Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 18, No. 3, (December 2019) pp. 233–243
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2019.18.3.233

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 18, No. 3, December 2019

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Received 28 April 2019; Revised 21 May 2019; Accepted 29 May 2019

Abstract. In this paper, we introduce the concept of d-fuzzy ideals and injective fuzzy ideals in a distributive lattice with respect to derivation. It is proved that the set of all d-fuzzy ideals forms a distributive lattice. A set of equivalent conditions are derived for a derivation d of L to became injective. Moreover, we proved that the set of all injective fuzzy ideals forms a complete distributive lattice.

2010 AMS Classification: 06D72, 06B10

Keywords: Derivation, d-ideal, Injective ideal, Fuzzy ideal, d- fuzzy ideal, Injective fuzzy ideal.

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1. Introduction

Bell and Kappe [3], and Kaya [6] have studied derivations in rings and prime rings after Posner [10] had given the definition of the derivation in ring theory. Szasz [14, 15]introduced and developed the theory of derivations in lattice structures. In particular, Szasz [15] observed that a derivation d of a lattice L is a lattice homomorphism and also preserves the minimum 0. Ferrari [5] extended these concepts to lattices and he embedded any lattice having some additional properties into the lattice of its derivations. Rao [11] introduced the concept of d-ideals and injective ideals in a distributive lattice with respect to derivations.

On the other hand, the notion of a fuzzy set initiated by Zadeh in [17]. Rosenfeld has developed the concept of fuzzy subgroups [12]. Since then, several authors have developed interesting results on fuzzy theory, like ([1],[2],[7],[9],[13],[16]).

In this paper, we introduce the concepts of d-fuzzy ideals and injective fuzzy ideals in a distributive lattice with respect to derivation. We prove that the set of all d-fuzzy ideals forms a distributive lattice. Set of equivalent conditions are given for a derivation d of L to became injective. We also characterized injective fuzzy ideals in terms of extension of fuzzy ideals. Finally, we prove that the set of all injective fuzzy ideals forms a complete distributive lattice.

2. Preliminaries

We refer to Birkhoff [4] for the elementary properties of lattices.

Now we recall the concept of derivation of a lattice L.

Definition 2.1 ([15]). A self-map $d : L \longrightarrow L$ is a derivation of L, if it satisfies the following conditions:
(i) $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$,
(ii) $d(x \vee y) = d(x) \vee d(y)$.

In [15], Szasz observed that a derivation d of a lattice L is a lattice homomorphism and also preserves the minimum 0.

Remark 2.2. In [5], Ferrari observed the condition (i) is redundant and is equivalent to

$$d(x \wedge y) = d(x) \wedge y = x \wedge d(y).$$

Lemma 2.3 ([5]). *Let d be a derivation of L. Then for any $x, y \in L$, we have*
(1) $d(0) = 0$,
(2) $d(x) \leq x$,
(3) $x \leq y \Rightarrow d(x) \leq d(y)$.

Remember that, for any set A, a function $\mu : A \to [0, 1]$ is called a fuzzy subset of A, where $[0, 1]$ is a unit interval, $\alpha \wedge \beta = min\{\alpha, \beta\}$ and $\alpha \vee \beta = max\{\alpha, \beta\}$ for all $\alpha, \beta \in [0, 1]$ [17].

Definition 2.4 ([12]). Let μ and θ be fuzzy subsets of a set A. Define the fuzzy subsets $\mu \cup \theta$ and $\mu \cap \theta$ of A as follows: for each $x \in A$,

$$(\mu \cup \theta)(x) = \mu(x) \vee \theta(x) \text{ and } (\mu \cap \theta)(x) = \mu(x) \wedge \theta(x).$$

Then $\mu \cup \theta$ and $\mu \cap \theta$ are called the union and intersection of μ and θ, respectively.

For any collection, $\{\mu_i : i \in I\}$ of fuzzy subsets of X, where I is a nonempty index set, the least upper bound $\bigcup_{i\in I} \mu_i$ and the greatest lower bound $\bigcap_{i\in I} \mu_i$ of the μ_i's are given by for each $x \in X$,

$$(\bigcup_{i\in I} \mu_i)(x) = \bigvee_{i\in I} \mu_i(x) \text{ and } (\bigcap_{i\in I} \mu_i)(x) = \bigwedge_{i\in I} \mu_i(x),$$

respectively.

We define the binary operations "+" and "$\cdot$" on the set of all fuzzy subsets of L as:

$$(\mu + \theta)(x) = Sup\{\mu(y) \wedge \theta(z) : y, z \in L,\ y \vee z = x\} \text{ and}$$
$$(\mu \cdot \theta)(x) = Sup\{\mu(y) \wedge \theta(z) : y, z \in L,\ y \wedge z = x\}.$$

If μ and θ are fuzzy ideals of L, then $\mu \cdot \theta = \mu \cap \theta$ and $\mu + \theta = \mu \vee \theta$.

