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Bernstein Stancu operator of rough I-core of triple sequences

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Abstract. We introduce and study some basic properties of Bernstein-Stancu polynomials of rough I-convergent of triple sequences and also study the set of all Bernstein-Stancu polynomials of rough I-limits of a triple sequence and relation between analytic ness and Bernstein-Stancu polynomials of rough I-core of a triple sequence.

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Corresponding Author: A. Esi (aesi23@hotmail.com)

1. Introduction

The idea of rough convergence was first introduced by Phu [11, 12, 13] in finite dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r.

Aytar [1] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [2] studied that the r-limit set of the sequence is equal to intersection of these sets and that r-core of the sequence is equal to the union of these sets. Dundar and Cakan [9] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence. The notion of I-convergence of a triple sequence which is based on the structure of the ideal I of subsets of $\mathbb{N}^3$, where $\mathbb{N}$ is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

In this paper we investigate some basic properties of rough I-convergence of a triple sequence in three dimensional matrix spaces which are not earlier. We study the set of all rough I-limits of a triple sequence and also the relation between analytic ness and rough I-core of a triple sequence. Let K be a subset of the set of positive integers $\mathbb{N}^3$ and let us denote the set $K_{ik\ell} = \{(m,n,k) \in K : m \leq i, n \leq j, k \leq \ell\}$. Then the natural density of K is given by

$$\delta_3(K) = \lim_{i,j,\ell \to \infty} \frac{|K_{ij\ell}|}{ij\ell},$$

where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

First applied the concept of (p,q)-calculus in approximation theory and introduced the (p,q)-analogue of Bernstein operators. Later, based on (p,q)-integers, some approximation results for Bernstein-stancu operators, Bernstein- Kantorovich operators, (p,q)-Lorentz operators, Bleimann-Butzer and Hahn operators and Bernstein-Shurer operators, etc.

Khan et al. [10] have given a nice application in computer-aided geometric design and applied these Bernstein basis for construction of (p,q)-Bezier curves and surfaces based on (p,q)-integers which is further generalization of q-Bezier curves and surfaces.

Motivated by the above mentioned work on (p,q)-approximation and its application, in this paper we study statistical approximation properties of Bernstein-Stancu operators based on (p,q)-integers.

Now we recall some basic definitions about (p,q)-integers. For any $u,v,w \in \mathbb{N}$, the (p,q)-integer $[uvw]_{p,q}$ is defined by

$$[0]_{p,q} := 0 \text{ and } [uvw]_{p,q} = \frac{p^{uvw} - q^{uvw}}{p-q} \text{ if } u,v,w \geq 1,$$

where $0 < q < p \leq 1$. The (p,q)-factorial is defined by

$$[0]_{p,q}! := 1 \text{ and } [uvw]!_{p,q} = [1]_{p,q}[2]_{p,q} \cdots [uvw]_{p,q} \text{ if } u,v,w \geq 1.$$

Also the (p,q)-binomial coefficient is defined by

$$\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q} = \frac{[u]!_{p,q}}{[m]!_{p,q}\,[u-m]!_{p,q}} \frac{[v]!_{p,q}}{[n]!_{p,q}\,[v-n]!_{p,q}} \frac{[w]!_{p,q}}{[k]!_{p,q}\,[w-k]!_{p,q}}$$

for all $u,v,w,m,n,k \in \mathbb{N}$ with $(u,v,w) \geq (m,n,k)$.

The formula for (p,q)-binomial expansion is as follows:

$$(ax+by)^{uvw}_{p,q} = \sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w} p^{\frac{(u-m)(u-m-1)+(v-n)(v-n-1)+(w-k)(w-k-1)}{2}} q^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}.$$

$$\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q} a^{(u-m)+(v-n)+(w-k)} b^{m+n+k} x^{(u-m)+(v-n)+(w-k)} y^{m+n+k},$$

$$(x+y)^{uvw}_{p,q} = (x+y)(px+qy)\left(p^2x+q^2y\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)}x+q^{(u-1)+(v-1)+(w-1)}y\right),$$

$$(1-x)^{uvw}_{p,q} = (1-x)\,(p-qx)\,\left(p^2-q^2x\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)}-q^{(u-1)+(v-1)+(w-1)}x\right),$$

and

$$(x)^{mnk}_{p,q} = x\,(px)\,\left(p^2x\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)}x\right) = p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}.$$

The Bernstein operator of order (r, s, t) is given by

$$B_{rst}(f,x) = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k} x^{m+n+k}(1-x)^{(m-r)+(n-s)+(k-t)}$$

where f is a continuous (real or complex valued) function defined on $[0, 1]$.

