P\}$ such that $P \subseteq P'$, then P is called a maximal element of ϑ_R.

(ii) ϑ_R^* denotes the mset of all maximal elements of ϑ_R. Since $\bigcup \vartheta_R^* = M$, ϑ_R^* is called the minimal complete cover of (M, τ_R) relative to the base ϑ_R.

4.2. **The properties of M-topological spaces induced by a reflexive M-relation.** In this section, we will investigate the properties of (M, τ_R) induced by a reflexive mset relation R on M.

Lemma 4.6. *Let R be a reflexive mset relation on M and R_s the transmitting expression of R. For each $m/x \in M$, put $L_x = \{n/y \in M : \text{for some} k(k/x)R_s(n/y)\}$. Then*
(1) $L_x \in \tau_R$,
(2) $\{L_x\}$ *is an open neighborhood base of m/x,*
(3) $\beta_R = \{L_x : m/x \in M\}$ *is a base for* (M, τ_R),

(4) L_x *is a compact submset of* (M, τ_R).

Proof. (1) Supposet $m/x \in \underline{R}(L_x)$. Then $(m/x)R \subseteq L_x$, i.e.,

$$\{n/y \in M : \exists\ k \text{ with } (k/x)R(n/y)\} \subseteq L_x.$$

Since R is reflexive, R_s is also reflexive. Thus $m/x \in L_x$, i.e., $\underline{R}(L_x) \subseteq L_x$.

Conversely, suppose $m/x \in L_x$. Then $(m/x)R_s(m/x)$. Thus $(m/x)R(m/x)$ or there exists $\{m_1/v_1, m_2/v_2, ..., m_n/v_n\} \subseteq M$ such that

$$(m/x)R(m_1/v_1), (m_1/v_1)R(m_2/v_2), ..., (m_n/v_n)R(m/x).$$

.

If $(m/x)R(m/x)$, then $m/x \in (m/x)R$. Thus $(m/x)R \subseteq L_x$, i.e, $m/x \in \underline{R}(L_x)$.

If there exists $\{m_1/v_1, m_2/v_2, ..., m_n/v_n\} \subseteq M$ such that

$$(m/x)R(m_1/v_1), (m_1/v_1)R(m_2/v_2), ..., (m_n/v_n)R(m/x),$$

then $m/x \in (m/x)R$. Thus $(m/x)R \subseteq L_x$, i.e, $m/x \in \underline{R}(L_x)$. So $L_x \subseteq \underline{R}(L_x)$. Hence $L_x \in \tau_R$.

(2) Since $L_x \in \tau_R$ and $m/x \in L_x$, $\{L_x\}$ is an open neighborhood base of m/x.

(3) Since R is reflexive, by Definition 2.10, $\bigcup_{m/x \in M}(L_x) = M$. Let $A \in \tau_R$ and $m/x \in A$. Then $(m/x)R \subseteq A$. Thus from (1), we get $L_x \subseteq A$. So β_R is a base for (M, τ_R).

(4) From Definition 4.5 and (3), the proof is immediately. □

Remark 4.7. (1) Let R be a binary mset relation on M. For all $m/x, n/y \in M$, if $(m/x)R(n/y)$ and $(n/y)R(m/x)$, then $L_x = L_y$.

(2) Let ϑ_R be a base for (M, τ_R). Then $\beta_R \subseteq \vartheta_R$. Otherwise, there exists $B \in \beta_R$ but $B \in \vartheta_R$. Notice that $B \in \beta_R$, there exists $m/x \in B$ such that $B = L_x$. Since ϑ_R is a base for (M, τ_R), there exists $\vartheta^*_R \subseteq \vartheta_R$ such that $B = \bigcup \vartheta^*_R$. Thus $m/x \in P \subseteq B$ for some $P \in \vartheta'_R$. By Lemma 4.6, $B \subseteq P$. So $B = P \in \vartheta_R$ and this is a contradiction. Hence $\beta_R \subseteq \vartheta_R$.

Theorem 4.8. *Let* (M, τ_R) *be the M-topological space induced by a reflexive mset relation* R *on* M. *Then*

(1) (M, τ_R) *is a first countable space,*

(2) (M, τ_R) *is a locally compact space,*

(3) *if* M *is countable, then* (M, τ_R) *is a second countable space.*

Theorem 4.9. *Let* R *be a reflexive mset relation on* M *and* R_s *the transmitting expression of* R. *Then* $(M, \tau_R) = (M, \tau_{R_s})$.

Lemma 4.10. *Let* (M, τ_R) *be the M-topological space induced by a reflexive mset relation* R *on* M, β^*_R *the minimal complete cover of* (M, τ_R) *relative to the base* β_R *. Then, for each* $F \in \beta^*_R$, $\bigcup(\beta_R - \{F\}) \neq M$ *and* $\bigcup(\beta^*_R - \{F\}) \neq M$.

Proof. Suppose that $\bigcup(\beta_R - \{F\}) = M$. Then there exists $\beta'_R \subseteq \beta_R - \{F\}$ such that $F \subseteq \bigcup \beta'_R$. Since $F \in \beta^*_R \subseteq \beta_R$, there exists $m/x \in M$ such that $F = L_x$. Thus $m/x \in F'$, for some $F' \in \beta^*_R$. By Lemma 4.6, $F = L_x \subseteq F'$. So F is not a maximal element of β_R and this implies a contradiction. Hence $\bigcup(\beta_R - \{F\}) \neq M$, since $\bigcup(\beta_R - \{F\}) \neq M$ and $\bigcup(\beta^*_R - \{F\}) \neq M$. □

Lemma 4.11. *Let (M, τ_R) be the M-topological space induced by a reflexive mset relation R on M, β_R^* the minimal complete cover of (M, τ_R) relative to the base β_R and ϑ_R an open cover of (M, τ_R). Then, for each $F \in \beta_R^*$, there exists $P \in \vartheta_R$ such that $F \subseteq P$.*

Proof. Since ϑ_R is an open cover of (M, τ_R), for each $F \in \beta_R^*$, there exists $\vartheta_R^{'} \subseteq \vartheta_R$ such that $F \subseteq \bigcup \vartheta_R^{'}$. Because $F \in \beta_R^* \subseteq \beta_R$, $F = L_x$, for some $m/x \in F$. Then there exists $P \in \vartheta_R^{'} \subseteq \vartheta_R$ such that $m/x \in P$. Thus by Lemma 4.6, $F \subseteq P$. □

Lemma 4.12. *Let (M, τ_R) be the M-topological space induced by a reflexive relation R on M, β_R^* the minimal complete cover of (M, τ_R) relative to the base β_R and ϑ_R an open cover of (M, τ_R), which is constituted by some elements of β_R^*. Then $\beta_R^* \subseteq \vartheta_R$.*

Proof. For each $F \in \beta_R^*$, we claim $F \in \vartheta_R$. Otherwise, $F \in \vartheta_R$. Since $\bigcup \vartheta_R = M$, $\bigcup \vartheta_R \ \{F\} = M$. Then $\bigcup \beta_R - \{F\} = M$. By Lemma 4.10, $\bigcup \beta_R - \{F\} \neq M$ and this is a contradiction. Thus $\beta_R^* \subseteq \vartheta_R$. □

Theorem 4.13. *Let (M, τ_R) be the topological space induced by a reflexive mest relation R on M, β_R^* is the minimal complete cover of (M, τ_R) relative to the base β_R. Then (M, τ_R) is a compact space if and only if β_R^* is a finite set.*

Proof. The if part follows from Lemma 4.11. We will prove the only if part. Suppose that (M, τ_R) is compact. Since β_R is an open cover of (M, τ_R), β_R has a finite subcover $\beta_R^{'}$. By Lemma 4.12, $\beta_R^* \subseteq \beta_R^{'}$. Then $| \beta_R^* | \leq | \beta_R^{'} |$. Thus β_R^* is a finite set. □

4.3. **The properties of M-topological spaces induced by a tolerance M-relation.** In this section, we will investigate the properties of (M, τ_R) induced by a tolerance mset relation R on M.

Lemma 4.14 ([1])**.** *Let (M, τ_R) be the M-topological space induced by a tolerance mset relation R on M. Then for any $A \subseteq M$, A is open if and only if A is closed.*

Theorem 4.15. *Let (M, τ_R) be the M-topological space induced by a tolerance mset relation R on M. Then (M, τ_R) is T_0 if and only if (M, τ_R) is discrete.*

Proof. The if part is obvious. We will prove the only if part. By Lemma 4.6, we know that if R is reflexive, then $\{L_x : m/x \in M\}$ is a base for (M, τ_R). We claim that $L_x = \{m/x\}$ for any $m/x \in M$. In fact. Suppose $L_x \neq \{m/x\}$, for some $m/x \in M$. By Proposition 4.4, R_s is an equivalence mset relation on M. Then $L_x = [x]R_s$. Pick $n/y \in [x]R_s$ such that $n/y \neq m/x$. Since (M, τ_R) is T_0, there exists an open submset U such that $m/x \in U$ and $n/y \in U$, or there exists an open submset V such that $n/y \in V$ and $m/x \in V$.

If there exists an open submset U such that $m/x \in U$ and $n/y \in U$, then $m/x \in L_z \subseteq U$, for some $k/z \in M$, by Lemma 4.6. It follows that $n/y \in L_z$. Since R_s is an equivalence mset relation on M, $[x]R_s = [z]R_s = L_z$. Thus $n/y \in [x]R_s = L_z$ is a contradiction.

If there exists an open submset V such that $n/y \in V$ and $m/x \in V$, then the proof is similar. So $\{m/x\}$ is open for any $m/x \in M$. Therefore, all submsets of M are open and this means that (M, τ_R) is discrete. □

Theorem 4.16. *Let (M, τ_R) be the M-topological space induced by a tolerance mset relation R on M. Then following are equivalent:*
(1) *M/R_s is countable,*
(2) *(M, τ_R) is a second countable space,*
(3) *(M, τ_R) is a separable space,*
(4) *(M, τ_R) is a Lindelof space.*

Proof. (1) $\Rightarrow$ (2): Since R_s is an equivalence mset relation on M, $\{L_x : m/x \in M\} = M/R_s$. By Lemma 4.6, (M, τ_R) is a second countable space.

(2) $\Rightarrow$ (1): Suppose that G is a countable base for (M, τ_R). Then for $m/x \in M$, there exists $g_x \in G$ such that $m/x \in g_x \subseteq L_x$. By Lemma 4.6, $m/x \in L_y \subseteq g_x$, for some $n/y \in M$. Since $L_x = [m/x]R_s = [n/y]R_s = L_y$, $g_x = [m/x]R_s$. We define $f : M/R_s \to G$ by $f([m/x]R_s) = gx$, then f is injective. Thus $|M/R_s| \leq |G|$. So M/R_s is countable.

(2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are obvious.

(3) $\Rightarrow$ (2): Suppose that C is a countable dense submset of (M, τ_R). Put $\mathcal{F} = \{L_x : m/x \in C\}$. Then $\mathcal{F}$ is countable. By Lemma 4.6, for each $m/x \in M$ and open submset U with $m/x \in U$, we have $m/x \in L_y \subseteq U$ for some $n/y \in M$. Since C is dense, $L_y \cap C \neq \phi$. Pick $k/z \in L_y \cap C$. Then $L_z \in \mathcal{F}$. Since R_s is an equivalence mset relation on M, $L_z = [k/z]R_s = [n/y]R_s = L_y$. It follows $m/x \in L_z \subseteq U$. Thus $\mathcal{F}$ is a base for (M, τ_R). So (M, τ_R) is a second countable space.

(4) $\Rightarrow$ (2): Suppose that M/R_s is not countable. Since R_s is an equivalence mset relation on M, $\{L_x : m/x \in M\} = M/R_s$. It is obvious that $\{L_x : m/x \in M\}$ is an open cover of (M, τ_R). But $\{L_x : m/x \in M\}$ do not have any countable subcover and we obtain a contradiction. □

Theorem 4.17. *Let (M, τ_R) be the topological space induced by a tolerance mset relation R on M. Then (M, τ_R) is a connected space if and only if $R_s = M \times M$.*

Proof. Suppose that (M, τ_R) is connected. If $R_s \neq M \times M$, then $(M \times M) - R_s \neq \phi$. Pick $(m/x, n/y) \in (M \times M) - R_s \neq \phi$. Then $n/y \in [x]R_s = L_x$. So $L_x \neq M$ and $L_x \neq \phi$. By Lemma 4.14, L_x is both open and closed. This gives a contradiction. Now suppose that $R_s = M \times M$. Then $M/R_s = \{M\}$. Thus $\tau_R = \{M, \phi\}$. So (M, τ_R) is connected. □

Theorem 4.18. *Let (M, τ_R) be the topological space induced by a tolerance mset relation R on M. Then*
(1) *(M, τ_R) is a locally connected space,*
(2) *(M, τ_R) is a locally separable space,*
(3) *(M, τ_R) is a regular space,*
(4) *(M, τ_R) is a normal space.*

5. Conclusions

In this paper, we studied GMA-spaces in terms of topological methods and gave further connections between M-topology and rough mset theory. we first characterized the compactness and connectedness of M-topological GMA-spaces and then extended all these properties to general GMA-spaces.
Secondly, we studied some M-topological structure of generalized rough msets, where

the M-topology is induced by a reflexive mset relation and a tolerance mset relation respectively. Also, we investigated approximating spaces and obtained sufficient and necessary conditions that M-topological spaces are approximating spaces. In future work, we will continue the study of M-topological properties of rough msets.

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El-Sayed A. Abo-Tabl (abotabl@yahoo.com)
Mathematics Department, Faculty of Science, Assiut University, Assiut, Egypt
Mathematics Department, Faculty of Science and Arts, Qassim University, Mithnab, KSA