Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 18, No. 3, (December 2019) pp. 209–231
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2019.18.3.209

@FMI

Hesitant fuzzy sets applied to BCK/BCI-algebras

J. Kim, P. K. Lim, J. G. Lee, K. Hur

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 18, No. 3, December 2019

Annals of Fuzzy Mathematics and Informatics
Volume 18, No. 3, (December 2019) pp. 209–231
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2019.18.3.209

@FMI

Hesitant fuzzy sets applied to BCK/BCI-algebras

J. Kim, P. K. Lim, J. G. Lee, K. Hur

Received 18 February 2019; Revised 16 March 2019; Accepted 21 April 2019

Abstract. We define a hesitant fuzzy BCK/BCI-algebra and obtain some of its properties. Next, we introduce the concept of hesitant fuzzy ideal and obtain some of its properties and give some examples. Finally, we define a hesitant fuzzy positive implicative ideal, a hesitant fuzzy implicative ideal, a hesitant fuzzy commutative ideal and investigate some of its properties, respectively and their relations. In particular, we give characterizations of hesitant fuzzy positive implicative ideal, a hesitant fuzzy implicative ideal and a hesitant fuzzy commutative ideal, respectively.

2010 AMS Classification: 06F35, 03G25, 03E72

Keywords: Hesitant fuzzy set, Hesitant fuzzy BCK-algebra, Hesitant fuzzy ideal, Hesitant fuzzy positive implicative ideal, Hesitant fuzzy positive ideal, Hesitant fuzzy commutative ideal.

Corresponding Author: J. G. Lee (jukolee@wku.ac.kr)

1. Introduction

In 2010, Torra [21] introduced the notion of a hesitant fuzzy set which can consider as the generalization of fuzzy sets (Refer to [18, 20]). After then, Jun et al. [8] studied hesitant fuzzy bi-ideals in semigroups. Xia and Xu [22] applied hesitant fuzzy set to decision making. Furthermore, Deepark and John [2] investigated hesitant fuzzy rough sets through hesitant fuzzy relations. Also They [3, 4, 5] studied homomorphisms of hesitant fuzzy subgroups, and hesitant fuzzy subrings and ideals. On the other hand, Alshehri and Alshehri [1] applied Hesitant anti-fuzzy soft sets to BCK-algebras. Jun and Ahn [7] studied hesitant ideals in BCK-algebras. Muhiuddin and Aldhafeeri [16] studied Subalgebras and ideals in BCK/BCI-algebras based on uni-hesitant fuzzy set theory. Rezaei and Saeid [17] introduced the notion of hesitant fuzzy filters, and obtained some of its properties and studied hesitant Fuzzy implicative filters in BE-algebras. Solariaju and Mahalakshmi [19] investigated hesitant intuitionistic fuzzy soft groups. Deepark and Mashinchi [6] studied hesitant L-fuzzy relations. Recently, Krishankumar et al. [14] studied a new two-stage decision-making framework for supplier outsourcing using hesitant fuzzy information and a new ranking method called three-way hesitant fuzzy VIKOR. Kim et al [12] introduced the category $\mathbf{HSet}(H)$ consisting of all hesitant H-fuzzy spaces and all morphisms between them and studied $\mathbf{HSet}(H)$ in the sense of a topological universe. Also they [13] investigated hesitant fuzzy subgroups and rings.

In this paper, we define a hesitant fuzzy BCK/BCI-algebra and obtain some of its properties. Next, we introduce the concept of hesitant fuzzy ideal and obtain some of its properties and give some examples. Finally, we define a hesitant fuzzy positive implicative ideal, a hesitant fuzzy implicative ideal, a hesitant fuzzy commutative ideal and investigate some of its properties, respectively and their relations. In particular, we give characterizations of hesitant fuzzy positive implicative ideal, a hesitant fuzzy implicative ideal and a hesitant fuzzy commutative ideal, respectively (See Theorems 5.6, 5.17 and 5.24).

2. Preliminaries

In this section, we list some basic definitions and some properties needed in the next sections.

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [10, 11]) and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ is called a BCI-algebra, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = 0,\ y * x = 0 \ \Rightarrow\ x = y$.

If a BCI-algebra X satisfies the following identity:
(V) $0 * x = 0$, for each $x \in X$,
then X is called a BCK-algebra.

Any BCK/BCI-algebra X satisfies the following conditions: for any $x,\ y,\ z \in X$,

$$x * 0 = x, \tag{2.1}$$

$$x \le y \Rightarrow x * z \le y * z,\ z * y \le z * x, \tag{2.2}$$

$$(x * y) * z = (x * z) * y, \tag{2.3}$$

$$(x * z) * (y * z) \le x * y, \tag{2.4}$$

where $x \le y$ if and only if $x * y = 0$.

Any BCI-algebra X satisfies the following conditions (see [9]): for any $x,\ y,\ z \in X$,

$$x * (x * (x * y)) = x * y, \tag{2.5}$$

$$0 * (x * y) = (0 * x) * (0 * y). \tag{2.6}$$

A BCK-algebra X is said to be positive implicative, if it satisfies the condition:

$$(\forall x,\ y,\ z \in X)((x * y) * z) = (x * z) * (y * z)). \tag{2.7}$$

A BCK-algebra X is said to be implicative, if it satisfies the condition:

$$(\forall x,\ y \in X)(x = x * (y * x)). \tag{2.8}$$

A BCK-algebra X is said to be commutative, if it satisfies the condition:

$$(\forall x,\ y \in X)(x * (x * y) = y * (y * x)). \tag{2.9}$$

Definition 2.1 ([12, 21])**.** Let X be a reference set and let $P[0,1]$ denote the power set of $[0,1]$. Then a mapping $h : X \to P[0,1]$ is called a hesitant fuzzy set in X.

The hesitant fuzzy empty [resp. whole] set, denoted by $h^{\mathbf{0}}$ [resp. $h^{\mathbf{1}}$], is a hesitant fuzzy set in X defined as: for each $x \in X$,

$$h^{\mathbf{0}}(x) = \phi \text{ [resp. } h^{\mathbf{1}}(x) = [0,1]].$$

In this case, we will denote the set of all hesitant fuzzy sets in X as $HS(X)$.

Definition 2.2 ([2])**.** Let $h_1,\ h_2 \in HS(X)$. Then

(i) we say that h_1 is a subset of h_2, denoted by $h_1 \subset h_2$, if $h_1(x) \subset h_2(x)$, for each $x \in X$,

(ii) we say that h_1 is equal to h_2, denoted by $h_1 = h_2$, if $h_1 \subset h_2$ and $h_2 \subset h_1$.

Definition 2.3 ([12])**.** Let $h_1,\ h_2 \in HS(X)$ and let $(h_j)_{j\in J} \subset HS(X)$. Then

(i) the intersection of h_1 and h_2, denoted by $h_1 \widetilde{\cap} h_2$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$(h_1 \widetilde{\cap} h_2)(x) = h_1(x) \cap h_2(x),$$

(ii) the intersection of $(h_j)_{j\in J}$, denoted by $\widetilde{\bigcap}_{j\in J} h_j$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$(\widetilde{\bigcap}_{j\in J} h_j)(x) = \bigcap_{j\in J} h_j(x),$$

(iii) the union of h_1 and h_2, denoted by $h_1 \widetilde{\cup} h_2$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$(h_1 \widetilde{\cup} h_2)(x) = h_1(x) \cup h_2(x),$$

(iv) the union of $(h_j)_{j\in J}$, denoted by $\widetilde{\bigcup}_{j\in J} h_j$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$(\widetilde{\bigcup}_{j\in J} h_j)(x) = \bigcup_{j\in J} h_j(x).$$

Definition 2.4 ([12])**.** Let X be a nonempty set and let $h \in HS(X)$. Then the complement of h, denoted by h^c, is a hesitant fuzzy set in X defined as: for each $x \in X$,

$$h^c(x) = h(x)^c = [0,1] \setminus h(x).$$

Definition 2.5 ([13])**.** Let $h \in HS(X)$. Then h is called a hesitant fuzzy point with the support $x \in X$ and the value $\phi \neq \lambda \in P[0,1]$, denoted by x_λ, if $x_\lambda : X \to P[0,1]$ is the mapping given by: for each $y \in X$,

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } y \neq x \\ \phi & \text{otherwise.} \end{cases}$$

We will denote the set of all hesitant fuzzy points in X as $H_P(X)$.

For any $x_\lambda,\ y_\mu \in H_P(X)$, we say that x_λ is equal to y_μ, denoted by $x_\lambda = y_\mu$, if $x = y$ and $\lambda = \mu$.

Definition 2.6 ([13])**.** Let $h \in HS(X)$ and let $x_\lambda \in H_P(X)$. Then x_λ is said to be belong to h, denoted by $x_\lambda \in h$, if $\lambda \subset h(x)$.

It is obvious that $h = \widetilde{\bigcap}_{x_\lambda \in A} x_\lambda$.

3. Hesitant fuzzy BCK-algebras

Definition 3.1. Let $(X; *, 0)$ be a BCI/BCK-algebra and let $h_1,\ h_2 \in HS(X)$. Then the hesitant fuzzy product of h_1 and h_2, denoted by $h_1 \circ h_2$ is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$(h_1 \circ h_2)(x) = \widetilde{\bigcup}_{y*z=x} [h_1(y) \cap h_2(z).$$

Example 3.2 (See Example 1 of 12 pages in [15])**.** Let $X = \{0, a, b, c\}$ be the BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Table 3.1

Consider two hesitant fuzzy sets h_1 and h_2 defined as follows:

$$h_1(0) = (0, 1),\ h_1(a) = [0.2, 0.6),\ h_1(b) = \{0.3, 0.7\},\ h_1(c) = (0.4, 0.8],$$

$$h_2(0) = (0, 1],\ h_2(a) = \{0.1, 0.5\},\ h_2(b) = (0.4, 0.9],\ h_2(c) = \{0.2, 0.7\}.$$

Since $0 = 0 * 0 = 0 * a = 0 * b = 0 * c = a * a = b * b = c * c$,

$$\begin{aligned}&(h_1 \circ h_2)(0)\\ &= (h_1(0) \cap h_2(0)) \cup (h_1(0) \cap h_2(a)) \cup (h_1(0) \cap h_2(b)) \cup (h_1(0) \cap h_2(c))\\ &= (0, 1).\end{aligned}$$

Similarly, we have

$(h_1 \circ h_2)(a) = [0.2, 0.6]$, $(h_1 \circ h_2)(b) = \{0.3, 0.7\}$, $(h_1 \circ h_2)(c) = (0.4, 0.8]$.