(p, q)-Bernstein operators are defined as follows:

$$B_{rst,p,q}(f,x) = \frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{2}}}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k} p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}} x^{m+n+k}.$$

$$\prod_{u_1=0}^{(r-m-1)}(p^{u_1}-q^{u_1}x)\prod_{u_2=0}^{(s-n-1)}(p^{u_2}-q^{u_2}x)\prod_{u_3=0}^{(t-k-1)}(p^{u_3}-q^{u_3}x)\cdot \tag{1.1}$$

$$f\left(\frac{p^{(r-m)}[m]_{p,q}+p^{(s-n)}[n]_{p,q}\,p^{(t-k)}[k]_{p,q}}{[r]_{p,q}+[s]_{p,q}+[t]_{p,q}}\right), x \in [0,1]$$

Also, we have

$$(1-x)_{p,q}^{rst} = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}(-1)^{m+n+k} p^{\frac{(r-m)(r-m-1)+(s-n)(s-n-1)+(t-k)(t-k-1)}{2}}.$$

$$q^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}\binom{r}{m}\binom{s}{n}\binom{t}{k} x^{m+n+k}.$$

$$S_{rst,p,q}(f,x) = \frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{2}}}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k} p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}} x^{m+n+k}.$$

$$\prod_{u_1=0}^{(r-m-1)}(p^{u_1}-q^{u_1}x)\prod_{u_2=0}^{(s-n-1)}(p^{u_2}-q^{u_2}x)\prod_{u_3=0}^{(t-k-1)}(p^{u_3}-q^{u_3}x)\cdot \tag{1.2}$$

$$f\left(\frac{p^{(r-m)}[m]_{p,q}+p^{(s-n)}[n]_{p,q}\,p^{(t-k)}[k]_{p,q}+\eta}{[r]_{p,q}+[s]_{p,q}+[t]_{p,q}+\mu}\right), x \in [0,1]$$

Note that for $\eta = \mu = 0$, (p, q)-Bernstein-Stancu operators given by (1.2) reduces into (p, q)-Bernstein operators. Also for $p = 1$, (p, q)-Bernstein-Stancu operators given by (1.1) turn out to be q-Bernstein-Stancu operators.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric (X, d). Consider a triple sequence of Bernstein stancu polynomials $(S_{rst,p,q}(f,x))$ such that $(S_{rst,p,q}(f,x)) \in \mathbb{R}$, $m, n, k \in \mathbb{N}$.

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st_3 - \lim S_{rst,p,q}(f,x) = f(x)$, provided that the set

$$K_\epsilon := \left\{(m,n,k) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - (f,x)| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein-Stancu polynomials. i.e., $\delta_3(K_\epsilon) = 0$. That is,

$$\lim_{r,s,t\to\infty}\frac{1}{rst}\left|\{m \leq r, n \leq s, k \leq t : |S_{rst,p,q}(f,x) - (f,x)| \geq \epsilon\}\right| = 0.$$

In this case, we write $\delta_3 - \lim S_{rst,p,q}(f,x) = (f,x)$ or $S_{rst,p,q}(f,x) \xrightarrow{st_3} (f,x)$.

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers , χ_A- the characteristic function of $A \subset \mathbb{N}$, $\mathbb{R}$ the set of all real numbers. A subset A of $\mathbb{N}$ is said to have asymptotic density $d(A)$ if

$$d_3(A) = \lim_{i,j,\ell \to \infty} \frac{1}{ij\ell} \sum_{m=1}^{i} \sum_{n=1}^{j} \sum_{k=1}^{\ell} \chi_A(K).$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers, respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [14, 15], Esi et al. [3, 4, 5, 6], Datta et al. [7], Subramanian et al. [16], Debnath et al. [8] and many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic, if

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3.

2. Definitions and Preliminaries

Throughout the paper, $\mathbb{R}^3$ denotes the real three dimensional case with the usual metric. Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}^3$; $m,n,k \in \mathbb{N}^3$. The following definition are obtained.

Definition 2.1. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is said to be statistically convergent to (f,x), denoted by $S_{rst,p,q}(f,x) \to^{st-\lim x} (f,x)$, if for any $\epsilon > 0$, we have $d(A(\epsilon)) = 0$, where

$$A(\epsilon) = \left\{(r,s,t) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - (f,x)| \geq \epsilon\right\}.$$

Definition 2.2. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is said to be statistically convergent to (f,x), denoted by $S_{rst,p,q}(f,x) \to^{st-\lim x} (f,x)$, provided that the set

$$\left\{(r,s,t) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - (f,x)| \geq \epsilon\right\},$$

has natural density zero, for every $\epsilon > 0$. In this case, (f,x) is called the statistical limit of the sequence of Berstein-Stancu polynomials.

