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Abstract. In this paper, some basic results in complete fuzzy cone metric space are studied and Cantor's Intersection Theorem is established in fuzzy setting. On the other hand some results in compact fuzzy cone metric spaces are developed.

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Keywords: Normal cone, α-fuzzy diameter, Cantor's Intersection Theorem, α-fuzzy compact set, α-fuzzy totally boundedness.

Corresponding Author: Tarapada Bag (tarapadavb@gmail.com)

1. Introduction

After introducing the concept of fuzzy set by Zadeh [10] in 1965, the research in fuzzy mathematics has been demonstrated in different directions such as fuzzy functional analysis, fuzzy topology, fuzzy control theory, fuzzy dynamical system etc. We know that metric space, normed linear space and inner product space are the main tools in functional analysis, so to develop fuzzy functional analysis, fuzzy metric space plays an important role. Several authors introduced the idea of fuzzy metric space in different approaches (for references please see ([3, 5, 6]).

Recently, Huang and Zhang [4] introduced a generalized idea of metric space which is called cone metric space and by using it many papers have been published (please see ([1, 2, 7, 8, 9]) in such space. In our earlier paper [7], following the definition of cone metric space introduced by Huang and Zhang [4], definition of fuzzy cone metric space is given and established some basic results.

In this paper, various properties specially completeness and compactness of fuzzy cone metric space have been defined by Majumder and Bag [7]. Cantor's Intersection Theorem is established in fuzzy setting. We have also introduced the concept of totally bounded set in fuzzy cone metric space.

The organization of the paper is as follows:
In Section 2, some preliminary results are given to be used in this paper. Some basic properties of fuzzy Cone metric spaces have been studied in Section 3. In Section 4, Cantor's Intersection Theorem is established in fuzzy cone metric spaces., Some results of α-fuzzy completeness, α-fuzzy compactness, α-fuzzy totally boundedness have been studied in Section 5.

2. Preliminaries

Throughout the paper, we denote a real Banach space by E and the zero element of E by θ.

Definition 2.1 ([4]). Let E be a real Banach space and P be a subset of E. Then P is called a cone, if it satisfies the following conditions:
(i) P is closed, nonempty and $P \neq \{\theta\}$,
(ii) $a, b \in R$; $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.
Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ iff $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ by $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$.

The cone P is called normal, if there is a number $K > 0$ such that $\forall x, y \in E$, with $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying above is called the normal constant of P.

Definition 2.2 ([7]). Let X be a non-empty set and E be a real Banach space with cone P, $*$ be a t- norm. A fuzzy subset $M_c : X \times X \times E \to [0, 1]$ is said to be a fuzzy cone metric, if the following conditions hold:
(CM1) $M_c(x, y, t) = 0 \ \forall t \preceq \theta$,
(CM2) $\forall t \succ \theta, M_c(x, y, t) = 1$ iff x=y,
(CM3) $M_c(x, y, t) = M_c(y, x, t) \ \forall t \succ \theta$,
(CM4) for $t, s \in E$, $\forall x, y, z \in X$, $M_c(x, z, s + t) \geq M_c(x, y, s) * M_c(y, z, t)$,
(CM5) $\lim\limits_{\|t\|\to\infty} M_c(x, y, t) = 1$.

Theorem 2.3 ([7]). *Let $(X, M_c, *)$ be a fuzzy cone metric space. Then $\{x_n\}$ in X converges iff $\lim\limits_{n\to\infty} M_c(x_n, x, t) = 1$, $\forall t \succ \theta$.*

Theorem 2.4 ([7]). *Let $(X, M_c, *)$ be a fuzzy cone metric space. Then $\{x_n\}$ in X is a Cauchy sequence iff $\lim\limits_{m,n\to\infty} M_c(x_n, x_m, t) = 1$, $\forall t \succ \theta$.*

Proposition 2.5. *If $(X, M_c, *)$ is a fuzzy cone metric space and P is a normal cone with normal constant $K(\geq 1)$, then for any $\epsilon \succ \theta$, we have $\|\epsilon\| > 0$.*

Proof. Since P is a normal cone with normal constant K, $\theta \prec \epsilon$. Then $\|\theta\| \leq K\|\epsilon\|$. Thus $\|\epsilon\| \geq 0$. So $\|\epsilon\| > 0$. □

Definition 2.6. Let $(X, M_c, *)$ be a fuzzy cone metric space. For $x, y \in X, M_c(x, y, .)$ is said to be continuous at $t \in E$, if $\lim\limits_{\|\epsilon\|\to 0} M_c(x, y, t \pm \epsilon) = M_c(x, y, t)$.

If it is true for any $t \in E$, then we say $M_c(x, y, .)$ is continuous on E.

3. Some basic properties

In this section some basic properties of fuzzy cone metric spaces have been studied.

Lemma 3.1. *Let $(X, M_c, *)$ be a fuzzy cone metric space with normal constant K, let $*$ be a continuous t-norm and let $M_c(x, y, .)$ be continuous on E, for each $x, y \in X$. Suppose $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that $x_n \to x_0$, $y_n \to y_0$, for some $x_0, y_0 \in X$. Then $\lim\limits_{n\to\infty} M_c(x_n, y_n, t) = M_c(x_0, y_0, t)$, $\forall t \succ \theta$.*

Proof. Choose $\epsilon \succ \theta$ arbitrary. Since $(X, M_c, *)$ is a fuzzy cone metric space with normal constant K, it follows that $\|\epsilon\| > 0$ is arbitrary.