Then we can see that $(h_1 \circ h_2)$ is a hesitant fuzzy set in X.

Proposition 3.3. *Let $(X; *, 0)$ be a BCI/BCK-algebra, let $h_1,\ h_2 \in HS(X)$ and let $x_\lambda,\ y_\mu \in H_P(X)$. Then*

(1) *$x_\lambda \circ y_\mu = (x * y)_{\lambda \cap \mu}$, in particular, $x_\lambda \circ y_\lambda = (x * y)_\lambda$,*

(2) *$h_1 \circ h_2 = \widetilde{\bigcup}_{x_\lambda \in h_1,\ y_\mu \in h_2} x_\lambda \circ y_\mu$.*

Proof. (1) Let $z \in X$ and suppose $z = x' * y'$. Then

$$(x_\lambda \circ y_\mu)(z) = \widetilde{\bigcup}_{z=x'*y'} [x_\lambda(x') \cap y_\mu(y')] = \lambda \cap \mu.$$

(2) Let $h = \widetilde{\bigcup}_{x_\lambda \in h_1,\ y_\mu \in h_2} x_\lambda \circ y_\mu$ and let $z \in X$. Then

$$\begin{aligned}(h_1 \circ h_2)(z) &= \widetilde{\bigcup}_{u*v=z}[h_1(u) \cap h_2(v)] \\ &\supset \widetilde{\bigcup}_{u*v=z}(\widetilde{\bigcup}_{x_\lambda \in h_1,\ y_\mu \in h_2}[x_\lambda(u) \cap y_\mu(v)] \\ &= \widetilde{\bigcup}_{x_\lambda \in h_1,\ y_\mu \in h_2}[x_\lambda \circ y_\mu] \\ &= h(z)\end{aligned}$$

and

$$\begin{aligned}h(z) &= \widetilde{\bigcup}_{x_\lambda \in h_1,\ y_\mu \in h_2}(\widetilde{\bigcup}_{u*v=z}[x_\lambda(u) \cap y_\mu(v)]) \\ &= \widetilde{\bigcup}_{u*v=z}(\widetilde{\bigcup}_{x_\lambda \in h_1,\ y_\mu \in h_2}[x_\lambda(u) \cap y_\mu(v)]) \\ &\supset \widetilde{\bigcup}_{u*v=z}[u_{h_1(u)}(u) \cap v_{h_2(v)}(v)] \\ &= \widetilde{\bigcup}_{u*v=z}[h_1(u) \cap h_2(v)] \\ &= (h_1 \circ h_2)(z).\end{aligned}$$

Thus $h_1 \circ h_2 = h$. So $h_1 \circ h_2 = \widetilde{\bigcup}_{x_\lambda \in h_1,\ y_\mu \in h_2} x_\lambda \circ y_\mu$. □

The followings are immediate results of Proposition 3.3 (1).

Proposition 3.4. *Let $(X; *, 0)$ be a BCI/BCK-algebra. Then $(H_P(X), \circ)$ satisfies the followings: for any $x,\ y,\ z \in X$,*

(1) $((x_\lambda \circ y_\mu) \circ (x_\lambda \circ z_\nu)) \circ (z_\nu \circ y_\mu) = 0_{\lambda \cap \mu \cap \nu}$,

(2) $(x_\lambda \circ (x_\lambda \circ y_\mu)) \circ y_\mu = 0_{\lambda \cap \mu}$,

(3) $x_\lambda \circ x_\mu = 0_{\lambda \cap \mu}$,

(4) *if* $x_\lambda \circ y_\mu = 0_\nu$ *and* $y_\mu \circ x_\lambda = 0_\rho$, *then* $x = y$ *and* $\nu = \rho = \lambda \cap \mu$, *in fact,* $x_{\lambda \cap \mu} = y_{\lambda \cap \mu}$ *but* $x_\lambda \neq y_\mu$ *in general,*

(5) $0_\lambda \circ x_\mu = 0_{\lambda \cap \mu}$.

For a fixed $\lambda \in P[0, 1]$, let $[H_P(X)]_\lambda$ denote the set of all hesitant fuzzy points with the value in X. Then by Proposition 3.3 (1), we have the following results.

Proposition 3.5. *Let $(X; *, 0)$ be a BCI/BCK-algebra and let $\lambda \in P[0, 1]$ be fixed. Then $([H_P(X)]_\lambda; \circ, 0_\lambda)$ satisfies the followings: for any $x_\lambda,\ y_\lambda,\ z_\lambda \in [H_P(X)]_\lambda$,*

HBCI-1 $((x_\lambda \circ y_\lambda) \circ (x_\lambda \circ z_\lambda)) \circ (z_\lambda \circ y_\lambda) = 0_\lambda$,

HBCI-2 $(x_\lambda \circ (x_\lambda \circ y_\lambda)) \circ y_\lambda = 0_\lambda$,

HBCI-3 $x_\lambda \circ x_\lambda = 0_\lambda$,

HBCI-4 $x_\lambda \circ y_\lambda = 0_\lambda$ *and* $y_\lambda \circ x_\lambda = 0_\lambda$ *imply* $x_\lambda = y_\lambda$,

HBCK-5 $0_\lambda \circ x_\lambda = 0_\lambda$.

In this case, $([H_P(X)]_\lambda; \circ, 0_\lambda)$ will be called a hesitant fuzzy pointwise BCI/BCK-algebra with the value λ induced by X.

For a BCK-algebra X, we define a binary relation $\leq$ on $([H_P(X)]_\lambda; \circ, 0_\lambda)$ as follows: for any $x_\lambda,\ y_\lambda \in [H_P(X)]_\lambda$,

$$x_\lambda \leq y_\lambda \text{ if and only if } x_\lambda \circ y_\lambda = 0_\lambda.$$

Then we have the following results.

Theorem 3.6. *Let $(X;*,0)$ be a BCK-algebra and let $\lambda \in P[0,1]$ be fixed. Then $([H_P(X)]_\lambda;\circ,0_\lambda)$ is a hesitant fuzzy BCK-algebra if and only if it satisfies the followings: for any $x,\ y,\ \in X$,*

(3.1) $$(x_\lambda \circ y_\lambda)\circ(x_\lambda \circ z_\lambda) \leq z_\lambda \circ y_\lambda,$$

(3.2) $$x_\lambda \circ (x_\lambda \circ y_\lambda)) \leq y_\lambda,$$

(3.3) $$x_\lambda \leq x_\lambda,$$

(3.4) $$x_\lambda \leq y_\lambda \text{ and } y_\lambda \leq x_\lambda \text{ imply } x_\lambda = y_\lambda,$$

(3.5) $$0_\lambda \leq x_\lambda.$$

Theorem 3.7. *Let $(X;*,0)$ be a BCK-algebra. Then X is positive implicative if and only if for any $x_\lambda,\ y_\mu,\ z_\nu \in H_P(X)$,*

(3.6) $$(x_\lambda \circ y_\mu)\circ z_\nu = (x_\lambda \circ z_\nu)\circ(y_\mu \circ z_\nu).$$

Proof. Suppose X is positive implicative and let $x_\lambda,\ y_\mu,\ z_\nu \in H_P(X)$. Then by Proposition 3.3 (1),

$$(x_\lambda \circ y_\mu)\circ z_\nu = (x*y)_{\lambda\cap\mu} * \circ z_\nu = ((x*y)*z)_{\lambda\cap\mu\cap\nu}$$

and

$$(x_\lambda \circ z_\nu)\circ(y_\mu \circ z_\nu) = (x*z)_{\lambda\cap\nu}\circ(y*z)_{\mu\cap\nu} = ((x*z)*(y*z))_{\lambda\cap\mu\cap\nu}.$$

Since X is positive implicative, by (2.7), $(x*y)*z = (x*z)*(y*z)$. Thus (3.6) holds.

The proof of the converse is easy. □

The following is an immediate result of Theorem 3.7.

Corollary 3.8. *Let $(X;*,0)$ be a BCK-algebra and let $\lambda \in P[0,1]$ be fixed. Then X is positive implicative if and only if $[H_P(X)]_\lambda$ is positive implicative, i.e., for any $x_\lambda,\ y_\lambda,\ z_\lambda \in [H_P(X)]_\lambda$,*

(3.7) $$(x_\lambda \circ y_\lambda)\circ z_\lambda = (x_\lambda \circ z_\lambda)\circ(y_\lambda \circ z_\lambda).$$

Theorem 3.9. *Let $(X;*,0)$ be a BCK-algebra and let $\lambda \in P[0,1]$ be fixed. Then X is implicative if and only if $[H_P(X)]_\lambda$ is implicative, i.e., for any $x_\lambda,\ y_\lambda \in [H_P(X)]_\lambda$,*

(3.8) $$x_\lambda = x_\lambda \circ (y_\lambda \circ x_\lambda).$$

Proof. Suppose X is implicative and let $x_\lambda,\ y_\mu \in [H_P(X)]_\lambda$. Then by Proposition 3.3 (1), $x_\lambda \circ (y_\lambda \circ x_\lambda) = x_\lambda \circ (y*x)_\lambda = (x*(y*x))_\lambda$. Since X is implicative, by (2.7), $x = x*(y*x)$. Thus (3.8) holds.

The proof of the converse is easy. □

Theorem 3.10. *Let $(X;*,0)$ be a BCK-algebra. Then X is commutative if and only if for any $x_\lambda,\ y_\mu \in H_P(X)$,*

(3.9) $$x_\lambda \circ (x_\lambda \circ y_\mu = y_\mu \circ (y_\mu \circ x_\lambda).$$

Proof. Suppose X is commutative and let $x_\lambda,\ y_\mu,\ z_\nu \in H_P(X)$. Then by Proposition 3.3 (1),

$$x_\lambda \circ (x_\lambda \circ y_\mu = x_\lambda \circ (x*y)_{\lambda\cap\mu} = (x*(x*y))_{\lambda\cap\mu}$$

and

$$y_\mu \circ (y_\mu \circ x_\lambda) = y_\mu \circ (y*x)_{\lambda\cap\mu} = (y*(y*x))_{\lambda\cap\mu}.$$

Since X is commutative, by (2.9), $(x*y)*z = (x*z)*(y*z)$. Thus (3.9) holds.

The proof of the converse is easy. □

The following is an immediate result of Theorem 3.11.