Definition 2.3. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ in a metric space $(X, |.,.|)$ and β be a non-negative real number is said to be β-convergent to (f,x), denoted by $S_{rst,p,q}(f,x) \to^{\beta} (f,x)$, if for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}^3$ such that for all $r,s,t \geq N_\epsilon$ we have

$$|S_{rst,p,q}(f,x) - (f,x)| < r + \epsilon.$$

In this case, $S_{rst,p,q}(f,x)$ is called an β- limit of (f,x).

Remark 2.4. We consider β-limit set $S_{rst,p,q}(f,x)$ which is denoted by $\text{LIM}^{\beta} S_{rst,p,q}(f,x)$ and is defined by

$$\text{LIM}^{\beta} S_{rst,p,q}(f,x) = \left\{ f : S_{rst,p,q}(f,x) \to^{\beta} (f,x) \right\}.$$

Definition 2.5. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is said to be β-convergent, if $\text{LIM}^{\beta} S_{rst,p,q}(f,x) \neq \phi$ and β is called a rough convergence degree of $S_{rst,p,q}(f,x)$. If $\beta = 0$, then it is ordinary convergence of triple sequence of Bernstein-Stancu polynomials.

Definition 2.6. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ in a metric space $(X, |.,.|)$ and β be a non-negative real number is said to be β- statistically convergent to (f,x), denoted by $S_{rst,p,q}(f,x) \to^{\beta - st_3} (f,x)$, if for any $\epsilon > 0$, we have $d(A(\epsilon)) = 0$, where

$$A(\epsilon) = \left\{ (r,s,t) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - (f,x)| \geq r + \epsilon \right\}.$$

In this case, (f,x) is called β- statistical limit of $S_{rst,p,q}(f,x)$. If $\beta = 0$, then it is ordinary statistical convergent of triple sequence of Bernstein-Stancu polynomials.

Definition 2.7. A class I of subsets of a nonempty set X is said to be an ideal in X, provided
(i) $\phi \in I$,
(ii) $A, B \in I$ implies $A \bigcup B \in I$,
(iii) $A \in I$, $B \subset A$ implies $B \in I$.
I is called a nontrivial ideal, if $X \notin I$.

Definition 2.8. A nonempty class F of subsets of a nonempty set X is said to be a filter in X, provided
(i) $\phi \in F$,
(ii) $A, B \in F$ implies $A \bigcap B \in F$,
(iii) $A \in F$, $A \subset B$ implies $B \in F$.

Definition 2.9. Let I be a non trivial ideal in X and $X \neq \phi$. Then the class

$$F(I) = \{ M \subset X : M = X \backslash A \text{ for some } A \in I \}$$

is a filter on X, called the filter associated with I.

Definition 2.10. A non trivial ideal I in X is called admissible, if $\{x\} \in I$, for each $x \in X$.

Note 2.11. *If I is an admissible ideal, then usual convergence in X implies I convergence in X.*

Remark 2.12. If I is an admissible ideal, then usual rough convergence implies rough I- convergence.

Definition 2.13. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ in a metric space $(X, |.,.|)$ and β be a non-negative real number is said to be rough ideal convergent or βI- convergent to (f,x), denoted by $S_{rst,p,q}(f,x) \to^{\beta I} (f,x)$, if for any $\epsilon > 0$, we have

$$\left\{(r,s,t) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - (f,x)| \geq \beta + \epsilon\right\} \in I.$$

In this case, $(S_{rst,p,q}(f,x))$ is called βI- convergent to (f,x) and a triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is called rough I- convergent to (f,x) with β as roughness of degree. If $\beta = 0$ then it is ordinary I- convergent.

Note 2.14. *Generally, let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(g,x))$ is not I-convergent in usual sense and $|S_{rst,p,q}(f,x) - S_{rst,p,q}(g,x)| \leq \beta$, for all $(r,s,t) \in \mathbb{N}^3$ or*

$$\left\{(r,s,t) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - S_{rst,p,q}(g,x)| \geq \beta\right\} \in I,$$

for some $\beta > 0$. Then the triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is βI- convergent.

Note 2.15. *It is clear that βI- limit of a sequence $S_{rst,p,q}(f,x)$ of Bernstein-Stancu polynomial is not necessarily unique.*

Definition 2.16. Consider βI- limit set of (f,x), which is denoted by

$$I - \mathrm{LIM}^{\beta}\, S_{rst,p,q}(f,x) = \left\{f : S_{rst,p,q}(f,x) \to^{\beta I} (f,x)\right\}.$$

Then the triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is said to be βI- convergent, if $I - \mathrm{LIM}^{\beta}\, S_{rst,p,q}(f,x) \neq \phi$ and β is called a rough I- convergence degree of $S_{rst,p,q}(f,x)$.