Now for $t \succ \theta$, we have

$$\begin{aligned} M_c(x_0, y_0, t+\epsilon) &\geq M_c(x_n, x_0, \tfrac{\epsilon}{2}) * M_c(x_n, y_0, t+\tfrac{\epsilon}{2}) \\ &\geq M_c(x_n, x_0, \tfrac{\epsilon}{2}) * M_c(x_n, y_n, t) * M_c(y_n, y_0, \tfrac{\epsilon}{2}). \end{aligned}$$

Now let $n \to \infty$. Then we have

$$\begin{aligned} M_c(x_0, y_0, t+\epsilon) &\geq \lim_{n\to\infty} M_c(x_n, x_0, \frac{\epsilon}{2}) * \lim_{n\to\infty} M_c(x_n, y_n, t) * \lim_{n\to\infty} M_c(y_n, y_0, \frac{\epsilon}{2}) \\ &= 1 * \lim_{n\to\infty} M_c(x_n, y_n, t) * 1 \\ &= \lim_{n\to\infty} M_c(x_n, y_n, t). \end{aligned}$$

Thus $\lim\limits_{\|\epsilon\|\to 0} M_c(x_0, y_0, t+\epsilon) \geq \lim\limits_{n\to\infty} M_c(x_n, y_n, t)$, i.e.,

$$M_c(x_0, y_0, t) \geq \lim_{n\to\infty} M_c(x_n, y_n, t), \ \forall t \succ \theta. \tag{3.1}$$

Next for $t \succ \theta$, choose ϵ such that $\theta \prec \frac{\epsilon}{2} \prec t$. Then we have

$$\begin{aligned} M_c(x_n, y_n, t) &\geq M_c(x_n, x_0, \tfrac{\epsilon}{4}) * M_c(x_0, y_n, t-\tfrac{\epsilon}{4}) \\ &\geq M_c(x_n, x_0, \tfrac{\epsilon}{4}) * M_c(x_0, y_0, t-\tfrac{\epsilon}{2}) * M_c(y_n, y_0, \tfrac{\epsilon}{4}). \end{aligned}$$

Now let $n \to \infty$. Then we have

$$\begin{aligned} \lim_{n\to\infty} M_c(x_n, y_n, t) &\geq \lim_{n\to\infty} M_c(x_n, x_0, \frac{\epsilon}{4}) * M_c(x_0, y_0, t-\frac{\epsilon}{2}) * \lim_{n\to\infty} M_c(y_n, y_0, \frac{\epsilon}{4}) \\ &= 1 * M_c(x_0, y_0, t-\tfrac{\epsilon}{2}) * 1 \\ &= M_c(x_0, y_0, t-\tfrac{\epsilon}{2}). \end{aligned}$$

Since $t \succ \theta$ is arbitrary and $\theta \prec \frac{\epsilon}{2} \prec t$, we have $\|\epsilon\| > 0$ is arbitrary. Thus we get

$$\lim_{n\to\infty} M_c(x_n, y_n, t) \geq \lim_{\frac{\|\epsilon\|}{2}\to 0} M_c(x_0, y_0, t-\frac{\epsilon}{2}) = M_c(x_0, y_0, t), \text{ i.e.,}$$

$$\lim_{n\to\infty} M_c(x_n, y_n, t) \geq M_c(x_0, y_0, t). \tag{3.2}$$

From (3.1) and (3.2), we have

$$\lim_{n\to\infty} M_c(x_n, y_n, t) = M_c(x_0, y_0, t), \ \forall t \succ \theta.$$

□

Definition 3.2. Let $(X, M_c, *)$ be a fuzzy cone metric space. We define the open ball $B_c(x, r, t)$ with centre $x \in X$ and radius r; $0 < r < 1$, $t \succ \theta$ as follows:

$$B_c(x, r, t) = \{y \in X : M_c(x, y, t) > 1 - r\}.$$

Proposition 3.3. *Let $(X, M_c, *)$ be a fuzzy cone metric space and $F \subset X$. If $x \in \bar{F}$, then for a given α, $0 < \alpha < 1$ and $t \succ \theta$, $\exists y \in F$ such that $M_c(x, y, t) > 1 - \alpha$, where $\bar{F}$ denotes the closure of F.*

Proof. Note that $\bar{F} = F \bigcup F'$, whereF'-derived set of F. Choose $x \in \bar{F}$.

(Case-I): If$x \in F$, then take y=x. Thus we have for any $\alpha \in (0,1)$ and any $\forall t \succ \theta$,

$$M_c(x,y,t) = M_c(x,x,t) = 1 > 1-\alpha.$$

(Case-II): If $x \in F'$ but $x \neq y$, then for each α, $0<\alpha<1$ and $t \succ \theta$, $\exists y \in F$ such that $y \in B_c(x,\alpha,t)$. Thus $M_c(x,y,t) > 1-\alpha$. □