Corollary 3.11. *Let $(X; *, 0)$ be a BCK-algebra and let $\lambda \in P[0,1]$ be fixed. Then X is commutative if and only if $[H_P(X)]_\lambda$ is commutative, i.e., for any $x_\lambda,\ y_\lambda, \in [H_P(X)]_\lambda$,*

$$x_\lambda \circ (x_\lambda \circ y_\lambda = y_\lambda \circ (y_\lambda \circ x_\lambda). \tag{3.10}$$

4. Hesitant fuzzy subalgebras and ideals

In what follows, let X denotes a BCK/BCI-algebra unless otherwise specified.

Definition 4.1 (See [7]). Let $h^{\mathbf{0}} \neq h \in HS(X)$. Then h is called a hesitant fuzzy subalgebra of X, if it satisfies the following condition:

(HFSA) $$h(x * y) \supset h(x) \cap h(y),\ \forall x,\ y \in X.$$

It is clear that $h^{\mathbf{1}}$ is a hesitant fuzzy subalgebra of X.

Example 4.2. (1) Let $X = \{0, a, b\}$ be the BCK-algebra with the following Cayley table: Let h be the hesitant fuzzy set in X given by:

$*$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Table 4.1

$$h(0) = (0, 1),\ h(a) = [0.3, 0.9),\ h(b) = (0.2, 0.8].$$

Then we can easily see that h is a hesitant fuzzy subalgebra of X.

(2) Let $X = \{0, a, b, c\}$ be the BCK-algebra given by Example 3.2. Let h be the hesitant fuzzy set in X given by:

$$h(0) = (0.2, 0.7],\ h(a) = [0.3, 0.6],\ h(b) = (0.2, 0.4] \cup \{0.5, 0.6\},\ \ h(c) = (0.3, 0.4) \cup \{0.6\}.$$

The we can easily see that h is a hesitant fuzzy subalgebra of X.

Theorem 4.3. *Let X be a BCK-algebra and let $h^{\mathbf{0}} \neq h \in HS(X)$. Then h is a hesitant fuzzy subalgebra of X if and only if it satisfies the following condition:*

$$(\forall\ x_\lambda,\ y_\mu \in H_P(X))(x_\lambda,\ y_\mu \in I \Rightarrow x_\lambda \circ y_\mu \in h) \tag{4.1}$$

Proof. Suppose h is a hesitant fuzzy subalgebra of X and for any $x_\lambda,\ y_\mu \in H_P(X)$, suppose $x_\lambda,\ y_\mu \in h$. Then $h(x) \supset \lambda$ and $h(y) \supset \mu$. Thus

$$\begin{aligned} h(x * y) &\supset h(x) \cap h(y) \text{ [Since } h \text{ is a hesitant fuzzy subalgebra of } X] \\ &\supset \lambda \cap \mu. \end{aligned}$$

So $x_\lambda \circ y_\mu = (x * y)_{\lambda \cap \mu} \in h$. Hence the condition (4.1) holds.

Conversely, suppose the necessary condition (4.1) holds and let $x,\ y \in X$. Let $h(x) = \lambda$ and $h(y) = \mu$. Then clearly, $x_\lambda,\ y_\mu \in h$. Thus by the condition (4.1), $(x * y)_{\lambda \cap \mu} = x_\lambda \circ y_\mu \in h$. So $h(x * y) \supset \lambda \cap \mu = h(x) \cap h(y)$. Hence h is a hesitant fuzzy subalgebra of X □

Definition 4.4 (See [7]). Let $h^{\mathbf{0}} \neq h \in HS(X)$ and let $\lambda \in P[0,1]$. Then the λ-level set of h, denoted by h_λ, is a subset of X defined by:

$$h_\lambda = \{x \in X : h(x) \supset \lambda\}.$$

It is obvious that either $h_\lambda = \phi$ or $h_\lambda \neq \phi$. In the future, we will assume $h_\lambda \neq \phi$, for each $\lambda \in P[0,1]$.

Theorem 4.5. *Let $h^{\mathbf{0}} \neq h \in HS(X)$ and let $h_\lambda \neq \phi$, for each $\lambda \in P[0,1]$. Then h is a hesitant fuzzy subalgebra of X if and only if h_λ is a subalgebra of X.*

In this case, h_λ will be called the λ-level subalgebra of X.

Proof. Suppose h is a hesitant fuzzy subalgebra of X and let $x,\ y \in h_\lambda$, for each $\lambda \in P[0,1]$. Then $h(x) \supset \lambda$ and $h(y) \supset \lambda$. Thus $h(x * y) \supset h(x) \cap h(y) \supset \lambda$. So $x * y \in h_\lambda$. Hence h_λ is a subalgebra of X

Conversely, suppose h_λ is a subalgebra of X, for each $\lambda \in P[0,1]$ and for any $x,\ y \in X$, let $h(x) \cap h(y) = \mu$. Then clearly, $x,\ y \in h_\mu$. Thus by the hypothesis, $x * y \in h_\mu$. So $h(x * y) \supset \mu = h(x) \cap h(y)$. Hence h is a hesitant fuzzy subalgebra of X. □

Proposition 4.6. *Let h be a hesitant fuzzy subalgebra of X. Then $h(0) \supset h(x)$, for each $x \in X$.*

Proof. Let $x \in X$. Then clearly, $x*x = 0$. Thus $h(0) = h(x*x) \supset h(x) \cap h(x) = h(x)$. So $h(0) \supset h(x)$. □

Proposition 4.7. *Let X be a BCI-algebra. if h is a hesitant fuzzy subalgebra of X, then $h(x * (0 * y)) \supset h(x) \cap h(y)$, for any $x,\ y \in X$.*

Proof. Let $x,\ y \in X$. Then by Definition 4.1 and Proposition 4.6,

$$h(x * (0 * y)) \supset h(x) \cap h(0 * y) \supset h(x) \cap h(0) \cap h(y) = h(x) \cap h(y).$$

□

Theorem 4.8. *Let h be a hesitant fuzzy subalgebra of X. Then*

$$(\forall x,\ y \in X)(h(x * y) \supset h(y) \Leftrightarrow h(x) = h(0)).$$

Proof. Suppose $h(x * y) \supset h(y)$, for any $x,\ y \in X$. Let $y = 0$. Then by (2.1), $h(x) = h(x * 0) \supset h(0)$. By Proposition 4.6, $h(0) \supset h(x)$, for each $x \in X$. Thus $h(x) = h(0)$, for each $x \in X$.

Conversely, suppose $h(x) = h(0))$, for each $x \in X$. Let $x,\ y \in X$. Then by the hypothesis and Proposition 4.6,

$$h(x * y) \supset h(x) \cap h(y) = h(0) \cap h(y) = h(y).$$

Thus $h(x * y) \supset h(y)$. □

Proposition 4.9. *Let h be a hesitant fuzzy subalgebra of X and let $\lambda \in P[0,1]$ such that $h_\lambda \neq \phi$. We define a mapping $h^* : X \to P[0,1]$ as follows: for each $x \in X$,*

$$h^*(x) = \begin{cases} h(x) & \text{if } x \in h_\lambda \\ \phi & \text{otherwise.} \end{cases}$$

Then h^* is a hesitant fuzzy subalgebra of X.

Proof. It is clear that h^* is a hesitant fuzzy set in X from the definition of h^*. From Theorem 4.5, it is obvious that h_λ is a subalgebra of X, for each $\lambda \in P[0,1]$ with $h_\lambda \neq \phi$. Let $x,\ y \in X$.

Suppose $x,\ y \in h_\lambda$. Then clearly, $x * y \in h_\lambda$. Thus

$$h^*(x * y) = h(x * y) \supset h(x) \cap h(y) = h^*(x) \cap h^*(y).$$

Suppose $x \notin h_\lambda$ or $y \notin h_\lambda$. Then $h^*(x) = \phi$ or $h^*(y) = \phi$. Thus

$$h^*(x * y) \supset \phi = h^*(x) \cap h^*(y).$$

So in ether cases, $h^*(x * y) \supset h^*(x) \cap h^*(y)$. Hence h^* is a hesitant fuzzy subalgebra of X. □

Let A be a nonempty subset of a set X, let $\lambda,\ \mu \in P[0,1]$ be fixed such that $\lambda \supsetneqq \mu$ and let $h_A,\ [h_A]_{(\lambda,\mu)} : X \to P[0,1]$ be two mappings defined as follows, respectively: for each $x \in X$,

$$h_A(x) = \begin{cases} [0,1] & \text{if } x \in A \\ \phi & \text{otherwise,} \end{cases}$$

$$[h_A]_{(\lambda,\mu)}(x) = \begin{cases} \lambda & \text{if } x \in A \\ \mu & \text{otherwise.} \end{cases}$$

Then clearly, h_A and $[h_A]_{(\lambda,\mu)}$ are hesitant fuzzy sets in X. In particular, $h_{\{0\}}$ is a special hesitant fuzzy point in X.

The followings are immediate results of definitions of h_A and $[h_A]_{(\lambda,\mu)}$.

Theorem 4.10. *Let A be a nonempty subset of a BCK-algebra X and let $\lambda,\ \mu \in P[0,1]$ be fixed such that $\lambda \supsetneqq \mu$. Then the followings are equivalent:*
(1) *A is a subalgebra of X,*
(2) *h_A is a hesitant fuzzy subalgebra of X,*
(3) *$[h_A]_{(\lambda,\mu)}$ is a hesitant fuzzy subalgebra of X.*

Definition 4.11. Let $h^{\mathbf{0}} \neq I \in HS(X)$. Then I is called a hesitant fuzzy ideal of X, if it satisfies the following conditions:

$$(HFI_1) \qquad I(0) \supset I(x),\ \forall x \in X,$$

$$(HFI_2) \qquad I(x) \supset I(x * y) \cap I(y),\ \forall x,\ y \in X.$$

It is obvious that $h^{\mathbf{1}}$ is a hesitant fuzzy ideal of X. We will call $h^{\mathbf{1}}$ is trivial. A hesitant fuzzy ideal I of X is said to be proper, if $I \neq h^{\mathbf{1}}$.

The followings are immediate results of h_A and $[h_A]_{(\lambda,\mu)}$.