Definition 2.17. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is said to be I-analytic, if there exists a positive real number M such that

$$\left\{(r,s,t) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x)|^{1/m+n+k} \geq M\right\} \in I.$$

Definition 2.18. Let f be a continuous function defined on the closed interval $[0,1]$. A point $L \in X$ is said to be an I- accumulation point of a Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ in a metric space (X,d), if for each $\epsilon > 0$, the set

$$\left\{(r,s,t) \in \mathbb{N}^3 : d\left(S_{rst,p,q}(f,x),(f,x)\right) = |S_{rst,p,q}(f,x) - (f,x)| < \epsilon\right\} \notin I.$$

We denote the set of all I-accumulation points of $(S_{rst,p,q}(f,x))$ by $I(\Gamma(S_{rst,p,q}(f,x)))$.

Definition 2.19. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is said to be rough I- convergent, if $I - \mathrm{LIM}^{\beta}\, S_{rst,p,q}(f,x) \neq \phi$.

It is clear that if $I - \mathrm{LIM}^{\beta}\, S_{rst,p,q}(f,x) \neq \phi$ for a triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ of real numbers, then we have

$$I - \mathrm{LIM}^{\beta}\, S_{rst,p,q}(f,x) = [I - \limsup S_{rst,p,q}(f,x) - \beta,\ I - \liminf S_{rst,p,q}(f,x) + \beta].$$

Definition 2.20. Let f be a continuous function defined on the closed interval $[0,1]$. The number of $\bar{\beta} = \inf\left\{\beta \geq 0 : I - \mathrm{LIM}^{\beta}\, S_{rst,p,q}(f,x) \neq \varnothing\right\}$ is said to be rough $I-$ *core* $S_{rst,p,q}(f,x)$ of triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$.

3. Main results

Theorem 3.1. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ of real numbers and $I \subset 2^{\mathbb{N}}$ be an admissible ideal, we have* $\operatorname{diam}\left(I - \mathrm{LIM}^{\beta} S_{rst,p,q}(f,x)\right) \leq 2\beta$. *In general,* $\operatorname{diam}\left(I - \mathrm{LIM}^{\beta} S_{rst,p,q}(f,x)\right)$ *has an upper bound.*

Proof. Assume that $\operatorname{diam}\left(\mathrm{LIM}^{\beta} S_{rst,p,q}(f,x)\right)$. Then, $\exists S_{rst,p,q}(p,x)$,

$$S_{rst,p,q}(q,x) \in \mathrm{LIM}^{\beta} S_{rst,p,q}(f,x) \ni: |S_{rst,p,q}(p,x) - S_{rst,p,q}(q,x)| > 2\beta.$$

Take $\epsilon \in \left(0, \frac{|S_{rst,p,q}(p,x) - S_{rst,p,q}(q,x)|}{2} - \beta\right)$. Because

$$S_{rst,p,q}(p,x), S_{rst,p,q}(q,x) \in I - \mathrm{LIM}^{\beta} S_{rst,p,q}(f,x),$$

we have $A_1(\epsilon) \in I$ and $A_2(\epsilon) \in I$ for every $\epsilon > 0$, where

$$A_1(\epsilon) = \left\{(u,v,w) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - S_{rst,p,q}(p,x)| \geq r + \epsilon\right\}$$

and

$$A_2(\epsilon) = \left\{(u,v,w) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - S_{rst,p,q}(q,x)| \geq r + \epsilon\right\}.$$

Using the properties $F(I)$, we get

$$\left(A_1(\epsilon)^c \bigcap A_2(\epsilon)^c\right) \in F(I).$$

Thus we write,

$$|S_{rst,p,q}(p,x) - S_{rst,p,q}(q,x)| \leq |S_{rst,p,q}(f,x) - S_{rst,p,q}(p,x)| + |S_{rst,p,q}(f,x) - S_{rst,p,q}(q,x)| < (\beta+\epsilon) + (\beta+\epsilon) < 2(\beta+\epsilon),$$

for all $(r,s,t) \in A_1(\epsilon)^c \bigcap A_2(\epsilon)^c$ which is a contradiction. So

$$\operatorname{diam}\left(\mathrm{LIM}^{\beta} S_{rst,p,q}(f,x)\right) \leq 2\beta.$$

Now, consider a triple sequence of Bernstein-Stancu polynomials of $(S_{rst,p,q}(f,x))$ of real numbers such that $I - \lim_{rst\to\infty} S_{rst,p,q}(f,x) = (f,x)$. Let $\epsilon > 0$. Then we can write

$$\left\{(r,s,t) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - (f,x)| \geq \epsilon\right\} \in I.$$