For each $t \in [0, 1]$, the set

$$\mu_t = \{x \in A : \mu(x) \geq t\}$$

is called the level subset of μ at t [17].

Note that a fuzzy subset μ of L is nonempty, if there exists $x \in L$ such that $\mu(x) \neq 0$.

Definition 2.5. [12] Let f be a function from X into Y, μ be a fuzzy subset of X and let θ be a fuzzy subset of Y.

(i) The image of μ under f, denoted by $f(\mu)$, is a fuzzy subset of Y defined by: for each $y \in Y$,

$$f(\mu)(y) = \begin{cases} Sup\{\mu(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{otherwise.} \end{cases}$$

(ii) The preimage of θ under f, denoted by $f^{-1}(\theta)$, is a fuzzy subset of X defined by: for each $x \in X$,

$$f^{-1}(\theta)(x) = \theta(f(x)).$$

Definition 2.6 ([12])**.** Let f be any function from a set X to a set Y and let μ be any fuzzy subset of X. Then μ is called f-invariant, if for any $x,\ y \in X$,

$$f(x) = f(y) \text{ implies } \mu(x) = \mu(y).$$

Definition 2.7 ([13])**.** A fuzzy subset μ of a bounded lattice L is said to be a fuzzy ideal of L, if for all $x,\ y \in L$,

(i) $\mu(0) = 1$,
(ii) $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$,
(iii) $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$.

In [13], Swamy and Raju observed that a fuzzy subset μ of a lattice L is a fuzzy ideal of L if and only if

$$\mu(0) = 1 \text{ and } \mu(x \vee y) = \mu(x) \wedge \mu(y), \text{ for all } x, y \in L.$$

Corollary 2.8 ([13])**.** *Let $\alpha \in [0,1]$. If μ be a fuzzy ideal and θ be a fuzzy filter of a lattice such that $\mu \cap \theta = \alpha$ (the constant fuzzy subset attaining α), then there exists a prime fuzzy ideal η of a lattice L such that*

$$\mu \subseteq \eta \text{ and } \eta \cap \theta = \alpha.$$

Let μ be a fuzzy subset of a lattice L. The smallest fuzzy ideal of L containing μ is called a fuzzy ideal of L induced by μ and denoted by $(\mu]$ and

$$(\mu] = \bigcap\{\theta \in FI(L) : \mu \subseteq \theta\}$$

Theorem 2.9 ([8])**.** *Let μ be a fuzzy subset of L. The fuzzy subset $\overline{\mu}$ of L define by $\overline{\mu}(x) = Sup\{t \in [0,1] : x \in (\mu_t]\}$ for all $x \in L$ is the fuzzy ideal induced by μ.*

The set of all fuzzy ideals of L is denoted by $FI(L)$.

3. d-fuzzy ideals

In this section, we introduce the concept of d-fuzzy ideals in a distributive lattice. We prove that the set of all d-fuzzy ideals forms a distributive lattice. Some properties of d-fuzzy ideals also studied.

Throughout the rest of this paper, L stands for a distributive lattice with 0 and d is a derivation of L.

Lemma 3.1. *Let μ be a fuzzy ideal of L. If $a, b \in L$ and $b \leq a$, then $d(\mu)(b) \geq d(\mu)(a)$.*

Proof. Let $a, b \in L$ such that $b \leq a$. If $d(\mu)(a) = 0$, then it holds trivially. Again, let $d(\mu)(a) = Sup\{\mu(r) : r \in d^{-1}(a)\} = k > 0$. For each $\epsilon > 0$, there is $x \in d^{-1}(a)$ such that $\mu(x) > k - \epsilon$. Take $y = x \wedge b$. We get that $y \in d^{-1}(b)$ and $\mu(y) > k - \epsilon$. This implies that for each $\epsilon > 0$, we can find $y \in d^{-1}(b)$ such that $\mu(y) > k - \epsilon$. Thus $d(\mu)(b) \geq d(\mu)(a)$. □

Theorem 3.2. *Let μ be a fuzzy ideal of L. Then*
(1) *$d(\mu)$ is a fuzzy ideal of L such that $d(\mu) \subseteq \mu$,*
(2) *$d^{-1}(\mu)$ is a fuzzy ideal of L.*

Proof. Suppose μ is a fuzzy ideal of L and $x,\ y \in L$. Clearly $d(\mu)(0) = 1$ and $d(\mu)(x \vee y) \geq d(\mu)(x) \wedge d(\mu)(y)$. Again, by the above lemma,

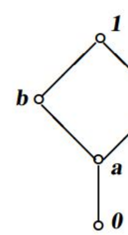
$$d(\mu)(x \wedge y) \geq d(\mu)(x) \vee d(\mu)(y).$$

Thus $d(\mu)$ is a fuzzy ideal of L.