Theorem 4.12. *Let A be a subset of a BCK-algebra X such that $0 \in A$ and let $\lambda,\ \mu \in P[0,1]$ be fixed such that $\lambda \supsetneqq \mu$. Then the followings are equivalent:*
(1) *A is an ideal of X,*
(2) *h_A is a hesitant fuzzy ideal of X,*
(3) *$[h_A]_{(\lambda,\mu)}$ is a hesitant fuzzy ideal of X.*

Example 4.13. (1) Let $X = \{0, a, b\}$ be the BCK-algebra given in Example 4.2 (1). Then we can see easily see that X is an implicative BCK-algebra. Moreover, by Theorem 4.12, $0_\lambda \in H_P(X)$, $h_{\{0\}}$, $h_{\{0,a\}}$, $h_{\{0,b\}}$, $[h_{\{0\}}]_{(\lambda,\mu)}$, $[h_{\{0,a\}}]_{(\lambda,\mu)}$, $[h_{\{0,b\}}]_{(\lambda,\mu)}$, $h^{\mathbf{1}}$ are hesitant fuzzy ideals of X.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
c	c	b	a	0

Table 4.2

(2) Let $X = \{0, a, b, c\}$ be the BCK-algebra with the following Cayley table: Then we can easily see that X is a commutative BCK-algebra. Moreover, by Theorem 4.12, $0_\lambda \in H_P(X)$, $h_{\{0\}}$, $[h_{\{0\}}]_{(\lambda,\mu)}$ and $h^{\mathbf{1}}$ are hesitant fuzzy ideals of X.

(3) Let $X = \{0, a, b, c, d\}$ a BCI-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	c	c
a	a	0	a	d	c
b	b	b	0	c	c
c	c	c	c	0	0
d	d	c	d	a	0

Table 4.3

Consider the mapping $I : X \to P[0, 1]$ defined by:

$I(0) = [0.1, 0.9],\ I(a) = [0.2, 0.4) \cup \{0.5, 0.8\},\ I(b) = (0.1, 0.3] \cup \{0.5\} \cup [0.7, 0.8],$
$I(c) = [0.1, 0.5] \cup (0.7, 0.8),\ I(d) = [0.2, 0.4] \cup \{0.5, 0.6, 0.7, 0.8\}.$

Then we can easily see that I is a hesitant fuzzy ideal of X.

Theorem 4.14. *Let $h^{\mathbf{0}} \neq I \in HS(X)$. Then I is a hesitant fuzzy ideal of X if and only if I_λ is an ideal of X, for each $\lambda \in P[0, 1]$ such that $I(0) \supset \lambda$.*

Proof. Suppose I is a hesitant fuzzy ideal of X. For each $x \in X$, let $I(x) = \lambda$. Then by (HFI_1), $I(0) \supset I(x)$. Thus $I(0) \supset \lambda$. So $0 \in I_\lambda$.

Now suppose $x * y \in I_\lambda$ and $y \in I_\lambda$. Then clearly, $I(x * y) \supset \lambda$ and $I(y) \supset \lambda$. Thus by (HFI_2), $I(x) \supset I(x * y) \cap I(y) \supset \lambda$. So $x \in I_\lambda$. Hence I_λ is an ideal of X.

Conversely, suppose I_λ is an ideal of X, for each $\lambda \in P[0, 1]$ such that $I(0) \supset \lambda$. For each $x \in X$, let $I(x) = \mu \subset [0, 1]$. Then clearly, $I(0) \supset \mu = I(x)$. Now for any $x,\ y \in X$, let $I(x * y) = \lambda$, $I(y) = \mu$ and let $\delta = \lambda \cap \mu$. Then clearly, $x * y \in I_\delta$ and $y \in I_\delta$. Thus $x \in I_\delta$. So $I(x) \supset \delta = \lambda \cap \mu = I(x * y) \cap I(y)$. Hence I is a hesitant fuzzy ideal of X. □

Theorem 4.15. *Let $h^{\mathbf{0}} \neq I \in HS(X)$ such that $I(0) = [0, 1]$. Then I is a hesitant fuzzy ideal of X if and only if it satisfies the following condition:*

(4.2) $$(\forall x_\lambda,\ y_\mu \in H_P(X))(x_\lambda \circ y_\mu \in I \text{ and } y_\mu \in I \Rightarrow x_{\lambda\cap\mu} \in I).$$

Proof. Suppose I is a hesitant fuzzy ideal of X and for any $x_\lambda,\ y_\mu \in H_P(X)$, let $x_\lambda \circ y_\mu \in I$ and $y_\mu \in I$. Then $I(y) \supset \mu$. Since $x_\lambda \circ y_\mu = (x * y)_{\lambda\cap\mu}$, $I(x * y) \supset \lambda \cap \mu$. Thus by the hypothesis, $I(x) \supset I(x * y) \cap I(y) \supset (\lambda \cap \mu) \cap \mu = \lambda \cap \mu$. So $x_{\lambda\cap\mu} \in I$. Hence the condition (4.2) holds.

Conversely, suppose the necessary condition (4.2) holds and for any $x,\ y \in X$, let $I(x * y) = \delta$ and $I(y) = \mu$. Then there is $\lambda \in P[0, 1]$ such that $x_\lambda \circ y_\mu = (x * y)_{\lambda\cap\mu}$

and $\lambda \cap \mu = \delta$, and $y_\mu \in I$. Thus $(x * y)_{\lambda\cap\mu} \in I$, i.e., $x_\lambda \circ y_\mu \in I$. By the condition (4.2), $x_{\lambda\cap\mu} \in I$. So $I(x) \supset \lambda \cap \mu = (\lambda \cap \mu) \cap \mu = \delta \cap \mu = I(x * y) \cap I(y)$. Since $I(0) = [0, 1]$, it is clear that $I(0) \supset I(x)$, for each $x \in X$. Hence I is a hesitant fuzzy ideal of X. □

Lemma 4.16. *Let I be a hesitant fuzzy ideal of X. Then*

$$(\forall x,\ y \in X)(x * y = 0 \Rightarrow I(x) \supset I(y)).$$

Proof. Suppose $x * y = 0$, for any $x,\ y \in X$. Then

$I(x) \supset I(x * y) \cap I(y)$ [By (HFI_2)]
$= I(0) \cap I(y)$ [By the hypothesis]
$= I(y)$. [By (HFI_1)] □

Since $y \leq x$ if and only if $y * x = 0$, the following is an immediate result of the above Lemma.

Corollary 4.17. *Let I be a hesitant fuzzy ideal of a BCK-algebra X. If for any $x,\ y \in X$, $y \leq x$, then $I(y) \supset I(x)$.*

Proposition 4.18. *Let I be a hesitant fuzzy ideal of X. Then*
(1) $(\forall x,\ y,\ z \in X)((I(x * y) \supset I(x * z) \cap I(z * y))$,
(2) $(\forall x,\ y \in X)((I(x * y) = I(0) \Rightarrow I(x) \supset I(y))$.

Proof. (1) Let $x,\ y,\ z \in X$. Since X is a BCK-algebra, by the axiom (I),

$$((x * y) * (x * z)) * (z * y) = 0.$$

Then by Lemma 4.16, $I((x * y) * (x * z)) \supset I(z * y)$. Thus

$I(x * y) \supset I((x * y) * (x * z)) \cap I(x * z)$ [By (HFI_2)]
$\supset I(z * y) \cap I(x * z)$
$= I(x * z) \cap I(z * y)$.

(2) Suppose $I(x * y) = I(0)$, for any $x,\ y \in X$. Then

$I(x) \supset I(x * y) \cap I(y)$ [By (HFI_2)]
$= I(0) \cap I(y)$ [By the hypothesis]
$= I(y)$. [By (HFI_1)] □

Theorem 4.19. *Let I be a hesitant fuzzy ideal of X. Then the followings are equivalent:*
(1) $(\forall x,\ y \in X)(I(x * y) \supset I((x * y) * y))$,
(2) $(\forall x,\ y,\ z \in X)(I((x * z) * (y * z)) \supset I((x * y) * z)))$.

Proof. (1)⇒(2): Suppose the condition (1) holds and let $x,\ y,\ z \in X$. Since X is a BCK-algebra, by (2.3), (2.4) and the definition of $\leq$,

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z,$$

$$\text{i.e., } (((x * (y * z)) * z) * z) * ((x * y) * z) = 0.$$

Then by Lemma 4.15,

$$I((x * z) * (y * z)) = I((x * (y * z)) * z) \supset I(((x * (y * z)) * z) * z) \supset I((x * y) * z).$$

(2)⇒(1): Suppose the condition (2) holds and let $y = z$ in (2). Then by (III) and (2.1),

$$I((x * z) * z))) \subset I((x * z) * (z * z)) = I((x * z) * 0) = I(x * z).$$

This completes the proof. □

Proposition 4.20. *Every hesitant fuzzy ideal of a BCK-algebra X is a hesitant fuzzy subalgebra of X.*

Proof. Let I be a hesitant fuzzy ideal of a BCK-algebra X and let $x,\ y \in X$. Then clearly, by (2.3) and (V), $(x * y) * x = 0$. Thus by Lemma 4.16 and (HFI_2),

$$I(x * y) \supset I(x) \supset I(x * y) \cap I(y) \supset I(x) \cap I(y).$$

So I is hesitant fuzzy subalgebra of X. □

Remark 4.21. If X is a BCI-algebra, then Proposition 4.20 is not hold, in general.

Example 4.22 (See Example 4 in [16])**.** Let $(Y, *, 0)$ be a BCI-algebra and let $(\mathbb{Z}, -, 0)$ be the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers. Let $X = Y \times \mathbb{Z}$ and let $\otimes$ be the operation on X defined as follows: for any $(x, m),\ (y, n) \in X$,

$$(x, m) \otimes (y, n) = (x * y, m - n).$$

Then clearly, X is a BCI-algebra. for a subset $A = Y \times (\mathbb{N} \cup \{0\})$ of X, let $I : X \to P[0, 1]$ be the hesitant fuzzy set in X given by: for each $(x, m) \in X$,

$$I(x) = \begin{cases} \lambda & \text{if } (x, m) \in A \\ \phi & \text{otherwise,} \end{cases}$$

where $\lambda \in P[0, 1]$ with $\lambda \neq \phi$. Then we can easily see that I is a hesitant fuzzy ideal of X. Consider $(0, 2),\ (0, 3) \in A$. Then clearly, $(0, 2) \otimes (0, 3) = (0, -1) \notin A$. Thus

$$I((0, 2) \otimes (0, 3)) = \phi \not\supseteq \lambda = I(0, 2) \cap I(0, 3).$$

So I is not a hesitant fuzzy subalgebra of X.