Thus we have

$$\begin{aligned} |S_{rst,p,q}(f,x) - S_{rst,p,q}(p,x)| &\leq |S_{rst,p,q}(f,x) - (f,x)| + |(f,x) - S_{rst,p,q}(p,x)| \\ &\leq |S_{rst,p,q}(f,x) - (f,x)| + \beta \leq \beta + \epsilon, \end{aligned}$$

for each $S_{rst,p,q}(p,x) \in \bar{S}_{\beta}((f,x)) := \left\{S_{rst,p,q}(p,x) \in \mathbb{R}^3 : |S_{rst,p,q}(p,x) - (f,x)| \leq \beta\right\}$. So we get

$$|S_{rst,p,q}(f,x) - S_{rst,p,q}(p,x)| < \beta + \epsilon$$

for each $(r,s,t) \in \left\{(r,s,t) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - (f,x)| < \epsilon\right\}$. Because the triple sequence of Bernstein-Stancu polynomials of $S_{rst,p,q}(f,x)$ is I- convergent to (f,x), we have

$$\left\{(r,s,t) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - (f,x)| < \epsilon\right\} \in F(I).$$

Hence we get $p \in I - \text{LIM}^r\, S_{rst,p,q}(f,x)$. Consequently, we can write

$$I - \text{LIM}^{\beta}\, S_{rst,p,q}(f,x) = \bar{S}_{\beta}((f,x)). \tag{3.1}$$

Because $\text{diam}\left(\bar{S}_{\beta}((f,x))\right) = 2\beta$, this shows that in general, the upper bound 2β of the diameter of the set $I - \text{LIM}^{\beta}\, S_{rst,p,q}(f,x)$ is not lower bound. □

Theorem 3.2. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ of real numbers, $I \subset 3^{\mathbb{N}}$ be an admissible ideal. For an arbitrary $(f,c) \in I(\Gamma_x)$, we have $|S_{rst,p,q}(f,x) - (f,c)| \le \beta$ for all $S_{rst,p,q}(f,x) \in I - \text{LIM}^{\beta}\, S_{rst,p,q}(f,x)$.*

Proof. Assume on the contrary that there exist a point $(f,c) \in I(\Gamma_x)$ and $S_{rst,p,q}(f,x) \in I - \text{LIM}^{\beta}\, S_{rst,p,q}(f,x)$ such that $|S_{rst,p,q}(f,x) - (f,c)| > \beta$. Define $\epsilon := \frac{|S_{rst,p,q}(f,x)-(f,c)|-\beta}{3}$. Then

$$\begin{aligned} &\left\{(r,s,t) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - (f,c)| < \epsilon\right\} \\ &\subseteq \left\{(r,s,t) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - (f,x)| \ge \beta + \epsilon\right\}. \end{aligned} \tag{3.2}$$

Since $(f,c) \in I(\Gamma_x)$, we have

$$\left\{(r,s,t) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - (f,c)| < \epsilon\right\} \notin I.$$

But from definition of I- convergence, since

$$\left\{(r,s,t) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - (f,x)| \ge \beta + \epsilon\right\} \in I,$$

by (3.2), we have

$$\left\{(r,s,t) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - (f,c)| < \epsilon\right\} \in I,$$

which contradicts the fact $(f,c) \in I(\Gamma_x)$. On the other hand, if $(f,c) \in I(\Gamma_x)$ i.e.,

$$\left\{(r,s,t) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - (f,c)| < \epsilon\right\} \notin I,$$

then

$$\left\{(r,s,t) \in \mathbb{N}^3 : |S_{rst,p,q}(f,x) - (f,x)| \ge \beta + \epsilon\right\} \notin I,$$

which contradicts the fact $(f,x) \in I - \text{LIM}^{\beta}\, S_{rst,p,q}(f,x)$. □

Theorem 3.3. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials*

$$(S_{rst,p,q}(f,x)) \to^{I} (f,x) \Longleftrightarrow I - \text{LIM}^{\beta}\, S_{rst,p,q}(f,x) = \bar{S}_{\beta}((f,x)).$$

Proof. **Necessity:** By Theorem 3.1.