Now we prove $d(\mu) \subseteq \mu$. Let $x \in L$. Then $d(\mu)(x) = \text{Sup}\ \{t = \mu(a) : a \in d^{-1}(x)\}$ and $\mu(x) = \text{Sup}\ \{k : x \in \mu_k\}$. Put $A = \{t = \mu(a) : a \in d^{-1}(x)\}$ and $B = \{k : x \in \mu_k\}$. To verify that $A \subseteq B$, let $t \in A$. Then $t = \mu(a)$, for some $a \in d^{-1}(x)$. This implies $a \in \mu_t$ and $d(a) = x$. Since μ_t is an ideal of L, $d(a) \in \mu_t$. Thus $A \subseteq B$. So $d(\mu) \subseteq \mu$. □

Definition 3.3. A fuzzy ideal μ of L is called a d- fuzzy ideal, if $\mu = d(\mu)$.

Example 3.4. Consider the distributive lattice $L = \{0, a, b, c, 1\}$



Define a self map $d : L \longrightarrow L$ as follows: for each $x \in L$,

$$d(x) = \begin{cases} 0, & \text{if } x = 0 \\ a, & \text{if } x = a, c \\ b, & \text{if } x = b, 1. \end{cases}$$

Then it can be easily verified that d is a derivation of L.

Now define a fuzzy subset μ of L as follows:

$$\mu(0) = 1,\ \mu(a) = \mu(b) = 0.5 \text{ and } \mu(c) = 0 = \mu(1).$$

Then μ is a fuzzy ideal of L and $d(\mu) = \mu$. Thus μ is a d-fuzzy ideal of L.

Note that for any fuzzy ideal μ of a lattice L, the ideals μ_t are called level ideals of μ. The level ideals of μ is denoted by F_μ and $F_\mu = \{\mu_t : t \in Im\ \mu\}$.

Theorem 3.5. *Let μ be a fuzzy ideal of L. If μ is d-invariant and d is onto, then the following are true*

(1) $F_{d(\mu)} = \{d(\mu_t),\ \ t \in Im\ \mu\}$,
(2) $F_{d^{-1}(\mu)} = \{d^{-1}(\mu_t),\ \ t \in Im\ \mu\}$.

$F_{d(\mu)}$ and $F_{d^{-1}(\mu)}$ denote the family of level ideals of $d(\mu)$ and $d^{-1}(\mu)$, respectively.

Proof. (1) Take any d-invariant fuzzy ideal μ of L. First, we observe that Im μ=Im $d(\mu)$.

$$\begin{aligned} t \in Im\ \mu &\Leftrightarrow \mu(x) = t \text{ for some } x \in L \\ &\Leftrightarrow (d^{-1}(d(\mu)))(x) = t \text{ since } \mu \text{ is } d\text{-invariant} \\ &\Leftrightarrow (d(\mu))(d(x)) = t \\ &\Leftrightarrow t \in Im\ d(\mu). \end{aligned}$$

Next, we proceed to show that $d(\mu_t) = (d(\mu))_t$. Let $y \in d(\mu_t)$. Then there exists $x \in \mu_t$ such that $d(x) = y$. Which implies $Sup\{\mu(a) : a \in d^{-1}(y)\} \geq t$. This shows that $y \in (d(\mu))_t$. Conversely, let $y \in (d(\mu))_t$. Then $d(\mu)(y) \geq t$. Since d is onto, $(d(\mu))(d(x)) \geq t$, for some $x \in L$ such that $y = d(x)$. This implies $(d^{-1}(d(\mu)))(x) \geq t$. Since μ is d-invariant, we get that $\mu(x) = (d^{-1}(d(\mu)))(x) \geq t$. This shows that $y = d(x) \in d(\mu_t)$. Thus $F_{d(\mu)} = \{d(\mu_t),\ \ t \in Im\ \mu\} = \{(d(\mu))_t,\ \ t \in Im\ \mu\}$.

(2) Take any fuzzy ideal μ of L. Then clearly, $Im\ \mu = Im\ d^{-1}(\mu)$. Now we proceed to show $(d^{-1}(\mu))_t = d^{-1}(\mu_t)$.

$$\begin{aligned} x \in (d^{-1}(\mu))_t &\Leftrightarrow d^{-1}(\mu)(x) \geq t \\ &\Leftrightarrow \mu(d(x)) \geq t \\ &\Leftrightarrow d(x) \in \mu_t \\ &\Leftrightarrow x \in d^{-1}(\mu_t). \end{aligned}$$

Thus $F_{d^{-1}(\mu)} = \{d^{-1}(\mu_t),\ \ t \in Im\ \mu\} = \{(d^{-1}(\mu))_t,\ \ t \in Im\ \mu\}$. □

Lemma 3.6. *Let μ and θ be any two fuzzy ideals of L. Then we have*

(1) $\mu \subseteq \theta \Rightarrow d(\mu) \subseteq d(\theta)$,
(2) $d(\mu \cap \theta) = d(\mu) \cap d(\theta)$,
(3) $d(\mu \vee \theta) = d(\mu) \vee d(\theta)$.