Proposition 4.23. *Let I be a hesitant fuzzy ideal of X. Then I satisfies the following condition:*

$$(4.3) \qquad (\forall x,\ y,\ z \in X)((x * y) * z = 0 \Rightarrow I(x) \supset I(y) \cap I(z)).$$

Proof. Suppose $(x*y)*z = 0$, for any $x,\ y,\ z \in X$. Then by (HFI_2) and Proposition 4.6,

$$I(x * y) \supset I((x * y) * z) \cap I(z) = I(0) \cap I(z) = I(z).$$

Thus for any $x,\ y,\ z \in X$,

$$I(x) \supset I(x * y) \cap I(y) \supset I(z) \cap I(y) = I(y) \cap I(z).$$

So (4.3) holds. □

The following give conditions for a hesitant fuzzy set to be a hesitant fuzzy ideal.

Proposition 4.24. *Let $h^{\mathbf{0}} \neq I \in HS(X)$ satisfies the following conditions:*
(1) *$I(0) \supset I(x)$, for each $x \in X$,*
(2) *for any $x,\ y,\ z \in X$, if $(x * y) * z = 0$, then $I(x) \supset I(y) \cap I(z)$.*
Then I is a hesitant fuzzy ideal of X

Proof. From (II) and the condition (2), the proof is clear. □

Theorem 4.25. *Let $h^{\mathbf{0}} \neq h \in HS(X)$. Then the followings are equivalent:*
(1) *h is a hesitant fuzzy ideal of X,*
(2) *the nonempty λ-level set h_λ of h is an ideal of X, for each $\lambda \in P[0,1]$.*

Proof. Suppose (1) holds and let $\lambda \in P[0,1]$ such that $h_\lambda \neq \phi$. Then there is $x \in X$ such that $h(x) \supset \lambda$. Thus $h(0) \supset h(x) \supset \lambda$. So $0 \in h_\lambda$. Let $x,\ y \in X$ such that $x * y \in h_\lambda$ and $y \in h_\lambda$. Then $h(x * y) \supset \lambda$ and $h(y) \supset \lambda$. Thus by (HFI_2),

$$h(x) \supset h(x * y) \cap h(y) \supset \lambda.$$

Hence $x \in h_\lambda$. Therefore h_λ is an ideal of X.

Conversely, suppose (2) holds. Then clearly, $0 \in h_\lambda$. Assume that there is $a \in X$ such that $h(0) \not\supset h(a)$. Then $h(0) \not\supset \lambda$, for $\lambda = h(a) \setminus h(0)$. Thus $0 \notin h_\lambda$. This is a contradiction. So $h(0) \supset h(x)$, for each $x \in X$.

Now let $x,\ y \in X$ such that $h(x * y) = \lambda_1 \in P[0,1]$ and $h(y) = \lambda_2 \in P[0,1]$. Let $\lambda = \lambda_1 \cap \lambda_2$. Then $x * y \in h_\lambda$ and $y \in h_\lambda$. Thus by the hypothesis, $x \in h_\lambda$. So

$$h(x) \supset \lambda = \lambda_1 \cap \lambda_2 = h(x * y) \cap h(y).$$

Hence h is a hesitant fuzzy ideal of X. □

5. Hesitant fuzzy (positive) implicative and commutative ideals

Definition 5.1. Let $(X, *, 0)$ be a BCK-algebra and let $h^{\mathbf{0}} \neq I \in HS(X)$. Then I is called a hesitant fuzzy positive implicative ideal (in short, HFPII) of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

(HFI_1) $$I(0) \supset I(x),$$

(HFI_3) $$I(x * z) \supset I((x * y) * z) \cap I(y * z).$$

It is obvious that $h^{\mathbf{1}}$ is a hesitant fuzzy positive implicative ideal of X.

We have easily the following results.

Theorem 5.2. *Let A be a subset of a BCK-algebra X such that $0 \in A$ and let $\lambda,\ \mu \in P[0,1]$ be fixed such that $\lambda \supsetneqq \mu$. Then the followings are equivalent:*
(1) *A is a positive implicative ideal of X,*
(2) *h_A is a hesitant fuzzy positive implicative ideal of X,*
(3) *$[h_A]_{(\lambda,\mu)}$ is a hesitant fuzzy positive implicative ideal of X.*

Example 5.3. Let $X = \{0, a, b, c, d\}$ a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	d	d	0

Table 5.1

Consider the mappings $I_1,\ I_2 : X \to P[0,1]$ defined by:

$I_1(0) = [0, 1],\ I_1(a) = [0.1, 0.4),\ I_1(b) = I_1(c) = I_1(d) = [0.1, 0.4],$
$I_2(0) = I_2(c) = [0, 1),\ I_2(a) = (0, 0.6],\ I_2(b) = (0, 0.5),\ I_2(d) = [0, 0.6) \cup \{0.8\}.$
Then we can easily see that I_1 is a hesitant fuzzy positive implicative ideal and I_{12} is a hesitant fuzzy ideal of X. But

$$I_2(b * a) = (0, 0.5) \not\supset [0, 0.6) \cup \{0.8\} = I_2((b * d) * a) \cap I_2(d * a).$$

Thus I_2 is not a hesitant fuzzy positive implicative ideal of X. Moreover, by Theorem 5.2, we can easily check that $h_{\{0,a,c\}}$, $h_{\{0,a,b,c\}}$, $[h_{\{0,a,c\}}]_{(\lambda,\mu)}$, $[h_{\{0,a,b,c\}}]_{(\lambda,\mu)}$ are hesitant fuzzy positive implicative ideal of X, and $h_{\{0\}}$, $h_{\{0,b\}}$, $h_{\{0,b,d\}}$, $[h_{\{0\}}]_{(\lambda,\mu)}$, $[h_{\{0,b\}}]_{(\lambda,\mu)}$, $[h_{\{0,b,d\}}]_{(\lambda,\mu)}$ are hesitant fuzzy ideal but not hesitant fuzzy positive implicative ideal of X.

Proposition 5.4. *Every hesitant fuzzy positive implicative ideal is an ideal but the converse is not true.*

Proof. Suppose I is a hesitant fuzzy positive implicative ideal of X and let $x,\ y \in X$. Then

$$\begin{aligned} I(x) &= I(x * 0) \text{ [By (2.1)]} \\ &\supset I((x * y) * 0) \cap I(y * 0) \text{ [By the hypothesis]} \\ &= I(x * y) \cap I(y). \text{ [By (2.1)]} \end{aligned}$$

Thus I is a hesitant fuzzy ideal of X. Moreover, form Example 5.3, we can see that the converse is not true. □

Theorem 5.5. *Let $I \in HS(X)$ such that $I(0) = [0, 1]$. Then I is a hesitant fuzzy positive implicative ideal of X if and only if it satisfies the following condition:*

(5.1) $(\forall x_\lambda,\ y_\mu,\ z_\nu \in H_P(X))((x_\lambda \circ y_\mu) \circ z_\nu \in I \text{ and } y_\mu \circ z_\nu \in I \Rightarrow x_{\lambda\cap\mu} \circ z_\nu \in I)$.

Proof. Suppose I is a hesitant fuzzy positive implicative ideal of X and for any $x_\lambda,\ y_\mu,\ z_\nu \in H_P(X)$, let $(x_\lambda \circ y_\mu) \circ z_\nu \in I$ and $y_\mu \circ z_\nu \in I$. Since $(x_\lambda \circ y_\mu) \circ z_\nu = ((x * y) * z)_{\lambda\cap\mu\cap\nu}$ and $y_\mu \circ z_\nu = (y * z)_{\mu\cap\nu}$. Thus $I((x * y) * z) \supset \lambda \cap \mu \cap \nu$ and $I(y * z) \supset \mu \cap \nu$. By the hypothesis, $I(x * z) \supset I((x * y) * z) \cap I(y * z)$. So $I(x*z) \supset (\lambda \cap \mu) \cap \nu$, i.e., $(x*z)_{(\lambda\cap\mu)\cap\nu} \in I$. Since $(x*z)_{(\lambda\cap\mu)\cap\nu} = x_{\lambda\cap\mu} \circ z_\nu$. Hence $x_{\lambda\cap\mu} \circ z_\nu \in I$.

Conversely, suppose the necessary condition (5.1) and for any $x,\ y,\ z \in X$, let $I((x * y) * z) = \delta,\ I(y * z) = \eta$. Then there are $\lambda,\ \mu,\ \nu \in P[0, 1]$ such that $(x_\lambda \circ y_\mu) \circ z_\nu = ((x * y) * z)_{\lambda\cap\mu\cap\nu} = ((x * y) * z)_\delta$ and $y_\mu \circ z_\nu = (y * z)_{\mu\cap\nu} = (y * z)_\eta$. Since $((x * y) * z)_\delta \in I$ and $(y * z)_\eta \in I$, $(x_\lambda \circ y_\mu) \circ z_\nu \in I$ and $y_\mu \circ z_\nu \in I$. Thus by the condition (5.1), $x_{\lambda\cap\mu} \circ z_\nu \in I$. It is clear that $x_{\lambda\cap\mu} \circ z_\nu = (x * z)_{\lambda\cap\mu\cap\nu}$. So

$$I(x * z) \supset \lambda \cap \mu \cap \nu = \delta \cap \eta = I((x * y) * z) \cap I(y * z).$$

Since $I(0) = [0, 1]$, it is obvious that $I(0) \supset I(x)$, for each $x \in X$. Hence I is a hesitant fuzzy positive implicative ideal of X. □

Theorem 5.6. *Let X be a BCK-algebra and let $h^{\mathbf{0}} \neq I \in HS(X)$. Then the followings are equivalent: for any $x,\ y,\ z \in X$,*

(1) *I is a hesitant fuzzy positive implicative ideal of X,*
(2) *I is a hesitant fuzzy ideal of X and $I(x * y) \supset I((x * y) * y)$,*
(3) *I is a hesitant fuzzy ideal of X and $I((x * z) * (y * z)) \supset I((x * y) * z)$,*
(4) *$I(0) \supset I(x)$ and $I(x * y) \supset I(((x * y) * y) * z) \cap I(z)$.*

Proof. (1)⇒(2): Suppose I is a hesitant fuzzy positive implicative ideal of X. Then by Proposition 5.4, I is a hesitant fuzzy ideal of X. Let $x,\ y \in X$. Then

$$\begin{aligned} I(x*y) &\supset I((x*y)*y) \cap I(y*y) \text{ [By the hypothesis]} \\ &= I((x*y)*y) \cap I(0) \text{ [By (III)]} \\ &= I((x*y)*y). \text{ [By } (HFI_1)] \end{aligned}$$

Thus (2) holds.

(2)⇔(3): From Theorem 4.19, it is obvious.