Sufficiency: Let $I - \text{LIM}^{\beta}\, S_{rst,p,q}(f,x) = \bar{S}_{\beta}((f,x)) (\neq \phi)$. Thus the triple sequence of Bernstein-Stancu polynomials of $(S_{rst,p,q}(f,x))$ is I- analytic. Suppose that (f,x) has another I- cluster point $\left(f^{'},x\right)$ different from (f,x). The point

$$(\bar{f},x) = (f,x) + \frac{\beta}{|(f,x)-(f^{'},x)|}\left((f,x) - \left(f^{'},x\right)\right)$$

$$\left(\bar{f},x\right)-\left(f^{'},x\right)=\left(f,x\right)-\left(f^{'},x\right)+\frac{\beta}{\left|\left(f,x\right)-\left(f^{'},x\right)\right|}\left[\left(\left(f,x\right)-\left(f^{'},x\right)\right)\right.$$
$$\left.-\left(\left(f^{'},x\right)-\left(f^{'},x\right)\right)\right]$$

$$\left|\left(\bar{f},x\right)-\left(f^{'},x\right)\right|=\left|\left(f,x\right)-\left(f^{'},x\right)\right|+\frac{\beta}{\left|\left(f,x\right)-\left(f^{'},x\right)\right|}\left|\left(f,x\right)-\left(f^{'},x\right)\right|$$

$$\left|\left(\bar{f},x\right)-\left(f^{'},x\right)\right|=\left|\left(f,x\right)-\left(f^{'},x\right)\right|+\beta>\beta.$$

Since $\left(f^{'},x\right)\in I\left(\Gamma_x\right)$, by Theorem 3.2, $\left(\bar{f},x\right)\notin I-\mathrm{LIM}^{\beta}\,S_{rst,p,q}\left(f,x\right)$. It is not possible as $\left|\left(\bar{f},x\right)-\left(f,x\right)\right|=\beta$ and $I-\mathrm{LIM}^{r}\,S_{rst,p,q}\left(f,x\right)=\bar{S}_{\beta}\left(\left(f,x\right)\right)$. Since $\left(f,x\right)$ is the unique-I- cluster point of $\left(f,x\right)$. Hence $\Longrightarrow S_{rst,p,q}\left(f,x\right)\rightarrow^{I}\left(f,x\right)$. □

Corollary 3.4. *If $\left(X,\left|.,.\right|\right)$ is a strictly convex spaces and let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $\left(S_{rst,p,q}\left(f,x\right)\right)\in X$, there exists $y_1,y_2\in I-\mathrm{LIM}^{\beta}\,S_{rst,p,q}\left(f,x\right)$ such that $\left|y_1-y_2\right|=2\beta$, then this triple sequence $\left(f,x\right)\rightarrow^{I}\frac{y_1+y_2}{2}$.*

Proof. Omitted. □

Theorem 3.5. *If $I-\mathrm{LIM}^{\beta}\neq\phi$, then $I-\limsup\ S_{rst,p,q}(f,x)$ and $I-\liminf\ S_{rst,p,q}(f,x)$ belong to the set $I-\mathrm{LIM}^{2\beta}\,S_{rst,p,q}\left(f,x\right)$.*

Proof. We know that $I-\mathrm{LIM}^{\beta}\,S_{rst,p,q}\left(f,x\right)\neq\phi$, a triple sequence of Bernstein-Stancu polynomials of $\left(S_{rst,p,q}(f,x)\right)$ is $I-$ analytic. The number $I-\liminf S_{rst,p,q}(f,x)$ is an I-cluster point of $\left(f,x\right)$ and consequently, we have

$$\left|\left(f,x\right)-I-\liminf S_{rst,p,q}\left(f,x\right)\right|\leq\beta\ \forall\ \left(f,x\right)\in I-\mathrm{LIM}^{\beta}\left(f,x\right).$$

Let $A=\left\{(r,s,t)\in\mathbb{N}^3:\left|\left(f,x\right)-S_{rst,p,q}\left(f,x\right)\right|\geq\beta+\epsilon\right\}$. Now if $(r,s,t)\notin A$, then

$$\left|S_{rst,p,q}\left(f,x\right)-\left(I-\liminf S_{rst,p,q}\left(f,x\right)\right)\right|\leq\left|S_{rst,p,q}\left(f,x\right)-\left(f,x\right)\right|$$
$$+\left|\left(f,x\right)-\left(I-\liminf S_{rst,p,q}\left(f,x\right)\right)\right|<2\beta+\epsilon.$$

Thus

$$I-\liminf S_{rst,p,q}\left(f,x\right)\in I-\mathrm{LIM}^{2\beta}\,S_{rst,p,q}\left(f,x\right).$$

Similarly, it can be shown that $I-\limsup\ x_{mnk}\in I-\mathrm{LIM}^{2\beta}\,x_{mnk}$. □

Corollary 3.6. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $\left(S_{rst,p,q}\left(f,x\right)\right)$ of real numbers, if $I-\mathrm{LIM}^{\beta}\,S_{rst,p,q}\left(f,x\right)\neq\phi$, then*

$$I-core\left\{\left(f,x\right)\right\}\subseteq I-\mathrm{LIM}^{2\beta}\,S_{rst,p,q}\left(f,x\right).$$

Proof. We have

$$I-\mathrm{LIM}^{\beta}\,S_{rst,p,q}(f,x)=\left[I-\limsup S_{rst,p,q}(f,x)-2\beta,I-\liminf S_{rst,p,q}(f,x)+2\beta\right].$$

Then the result follows from Theorem 3.5. □

Theorem 3.7. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ of real numbers. Then the* diam $(I-core\{S_{rst,p,q}(f,x)\})$ *of the set*

$$I-core\{S_{rst,p,q}(f,x)\}=\beta \Longleftrightarrow I-core\{(f,x)\}=I-\mathrm{LIM}^{\beta} S_{rst,p,q}(f,x)$$

Proof.

$$\begin{aligned}
&\operatorname{diam}(I-core\{S_{rst,p,q}(f,x)\})=\beta\\
\Longleftrightarrow&(I-\limsup\ S_{rst,p,q}(f,x))-(I-\liminf x_{mnk})=\beta\\
\Longleftrightarrow&I-core\{x_{mnk}\}=[I-\liminf x_{mnk}, I-\limsup\ S_{rst,p,q}(f,x)]\\
&=[I-I-\limsup\ S_{rst,p,q}(f,x)-\beta, I-\liminf S_{rst,p,q}(f,x)+r]\\
&=I-\mathrm{LIM}^{\beta} S_{rst,p,q}(f,x).
\end{aligned}$$

Also it is easy to see that

(i) $\beta>\operatorname{diam}(I-core\{S_{rst,p,q}(f,x)\})\Leftrightarrow I-core\{S_{rst,p,q}(f,x)\}\subset I-\mathrm{LIM}^{\beta} S_{rst,p,q}(f,x)$,

(ii) $\beta<\operatorname{diam}(I-core\{S_{rst,p,q}(f,x)\})\Leftrightarrow I-\mathrm{LIM}^{\beta} S_{rst,p,q}(f,x)\subset I-core\{S_{rst,p,q}(f,x)\}$.

□

Theorem 3.8. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ of real numbers, if $\bar{\beta}=\inf\left\{\beta\geq 0: I-\mathrm{LIM}^{\beta} S_{rst,p,q}(f,x)\neq\phi\right\}$, then $\bar{\beta}=$* radius $(I-core\{S_{rst,p,q}(f,x)\})$.

Proof. If the set I–$core\{S_{rst,p,q}(f,x)\}$ is single ton, then radius$(I-core\{S_{rst,p,q}(f,x)\})=0$ and the triple sequence of Bernstein-Stancu polynomials is $I-$ convergent, i.e., $I-\mathrm{LIM}^{0} S_{rst,p,q}(f,x)\neq\phi$. Hence we get $\bar{\beta}=$ radius $(I-core\{S_{rst,p,q}(f,x)\})=0$.

Now assume that the set $I-core\{S_{rst,p,q}(f,x)\}$ is not a single ton. We can write $I-core\{S_{rst,p,q}(f,x)\}=[a,b]$, where $a=I-\liminf S_{rst,p,q}(f,x)$ and $b=I-\limsup\ S_{rst,p,q}(f,x)$.

Now let us assume that $\bar{\beta}\neq$ radius $(I-core\{S_{rst,p,q}(f,x)\})$.

If $\bar{\beta}<$ radius $(I-core\{x_{mnk}\})$, then define $\bar{\epsilon}=\frac{\frac{b-a}{2}-\bar{\beta}}{3}$. Now, be definition of $\bar{\beta}$ implies that $I-\mathrm{LIM}^{\bar{\beta}+\bar{\epsilon}} S_{rst,p,q}(f,x)\neq\phi$, given $\epsilon>0$

$$\exists\, l\in\mathbb{R}: A=\left\{(r,s,t)\in\mathbb{N}^3: |S_{rst,p,q}(f,x)-(f,x)|\geq\left(\bar{\beta}+\bar{\epsilon}\right)+\epsilon\right\}\in I.$$

Since $\bar{\beta}+\bar{\epsilon}<\frac{b-a}{2}$ which is a contradiction of the definition of a and b.