Proposition 3.4. *Let $(X, M_c, *)$ be a fuzzy cone metric space and $F \subset X$. Then for any $x \in \bar{F}$ and for each $t \succ \theta$, there exists a sequence $\{x_n\}$ in F such that*

$$\lim_{n\to\infty} M_c(x_n, x, t) = 1.$$

Proof. From the above proposition, it follows that for $x \in \bar{F}$ and for a given $t \succ \theta$ and $0<\alpha<1$, $\exists y \in F$ such that $M_c(y,x,t) > 1-\alpha$. Choose $t_0 \in E$ with $t_0 \succ \theta$ be fixed and consider a sequence $\{\alpha_n\}$ in (0,1) such that $\alpha_n \to 0$ as $n \to \infty$. Then for each α_n, $\exists x_n \in F$ such that $M_c(x_n,x,t_0) > 1-\alpha_n$. Thus $\lim\limits_{n\to\infty} M_c(x_n,x,t_0) \geq 1-\lim\limits_{n\to\infty}\alpha_n = 1$. So $\lim\limits_{n\to\infty} M_c(x_n,x,t_0) = 1$. Hence $\exists$ a sequence $\{x_n\}$ in F such that $\lim\limits_{n\to\infty} M_c(x_n,x,t) = 1$, for each $t \succ \theta$. □

4. Cantor's Intersection Theorem

In this section, Cantor's Intersection Theorem is established in fuzzy cone metric spaces.

Definition 4.1. Let $(X, M_c, *)$ be a fuzzy cone metric space. Let $A \subset X$ and $\alpha \in (0,1)$. We define α-fuzzy diameter of A by:

$$\alpha - \delta(A) = \bigvee_{x,y\in A} \bigwedge \{\|t\| > 0 : M_c(x,y,t) \geq 1-\alpha\}.$$

Lemma 4.2. *Let $(X, M_c, *)$ be a fuzzy cone metric space, where $* = min$ with normal cone P and normal constant $K(\geq 1)$. Then for any nonempty subset A of X,*

$$\alpha - \delta(A) = \alpha - \delta(\bar{A}), \quad \forall \alpha \in (0,1).$$

Proof. Choose $\alpha_0 \in (0,1)$ be arbitrary. Then clearly,

$$\alpha_0 - \delta(A) = \bigvee_{x,y\in A} \bigwedge \{\|t\| > 0 : M_c(x,y,t) \geq 1-\alpha_0\}.$$

Since $A \subset \bar{A}$, we have

$$\bigvee_{x,y\in \bar{A}} \bigwedge \{\|t\| > 0 : M_c(x,y,t) \geq 1-\alpha_0\} \geq \bigvee_{x,y\in A} \bigwedge \{\|t\| > 0 : M_c(x,y,t) \geq 1-\alpha_0\}.$$

So

$$\alpha_0 - \delta(\bar{A}) \geq \alpha_0 - \delta(A). \tag{4.1}$$

Next, suppose that $\alpha_0 - \delta(A) < \|t_0\|$, for some $t_0(\neq \theta) \in E$. Then

$$\bigvee_{x,y\in A} \bigwedge \{\|t\| > 0 : M_c(x,y,t) \geq 1-\alpha_0\} \leq \|t_0\|$$
$$\Rightarrow \bigwedge \{\|t\| > 0 : M_c(x,y,t) \geq 1-\alpha_0\} \leq \|t_0\|, \forall x,y \in A$$

$$\Rightarrow \bigwedge\{\|t\| > 0 : M_c(x, y, t) > 1 - \alpha_0\} \leq \|t_0\|, \ \forall x, y \in A.$$

Thus

$$M_c(x, y, t_0) > 1 - \alpha_0, \ \forall x, y \in A. \tag{4.2}$$

Choose $\epsilon \succ \theta$ be arbitrary. Since P is a normal cone with normal constant $K(\geq 1)$, it follows that $\|\epsilon\| > 0$ is arbitrary. (i.e. $\|\epsilon\| \to 0$).

Choose $x_0, y_0 \in \bar{A}$ be arbitrary. Then $\exists x_1, y_1 \in A$ such that

$$M_c(x_0, x_1, \frac{\epsilon}{2}) > 1 - \alpha_0 \text{ and } M_c(y_0, y_1, \frac{\epsilon}{2}) > 1 - \alpha_0.$$

By (4.2), $M_c(x_0, y_0, t_0 + \epsilon) \geq M_c(x_0, x_1, \frac{\epsilon}{2}) * M_c(x_1, y_1, t_0) * M_c(y_0, y_1, \frac{\epsilon}{2})$. Thus $M_c(x_0, y_0, t_0) = \lim\limits_{\|\epsilon\| \to 0} M_c(x_0, y_0, t_0 + \epsilon) \geq 1 - \alpha_0$. So $M_c(x_0, y_0, t_0) \geq 1 - \alpha_0$. Since $x_o, y_0 \in \bar{A}$ arbitrary,

$$\begin{aligned}
&M_c(x, y, t_0) \geq 1 - \alpha_0, \quad \forall x, y \in \bar{A} \\
&\Rightarrow \bigwedge\{\|t\| > 0 : M_c(x, y, t) \geq 1 - \alpha_0\} \leq \|t_0\|, \quad \forall x, y \in \bar{A} \\
&\Rightarrow \bigvee_{x,y \in \bar{A}} \bigwedge\{\|t\| > 0 : M_c(x, y, t) \geq 1 - \alpha_0\} \leq \|t_0\| \\
&\Rightarrow \alpha_0 - \delta(\bar{A}) \leq \|t_0\|.
\end{aligned}$$

Hence

$$\alpha_0 - \delta(A) \geq \alpha_0 - \delta(\bar{A}). \tag{4.3}$$

Therefore from (4.1) and (4.3), we get $\alpha_0 - \delta(A) = \alpha_0 - \delta(\bar{A})$. □

Theorem 4.3. (Cantor's Intersection Theorem) *Let $(X, M_c, *)$ be a fuzzy cone metric space and P be a normal cone with normal constant K(=1). A necessary and sufficient condition that $(X, M_c, *)$ be complete is that every nested sequence of nonempty closed subset F_i with α-fuzzy diameter tending to 0 for each $\alpha \in (0, 1)$ as $i \to \infty$ be such that $\bigcap\limits_{i=1}^{\infty} F_i$ contains exactly one point.*

Proof. First we suppose that $(X, M_c, *)$ is a complete fuzzy cone metric space. Consider a sequence of closed subsets F_i such that

$$F_1 \supset F_2 \supset F_3 \ \dots \text{ with } \alpha - \delta_c(F_n) \to 0 \text{ as } n \to \infty, \quad \forall \alpha \in (0, 1).$$

Choose $x_n \in F_n$, for each $n = 1, 2, 3, \dots$. Then we obtain a sequence $\{x_n\}$ in X. Now we shall verify that $\{x_n\}$ is a Cauchy sequence.

We have $x_n \in F_n$ and $x_{n+p} \in F_{n+p} \subset F_n$, $\forall n$ and $p = 1, 2, 3, \dots$. Then for each n and $p = 1, 2, 3, \dots$ and for each $\alpha \in (0, 1)$,

$$\bigwedge\{\|t\| > 0 : M_c(x_n, x_{n+p}, t) \geq \alpha\} \leq \alpha - \delta_c(F_n).$$

Thus $\lim\limits_{n \to \infty} \bigwedge\{\|t\| > 0 : M_c(x_n, x_{n+p}, t) \geq \alpha\} = 0$, for $p = 1, 2, 3, \dots$ and $\forall \alpha \in (0, 1)$.

Since P is a normal cone with normal constant K=1, for each $\epsilon \succ \theta$,

$\exists N(\alpha, \epsilon)$ such that

$$\begin{aligned}
&\bigwedge\{\|t\| > 0 : M_c(x_n, x_{n+p}, t) \geq \alpha\} < \|\epsilon\|, \forall \alpha \in (0, 1), \forall n \geq N(\alpha, \epsilon), p = 1, 2, 3, \dots \\
&\Rightarrow M_c(x_n, x_{n+p}, \epsilon) \geq \alpha \quad \forall \alpha \in (0, 1), \forall n \geq N(\alpha, \epsilon), p = 1, 2, 3, \dots
\end{aligned}$$

$\Rightarrow \lim_{n\to\infty} M_c(x_n, x_{n+p}, t) = 1$, for each $p = 1,\ 2,\ 3,\ ....$
Since $\epsilon \succ \theta$ is arbitrary, it follows that

$$\lim_{n\to\infty} M_c(x_n, x_{n+p}, \epsilon) = 1,\ \forall t \succ \theta \text{ and for each } p = 1,\ 2,\ 3,\ ....$$

So $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $x_n \to x$, for some $x \in X$.

Let k be an arbitrary positive integer. Then each member of the sequence $\{x_k, x_{k+1}, x_{k+2}, ...\}$ lies in F_k. Since F_k is closed, it follows that $x \in F_k$ and as k is arbitrary, we have $x \in \bigcap_{i=1}^{\infty} F_i$.

Uniqueness: If possible, suppose that $\exists y \in X$ and $y \neq x$ such that $y \in \bigcap_{i=1}^{\infty} F_i$.
Then for $x, y \in F_k$ for $k = 1,\ 2,\ 3,\ \dots$, we have

$\bigwedge \{\|t\| > 0 : M_c(x, y, t) \geq \alpha\} \leq \alpha - \delta_c(F_k)$, $\forall \alpha \in (0, 1)$ and $k = 1,\ 2,\ 3,\ \dots$
$\Rightarrow \bigwedge \{\|t\| > 0 : M_c(x, y, t) \geq \alpha\} = 0$, since $\alpha - \delta_c(F_k) \to 0$ as $k \to \infty$.
$\Rightarrow M_c(x, y, t) \geq \alpha,\ \ \forall \alpha \in (0, 1), \forall t \succ \theta$
$\Rightarrow M_c(x, y, t) = 1,\ \ \forall t \succ \theta$
$\Rightarrow x = y$.

Conversely, suppose that the condition of the theorem is satisfied. we shall show that X is complete. Let $\{x_n\}$ be a Cauchy sequence in X. Let $H_n = \{x_n, x_{n+1}, x_{n+2}, ...\}$.
Then we have $\lim_{n\to\infty} M_c(x_n, x_{n+p}, t) = 1,\ \ \forall t > 0$ and for $p = 1,\ 2,\ 3,\ \dots$.
Thus $\lim_{n\to\infty} M_c(x_n, x_{n+p}, t) > \alpha$, $\forall t > 0$, for $p = 1,\ 2,\ 3,\ \dots$, $\forall \alpha \in (0, 1)$.
Choose $t_0 > 0$ be arbitrarily. Then for each $\alpha \in (0, 1)$, there exists a positive integer $N(\alpha)$ such that

$M_c(x_n, x_{n+p}, t_0) > \alpha$, $\forall n \geq N(\alpha)$, $\forall \alpha \in (0, 1)$, $p = 1,\ 2,\ 3,\ \dots$
$\Rightarrow \bigwedge \{\|t\| > 0 : M_c(x_n, x_{n+p}, t) > \alpha\} \leq \|t_o\|$, $\forall n \geq N(\alpha)$, $\forall \alpha \in (0, 1)$ and $p = 1,\ 2,\ 3,\ \dots$
$\Rightarrow \bigvee_{x_n \in Hn} \bigwedge \{\|t\| > 0 : M_c(x_n, x_{n+p}, t) \geq \alpha\} \leq \|t_o\|$, $\forall n \geq N(\alpha)$, $\forall \alpha \in (0, 1)$
$\Rightarrow \alpha - \delta_c(H_n) \leq \|t_o\|$, $\forall n \geq N(\alpha)$, $\forall \alpha \in (0, 1)$
$\Rightarrow \alpha - \delta_c(\bar{H_n}) \leq \|t_o\|$, $\forall n \geq N(\alpha)$, $\forall \alpha \in (0, 1)$ [by Lemma 4.2].

Since $t_0 \succ \theta$ is arbitrary, we have $\alpha - \delta_c(\bar{H_n}) = 0$ as $n \to \infty\quad \forall \alpha \in (0, 1)$.

On the other hand, by the definition of H_n, it is clear that $H_{n+1} \subset H_n$, for each n. Then $\bar{H}_{n+1} \subset \bar{H_n}$, $\forall n$. Thus $\{\bar{H_n}\}$ constitutes a closed, nested sequence of non-empty sets in X, where $\alpha - \delta_c(\bar{H_n}) \to 0$ as $n \to \infty$. By hypothesis, there exists a unique $x \in \bigcap_{n=1}^{\infty} \bar{H_n}$. Since $x_n \in H_n \subset \bar{H_n}$, $x \in \bar{H_n}$. So

$\bigwedge \{\|t\| > 0 : M_c(x_n, x, t) \geq \alpha\} \leq \alpha - \delta_c(\bar{H_n}),\ \ \forall \alpha \in (0, 1)$
$\Rightarrow \lim_{n\to\infty} \bigwedge \{\{\|t\| > 0 : M_c(x_n, x, t) \geq \alpha\} = 0$.

Choose $\epsilon \succ \theta$. Then there exists $N(\alpha, \epsilon)$ such that

$\bigwedge \{\|t\| > 0 : M_c(x_n, x, t) \geq \alpha\} < \|\epsilon\|,\ \ \forall \alpha \in (0, 1)$, $\forall n \geq N(\alpha, \epsilon)$
$\Rightarrow M_c(x_n, x, \epsilon) \geq \alpha$, $\forall \alpha \in (0, 1)$, $\forall n \geq N(\alpha, \epsilon)$

$$\Rightarrow \lim_{n\to\infty} M_c(x_n, x, \epsilon) = 1.$$

Since $\epsilon \succ \theta$ is arbitrary, $\lim_{n\to\infty} M_c(x_n, x, t) = 1$, $\forall t \succ \theta$. Thus $x_n \to x$. So X is complete. □

5. Compact fuzzy cone metric space

In this section, some results of α-fuzzy completeness, α-fuzzy compactness, α-fuzzy totally boundedness have been studied.

Definition 5.1. Let $(X, M_c, *)$ be a fuzzy cone metric space and $\alpha \in (0,1)$ be given. A sequence $\{x_n\}$ in X is said to be α-Cauchy sequence, if

$$\lim_{m,n\to\infty} M_c(x_n, x_m, t) > 1-\alpha,\ \forall t \succ \theta.$$

Definition 5.2. Let $(X, M_c, *)$ be a fuzzy cone metric space and $\alpha \in (0,1)$ be given. Then $\{x_n\}$ is said to be α-convergent and converges to x, if

$$\lim_{n\to\infty} M_c(x_n, x, t) > 1-\alpha,\ \forall t \succ \theta.$$

Definition 5.3. Let $(X, M_c, *)$ be a fuzzy cone metric space and $\alpha \in (0,1)$ be given. A subset $A(\subset X)$ is said to be α-fuzzy compact, if every sequence in A has an α-convergent subsequence which converges to some element in A.

Proposition 5.4. *For every α-fuzzy compact cone metric space $(X, M_c, *)$, where $*$ is a continuous t-norm, there exists $\beta > \alpha$, such that $(X, M_c, *)$ is β-fuzzy complete.*

Proof. Let $\{x_n\}$ be an α-Cauchy sequence in X. Then

$$\lim_{m,n\to\infty} M_c(x_n, x_m, t) > 1-\alpha,\ \forall t \succ \theta.$$

Thus for a given $t \succ \theta$, there exists a natural number say N_0 such that

$$M_c(x_n, x_m, \frac{t}{3}) > 1-\alpha,\ \forall m,\ n \geq N_0.$$

In particular,

$$M_c(x_n, x_{N_0}, \frac{t}{3}) > 1-\alpha,\ \forall n \geq N_0. \tag{5.1}$$

Since X is α-fuzzy compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is α-convergent to some $x \in X$. So there exists $m_0 \geq N_0$ such that

$$M_c(x_{n_m}, x, \frac{t}{3}) > 1-\alpha,\ \forall m \geq m_0,\ \text{i.e.,}$$

$$M_c(x_{n_{m_0}}, x, \frac{t}{3}) > 1-\alpha. \tag{5.2}$$

Since $n_{m_0} \geq m_0 \geq N_0$, we have

$$M_c(x_{N_0}, x_{n_{m_0}}, \frac{t}{3}) > 1-\alpha. \tag{5.3}$$

Let $n \geq N_0$. Then from (5.1),(5.2) and (5.3), we get

$$M_c(x_n, x, t) \geq M_c(x_n, x_{N_0}, \tfrac{t}{3}) * M_c(x_{N_0}, x_{n_{m_0}}, \tfrac{t}{3}) * M_c(x_{n_{m_0}}, x, \tfrac{t}{3})$$

$$\geq (1-\alpha) * (1-\alpha) * (1-\alpha).$$

Thus

$$\lim_{n\to\infty} M_c(x_n, x, t) \geq (1-\alpha) * (1-\alpha) * (1-\alpha),\ \forall \epsilon \succ \theta. \tag{5.4}$$

Since $*$ is continuous, there exists $\beta \in (0,1)$ such that

$$(1-\alpha) * (1-\alpha) * (1-\alpha) > 1-\beta.$$

From (5.4), $\lim_{n\to\infty} M_c(x_n, x, t) > 1-\beta,\quad \forall \epsilon \succ \theta$. So $\{x_n\}$ is β-fuzzy convergent and converges to x. Since $\{x_n\}$ is an arbitrary Cauchy sequence, X is β-fuzzy complete. □

Definition 5.5. Let $(X, M_c, *)$ be a fuzzy cone metric space. A subset A of X is said to be fuzzy bounded, if there exists $t \succ \theta (\in E)$ and $0 < r < 1$ such that

$$M_c(x, y, t) > 1-r,\ \forall x,\ y \in A.$$

Definition 5.6. Let $(X, M_c, *)$ be a fuzzy cone metric space and $A \subset X$ and $\alpha \in (0,1)$ be given. Let $\epsilon \succ \theta$ be an element of E. A set $B \subset X$ is said to be an α-fuzzy ϵ-net for the set A, if for any $x \in A$, there exists $y \in B$ such that

$$M_c(x, y, \epsilon) > 1-\alpha.$$

B may be finite or infinite.

Definition 5.7. A set A of $(X, M_c, *)$ is said to be α-fuzzy totally bounded, if for a given $\alpha \in (0,1)$, for any $\epsilon \succ \theta$, there exists a finite α-fuzzy ϵ-net for the set A.

Theorem 5.8. *Let $(X, M_c, *)$ be a fuzzy cone metric space and $*$ is a continuous t-norm and $A(\subseteq X)$ be a nonempty subset of X. If A is α-fuzzy totally bounded in X, then A is fuzzy bounded in X.*

Proof. Let $(X, M_c, *)$ be a fuzzy cone metric space, where $*$ is a continuous t-norm and A is an α-fuzzy totally bounded set in X. Then for $\epsilon(\succ \theta) \in E$, $\exists$ is a finite α-fuzzy ϵ-net B (say)for the set A.

Let $x,\ y$ be any two elements in A. Then $\exists x_1, y_1 \in B$ such that

$$M_c(x, x_1, \epsilon) > 1-\alpha \text{ and } M_c(y, y_1, \epsilon) > 1-\alpha.$$

We can write $min\{M_c(x', y', \epsilon) : x', y', \epsilon B\} > 1-\beta$ for some $\beta \in (0,1)$. Now,

$$\begin{aligned} M_c(x, y, 3\epsilon) &\geq M_c(x, x_1, \epsilon) * M_c(x_1, y, 2\epsilon) \\ &\geq M_c(x, x_1, \epsilon) * M_c(x_1, y_1, \epsilon) * M_c(y_1, y, \epsilon) \\ &> (1-\alpha) * (1-\beta) * (1-\alpha) > 1-r_0, \text{ say, for some } 0 < r_0 < 1. \end{aligned}$$

Take $t_0 = 3\epsilon$. Then $M_c(x, y, t_0) > 1-r_0,\quad \forall x, y \in A,\ t_0 \succ \theta$. Thus A is fuzzy bounded in X. □

The converse result of the above theorem may not be true. We justify it by the following example.

Example 5.9. Take $E = R^2$,and consider the metric space (X, ρ), where $X = l_2$ and $\rho(x, y) = (\sum_{i=1}^{\infty} |\xi_i - \eta_i|^2)^{\frac{1}{2}}$, $x = (\xi_1, \xi_2, ...)$, $y = (\eta_1, \eta_2, ...)$, $P = \{(k_1, k_2) : k_1, k_2 \geq 0\} \subset E$.

Let $'\preceq'$ be the usual ordering $\leq$ in E w.r.t P. Take $a*b = ab$. Define $M_c : X\times X\times E$ by

$$M_c(x,y,t) \ = \begin{cases} \frac{\|t\|}{\|t\|+\rho(x-y)} & \text{if } t\succ\theta \\ 0 & \text{otherwise.} \end{cases}$$

Then P is a normal cone with normal constant $K = 1$, M_c is a fuzzy cone metric and $(X, M_c, *)$ is a fuzzy cone metric space.

Solution. First we show that P is a normal cone with normal constant $K = 1$. It is easy to verify that P is a cone. Now,

$$\begin{aligned} &(x_1, x_2) \leq (y_1, y_2) \\ \Rightarrow\ & x_1 \leq y_1, x_2 \leq y_2 \\ \Rightarrow\ & x_1^2 \leq y_1^2 \text{ and } x_2^2 \leq y_2^2 \\ \Rightarrow\ & \sum_{i=1}^{2} |x_i|^2 \leq \sum_{i=1}^{2} |y_i|^2 \\ \Rightarrow\ & (\sum_{i=1}^{2} |x_i|^2)^{\frac{1}{2}} \leq (\sum_{i=1}^{2} |y_i|^2)^{\frac{1}{2}} \\ \Rightarrow\ & \|x\| \leq 1.\|y\|. \end{aligned}$$

Then P is a normal cone with normal constant $K = 1$.

1. From definition, $M_c(x,y,t) = 0, \quad \forall t \preceq \theta$. Thus (CM1) holds.
2. For any $t \succ \theta$,

$$\begin{aligned} &M_c(x,y,t) = 1 \\ \Rightarrow\ & \frac{\|t\|}{\|t\|+\rho(x,y)} = 1 \\ \Rightarrow\ & \rho(x,y) = 0 \\ \Rightarrow\ & x = y. \end{aligned}$$

Again,

$$\begin{aligned} &x = y \\ \Rightarrow\ & \rho(x,y) = 0 \\ \Rightarrow\ & \frac{\|t\|}{\|t\|+\rho(x,y)} = 1, \quad \forall t \succ \theta. \end{aligned}$$

Then $M_c(x,y,t) = 1, \quad \forall t \succ \theta$. Thus $M_c(x,y,t) = 1 \quad$ iff $x = y$. So (CM2) holds.

3. $M_c(x,y,t) = \frac{\|t\|}{\|t\|+\rho(x,y)} = \frac{\|t\|}{\|t\|+\rho(y,x)} = M_c(y,x,t), \ \forall t(\succ\theta) \in E$. Then (CM3) holds.

4. $E = R^2$. Since P is a normal cone with normal constant $K = 1$, we have $s \preceq t+s$ and $t \preceq t+s$ which imply

$$\|s\| \leq \|t+s\|, \|t\| \leq \|t+s\|.$$

Since $\frac{\|t+s\|}{\|s\|} \geq 1$ and $\frac{\|t+s\|}{\|t\|} \geq 1$, we can write

$$\begin{aligned} \rho(x,z) &\leq \rho(x,y) + \rho(y,z) \\ &\leq \frac{\|t+s\|}{\|t\|}\rho(x,y) + \frac{\|t+s\|}{\|s\|}\rho(y,z). \end{aligned}$$

Then $\frac{\rho(x,z)}{\|t+s\|} \leq \frac{\rho(x,y)}{\|t\|} + \frac{\rho(y,z)}{\|s\|}$. Now,

$$\begin{aligned} M_c(x,z,t+s) &= \frac{\|t+s\|}{\|t+s\|+\rho(x,z)} \\ &= \frac{1}{1+\frac{\rho(x,z)}{\|t+s\|}} \\ &\geq \frac{1}{1+\frac{\rho(x,y)}{\|t\|}+\frac{\rho(y,z)}{\|s\|}} \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{1+\frac{\rho(x,y)}{\|t\|}+\frac{\rho(y,z)}{\|s\|}+\frac{\rho(x,y)\rho(y,z)}{\|t\|\|s\|}}\\
&= \frac{1}{(1+\frac{\rho(x,y)}{\|t\|})}\cdot\frac{1}{(1+\frac{\rho(y,z)}{\|s\|})}\\
&= (\frac{\|t\|}{\|t\|+\rho(x,y)})(\frac{\|s\|}{\|s\|+\rho(y,z)})\\
&= M_c(x,y,t)*M_c(y,z,s).
\end{aligned}$$

Thus (CM4) holds.

On the other hand,

$$\begin{aligned}
\lim_{\|t\|\to\infty} M_c(x,y,t) &= \lim_{\|t\|\to\infty}\frac{\|t\|}{\|t\|+\rho(x,y)}.\\
&= \lim_{\|t\|\to\infty}\frac{1}{1+\frac{\rho(x,y)}{\|t\|}}\\
&= \tfrac{1}{1}=1.
\end{aligned}$$

So (CM5) holds.

Now consider the subset A of $X(=l_2)$ consisting of the elements:

$$x_1=(1,0,0,...),\ x_2=(0,1,0,...),\ x_3=(0,0,1,...),\ ...\ .$$

Then $\rho(x_i,x_j)=\sqrt{2}$, for $i\neq j$. Moreover, for any $x,y\in A$, for some $t_0\succ\theta$,

$$M_c(x,y,t_0)=\frac{\|t_0\|}{\|t_0\|+\rho(x,y)}=\frac{\|t_0\|}{\|t_0\|+\sqrt{2}}.$$

Take $\|t_0\|=\sqrt{2}$. Then $M_c(x,y,t_0)=\frac{\sqrt{2}}{\sqrt{2}+\sqrt{2}}=\frac{\sqrt{2}}{2\sqrt{2}}=\frac{1}{2}>\frac{1}{3}=1-\frac{2}{3}=1-r_0(r_0=\frac{2}{3})$. Thus $\forall x,y\in A,t_0\succ\theta$, $M_c(x,y,t_0)>1-r_0(0<r_0<1)$.
So A is fuzzy bounded in X.

We now verify that A is not α-fuzzy totally bounded. Choose $\epsilon\in E(\epsilon\succ\theta)$ such that $0<\|\epsilon\|<\frac{1}{2}$ and $0<\alpha<1$ $\ ((1-\alpha)^2=0.6=\frac{3}{5},\alpha=1-\sqrt{\frac{3}{5}})$. If possible, suppose that N is a finite ϵ-net for the set A. Then for x_i and $x_j(i\neq j)$, there must exist y_i and y_j from N such that

$$M_c(x_i,y_i,\epsilon)>1-\alpha \text{ and } M_c(x_j,y_j,\epsilon)>1-\alpha.$$

Now x_i is distinct from x_j and their number is infinite and the set N consists only a finite number of elements. Thus some y_i and $y_j(i\neq j)$ must be equal. If $y_i=y_j(i\neq j)$, then

$$\begin{aligned}
M_c(x_i,x_j,2\epsilon) &\geq M_c(x_i,y_i,\epsilon)*M_c(y_i,x_j,\epsilon)\\
&= M_c(x_i,y_i,\epsilon)\cdot M_c(y_j,x_j,\epsilon)\\
&> (1-\alpha)\cdot(1-\alpha)\\
&= (1-\alpha)^2\\
&> \tfrac{1}{2}.
\end{aligned}$$

So

$$\begin{aligned}
&M_c(x_i,x_j,2\epsilon)>\tfrac{1}{2}\\
&\Rightarrow \frac{2\|\epsilon\|}{2\|\epsilon\|+\rho(x_i,x_j)}>\frac{1}{2}\\
&\Rightarrow 4\|\epsilon\|>2\|\epsilon\|+\rho(x_i,x_j)\\
&\Rightarrow \rho(x_i,x_j)<2\|\epsilon\|<2\cdot\tfrac{1}{2}=1 \quad [\because 0<\|\epsilon\|<\tfrac{1}{2}]\\
&\Rightarrow \rho(x_i,x_j)<1.
\end{aligned}$$

Contradicting the fact that $\rho(x_i,x_j)=\sqrt{2}$. Hence there exists no finite α-fuzzy

ϵ-net for the set A, when $0 < \|\epsilon\| < \frac{1}{2}$ and $0 < \alpha < 1 (\alpha = 1 - \sqrt{\frac{3}{5}})$.
This shows that A cannot be α-fuzzy totally bounded.

Theorem 5.10. *Let $(X, M_c, *)$ be a fuzzy cone metric space and $A \subset X$, $\alpha \in (0, 1)$. If A is α-fuzzy compact in X, then A is α-fuzzy totally bounded.*

Proof. We assume that A is α-fuzzy compact in X. Let $\epsilon (\succ \theta)$ be an arbitrary element of E and x_1 be an arbitrary element of X. If $M_c(x, x_1, \epsilon) > 1 - \alpha,\ \forall x \in A$, then there exists a finite ϵ-net B for A, i.e, $B = \{x_1\}$. If not, there exists a point $x_2 \in A$ such that $M_c(x_1, x_2, \epsilon) \leq 1 - \alpha$.

If for every $x \in A$, either $M_c(x, x_1, \epsilon) > 1 - \alpha$ or $M_c(x, x_2, \epsilon) > 1 - \alpha$, then there exists a finite ϵ-net B for A, i.e $B = \{x_1, x_2\}$.
Continuing in this way, we get elements $x_1, x_2, ...x_n$; $x_1 \in X$, $x_i \in A$, $2 \leq i \leq n$ for which $M_c(x_i, x_j, \epsilon) \leq 1 - \alpha$, for $i \neq j$.

Now, two cases can occur.

Case I: The procedure stops after the k^{th} step. Then we obtain points $x_1, x_2, ...x_k$ such that for every $x \in A$, at least one of the inequalities
$M_c(x_i, x, \epsilon) > 1 - \alpha, \quad i = 1, 2, ..., k$ holds and $B = \{x_1, x_2, ..., x_n\}$ is a finite ϵ-net for A, proving that A is α-fuzzy totally bounded.

Case II: The process continues infinitely. Then we obtain an infinite sequence $\{x_n\}$, $x_1 \in X$ and $x_i \in A$, $i > 1$ such that $M_c(x_i, x_j, \epsilon) \leq 1 - \alpha$, for $i \neq j$.
Above relation shows that neither $\{x_n\}$ nor any of its subsequence is α-convergent, contradicting that A is α-fuzzy compact in X. Thus $A(\subset X)$ is α-fuzzy totally bounded. □

6. Conclusion.

Some basic results on completeness and compactness in fuzzy cone metric spaces have been studied. Cantor's intersection theorem has been established in fuzzy setting. Notion of totally fuzzy bounded set is introduced and we have studied some relation with compact fuzzy cone metric space. We think that researchers in the field of fuzzy cone metric space will be benefited by using the results of this paper.

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Arunima Majumder (arunimasamkhya1983@gmail.com)
Department of Mathematics, Visva-Bharati, Santiniketan-731235, West Bengal, India

Tarapada Bag (tarapadavb@gmail.com)
Department of Mathematics, Visva-Bharati, Santiniketan-731235, West Bengal, India