(3)⇒(4): Suppose (3) holds. Then clearly, $I(0) \supset I(x)$, for each $x \in X$. Let $x,\ y,\ z \in X$. Then

$$\begin{aligned} I(((x*y)*y)*z) \cap I(z) &\subset [I((x*y)*y) \cap I(z)] \cap I(z) \text{ [By Proposition 4.20]} \\ &= [I(x*y) \cap I(y)] \cap I(z) \text{ [By Proposition 4.20]} \\ &= I(x*y) \cap [I(y)] \cap I(z)] \\ &\subset I(x*y). \end{aligned}$$

Thus (4) holds.

(4)⇒(1): Suppose (4) holds and let $x,\ y \in X$. Then by the second condition of (4), $I(((x*0)*0)*y) \subset I(0*x)$. Thus by (2.1), $I(x*y) \subset I(x)$. By the first condition of (4), $I(0) \supset I(x)$, for each $x \in X$. So I is a hesitant fuzzy ideal of X.

Now let $x,\ y,\ z \in X$. Then

$$\begin{aligned} I(x*z) &\supset I(((x*z)*z)*(y*z)) \text{ [By the condition (4)]} \\ &\supset I((x*z)*z)) \cap I(y*z). \text{ [By Proposition 4.20]} \end{aligned}$$

Thus I is a hesitant fuzzy positive implicative ideal of X. □

Theorem 5.7. *Let $h^{\mathbf{0}} \neq I \in HS(X)$. Then I is a hesitant fuzzy positive implicative ideal of X if and only if I_λ is a positive implicative ideal of X, for each $\lambda \in P[0,1]$ such that $I(0) \supset \lambda$.*

Proof. Suppose I is a hesitant fuzzy positive implicative ideal of X. Then from the proof of Theorem 4.14, it is clear that $0 \in I_\lambda$. For any $x,\ y,\ z \in X$, suppose $(x*y)*z \in I_\lambda$ and $y*z \in I_\lambda$. Then clearly, $I((x*y)*z) \supset \lambda$ and $I(y*z) \supset \lambda$. Thus by (HFI_3), $I(x*z) \supset I((x*y)*z) \cap I(x*y) \supset \lambda$. So $x*z \in I_\lambda$. Hence I_λ is a positive implicative of X.

Conversely, suppose I_λ is a positive implicative ideal of X, for each $\lambda \in P[0,1]$ such that $I(0) \supset \lambda$. Then from the proof of Theorem 4.14, it is obvious that $I(0) \supset \mu = I(x)$. For any $x,\ y,\ z \in X$, let $I((x*y)*z) = \lambda$ and $I(x*y) = \mu$, say $\lambda \subset \mu$. Then clearly, $(x*y)*z \in I_\mu$ and $x*y \in I_\mu$. Thus by the hypothesis, $x*z \in I_\mu$. So $I(x*z) \supset \mu \supset I((x*y)*z) \cap I(x*y)$. Hence I is a hesitant fuzzy positive implicative ideal of X. □

Definition 5.8. Let $(X,*,0)$ be a BCK-algebra and let $h^{\mathbf{0}} \neq I \in HS(X)$. Then I is called a hesitant fuzzy implicative ideal (in short, HFII) of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

(HFI_1) $$I(0) \supset I(x),$$

(HFI_4) $$I(x) \supset I((x*(y*x))*z) \cap I(z).$$

It is obvious that $h^{\mathbf{1}}$ is a hesitant fuzzy implicative ideal of X, which will be called the trivial hesitant fuzzy implicative ideal.

We have easily the following results.

Theorem 5.9. *Let A be a subset of a BCK-algebra X such that $0 \in A$ and let $\lambda,\ \mu \in P[0,1]$ be fixed such that $\lambda \supsetneqq \mu$. Then the followings are equivalent:*
(1) *A is an implicative ideal of X,*
(2) *h_A is a hesitant fuzzy implicative ideal of X,*
(3) *$[h_A]_{(\lambda,\mu)}$ is a hesitant fuzzy implicative ideal of X.*

Example 5.10. Let $X = \{0, a, b, c, d\}$ a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	0
c	c	c	c	0	0
d	d	d	d	d	0

Table 5.2

Consider the mapping $I : X \to P[0,1]$ defined by:
$I(0) = [0,1],\ I(a) = (0,0.9],\ I(b) = [0,0.8],\ I(c) = [0.1,0.8],\ I(d) = (0.1,0.7].$
Then we can easily see that I is a hesitant fuzzy implicative ideal of X. Furthermore, by Theorem 5.9, we can easily check that $h_{\{0,a,b,c\}}$ and $[h_{\{0,a,b,c\}}]_{(\lambda,\mu)}$ are hesitant fuzzy ideals but not hesitant fuzzy implicative ideals of X.

Theorem 5.11. *Let $h^{\mathbf{0}} \neq I \in HS(X)$ such that $I(0) = [0,1]$. Then I is a hesitant fuzzy implicative ideal of X if and only if it satisfies the following condition:*

(5.2) $(\forall x_\lambda,\ y_\mu,\ z_\nu \in H_P(X))((x_\lambda \circ (y_\mu \circ x_\lambda)) \circ z_\nu \in I$ and $z_\nu \in I \Rightarrow x_{\lambda\cap\mu\cap\nu} \in I)$.

Proof. Suppose I is a hesitant fuzzy implicative ideal of X and for any $x_\lambda,\ y_\mu,\ z_\nu \in H_P(X)$, let $(x_\lambda \circ (y_\mu \circ x_\lambda)) \circ z_\nu \in I$ and $z_\nu \in I$. Since

$$(x_\lambda \circ (y_\mu \circ x_\lambda)) \circ z_\nu = (x * (y * x)) * z)_{\lambda\cap\mu\cap\nu},$$

$I((x * (y * x)) * z) \supset \lambda \cap \mu \cap \nu$ and $I(z) \supset \nu$. Thus
$I(x) \supset I((x * (y * x)) * z) \cap I(z)$ [By (HFI_4)]
$\supset \lambda \cap \mu \cap \nu$.
So $x_{\lambda\cap\mu\cap\nu} \in I$. Hence the condition (5.2) holds.

Conversely, suppose the necessary condition (5.2) and for any $x,\ y,\ z \in X$, let $I((x * (y * x)) * z) = \delta,\ I(z) = \eta$. Then there are $\lambda,\ \mu,\ \nu \in P[0,1]$ such that $(x_\lambda \circ (y_\mu \circ x_\lambda)) \circ z_\nu = ((x * (y * x)) * z)_{\lambda\cap\mu\cap\nu} = ((x * (y * x)) * z)_\delta$ and $I(z) = \eta = \nu$. Since $((x * (y * x)) * z)_\delta \in I$, $(x_\lambda \circ (y_\mu \circ x_\lambda)) \circ z_\nu \in I$ and $z_\eta \in I$. Thus by the condition (5.2), $x_{\lambda\cap\mu} \circ z_\nu \in I$. So

$$I(x) \supset \lambda \cap \mu \cap \nu = \delta \cap \eta = I((x * y) * z) \cap I(y * z).$$

Since $I(0) = [0,1]$, it is obvious that $I(0) \supset I(x)$, for each $x \in X$. Hence I is a hesitant fuzzy implicative ideal of X. □

Proposition 5.12. *Let X be a implicative BCK-algebra. Then every hesitant fuzzy ideal is a hesitant fuzzy implicative ideal.*

Proof. Let I be a hesitant fuzzy ideal of X and let $x,\ y,\ z \in X$. Then

$$I(((x*(y*x))*z)\cap I(z) = I(x*z)\cap I(z) \text{ [By (2.8)]}$$
$$\subset I(x). \text{ [By } (HFI_2)]$$

Thus I is a hesitant fuzzy implicative ideal. □

Proposition 5.13. *Every hesitant fuzzy implicative ideal is a hesitant fuzzy ideal, but the converse is not true.*

Proof. Let I be a hesitant fuzzy implicative ideal and let $x,\ y \in X$. Then

$$I(x) \supset I((x*(x*x))*y)\cap I(y) \text{ [By the condition } (HFI_4)]$$
$$= I(x*y)\cap I(y). \text{ [By (III) and (2.1)]}$$

Thus I is a hesitant fuzzy ideal.

The converse is shown by the following example. □

Example 5.14. Let $X = \{0, a, b, c, d\}$ a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	d	d	0

Table 5.3

Consider the mappings $I_1,\ I_2 : X \to P[0,1]$ defined by:

$I_1(0) = [0,1],\ I_1(b) = [0,0.8],\ I_1(a) = I_1(c) = I_1(d) = \phi,$

$I_2(0) = [0,1],\ I_2(a) = I_2(c) = [0.1,0.8),\ I_2(b) = I_2(d) = \phi.$

Then we can easily see that I_1 is a hesitant fuzzy ideal and I_2 is a hesitant fuzzy positive implicative ideal of X, respectively. But

$I_1(a) = \phi \not\supseteq [0,1] = I_1(((a*(b*a))*0)\cap I_1(0),$

$I_2(b) = \phi \not\supseteq [0,1] = I_2(((b*(c*b))*0)\cap I_2(0).$

Thus I_1 and I_2 are not hesitant fuzzy implicative ideal of X. Furthermore, by Theorem 5.9, we can easily check that $h_{\{0,b\}}$ and $[h_{\{0,b\}}]_\lambda,\mu)$ are hesitant fuzzy ideals but not hesitant fuzzy positive implicative ideal of X.

Proposition 5.15. *A hesitant fuzzy implicative ideal is positive implicative, but a hesitant fuzzy positive implicative ideal need not implicative*

Proof. Suppose I be a hesitant fuzzy implicative ideal of X and let $x,\ y,\ z \in X$. Then by (I) and (2.3),

$$((x*z)*z)*(y*z) \leq (x*z)*y = (x*y)*z. \tag{5.3}$$

Thus by Proposition 5.4 and Corollary 4.17,

$$I(((x*z)*z)*(y*z)) \supset I((x*z)*y) = I((x*y)*z). \tag{5.4}$$

Thus I is a hesitant fuzzy ideal, by (HFI_2),

$$I((x*z)*z) \supset I(((x*z)*z)*(y*z))\cap I(y*z). \tag{5.5}$$

By (2.3) and (2.5),

$$(5.6)\qquad (x*z)*(x*(x*z)) = (x*(x*(x*z)))*z = (x*z)*z.$$

So

$I(x*z) \supset I(((x*z)*(x*(x*z))*0) \cap I(0)$ [By (HFI_4)]
$= I(((x*z)*(x*(x*z))$ [By (HFI_1)]
$= I((x*z)*z)$ [By (5.6)]
$\supset I(((x*z)*z)*(y*z)) \cap I(y*z)$ [By (5.5)]
$\supset I((x*y)*z) \cap I(y*z)$. [By (5.5)]

Hence $I(x*z) \supset I((x*y)*z) \cap I(y*z)$. Therefore I is a hesitant fuzzy positive implicative ideal of X.