If $\bar{\beta}>$ radius $(I-core\{S_{rst,p,q}(f,x)\})$, then define $\bar{\epsilon}=\frac{\bar{\beta}-\frac{b-a}{2}}{3}$ and $\beta^{'}=\bar{\beta}-2\bar{\epsilon}$. It is clear that $0\leq\beta^{'}\leq\bar{\beta}$ and by definitions of a and b, the number $\frac{b-a}{2}\in I-\mathrm{LIM}^{\beta^{'}} S_{rst,p,q}(f,x)$. Then we get

$$\bar{\beta}\in\left\{\beta\geq 0: I-\mathrm{LIM}^{\beta} S_{rst,p,q}(f,x)\neq\phi\right\},$$

which contradicts the equality

$$\bar{\beta}=\inf\left\{\beta\geq 0: I-\mathrm{LIM}^{\beta} S_{rst,p,q}(f,x)\neq\phi\right\}\ \text{as}\ \beta^{'}<\beta.$$

□

Corollary 3.9. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ of real numbers, then $I-core\{S_{rst,p,q}(f,x)\} = I - \mathrm{LIM}^{2\bar{\beta}} S_{rst,p,q}(f,x)$.*

Proof. It follows that Theorem 3.7 and Theorem 3.8. □

Example 3.10. With the help of Matlab, we show comparisons and some illustrative graphics for the convergence of operators (1.2) to the function $f(x) = (x-3)(x-5)(x-6)$ under different parameters.

From Figure 1(a), it can be observed that as the value the q and p approaches towards 1 provided $0 < q < p \leq 1$, (p,q)-Bernstein-Stancu operators given by (1.2) converges towards the function $f(x) = (x-3)(x-5)(x-6)$. From Figure 1(a) and (b), it can be observed that for $\eta = \mu = 0$, as the value the (r,s,t) increases, (p,q)-Bernstein-Stancu operators given by (1.2) converges towards the function. Similarly from Figure 1(c), it can be observed that for $\eta = \mu = 5$, as the value the q and p approaches towards 1 or some thing else provided $0 < q < p \leq 1$, (p,q)-Bernstein-Stancu operators given by (1.2) converges towards the function. From Figure 1(c) and (d), it can be observed that as the value the $[r,s,t]$ increases, (p,q)-Bernstein-Stancu operators given by $f(x) = (x-3)(x-5)(x-6)$ converges towards the function.

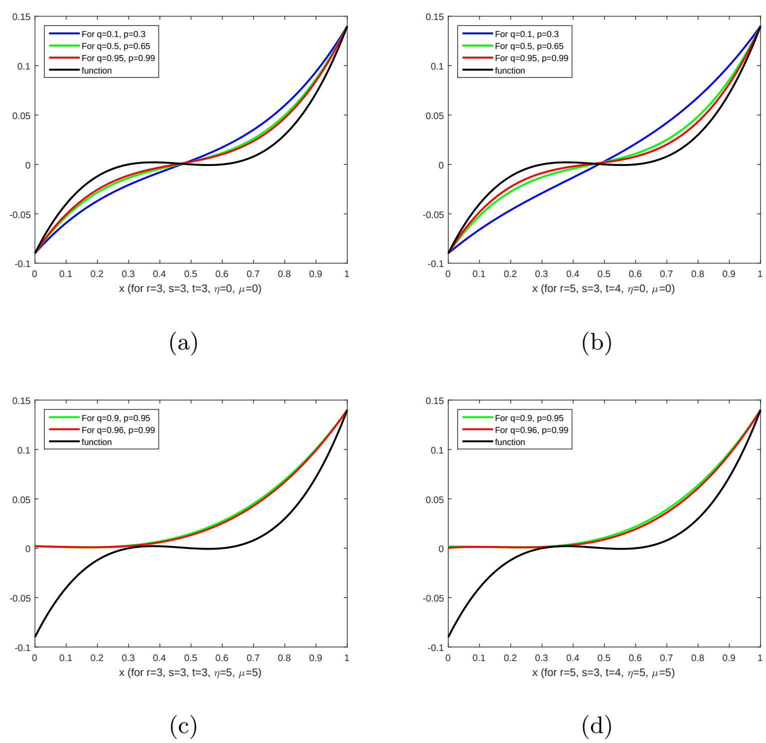

FIGURE 1. (p,q)-Bernstein-Stancu operators

Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

4. Conclusion

We introduced and studied some basic properties of Bernstein-Stancu polynomials of rough I-convergent of triple sequences and also studied the set of all Bernstein-Stancu polynomials of rough I-limits of a triple sequence and relation between analyticness and Bernstein-Stancu polynomials of rough I-core of a triple sequence.

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N. Subramanian (nsmaths@yahoo.com)
Department of Mathematics, SASTRA University, Thanjavur-613 401, India

A. Esi (aesi23@hotmail.com)
Department of Mathematics, Adiyaman University, 02040, Adiyaman, Turkey

M. K. Ozdemir (kozdemir73@gmail.com)
Department of Mathematics, Inonu University, 44280, Malatya, Turkey