Proof. (1) the proof is straightforward.

(2) Since $\mu \cap \theta \subseteq \mu$ and $\mu \cap \theta \subseteq \theta$, by (1), we get $d(\mu \cap \theta) \subseteq d(\mu) \cap d(\theta)$. For any $x \in L$,

$$d(\mu)(x) \wedge d(\theta)(x) = Sup\{\mu(a) : a \in d^{-1}(x)\} \wedge Sup\{\theta(b) : b \in d^{-1}(x)\}.$$

Since $d(a) = x$ and $d(b) = x$, $d(a \wedge b) = x$. Using this fact, we have

$$\begin{aligned} d(\mu)(x) \wedge d(\theta)(x) &\leq Sup\{\mu(a \wedge b) : a \wedge b \in d^{-1}(x)\} \wedge Sup\{\theta(a \wedge b) : a \wedge b \in d^{-1}(x)\} \\ &= Sup\{\mu(a \wedge b) \wedge \theta(a \wedge b) : a \wedge b \in d^{-1}(x)\} \\ &= Sup\{(\mu \cap \theta)(a \wedge b) : a \wedge b \in d^{-1}(x)\} \\ &\leq Sup\{(\mu \cap \theta)(c) : c \in d^{-1}(x)\} \\ &= d(\mu \cap \theta)(x). \end{aligned}$$

Then $d(\mu \cap \theta) = d(\mu) \cap d(\theta)$.

(3) By (1), we have $d(\mu) \vee d(\theta) \subseteq d(\mu \vee \theta)$. For any $x \in L$,

$$d(\mu \vee \theta)(x) = Sup\{(\mu \vee \theta)(a) : a \in d^{-1}(x)\}$$

$$= Sup\{Sup\{\mu(r_1) \vee \theta(r_2) : a = r_1 \vee r_2\}; a \in d^{-1}(x)\}$$

and

$$\begin{aligned}
&(d(\mu) \vee d(\theta))(x) \\
&= Sup\{d(\mu)(b_1) \wedge d(\theta)(b_2) : x = b_1 \vee b_2\} \\
&= Sup\{Sup\{\mu(c_1) : c_1 \in d^{-1}(b_1)\} \wedge Sup\{\theta(c_2) : c_2 \in d^{-1}(b_2)\}; x = b_1 \vee b_2\} \\
&= Sup\{Sup\{\mu(c_1) \wedge \theta(c_2) : c_1 \in d^{-1}(b_1),\ c_2 \in d^{-1}(b_2)\}\};\ x = b_1 \vee b_2\}.
\end{aligned}$$

Put $A = \{(r_1, r_2) \in L \times L : a = r_1 \vee r_2,\ a \in d^{-1}(x)\}$ and $B = \{(x_1, x_2) \in L\times L : x_1 \in d^{-1}(b_1),\ x_2 \in d^{-1}(b_2),\ x = d(b_1\vee b_2),\ b_1\vee b_2 \in d^{-1}(x)\}$. If $(x_1, x_2) \in B$, then $x_1 \in d^{-1}(b_1)$, $x_2 \in d^{-1}(b_2)$ and $x_1 \vee x_2 \in d^{-1}(x)$. Thus $(x_1, x_2) \in A$ and $B \subseteq A$. So $d(\mu\vee\theta) \subseteq (d(\mu)\vee d(\theta))$. Hence $d(\mu\vee\theta) = d(\mu)\vee d(\theta)$. □

For any derivation of a distributive lattice L, let us denote the class of all d-fuzzy ideals of L by $FI_d(L)$.

Theorem 3.7. *The set $FI_d(L)$ is a distributive lattice with respect to set inclusion. Moreover, if d is onto, then $FI_d(L)$ is a complete distributive lattice.*

Proof. Clearly, $(FI_d(L), \subseteq)$ a partially ordered set. By the above lemma, $(FI_d(L), \cap, \vee)$ is a lattice and sublattice of $FI(L)$. Since $FI(L)$ is a distributive lattice, $FI_d(L)$ is a distributive lattice.

Suppose d is onto. Then χ_L is greatest element of $FI_d(L)$. Let $\{\mu_\alpha : \alpha \in \Delta\}$ be a subset of a d-fuzzy ideal of L. Then $\bigcap_{\alpha\in\Delta} \mu_\alpha$ is a fuzzy ideal of L and $d(\bigcap_{\alpha\in\Delta} \mu_\alpha) \subseteq \bigcap_{\alpha\in\Delta} \mu_\alpha$. Since d is onto, for any $x \in L$, $d(x) = x$. Now,

$$\begin{aligned}
d(\bigcap_{\alpha\in\Delta} \mu_\alpha)(x) &= Sup\{\bigcap_{\alpha\in\Delta} \mu_\alpha(a) : a \in d^{-1}(x)\} \\
&\geq \bigcap_{\alpha\in\Delta} \mu_\alpha(x).
\end{aligned}$$

Thus $d(\bigcap_{\alpha\in\Delta} \mu_\alpha) = \bigcap_{\alpha\in\Delta} \mu_\alpha$. So $(FI_d(L), \cap, \vee)$ is a complete distributive lattice. □

Theorem 3.8. *A fuzzy subset μ of L is a d-fuzzy ideal if and only if μ_t is a d-ideal of L, for each $t \in (0, 1]$.*

Proof. Let μ be a d-fuzzy ideal. Then $\mu_t = (d(\mu))_t$. Since μ is a fuzzy ideal, μ_t is an ideal of L, for each $t \in [0, 1]$ and $d(\mu_t) \subseteq \mu_t$. Let $x \in \mu_t$ and $t \in (0, 1]$. Then $d(\mu)(x) = Sup\{\mu(a) : a \in d^{-1}(x)\} \geq t > 0$ and $x = d(x)$. Which implies $x \in d(\mu_t)$. Thus μ_t is a d-ideal of L, for each $t \in (0, 1]$.

Conversely, suppose that every proper level subset of μ is a d-ideal of L. Then μ is a fuzzy ideal of L and $d(\mu) \subseteq \mu$. Let $x \in L$. If $\mu(x) = 0$, then $d(\mu)(x) \geq \mu(x)$. Suppose $\mu(x) \neq 0$. Then

$$\mu(x) = Sup\{t \in (0, 1] : x \in d(\mu_t)\} \leq Sup\{t \in (0, 1] : x \in d(\mu)_t\} = d(\mu)(x).$$

Thus $\mu = d(\mu)$. So μ is a d- fuzzy ideal of L. □

Lemma 3.9. *If I is an ideal of L, then $d(\chi_I) = \chi_{d(I)}$.*

Proof. Let I be an ideal of L and $x \in L$. If $x \in d(I)$, then $x \in I$ and there is $a \in I$ such that $d(a) = x$. Thus $\chi_{d(I)}(x) = 1 = d(\chi_I)(x)$. Again, if $x \notin d(I)$, then there is no $a \in I$ such that $d(a) = x$ and $\chi_{d(I)}(x) = 0$. Assume that $d(\chi_I)(x) \neq 0$. Then

$d(\chi_I)(x) = 1$. This implies there is $a \in I$ such that $d(a) = x$ and $x \in d(I)$. Which is a contradiction. Thus $d(\chi_I)(x) = 0 = \chi_{d(I)}(x)$. So $d(\chi_I) = \chi_{d(I)}$. □

Theorem 3.10. *A nonempty subset I of L is a d-ideal of L if and only if χ_I is a d-fuzzy ideal.*

Proof. Suppose I is a d-ideal of L, by the above lemma, χ_I is a d- fuzzy ideal of L.

Conversely, suppose that χ_I is a d- fuzzy ideal of L. Then I is an ideal of L and $d(I) \subseteq I$. Let $x \in I$. Then $d(\chi_I)(x) = 1$. This implies that there is $a \in I$ such that $a \in d^{-1}(x)$. Thus $I \subseteq d(I)$. So I is a d-ideal. □

Lemma 3.11. *If d is onto, then every fuzzy ideal of L is a d-fuzzy ideal.*

Proof. Let μ be a fuzzy ideal of L and d is onto. Then for any $x \in L$, $d(x) = x$. To show that $\mu \subseteq d(\mu)$, let $x \in L$. Then $d(\mu)(x) = Sup\{\mu(a) : a \in d^{-1}(x)\} \geq \mu(x)$. Thus $\mu \subseteq d(\mu)$. So μ is d-fuzzy ideal. □

Theorem 3.12. *Let μ be any fuzzy ideal of L. If μ is a d-fuzzy ideal, then*

$$\mu = \cup_{\theta \subseteq \mu}(d(\theta)], \text{ where } \theta \text{ is a fuzzy subset of } L.$$

Lemma 3.13. *Let μ be any fuzzy ideal of L. If for each fuzzy ideal $\theta \subseteq \mu$ of L, there exists a fuzzy ideal $\eta \subseteq \mu$ of L such that $\theta = d(\eta)$, then μ is a d-fuzzy ideal.*

Proof. Assume that for each fuzzy ideal $\theta \subseteq \mu$ of L, there exists a fuzzy ideal $\delta \subseteq \mu$ of L such that $\theta = d(\delta)$. Since $\mu \subseteq \mu$, by the assumption, there is a fuzzy ideal $\beta \subseteq \mu$ of L such that $\mu = d(\beta)$. Since $d(\beta) \subseteq \beta$, we get that $\mu \subseteq \beta$. Thus $\mu = \beta$ and $\mu = d(\mu)$. So μ is a d-fuzzy ideal. □

4. Injective fuzzy ideals

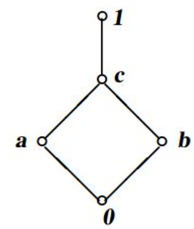
In this section, we introduce the concept of injective fuzzy ideals in a distributive lattice with respect to derivation. A set of equivalent conditions are derived for a derivation d of L to became injective. Finally, we proved that the set of all injective fuzzy ideals forms a complete distributive lattice.

Definition 4.1. A fuzzy ideal μ of L is called an injective fuzzy ideal with respect to d, if for $x, y \in L, d(x) = d(y)$, then $\mu(x) = \mu(y)$.

Theorem 4.2. *For any nonempty fuzzy subset μ of L, μ is an injective fuzzy ideal if and only if each level subset of μ is an injective ideal of L with respect to d. (In particular, a nonempty subset I of L is an injective ideal of L if and only if χ_I is an injective fuzzy ideal of L).*

The following examples demonstrates the independence between the class of all d-fuzzy ideals and injective fuzzy ideals.

Example 4.3. Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given below:



Define a self map $d : L \longrightarrow L$ as follows: for each $x \in L$,

$$d(x) = \begin{cases} a, & x = a,\ c,\ 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then it can be easily verified that d is a derivation of L.

Now define a fuzzy subset μ of L as follows:

$$\mu(0) = 1 = \mu(b) \text{ and } \mu(a) = \mu(c) = \mu(1) = 0.4.$$

Then μ is an injective fuzzy ideal but not a d-fuzzy ideal of L.

Example 4.4. Considering the distributive lattice $L = \{0, a, b, c, 1\}$ given in the Example 3.4 mentioned earlier, we have seen that μ is a d-fuzzy ideal and $d(a) = d(c)$, but $\mu(a) \neq \mu(c)$. From this, we can simply observe that μ is a d-fuzzy ideal but not injective with respect to d.

Lemma 4.5. *The characteristic function of $Ker\ d$ is the smallest injective fuzzy ideal of L.*

Proof. Since $Ker\ d$ is an injective ideal of L, by Theorem 4.2, $\chi_{Ker\ d}$ is an injective fuzzy ideal. To show $\chi_{Ker\ d}$ is the smallest injective fuzzy ideal, let us take any injective fuzzy ideal μ of L with respect to d. Suppose $x \in Ker\ d$. Then $d(x) = 0 = d(0)$. Since μ is injective with respect to d and $\mu(0) = 1$, we get that $\mu(x) = 1$. Thus $Ker\ d \subseteq \mu$. So $Ker\ d$ is the smallest injective fuzzy ideal of L. □

In the following theorem we established a set of equivalent conditions for $\chi_{\{0\}}$ to become an injective fuzzy ideal.

Theorem 4.6. *The followings are equivalent in L:*

(1) *$\chi_{\{0\}}$ is an injective fuzzy ideal of L with respect to d,*
(2) *$\chi_{Ker\ d} = \chi_{\{0\}}$,*
(3) *$d(x) = 0$ implies that $x = 0$, for all $x \in L$.*

Proof. (1) ⇒ (2): Suppose $\chi_{\{0\}}$ is an injective fuzzy ideal of L with respect to d. Then clearly, $\chi_{\{0\}} \subseteq \chi_{Ker\ d}$. By Lemma 4.5, $\chi_{Ker\ d}$ is the smallest injective fuzzy ideal of L. Thus we get that $\chi_{Ker\ d} \subseteq \chi_{\{0\}}$. So $\chi_{Ker\ d} = \chi_{\{0\}}$.

(2) ⇒ (3): Suppose $\chi_{Ker\ d} = \chi_{\{0\}}$. Then $Ker\ d = \{0\}$. Thus fixed any $x \in L$ satisfying $d(x) = 0$, $x \in Ker\ d$. By our hypothesis, we get that $x \in \{0\}$. This shows that $x = 0$, for all $x \in L$ such that $d(x) = 0$.

(3) ⇒ (1): Suppose the condition (3) holds. To prove $\chi_{\{0\}}$ is an injective fuzzy ideal, it suffices to show that $\{0\}$ is an injective ideal. Let $x, y \in L$ such that $d(x) = d(y)$ and $x \in \{0\}$. Then $d(y) = 0$. Thus by the condition (3), $y = 0$ and $y \in \{0\}$. So $\chi_{\{0\}}$ is an injective fuzzy ideal of L with respect to d. □

In the following theorem a set of equivalent conditions is given for a derivation d of L to become injective.

Theorem 4.7. *The following conditions are equivalent in L:*

(1) *d is injective,*
(2) *Every fuzzy ideal is injective with respect to d,*
(3) *Every prime fuzzy ideal is an injective fuzzy ideal with respect to d.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): The proof is straightforward.

(3) $\Rightarrow$ (1): suppose every prime fuzzy ideal of L is an injective fuzzy ideal. Let $x, y \in L$ such that $d(x) = d(y)$. Assume that $x \neq y$. Without loss of generality, we can assume that $(x] \cap [y) = \phi$. $\chi_{(x]}$ and $\chi_{[y)}$ are fuzzy ideal and fuzzy filter of L, respectively such that $\chi_{(x]} \cap \chi_{[y)} = \chi_{\phi}$ (the constant fuzzy subset attaining, value 0), by Corollary 2.8, there exists a prime fuzzy ideal θ of L such that

$$\chi_{(x]} \subseteq \theta \text{ and } \theta \cap \chi_{[y)} = \chi_{\phi}$$

Since $\chi_{(x]} \subseteq \theta$, $\theta(x) = 1$. Again, $\theta(y) \cap \chi_{[y)}(y) = 0$. Since 0 is meet irreducible in $[0, 1]$, $\theta(y) = 0$. Which is a contradiction. Thus d is injective. □

Theorem 4.8. *A fuzzy ideal μ of L is injective with respect to d if and only if for any $x \in L,\ \mu(x) = \mu(d(x))$.*

Now we are going to discuss about the extension of fuzzy ideals. We also characterize injective fuzzy ideals in terms of extension of fuzzy ideals. That is, for any given fuzzy ideal μ of L always there exists a smallest injective fuzzy ideal containing μ.

Definition 4.9. For any fuzzy ideal μ of L, an extension of μ is defined as:

$$\mu^{'}(x) = \text{Sup}\{\mu(a) : d(x) \in (d(a)]\}.$$

Lemma 4.10. *For any fuzzy ideal μ of L, $\mu^{'}$ is a fuzzy ideal of L.*

Proof. Let μ be a fuzzy ideal of L. Then $\mu^{'}(0) = 1$. For any $x, y \in L$,

$$\begin{aligned}
\mu^{'}(x) \wedge \mu^{'}(y) &= Sup\{\mu(a) : d(x) \in (d(a)]\} \wedge Sup\{\mu(b) : d(y) \in (d(b)]\} \\
&= Sup\{\mu(a) \wedge \mu(b) : d(x) \in (d(a)],\ d(y) \in (d(b)]\} \\
&= Sup\{\mu(a \vee b) : d(x) \in (d(a)],\ d(y) \in (d(b)]\} \\
&= Sup\{\mu(a \vee b) : d(x) \vee d(y) \in (d(a)] \vee (d(b)]\} \\
&= Sup\{\mu(a \vee b) : d(x \vee y) \in (d(a) \vee d(b)]\} \\
&= Sup\{\mu(a \vee b) : d(x \vee y) \in (d(a \vee b)]\} \\
&\leq Sup\{\mu(c) : d(x \vee y) \in (d(c)]\} \\
&= \mu^{'}(x \vee y).
\end{aligned}$$

And, $\mu^{'}(x) = Sup\{\mu(a) : d(x) \in (d(a)]\} \leq Sup\{\mu(a) : d(x \wedge y) \in (d(a)]\} = \mu^{'}(x \wedge y)$. Similarly, $\mu^{'}(x) \leq \mu^{'}(x \wedge y)$. This shows that $\mu^{'}(x \wedge y) \geq \mu^{'}(x) \vee \mu^{'}(y)$. Thus $\mu^{'}$ is a fuzzy ideal of L. □

Example 4.11. Considering the distributive lattice $L = \{0, a, b, c, 1\}$ given in the Example 4.3. Define a fuzzy subset μ of L as follows:

$$\mu(0) = 1, \mu(b) = 0.5 \text{ and } \mu(a) = \mu(c) = \mu(1) = 0.4.$$

Then μ is a fuzzy ideal of L. Now we can easily show that $\mu^{'}(0) = \mu^{'}(b) = 1$ and $\mu^{'}(a) = \mu^{'}(c) = \mu^{'}(1) = 0.4$.

Lemma 4.12. *For any two fuzzy ideals μ and θ of L, we have the following:*

(1) $\mu \subseteq \mu^{'}$,
(2) $\mu \subseteq \theta \Rightarrow \mu^{'} \subseteq \theta^{'}$,
(3) $\mu^{'} \cap \theta^{'} = (\mu \cap \theta)^{'}$.

Proof. The proofs of (1) and (2) are straightforward. Now we proceed to prove the following. Let μ and θ be fuzzy ideals of L. Then by (2), we get $(\mu \cap \theta)^{'} \subseteq \theta^{'} \cap \mu^{'}$. For any $x \in L$, $\mu^{'}(x) \wedge \theta^{'}(x) = Sup\{\mu(a) : d(x) \in (d(a)]\} \wedge Sup\{\theta(b) : d(x) \in (d(b)]\}$.

If $d(x) \in (d(a)]$ and $d(x) \in (d(b)]$, then we get $d(x \wedge a) = d(x)$, $d(x \wedge b) = d(x)$ and $d(x \wedge (a \wedge b)) = d(x)$. Thus based on this fact we have,

$$\begin{aligned} \mu^{'}(x) \wedge \theta^{'}(x) &\leq Sup\{\mu(x \wedge (a \wedge b)) : d(x) \in (d(x \wedge (a \wedge b)]\} \\ &\quad \wedge Sup\{\theta(x \wedge (a \wedge b)) : d(x) \in (d(x \wedge (a \wedge b)]\} \\ &= Sup\{\mu(x \wedge (a \wedge b)) \wedge \theta(x \wedge (a \wedge b)) : d(x) \in (d(x \wedge (a \wedge b)]\} \\ &= Sup\{(\mu \cap \theta)(x \wedge (a \wedge b)) : d(x) \in (d(x \wedge (a \wedge b)]\} \\ &\leq Sup\{(\mu \cap \theta)(c) : d(x) \in (d(c)]\} \\ &= (\mu \cap \theta)^{'}(x). \end{aligned}$$

So $(\mu \cap \theta)^{'} = \mu^{'} \cap \theta^{'}$. □

Proposition 4.13. *For any fuzzy ideal μ of L, $\mu^{'}$ is the smallest injective fuzzy ideal of L with respect to d such that $\mu \subseteq \mu^{'}$.*

Proof. Let μ be any fuzzy ideal of L. Then $\mu^{'}$ is injective. To show $\mu^{'}$ is the smallest ideal containing μ, suppose θ is any injective fuzzy ideal of L containing μ. Let $x \in L$. Then $\mu^{'}(x) = Sup\{\mu(a) : d(x) \in (d(a)]\}$. This implies $d(x \wedge a) = d(x)$, for all $a \in L$ such that $d(x) \in (d(a)]$. Since θ is injective and $\mu \subseteq \theta$, we have $\mu(a) \leq \theta(x)$, for all a such that $d(x) \in (d(a)]$. This implies $\theta(x)$ is an upper bound of $\{\mu(a) : d(x) \in (d(a)]\}$. Thus $\mu^{'}(x) \leq \theta(x)$. So $\mu^{'}$ is the smallest injective fuzzy ideal containing μ. □

Corollary 4.14. *If μ is an injective fuzzy ideal of L, then $\mu = \mu^{'}$.*

Theorem 4.15. *The set $IFI(L)$ of all injective fuzzy ideals of L with respective to a given derivation d of L forms a complete distributive lattice.*

Proof. Since L is injective ideal, χ_L is a largest injective fuzzy ideal of L. For $\mu, \theta \in IFI(L)$, define the operations $\wedge$ and $\sqcup$ such that $\mu \wedge \theta = \mu \cap \theta$ and $\mu \sqcup \theta = (\mu \vee \theta)^{'}$. Then clearly, $\mu \wedge \theta,\ \mu \sqcup \theta \in IFI(L)$ and $(IFI(L), \wedge, \sqcup)$ is a lattice. Now for any $\mu, \theta, \eta \in IFI(L)$, we have $\mu \sqcup (\theta \wedge \eta) = (\mu \sqcup \theta) \wedge (\theta \sqcup \eta)$. Thus $(IFI(L), \wedge, \sqcup)$ is a distributive lattice. To show $IFI(L)$ is complete, let $\{\mu_\alpha : \alpha \in I\}$ be a subfamily of $IFI(L)$. Then $\bigcap_{\alpha \in I} \mu_\alpha$ is fuzzy ideal of L. Let $x, y \in L$ such that $d(x) = d(y)$. Then $\bigcap_{\alpha \in I} \mu_\alpha(x) = inf\{\mu_\alpha(x) : \alpha \in I\} = inf\{\mu_\alpha(y) : \alpha \in I\} = \bigcap_{\alpha \in I} \mu_\alpha(y)$. Thus $(IFI(L), \wedge, \sqcup)$ is a complete distributive lattice. □

5. Conclusions

In this work, we introduce the concept of d-fuzzy ideals and injective fuzzy ideals in a distributive lattice with respect to derivation and investigate some their properties. We have shown that the class of d-fuzzy ideals forms a distributive lattice. Moreover, the class of d-fuzzy ideals can be made a complete distributive lattice whenever d is onto. Furthermore, a set of equivalent conditions also derived for a derivation d to become injective. Our future work will focus on studding a d-fuzzy ideals of an almost distributive lattice.

Acknowledgements. The authors would like to thank the referees for their valuable comments and constructive suggestions.

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