From Example 5.14, we can see that I_2 is a hesitant fuzzy positive implicative ideal of X but not implicative. □

Theorem 5.16. *Let I be a hesitant fuzzy ideal of X. Then I is implicative if and only if for any $x,\ y \in X$,*

$$(5.7)\qquad I(x) \supset I(x*(y*x)).$$

Proof. Suppose I is implicative and let $x,\ y \in G$. Then
$I(x) \supset I((x*(y*x))*x) \cap I(x)$ [By (HFI_4)]
$\supset I(x*(y*x)) \cap I(x)$. [Since I is a hesitant fuzzy algebra of X]

Thus $I(x) \supset I(x*(y*x))$.

Conversely, suppose the necessary condition (5.7) holds and let $x,\ y,\ z \in X$. Then
$I(x) \supset I(x*(y*x))$ [By the hypothesis]
$\supset I((x*(y*x))*z) \cap I(z)$. [By ($HFI_2$)]

Thus I is implicative. This completes the proof. □

Theorem 5.17. *Let I be a hesitant fuzzy ideal of X. Then I is implicative if and only if for any $x,\ y \in X$,*

$$(5.8)\qquad I(x*(x*y)) \supset I(y*(y*x)).$$

Proof. Suppose I is implicative and let $x,\ y \in G$. Then we can see that the following holds (See the proof of Theorem 4.4 in 75 page of [15]):

$$(5.9)\qquad (x*(x*y))*(y*(x*(x*y))) \leq y*(y*x).$$

Thus by Proposition 5.4 and Corollary 4.17,

$$(5.10)\qquad I(x*(x*y))*(y*(x*(x*y))) \supset I(y*(y*x)).$$

So

$I(x*(x*y)) \supset I(((x*(y*x))*(y*(x*(x*y)))*0) \cap I(0)$ [By (HFI_4)]
$\supset I((x*(y*x))*(y*(x*(x*y))) \cap I(0)$
[By Proposition 4.19 and Definition 4.1]
$= I((x*(y*x))*(y*(x*(x*y)))$ [By (HFI_1)]
$\supset I(y*(y*x))$. [By (5.10)]

Thus $I(x*(x*y)) \supset I(y*(y*x))$.

Conversely, suppose the necessary condition (5.8) holds and let $x,\ y,\ z \in X$. Since I is a hesitant fuzzy ideal of X,

$$I(x*(y*x)) \supset I((x*(y*x))*z) \cap I(z). \tag{5.11}$$

Moreover, from the proof of Theorem 3.4 in 75 page of [15], we can see the followings hold:

$$(y*(y*x))*(y*x) \leq x*(y*x), \tag{5.12}$$

$$(x*y)*z \leq x*y \leq x*(y*x). \tag{5.13}$$

Then by Proposition 5.4 and Corollary 4.17,

$$I((y*(y*x))*(y*x)) \supset I(x*(y*x)), \tag{5.14}$$

$$I((x*y)*z) \supset I(x*y) \supset I(x*(y*x)). \tag{5.15}$$

Since I is a hesitant fuzzy positive implicative ideal of X, by Theorem 5.5 (2),

$$I(y*(y*x)) \supset I((y*(y*x))*(y*x)). \tag{5.16}$$

Thus $I(x) \supset I(x*(x*y)) \cap I(x*y)$ [By Proposition 5.4 and (HFI_2)]
$\supset I(y*(y*x)) \cap I(x*y)$ [By the condition (5.8)]
$\supset I((y*(y*x))*(y*x)) \cap I(x*y)$ [By (5.16)]
$\supset I(x*(y*x)) \cap I(x*y)$ [By (5.14)]
$= I(x*(y*x))$ [By (5.15)]
$\supset I((x*(y*x))*z) \cap I(z)$. [By (5.11)]
So I is implicative. This completes the proof. □

Theorem 5.18. *Let $h^{\mathbf{0}} \neq I \in HS(X)$. Then I is a hesitant fuzzy implicative ideal of X if and only if I_λ is an implicative ideal of X, for each $\lambda \in P[0,1]$ such that $I(0) \supset \lambda$.*

Proof. Suppose I is a hesitant fuzzy implicative ideal of X. Then from the proof of Theorem 4.14, it is clear that $0 \in I_\lambda$. For any $x,\ y,\ z \in X$, suppose $(x*(y*x))*z \in I_\lambda$ and $z \in I_\lambda$. Then clearly, $I((x*(y*x))*z) \supset \lambda$ and $I(z) \supset \lambda$. Thus by (HFI_4), $I(x) \supset I((x*(y*x))*z) \cap I(z) \supset \lambda$. So $x \in I_\lambda$. Hence I_λ is an implicative ideal of X.

Conversely, suppose I_λ is a implicative ideal of X, for each $\lambda \in P[0,1]$ such that $I(0) \supset \lambda$. Then from the proof of Theorem 4.14, it is obvious that $I(0) \supset \mu = I(x)$. For any $x,\ y,\ z \in X$, let $I((x*(y*x))*z) = \lambda$ and $I(z) = \mu$, say $\lambda \subset \mu$. Then clearly, $(x*(y*x))*z \in I_\mu$ and $z \in I_\mu$. Thus by the hypothesis, $x \in I_\mu$. So $I(x) \supset \mu \supset I((x*(y*x))*z) \cap I(z)$. Hence I is a hesitant fuzzy implicative ideal of X. □

Definition 5.19. Let $(X, *, 0)$ be a BCK-algebra and let $h^{\mathbf{0}} \neq I \in HS(X)$. Then I is called a hesitant fuzzy commutative ideal (in short, HFCI) of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

$$I(0) \supset I(x), \tag{HFI_1}$$

$$I(x*(y*(y*x))) \supset I((x*y)*z) \cap I(z). \tag{HFI_5}$$

It is obvious that $h^{\mathbf{1}}$ is always a hesitant fuzzy commutative ideal of X, which will be called the hesitant fuzzy trivial commutative ideal.

We have easily the following result.

Theorem 5.20. *Let A be a subset of a BCK-algebra X such that $0 \in A$ and let $\lambda,\ \mu \in P[0,1]$ be fixed such that $\lambda \supsetneqq \mu$. Then the followings are equivalent:*
(1) *A is a commutative ideal of X,*
(2) *h_A is a hesitant fuzzy commutative ideal of X,*
(3) *$[h_A]_{(\lambda,\mu)}$ is a hesitant fuzzy commutative ideal of X.*

Example 5.21. Let I_1 and I_2 be the hesitant fuzzy sets in X given in Example 5.14. Then we can easily see that I_1 is a hesitant fuzzy commutative ideal and I_2 is a hesitant fuzzy positive implicative ideal. But
$$I_1(a*b) = I_1(a) = \phi \not\supseteq [0,0.8] = I_1((a*c)*b) \cap I_1(b),$$
$$I_2(b*(c*(c*b))) = I_2 = \phi \not\supseteq [0,1] = I_2((b*c)*0) \cap I_2(0).$$
Thus I_1 is not positive implicative and I_2 is not commutative. Furthermore, by Theorem 5.20, we can easily check that $h_{\{0,b\}}$, $h_{\{0,b,d\}}$, $[h_{\{0,b\}}]_{(\lambda,\mu}$, $[h_{\{0,b,d\}}]_{(\lambda,\mu}$ are hesitant fuzzy commutative ideals but not hesitant fuzzy positive implicative ideals; $h_{\{0,a,c\}}$ and $[h_{\{0,a,c\}}]_{(\lambda,\mu)}$ are hesitant fuzzy positive implicative ideals but not hesitant fuzzy commutative ideals; $h_{\{0,a,b,c\}}$ and $[h_{\{0,a,b,c\}}]_{(\lambda,\mu)}$ are hesitant fuzzy positive ideals.

Proposition 5.22. *A hesitant fuzzy commutative ideal is a hesitant fuzzy ideal but the converse does not hold.*

Proof. Suppose I is a hesitant fuzzy commutative ideal of X and let $x,\ y \in X$. Then
$$\begin{aligned} I(x) &= I(x*(0*(0*x))) \text{ [By (V)]} \\ &\supset I((x*0)*y) \cap I(y) \text{ [By the hypothesis]} \\ &= I(x*y)]capI(y). \text{ [By (2.1)]} \end{aligned}$$
Thus I is a hesitant fuzzy ideal of X.

From Example 5.18, we can easily see that the converse does not hold. □

Theorem 5.23. *Let I be a hesitant fuzzy ideal of X. Then I is commutative if and only if for any $x,\ y \in X$,*
$$I(x*(y*(y*x))) \supset I(x*y). \tag{5.17}$$

Proof. Suppose I is commutative and let $x,\ y \in X$. Then
$$\begin{aligned} I(x*(y*(y*x))) &\supset I((x*y)*0) \cap I(0) \text{ [By the hypothesis]} \\ &= I(x*y) \cap I(0) \text{ [By Proposition 4.20 and Definition 4.1]} \\ &= I(x*y). \text{ [By } (HFI_1)] \end{aligned}$$
Thus the condition (5.17) holds.

Conversely, suppose the condition (5.17) holds and let $x,\ y,\ z \in X$. Since I is a hesitant fuzzy ideal of X,
$$I(x*y) \supset I((x*y)*z) \cap I(z). \tag{5.18}$$
Thus
$$\begin{aligned} I(x*(y*(y*x))) &\supset I(x*y) \text{ [By the condition (5.17)]} \\ &\supset I((x*y)*z) \cap I(z). \text{ [By (5.18)]} \end{aligned}$$
So I is commutative. □

Theorem 5.24. *Let X be a BCK-algebra and let $\phi \neq I \in HS(X)$. Then I is a hesitant fuzzy implicative ideal if and only if it is both commutative and positive implicative.*

Proof. Suppose I is a hesitant fuzzy implicative ideal. Then by Proposition 5.15, I is positive implicative. Let $x,\ y,\ z \in X$ and let $u = x * (y * (y * x))$. Then from the proof of Theorem 4.4 in 78 page of [15], we can see the followings hold:

$$(u * (y * u)) \leq x * y. \tag{5.19}$$

Thus by Corollary 4.17,

$$I((u * (y * u))) \supset I(x * y). \tag{5.20}$$

Since I is implicative, by Theorem 5.16,

$$I(u) \supset I(u * (y * u)). \tag{5.21}$$

By (5.21) and (5.20),

$$I(x * (y * (y * x))) = I(u) \supset I(u * (y * u)) \supset I(x * y).$$

So the condition (5.17) holds. Hence by Theorem 5.23, I is commutative.

Conversely, suppose the necessary conditions hold and let $x,\ y \in X$. Then we have the following inequalities:

$$(y * (y * x)) * (y * x) \leq x * (y * x),\ x * y \leq x * (y * x). \tag{5.22}$$

Thus by Corollary 4.17,

$$I((y * (y * x)) * (y * x)) \supset I(x * (y * x)),\ I(x * y) \supset I(x * (y * x)). \tag{5.23}$$

Since I is commutative, by the condition (5.18),

$$I(x * (y * (y * x))) \supset I(x * y). \tag{5.24}$$

So

$I(x) \supset I(x * (y * (y * x))) \cap I(y * (y * x))$ [Since I is a hesitant fuzzy ideal]
$\supset I(x * y) \cap I(y * (y * x))$ [By the condition (5.24)]
$\supset I(x * y) \cap I((y * (y * x)) * (y * x))$ [By Theorem 5.6 (2)]
$\supset I(x * (y * x))$. [By (5.23)]

Hence by Theorem 5.14, I is implicative. □

Theorem 5.25. *Let $h^{\mathbf{0}} \neq I \in HS(X)$. Then I is a hesitant fuzzy commutative ideal of X if and only if I_λ is a commutative ideal of X, for each $\lambda \in P[0,1]$ such that $I(0) \supset \lambda$.*

Proof. Suppose I is a hesitant fuzzy implicative ideal of X. Then from the proof of Theorem 4.14, it is clear that $0 \in I_\lambda$. For any $x,\ y,\ z \in X$, suppose $(x * y) * z \in I_\lambda$ and $z \in I_\lambda$. Then clearly, $I((x * y) * z) \supset \lambda$ and $I(z) \supset \lambda$. Thus by (HFI_5), $I(x * (y * (y * x))) \supset I((x * y) * z) \cap I(z) \supset \lambda$. So $x * (y * (y * x)) \in I_\lambda$. Hence I_λ is a commutative ideal of X.

Conversely, suppose I_λ is a commutative ideal of X, for each $\lambda \in P[0,1]$ such that $I(0) \supset \lambda$. Then from the proof of Theorem 4.14, it is obvious that $I(0) \supset \mu = I(x)$. For any $x,\ y,\ z \in X$, let $I((x * y) * z) = \lambda$ and $I(z) = \mu$, say $\lambda \subset \mu$. Then clearly, $(x * y) * z \in I_\mu$ and $z \in I_\mu$. Thus by the hypothesis, $x * (y * (y * x)) \in I_\mu$. So

$I(x*(y*(y*x))) \supset \mu \supset I((x*y)*z) \cap I(z)$. Hence I is a hesitant fuzzy commutative ideal of X. □

Theorem 5.26. *Let $h^{\mathbf{0}} \neq I \in HS(X)$ such that $I(0) = [0,1]$ and let $\lambda \in P[0,1]$ be fixed. Then I is a hesitant fuzzy commutative ideal of X if and only if it satisfies the following condition:*
(5.25)
$(\forall x_\lambda,\ y_\lambda,\ z_\lambda \in [H_P(X)]_\lambda)((x_\lambda \circ y_\lambda) \circ z_\lambda \in I$ and $z_\lambda \in I \Rightarrow x_\lambda \circ (y_\lambda \circ (y_\lambda \circ x_\lambda)) \in I)$.

Proof. Suppose I is a hesitant fuzzy implicative ideal of X and for any $x_\lambda,\ y_\lambda,\ z_\lambda \in [H_P(X)]_\lambda$, let $(x_\lambda \circ y_\lambda) \circ z_\lambda \in I$ and $z_\lambda \in I$. Since

$$(x_\lambda \circ y_\lambda) \circ z_\lambda = ((x*y)*z)_\lambda,$$

$I((x*y)*z) \supset \lambda$ and $I(z) \supset \nu$. Thus by (HFI_5),

$$I(x*(y*(y*x))) \supset I((x*y)*z) \cap I(z) \supset \lambda.$$

So $(x*(y*(y*x)))_\lambda \in I$ Hence the condition (5.25) holds.

Conversely, suppose the necessary condition (5.25) and for any $x,\ y,\ z \in X$, let $I((x*(y*x))*z) = I(z) = \lambda$. Then clearly, $((x*(y*x))*z)_\lambda \in I$ and $z_\lambda \in I$. Since $(x_\lambda \circ y_\lambda) \circ z_\lambda = ((x*y)*z))_\lambda$, $(x_\lambda \circ y_\lambda) \circ z_\lambda \in I$ and $z_\lambda \in I$. Thus by the condition (5.25), $x_\lambda \circ (y_\lambda \circ (y_\lambda \circ x_\lambda)) \in I$, i.e., $x*(y*(y*x)) \in I$. So

$$I(x*(y*(y*x))) \supset \lambda = I((x*(y*x))*z) \cap I(z).$$

Since $I(0) = [0,1]$, it is obvious that $I(0) \supset I(x)$, for each $x \in X$. Hence I is a hesitant fuzzy commutative ideal of X. □

6. Conclusions

We proved that $([H_P(X)]_\lambda, \circ, 0_\lambda)$ forms a BCK/BCI-algebra (will be called a hesitant fuzzy BCK/BCI-algebra with the value λ induced by X) for a given BCK/BCI-algebra $(X,*,0)$ and a fixed $\lambda \in P[0,1]$. We introduced the concepts of a hesitant fuzzy algebra, a hesitant fuzzy ideal, a hesitant fuzzy posiyive implicative ideal, a hesitant fuzzy implicative ideal and a hesitant fuzzy commutative ideal and obtained some of its properties, respectively. In particular, we gave each characterizations (See Theorems 4.4, 4.14, 5.6, 5.17 and 5.24). In the future, we will try to deal with hesitant fuzzy congruences in BCK/BCI-algebras.

Funding: This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2018R1D1A1B07049321)

References

[1] Halimah Alshehri and Noura Alshehri, Hesitant anti-fuzzy soft set in BCK-algebras, Mathematical Problems in Engineering 2017, Article ID 3634258, 13 pages. Carlin.
343–351.
[2] D. Divakaran and Sunil Jacob John Hesitant fuzzy rough sets through hesitant fuzzy relations, Ann. Fuzzy Math. Inform. 8 (1) (2014) 33–46.
[3] D. Divakaran and Sunil Jacob John Homomorphism of hesitant fuzzy subgroups, International Journal of Scientific and Engineering Research 5 (9) (2014) 9–14.

[4] D. Divakaran and Sunil Jacob John Hesitant fuzzy subgroups, Journal of New Theory 11 (2016) 54–68.
[5] D. Divakaran and Sunil Jacob John Dual hesitant fuzzy subrings and ideals, Ann. Fuzzy Math. Inform. 13 (3) (2017) 437–448.
[6] D. Divakara and M. Mashinchi Hesitant L-fuzzy relations, 2018 6th Iranian Joint Congress on Fuzzy and Intelligent Systems (CFIS) 102–104.
[7] Y. B. Jun and S. S. Ahn Hesitant fuzzy set theory applied to BCK/BCI-algebras, J. Computationa Analysis and Applications 20 (4) (2016) 635–646.
[8] Y. B. Jun, K. J. Lee and S. Z. Song Hesitant fuzzy bi-ideals in semigroups, Commun. Korean Math. Soc. 30 (3) (2015) 143–154.
[9] Y. S. Huang, BCI-algebra, Beijing: Science Press 2006.
[10] K. Iséki, On BCI-algebras, Math. Seminar Notes 8 (1980), 125–130.
[11] K. Iséki and S. Tanaka, An introduction to the theory of BCK-algebras, Math. Japon. 23 (1978), 1–26.
[12] J. Kim, Y. B. Jun, P. K. Lim, J. G. Lee, K. Hur, The category of hesitant H-fuzzy sets, Ann. Fuzzy Math. Inform. 18 (1) (2019) 437–448.
[13] J. Kim, P. K. Lim, J. G. Lee, K. Hur, Hesitant fuzzy subgroups and subrings, Ann. Fuzzy Math. Inform. 18 (2) (2019) 105–122.
[14] R. Krishankumar, K. S. Ravichandran, K. K. Murthy and A. Borumand Saeid, A scientific decision-making framework for supplier outsourcing using hesitant fuzzy information, Soft Computing 22 (2018) 7445–7461.
[15] J. Meng and Y. B. Jun, BCK-algebras, Kyungmoon Sa Co. Seoul 1996
[16] G. Muhiuddin and Shuaa Aldhafeeri, Subalgebras and ideals in BCK/BCI-algebras based on uni-hesitant fuzzy set theory, Eur. J. Pure Appl. Math. 11 (2) (2018) 417–430.
[17] A. Rezaei and A. Borumand Saeid, Hesitant fuzzy filters in BE-algebras, International Journal of Computational Intelligence Systems 9 (1) (2016) 110-119.
[18] Zheng Pei and Liangzhong Yi A note on operations of hesitant fuzzy set, International Journal of Computational Intelligence Systems 8 (2) (2015) 226–239.
[19] A. Solariaju and S. Mahalakshmi Hesitant intuitionistic fuzzy soft groups, International Journal of Pure and Applied Mathematics 118 (10 (2018) 223–232.
[20] G. S. Thakur, Rekha Thakur and Ravi Singh New Hesitant fuzzy operatos, Fuzzy Inf. Eng. 6 (2014) 379–392.
[21] V. Torra, Hesitant fuzzy sets, International Journal of Intelligent Systems 25 (2010) 529–539.
[22] M. Xia and Z. Xu Hesitant fuzzy information aggregation in decision making, Internat. J. Approx. Reason. 52 (3) (2011) 395–407.
Fuzzy sets, Information and Control 8 (1965) 338–353.

J. Kim (junhikim@wku.ac.kr)
Department of Mathematics Education, Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea

P. K. Lim (pklim@wku.ac.kr)
Department of Applied Mathematics, Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea

J. G. Lee (jukolee@wku.ac.kr)
Department of Applied Mathematics, Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea

K. Hur (kulhur@wku.ac.kr)
Department of Applied Mathematics, Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea