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Abstract. In the present paper, we continue the study on generalized fuzzy soft topological spaces. We introduce the notion of separation axioms T_i $(i = 0, 1, 2, 3, 4)$ in generalized fuzzy soft topological spaces and study some of its properties. By using this notions, we also give some basic theorems of separation axioms in generalized fuzzy soft topological spaces. Finally, we discuss hereditary property and topological property in generalized fuzzy soft topological spaces.

2010 AMS Classification: 54A40, 54D10, 54D15

Keywords: Soft set, Fuzzy soft set, Generalized fuzzy soft set, Generalized fuzzy soft topology, Generalized fuzzy soft separation axioms, Generalized fuzzy soft T_i-spaces $(i = 0, 1, 2)$, Generalized fuzzy soft regular, Generalized fuzzy soft normal, Generalized fuzzy soft topological property.

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1. Introduction

The concept of soft sets was first introduced by Molodtsov [19] as a general mathematical tool for dealing with uncertain objects. Cagman et al. [4] and Shabir et al. [23] introduced soft topological space independently. Moreover, many authors studied soft topology and its applications [3, 10, 11, 18, 21, 22, 23, 24, 28]. Maji et al. [16] introduced the concept of fuzzy soft set and some of its properties. Tanay and Kandemir [26] introduced the definition of fuzzy soft topology over a subset of the initial universe set. Later, Roy and Samanta [20] gave the definition of fuzzy soft topology over the initial universe set. Ahmed and Karal [1] defined the notion of a mapping on classes of fuzzy soft sets In 2010, Majumdar and Samanta [17] introduced the notion of generalized fuzzy soft set as a generalization of fuzzy soft sets and some of its basic properties. Khedr et al. [12] introduced the concept of a generalized fuzzy soft point, a generalized fuzzy soft base (subbase) and a generalized fuzzy soft subspace. The concept of separation axioms is one of the most important concepts in topological spaces. In fuzzy setting, it had been studied by many authors such as: Das et al.[6], Saha et al. [7] and Hutton et al. [9]. In soft setting, it has been studied by Shabir et al. [25] and Gocur et al. [8]. In fuzzy soft topological space has been studied by Mahanta et al. [15] Khedr et al. [14] and Atmaca. [2]

In our present article, we introduce generalized fuzzy soft separation axioms $T_i(i = 0, 1, 2, 3, 4)$. By using this notions, we also give some basic theorems which are important for separation axioms and taking place in classical topological spaces.

2. Preliminaries

In this section, we will give some fundamental definitions and theorems about generalized fuzzy soft sets, generalized fuzzy soft topology and generalized fuzzy soft continuous mappings which will be needed in the sequel.

Definition 2.1 ([27])**.** Let X be a non-empty set. A fuzzy set A in X is defined by a membership function $\mu_A : X \to [0, 1]$ whose value $\mu_A(x)$ represents the 'grade of membership' of x in A for $x \in X$. The set of all fuzzy sets in a set X is denoted by I^X, where I is the closed unit interval $[0, 1]$

Definition 2.2 ([19])**.** Let X be an initial universe set and E be a set of parameters. Let $P(X)$ denotes the power set of X and $A \subseteq E$. A pair (f, A) is called a soft set over X, if f is a mapping from A into $P(X)$, i.e., $f : A \to P(X)$. In other words, a soft set is a parameterized family of subsets of the set X. For $e \in A$, $f(e)$ may be considered as the set of $e-$approximate elements of the soft set (f, A).

Definition 2.3 ([20])**.** Let X be an initial universe set and E be a set of parameters. Let $A \subseteq E$. A fuzzy soft set f_A over X is a mapping from E to I^X, i.e., $f_A : E \to I^X$, where $f_A(e) \neq \bar{0}$ if $e \in A \subset E$ and $f_A(e) = \bar{0}$ if $e \notin A$, where $\bar{0}$ is denotes empty fuzzy set in X.

Definition 2.4 ([17])**.** Let X be a universal set of elements and E be a universal set of parameters for X. Let $F : E \to I^X$ and μ be a fuzzy subset of E, i.e., $\mu : E \to I$. Let F_μ be the mapping $F_\mu : E \to I^X \times I$ defined as follows: $F_\mu(e) = (F(e), \mu(e))$, where $F(e) \in I^X$ and $\mu(e) \in I$. Then F_μ is called a generalized fuzzy soft set ($GFSS$, in short) over (X, E). The family of all generalized fuzzy soft sets ($GFSSs$, in short) over (X, E) is denotes by $GFSS(X, E)$.

Definition 2.5 ([17])**.** Let F_μ and G_δ be two $GFSSs$ over (X, E). F_μ is said to be a GFS subset of G_δ, denoted by $F_\mu \sqsubseteq G_\delta$, if

(i) μ is a fuzzy subset of δ,

(ii) $F(e)$ is also a fuzzy subset of $G(e)$, $\forall e \in E$.

Definition 2.6 ([17])**.** Let F_μ be a $GFSS$ over (X, E). The generalized fuzzy soft complement of F_μ, denoted by F_μ^c, is defined by $F_\mu^c{=}G_\delta$, where $\delta(e) = \mu^c(e)$ and $G(e) = F^c(e), \forall e \in E$. Obviously, $(F_\mu^c)^c{=}F_\mu$.

Definition 2.7 ([5])**.** Let F_μ and G_δ be two $GFSSs$ over (X, E).

(i) The union of F_μ and G_δ, denoted by $F_\mu \sqcup G_\delta$, is a $GFSS$ H_ν, defined as $H_\nu : E \longrightarrow I^X \times I$ such that $H_\nu(e) = (H(e), \nu(e))$, where $H(e) = F(e) \vee G(e)$ and $\nu(e) = \mu(e) \vee \delta(e), \forall e \in E$.

(ii) The Intersection of F_μ and G_δ, denoted by $F_\mu \sqcap G_\delta$, is a $GFSS$ M_σ, defined as $M_\sigma : E \longrightarrow I^X \times I$ such that $M_\sigma(e) = (M(e), \sigma(e))$, where $M(e) = F(e) \wedge G(e)$ and $\sigma(e) = \mu(e) \wedge \delta(e)$, $\forall e \in E$.

Definition 2.8 ([17])**.** A $GFSS$ is said to be a generalized null fuzzy soft set, denoted by $\widetilde{0}_\theta$, if $\widetilde{0}_\theta : E \longrightarrow I^X \times I$ such that $\widetilde{0}_\theta(e) = (\widetilde{0}(e), \theta(e))$, where $\widetilde{0}(e) = \overline{0}, \forall e \in E$ and $\theta(e) = 0$, $\forall e \in E$ (where $\overline{0}(x) = 0$, $\forall x \in X$).

Definition 2.9 ([17])**.** A $GFSS$ is said to be a generalized absolute fuzzy soft set, denoted by $\widetilde{1}_\triangle$, if $\widetilde{1}_\triangle : E \longrightarrow I^X \times I$, where $\widetilde{1}_\triangle(e) = (\widetilde{1}(e), \triangle(e))$ is defined by $\widetilde{1}(e) = \overline{1}$, $\forall e \in E$ and $\triangle(e) = 1$, $\forall e \in E$ (where $\overline{1}(x) = 1, \forall x \in X$).

Definition 2.10 ([5])**.** Let T be a collection of generalized fuzzy soft sets over (X, E). Then T is said to be a generalized fuzzy soft topology ($GFST$, in short) over (X, E), if the following conditions are satisfied:
(i) $\widetilde{0}_\theta$ and $\widetilde{1}_\triangle$ are in T,
(ii) arbitrary unions of members of T belong to T,
(iii) Finite intersections of members of T belong to T.
The triple (X, T, E) is called a generalized fuzzy soft topological space ($GFST$-space, in short) over (X, E). The member of T are called generalized fuzzy soft open set (GFS open, in short) in (X, T, E) and their generalized fuzzy soft complements are called GFS closed sets in (X, T, E). The family of all GFS closed sets in (X, T, E) is denotes by T^c.

Definition 2.11 ([5])**.** Let (X, T, E) be a $GFST-$space and $F\mu \in GFSS(X, E)$. The generalized fuzzy soft closure of $F\mu$, denoted by $cl(F\mu)$, is the intersection of all GFS closed supper sets of $F\mu$. i.e., $cl(F\mu) = \sqcap\{H_\nu : H_\nu \in T^c, F\mu \sqsubseteq H_\nu\}$. Clearly, $cl(F\mu)$ is the smallest GFS closed set over (X, E) which contains $F\mu$.

Definition 2.12 ([12])**.** Let (X, T, E) be a $GFST$-space and $Y \subseteq X$. Let H^Y_ν be a $GFSS$ over (Y, E), where $H^Y_\nu : E \longrightarrow I^X \times I$ such that $\forall e \in E, H^Y_\nu(e) = (H^Y(e), \nu(e))$,

$$H^Y(e)(x) = \begin{cases} 1 & x \in Y \\ 0 & x \notin Y \end{cases}, \ \nu(e) = 1,$$

$$\text{i.e., } H^Y(e) = Y, \ \ \forall e \in E, \ \nu(e) = 1.$$

Let $T_Y = \{H^Y_\nu \sqcap G_\delta : G_\delta \in T\}$. Then T_Y is a generalized fuzzy soft topology (GFS topology for short) over (X, E) called a generalized fuzzy soft subspace topology (GFS subspace topology, in short) over (Y, E) and (Y, T_Y, E) is called a GFS subspace of (X, T, E). If $H^Y_\nu \in T$ (resp. $H^Y_\nu \in T^c$) then (Y, T_Y, E) is called generalized fuzzy soft open (resp. closed) subspace of (X, T, E).

Definition 2.13 ([12])**.** The generalized fuzzy soft set $F_\mu \in GFS(X, E)$ is called a generalized fuzzy soft point (GFS point for short) over (X, E), if there exist $e \in E$ and $x \in X$ such that
(i) $F(e)(x) = \alpha(0 < \alpha \leq 1)$ and $F(e)(y) = 0$, for all $y \in X - \{x\}$,
(ii) $\mu(e) = \lambda(0 < \lambda \leq 1)$ and $\mu(e') = 0$, for all $e' \in E - \{e\}$.

We denote this generalized fuzzy soft point $F_\mu = (e_\lambda, x_\alpha)$. (e, x) and (λ, α) are called respectively, the support and the value of (e_λ, x_α). The class of all GFS points in (X, E), denoted by $GFSP(X, E)$.

Definition 2.14 ([12]). Let F_μ be a $GFSS$ over (X, E). We say that $(e_\lambda, x_\alpha)\widetilde{\in} F_\mu$ read as (e_λ, x_α) belongs to the $GFSS$ F_μ, if for the element $e \in E$, $\alpha \leq F(e)(x)$ and $\lambda \leq \mu(e)$.

Theorem 2.15 ([12]). *A GFS point (e_λ, x_α) over (X, E) satisfies the following properties:*

(1) *if $(e_\lambda, x_\alpha)\widetilde{\in} F_\mu$, then (e_λ, x_α) may or may not belong to F_μ^c,*
(2) *if $(e_\lambda, x_\alpha)\widetilde{\in} F_\mu$, then $(e_\lambda, x_\alpha)^c\widetilde{\in} F_\mu^c$ does not hold,*
(3) *the union of all the GFS points of a GFSS is equal to the GFSS.*

Definition 2.16 ([13]). Let $GFSS(X, E)$ and $GFSS(Y, K)$ be the families of all $GFSSs$ over (X, E) and (Y, K), respectively. Let $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be two mappings. Then a mapping $f_{up} : GFSS(X, E) \longrightarrow GFSS(Y, K)$ is defined as follows: for a $GFSS$ $F_\mu \in GFSS(X, E)$, $\forall e' \in p(E) \sqsubseteq K$ and $y \in Y$,we have

$$f_{up}(F_\mu)(e')(y) = \begin{cases} (\bigvee_{x\in u^{-1}(y)} \bigvee_{e\in p^{-1}(e')} F(e)(x), \bigvee_{e\in p^{-1}(e')} \mu(e)), \\ \qquad\qquad \text{if } u^{-1}(y) \neq \phi, p^{-1}(k) \neq \phi, \\ (0, 0), \qquad\qquad \text{otherwise.} \end{cases}$$

f_{up} is called the generalized fuzzy soft mapping [GFS mapping for short] and $f_{up}(F_\mu)$ is called the generalized fuzzy soft image (GFS image for short) of a $GFSS$ F_μ.

Definition 2.17 ([13]). Let $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be mappings.

Let $f_{up} : GFSS(X, E) \longrightarrow GFSS(Y, K)$ be a GFS mapping and $G_\delta \in GFSS(Y, K)$. Then $f_{up}^{-1}(G_\delta) \in GFSS(X, E)$ is defined as follows:

$$f_{up}^{-1}(G_\delta)(e)(x) = (G(p(e)(u(x)), \delta(p(e)))\,, \text{ for } e \in E, x \in X.$$

$f_{up}^{-1}(G_\delta)$ is called the GFS inverse image of G_δ.

If u and p are injective, then the generalized fuzzy soft mapping f_{up} is said to be generalized fuzzy soft injective (GFS injective for short). If u and p are surjective, then the generalized fuzzy soft mapping f_{up} is said to be generalized fuzzy soft surjective (GFS surjective for short). The generalized fuzzy soft mapping f_{up} is called generalized fuzzy soft constant (GFS constant for short), if u and p are constant. f_{up} is said to be generalized fuzzy soft bijective (GFS bijective for short), if f_{up} is GFS injective and GFS surjective mapping.

Definition 2.18 ([13]). Let (X, T_1, E) and (Y, T_2, K) be two $GFST$-spaces, and $f_{up} : (X, T_1, E) \longrightarrow (Y, T_2, K)$ be a GFS mapping. Then f_{up} is called:

(i) generalized fuzzy soft continuous (GFS-continuous, in short), if $f_{up}^{-1}(G_\delta) \in T_1$, for all $G_\delta \in T_2$,

(ii) generalized fuzzy soft open (GFS open, in short), if $f_{up}(F_\mu) \in T_2$, for each $F_\mu \in T_1$.

(iii) generalized fuzzy soft homeomorphism (GFS homeomorphism, for short), if f_{up} is GFS bijection, GFS continuous and GFS open.

Theorem 2.19 ([13]). *Let (X, T_1, E) and (Y, T_2, K) be two $GFST$-spaces and $f_{up} : (X, T_1, E) \longrightarrow (Y, T_2, K)$ be a GFS mapping. Then the following are equivalent:*
(1) *f_{up} GFS-continuous,*
(2) *$f_{up}^{-1}(G_\delta) \in T_1^c$, for each $G_\delta \in T_2^c$.*

Theorem 2.20 ([13]). *Let $GFSS(X, E)$ and $GFSS(Y, K)$ be two families of $GFSSs$. For the GFS mapping $f_{up} : GFSS(X, E) \longrightarrow GFSS(Y, K)$, we have the following properties.*
(1) *$f_{up}^{-1}(G_\delta)^c = (f_{up}^{-1}(G_\delta))^c, \forall G_\delta \in GFSS(Y, K)$.*
(2) *$f_{up}(f_{up}^{-1}(G_\delta)) \sqsubseteq G_\delta, \forall G_\delta \in GFSS(Y, K)$. If f_{up} is GFS surjective, then the equality holds.*
(3) *$F_\mu \sqsubseteq f_{up}^{-1}(f_{up}(F_\mu)), \forall F_\mu \in GFSS(X, E)$. If f_{up} is GFS injective, then the equality holds.*
(4) *$f_{up}(\widetilde{0}_{\theta_X}) = \widetilde{0}_{\theta_Y}$ and $f_{up}(\widetilde{1}_{\triangle_X}) \sqsubseteq \widetilde{1}_{\triangle_Y}$. If f_{up} is GFS injective, then the equality holds.*
(5) *$f_{up}^{-1}(\widetilde{0}_{\theta_Y}) = \widetilde{0}_{\theta_X}$ and $f_{up}^{-1}(\widetilde{1}_{\triangle_Y}) = \widetilde{1}_{\triangle_X}$.*
(6) *If $F_\mu \sqsubseteq H_\nu$, then $f_{up}(F_\mu) \sqsubseteq f_{up}(H_\nu), \forall F_\mu, H_\nu \in GFSS(X, E)$.*
(7) *If $G_\delta \sqsubseteq J_\sigma$, then $f_{up}^{-1}(G_\delta) \sqsubseteq f_{up}^{-1}(J_\sigma), \forall G_\delta, J_\sigma \in GFSS(Y, K)$.*
(8) *$f_{up}^{-1}(\sqcup_{i\in J}(G_\delta)_i) = \sqcup_{i\in J} f_{up}^{-1}(G_\delta)_i$ and $f_{up}^{-1}(\sqcap_{i\in J} G_\delta)_i) = \sqcap_{i\in J} f_{up}^{-1}(G_\delta)_i, \forall (G_\delta)_i \in GFSS(Y, K)$.*
(9) *$f_{pu}(\sqcup_{i\in J}(F_\mu)_i) = \sqcup_{i\in J} f_{up}(F_\mu)_i$ and $f_{up}(\sqcap_{i\in J}(F_\mu)_i) \sqsubseteq \sqcap_{i\in J} f_{up}(F_\mu)_i, \forall (F_\mu)_i \in GFSS(X, E)$. If f_{up} is GFS injective, then the equality holds.*

3. Generalized fuzzy soft separation axioms

Definition 3.1. Two GFS points (e_λ, x_α) and (e'_γ, y_β) are said to be distinct, if $e \neq e'$.

The proof of the following theorem follows directly from the definition of GFS point and therefore omitted.

Theorem 3.2. *A GFS point (e_λ, x_α) satisfies the following properties.*
(1) *If $(e_\lambda, x_\alpha) \sqcap F_\mu = \widetilde{0}_\theta$, then $(e_\lambda, x_\alpha) \widetilde{\notin} F_\mu$ and $(e_\lambda, x_\alpha) \widetilde{\in} F_\mu^c$.*
(2) *If $(e_\lambda, x_\alpha) \widetilde{\in} F_\mu$ and $\alpha > 0.5$ or $\lambda > 0.5$, then $(e_\lambda, x_\alpha) \widetilde{\notin} F_\mu^c$.*

Definition 3.3. A $GFST-$space (X, T, E) is said to be a generalized fuzzy soft T_0-space ($GFST_0-$space, in short), if for every pair of distinct GFS points (e_λ, x_α), (e'_γ, y_β), there exists a GFS open set containing one of the points but not the other.

Example 3.4. Let $X = \{x^1, x^2\}$, $E = \{e^1, e^2\}$ and $T = \{\widetilde{0}_\theta, \widetilde{1}_\triangle, F_\mu\}$, where

$$F_\mu = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1), (e^2 = \{\frac{x^1}{0}, \frac{x^2}{0}\}, 0)\}.$$

Then clearly, T is GFS topology over (X, E). Also for every pair of distinct GFS points, there exists GFS open set containing one of the points but not the other. Thus (X, T, E) is a $GFST_0-$space.

Example 3.5. Let $X = \{x^1, x^2\}$, $E = \{e^1, e^2\}$ and $T = \{\widetilde{0}_\theta, \widetilde{1}_\triangle, F_\mu, , G_\delta, H\nu\}$, where

$$F_\mu = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{0}\}, 1), (e^2 = \{\frac{x^1}{0}, \frac{x^2}{1}\}, 1)\},$$
$$G_\delta = \{(e^1 = \{\frac{x^1}{0}, \frac{x^2}{1}\}, 1), (e^2 = \{\frac{x^1}{1}, \frac{x^2}{0}\}, 1)\},$$
$$H\nu = \{(e^1 = \{\frac{x^1}{0}, \frac{x^2}{0}\}, 1), (e^2 = \{\frac{x^1}{0}, \frac{x^2}{0}\}, 1)\}.$$

Then T is a GFS topology over (X, E) but not a $GFST_0-$space.

Example 3.6. The discrete $GFST-$space is a $GFST_0-$space, but the indiscrete $GFST-$ space is not $GFST_0$.

Theorem 3.7. *A GFS subspace (X, T_Y, E) of $GFST_0-$ space (X, T, E) is $GFST_0$.*

Proof. Let (e_λ, x_α),(e'_γ, y_β) be two distinct GFS points in (Y, E). Then these GFS points are also in (X, E). Thus there exists a GFS open set F_μ in T containing one of the points say (e_λ, x_α), but not (e'_γ, y_β). So $H^Y_\nu \sqcap F_\mu$ is a GFS open set in T_Y containing (e_λ, x_α) but not (e'_γ, y_β). Hence (X, T_Y, E) is $GFST_0$. □

Definition 3.8. A $GFST-$space (X, T, E) is said to be a generalized fuzzy soft T_1-space ($GFST_1-$space, in short), if for every pair of distinct GFS points (e_λ, x_α), (e'_γ, y_β), there exist GFS open sets F_μ and G_δ such that $(e_\lambda, x_\alpha)\widetilde{\in}F_\mu$,$(e'_\gamma, y_\beta)\widetilde{\notin}F_\mu$ and $(e'_\gamma, y_\beta)\widetilde{\in}G_\delta$, $(e_\lambda, x_\alpha)\widetilde{\notin}G_\delta$.

Example 3.9. Let $X = \{x^1, x^2\}$, $E = \{e^1, e^2\}$ and
$T = \{\widetilde{0}_\theta, \widetilde{1}_\triangle, F_\mu, G_\delta, H\nu, K_\gamma, M\psi, N\eta, J_\sigma, L_\rho\}$, where

$$F_\mu = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1), (e^2 = \{\frac{x^1}{0}, \frac{x^2}{0}\}, 0)\},$$
$$G_\delta = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{0}\}, 1), (e^2 = \{\frac{x^1}{0}, \frac{x^2}{1}\}, 1)\},$$
$$H\nu = \{(e^1 = \{\frac{x^1}{0}, \frac{x^2}{1}\}, 1), (e^2 = \{\frac{x^1}{1}, \frac{x^2}{0}\}, 1)\},$$
$$K_\gamma = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1), (e^2 = \{\frac{x^1}{1}, \frac{x^2}{0}\}, 1)\},$$
$$M\psi = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1), (e^2 = \{\frac{x^1}{0}, \frac{x^2}{1}\}, 1)\},$$
$$N\eta = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{0}\}, 1), (e^2 = \{\frac{x^1}{0}, \frac{x^2}{0}\}, 0)\},$$
$$J_\sigma = \{(e^1 = \{\frac{x^1}{0}, \frac{x^2}{1}\}, 1), (e^2 = \{\frac{x^1}{0}, \frac{x^2}{0}\}, 0)\},$$
$$L_\rho = \{(e^1 = \{\frac{x^1}{0}, \frac{x^2}{0}\}, 1), (e^2 = \{\frac{x^1}{0}, \frac{x^2}{0}\}, 1)\}.$$

Then (X, T, E) is a $GFST_1-$space.

Example 3.10. The discrete $GFST-$space is a $GFST_1-$ space, but the indiscrete $GFST-$space is not $GFST_1$.

Theorem 3.11. *A GFS subspace* (X, T_Y, E) *of a* $GFST_1-$ *space* (X, T, E) *is* $GFST_1$.

Proof. It similar to the proof of Theorem 3.7. □

Theorem 3.12. *If every GFS point* (e_λ, x_α) *of a* $GFST-$*space* (X, T, E) *is GFS closed such that* $\alpha > 0.5$ *or* $\lambda > 0.5$*, then* (X, T, E) *is* $GFST_1$.

Proof. Suppose that (e_λ, x_α) ,(e'_γ, y_β) are two distinct GFS points over (X, E). Then by hypothesis, (e_λ, x_α) and (e'_γ, y_β) are GFS closed sets such that $\alpha > 0.5$ or $\lambda > 0.5$ and $\beta > 0.5$ or $\gamma > 0.5$. Thus $(e_\lambda, x_\alpha)^c$ and $(e'_\gamma, y_\beta)^c$ are GFS open sets where $(e_\lambda, x_\alpha)\widetilde{\in}(e'_\gamma, y_\beta)^c$, $(e'_\gamma, y_\beta)\widetilde{\notin}(e'_\gamma, y_\beta)^c$ and $(e_\lambda, x_\alpha)\widetilde{\notin}(e_\lambda, x_\alpha)^c$,$(e'_\gamma, y_\beta)\widetilde{\in}(e_\lambda, x_\alpha)^c$. So (X, T, E) is $GFST_1$. □

The condition $\alpha > 0.5$ or $\lambda > 0.5$, is necessary as shown by the following example:

Example 3.13. Let $X = \{x^1, x^2\}$ and $E = \{e^1, e^2\}$. Consider the collection T of $GFSSs$ over (X, E) as the following:

$T = \{\widetilde{0}_\theta, \widetilde{1}_\Delta, (F_\mu)_1, (F_\mu)_2, (F_\mu)_3, (F_\mu)_4, (F_\mu)_5, (F_\mu)_6, (F_\mu)_7, (F_\mu)_8, (F_\mu)_9, (F_\mu)_{10}, (F_\mu)_{11}, (F_\mu)_{12}, (F_\mu)_{13}, (F_\mu)_{14}\}$, where $(F_\mu)'_i s$ are as follows:

$(F_\mu)_1 = \{(e^1 = \{\frac{x^1}{1-\alpha}, \frac{x^2}{1}\}, 1-\lambda), (e^2 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1)\}$,

$(F_\mu)_2 = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{1-\beta}\}, 1-\gamma), (e^2 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1)\}$,

$(F_\mu)_3 = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1), (e^2 = \{\frac{x^1}{1-\alpha'}, \frac{x^2}{1}\}, 1-\lambda')\}$,

$(F_\mu)_4 = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1), (e^2 = \{\frac{x^1}{1}, \frac{x^2}{1-\beta'}\}, 1-\gamma')\}$,

$(F_\mu)_5 = \{(e^1 = \{\frac{x^1}{1-\alpha}, \frac{x^2}{1-\beta}\}, 1-\lambda \wedge 1-\gamma), (e^2 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1)\}$,

$(F_\mu)_6 = \{(e^1 = \{\frac{x^1}{1-\alpha}, \frac{x^2}{1}\}, 1), (e^2 = \{\frac{x^1}{1-\alpha'}, \frac{x^2}{1}\}, 1-\lambda')\}$,

$(F_\mu)_7 = \{(e^1 = \{\frac{x^1}{1-\alpha}, \frac{x^2}{1}\}, 1-\lambda), (e^2 = \{\frac{x^1}{1}, \frac{x^2}{1-\beta'}\}, 1-\gamma')\}$,

$(F_\mu)_8 = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{1-\beta}\}, 1-\gamma), (e^2 = \{\frac{x^1}{1-\alpha'}, \frac{x^2}{1}\}, 1-\lambda')\}$,

$(F_\mu)_9 = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1), (e^2 = \{\frac{x^1}{1-\alpha'}, \frac{x^2}{1-\beta'}\}, 1-\lambda' \wedge 1-\gamma')\}$,

$(F_\mu)_{10} = \{(e^1 = \{\frac{x^1}{1-\alpha}, \frac{x^2}{1-\beta}\}, 1-\lambda \wedge 1-\gamma), (e^2 = \{\frac{x^1}{1-\alpha'}, \frac{x^2}{1}\}, 1-\lambda')\}$,

$(F_\mu)_{11} = \{(e^1 = \{\frac{x^1}{1-\alpha}, \frac{x^2}{1-\beta}\}, 1-\lambda \wedge 1-\gamma), (e^2 = \{\frac{x^1}{1}, \frac{x^2}{1-\beta'}\}, \gamma')\}$,

$(F_\mu)_{12} = \{(e^1 = \{\frac{x^1}{1-\alpha}, \frac{x^2}{1}\}, 1-\lambda), (e^2 = \{\frac{x^1}{1-\alpha'}, \frac{x^2}{1-\beta'}\}, 1-\lambda' \wedge 1-\gamma')\}$,

$(F_\mu)_{13} = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{1-\beta}\}, 1-\gamma), (e^2 = \{\frac{x^1}{1-\alpha'}, \frac{x^2}{1-\beta'}\}, 1-\lambda' \wedge 1-\gamma')\}$,

$(F_\mu)_{14} = \{(e^1 = \{\frac{x^1}{1-\alpha}, \frac{x^2}{1-\beta}\}, 1-\lambda \wedge 1-\gamma), (e^2 = \{\frac{x^1}{1-\alpha'}, \frac{x^2}{1-\beta'}\}, 1-\lambda' \wedge 1-\gamma')\}$.

Then T is a GFS topology over (X, E) and $(e^1_\lambda, x^1_\alpha)$, $(e^2_{\lambda'}, x^1_{\alpha'})$ are two distinct GFS points in (X, E) such that at $\alpha < 0.5$ and $\lambda < 0.5$ any GFS open set which containing $(e^1_\lambda, x^1_\alpha)$ also containing $(e^2_{\lambda'}, x^1_{\alpha'})$. Thus (X, T, E) is not $GFST_1$.

Remark 3.14. If (X, T, E) is a $GFST_1-$ space, then (e_λ, x_α) may not a GFS closed set as the following example shows.

Example 3.15. In Example 3.9, (X, T, E) is a $GFST_1-$ space, but $(e^1_\lambda, x^1_\alpha)$ is not GFS closed set. To show that, let $(e^1_\lambda, x^1_\alpha) = \{(e^1 = \{\frac{x^1}{\alpha}, \lambda)\}$. Then

$$(e^1_\lambda, x^1_\alpha)^c = \{(e^1 = \{\frac{x^1}{1-\alpha}, \frac{x^2}{1}\}, 1-\lambda), (e^2 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1)\}$$

is not GFS open set i.e.,$(e^1_\lambda, x^1_\alpha)$ is not GFS closed set.

Definition 3.16. A $GFST-$space (X, T, E) is said to be a generalized fuzzy soft T_2-space ($GFST_2-$space, in short), if for every pair of distinct GFS points (e_λ, x_α), (e'_γ, y_β), there exist disjoint GFS open sets F_μ and G_δ such that $(e_\lambda, x_\alpha)\widetilde{\in}F_\mu$, and $(e'_\gamma, y_\beta)\widetilde{\in}G_\delta$.

Example 3.17. Let $X = \{x^1, x^2\}$, $E = \{e^1, e^2\}$. Consider the collection T of $GFSSs$ over (X, E), $T = \{\widetilde{0}_\theta, \widetilde{1}_\triangle, F_\mu, G_\delta\}$, where F_μ and G_δ are as follows:

$$F_\mu = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1), (e^2 = \{\frac{x^1}{0}, \frac{x^2}{0}\}, 0)\},$$

$$G_\delta = \{(e^1 = \{\frac{x^1}{0}, \frac{x^2}{0}\}, 0), (e^2 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1)\}.$$

Then clearly, T is a GFS topology over (X, E). Also, for every pair of distinct GFS points, there exist disjoint GFS open sets over (X, E) containing them. Thus (X, T, E) is a $GFST_2-$space.

Example 3.18. The discrete $GFST-$space is a $GFST_2-$ space, but the indiscrete $GFST-$space is not a $GFST_1$.

Theorem 3.19. *A GFS subspace (X, T_Y, E) of a $GFST_2-$ space (X, T, E) is $GFST_2$.*

Proof. Let (e_λ, x_α),(e'_γ, y_β) be two distinct GFS points in (Y, E). Then these GFS points are also in (X, E). Thus there exists disjoint GFS open set F_μ and G_δ in T such that $(e_\lambda, x_\alpha)\widetilde{\in}F_\mu$ and $(e'_\gamma, y_\beta)\widetilde{\in}G_\delta$. So $H^Y_\nu \sqcap F_\mu$ and $H^Y_\nu \sqcap G_\delta$ are disjoint GFS open sets F_μ and G_δ in T_Y such that $(e_\lambda, x_\alpha)\widetilde{\in}H^Y_\nu \sqcap F_\mu$ and $(e'_\gamma, y_\beta)\widetilde{\in}H^Y_\nu \sqcap G_\delta$. Hence (X, T_Y, E) is a $GFST_2-$ space. □

Remark 3.20. From definitions one deduce the following implication hold:
$GFST_2 \Longrightarrow GFST_1 \Longrightarrow GFST_0$.

The inverse implications may not be true as shows is by the following examples.

Example 3.21. In Example 3.4, (X, T, E) is a $GFST_0 - space$ but not $GFST_1$. Since $(e^1_\lambda, x^1_\alpha)$, (e^2_γ, x^2_β) are distinct GFS points in (X, E), but there does not GFS open set containing $(e^1_\lambda, x^1_\alpha)$. Then (X, T, E) is not $GFST_1$.

In Example 3.9, (X, T, E) is a $GFST_1-$space but not $GFST_2$. Since $(e^1_\lambda, x^1_\alpha)$, (e^2_γ, x^1_β) are distinct GFS points and the only GFS open sets which containing $(e^1_\lambda, x^1_\alpha)$, (e^2_γ, x^1_β) are F_μ,H_ν, respectively, but they are not disjoint. Then (X, T, E) is not $GFST_2$.

Definition 3.22. Let (X, T, E) be a $GFST-$ space. If for every GFS closed set H_ν and every GFS point (e_λ, x_α) such that $(e_\lambda, x_\alpha) \sqcap H_\nu = \widetilde{0}_\theta$, there exist disjoint GFS open sets F_μ and G_δ such that $(e_\lambda, x_\alpha)\widetilde{\in}F_\mu$ and $H_\nu \sqsubseteq G_\delta$. Then (X, T, E) is called generalized fuzzy soft regular space (GFS regular space, in short).

Definition 3.23. A $GFST-$space (X, T, E) is called a generalized fuzzy soft T_3-space ($GFST_3-$space, in short), if it is $GFST_1$ and GFS regular.

Example 3.24. Let $X = \{x^1, x^2\}$, $E = \{e^1, e^2\}$ and $T = \{\widetilde{0}_\theta, \widetilde{1}_\triangle, F_\mu, G_\delta\}$, where

$$F_\mu = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1)\},\ G_\delta = \{(e^2 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1)\}.$$

Then T is GFS topology over (X, E).

Now, let $T^c = \{\widetilde{1}_\triangle, \widetilde{0}_\theta, F^c_\mu, G^c_\delta\}$, where

$$F^c_\mu = \{(e^2 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1)\},\ G^c_\delta = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1)\}.$$

Then by used GFS regularity on GFS closed sets as follows:

$(e^1_\lambda, x^1_\alpha) \sqcap F^c_\mu = \widetilde{0}_\theta \Longrightarrow \exists G^c_\delta, G_\delta \in T$ such that $(e^1_\lambda, x^1_\alpha) \widetilde{\in} G^c_\delta$, $F^c_\mu \sqsubseteq G_\delta$ and $G^c_\delta \sqcap G_\delta = \widetilde{0}_\theta$,

$(e^1_\lambda, x^2_\beta) \sqcap F^c_\mu = \widetilde{0}_\theta \Longrightarrow \exists G^c_\delta, G_\delta \in T$ such that $(e^1_\lambda, x^2_\beta) \widetilde{\in} G^c_\delta$, $F^c_\mu \sqsubseteq G_\delta$ and $G^c_\delta \sqcap G_\delta = \widetilde{0}_\theta$,

$(e^2_\gamma, x^1_{\alpha'}) \sqcap G^c_\delta = \widetilde{0}_\theta \Longrightarrow \exists F^c_\mu, F_\mu \in T$ such that $(e^2_\gamma, x^1_{\alpha'}) \widetilde{\in} F^c_\mu, G^c_\delta \sqsubseteq F_\mu$ and $F^c_\mu \sqsubseteq F_\mu = \widetilde{0}_\theta$,

$(e^2_\gamma, x^2_\alpha) \sqcap G^c_\delta = \widetilde{0}_\theta \Longrightarrow \exists F^c_\mu, F_\mu \in T$ such that $(e^2_\gamma, x^2_\alpha) \widetilde{\in} F^c_\mu, G^c_\delta \sqsubseteq F_\mu$ and $F^c_\mu \sqsubseteq F_\mu = \widetilde{0}_\theta$.

Thus (X, T, E) is a GFS regular space, but not a $GFST_1-$ space and so not $GFST_3$.

Example 3.25. Let $X = \{x^1\}$, $E = \{e^1, e^2\}$ and $T = \{\widetilde{0}_\theta, \widetilde{1}_\triangle, F_\mu, G_\delta\}$, where

$$F_\mu = \{(e^1 = \{\frac{x^1}{1}\}, 1)\},\ G_\delta = \{(e^2 = \{\frac{x^1}{1}\}, 1)\}.$$

Then T is a GFS topology over (X, E). Moreover, $T^c = \{\widetilde{1}_\triangle, \widetilde{0}_\theta, F^c_\mu, G^c_\delta\}$, where

$$F^c_\mu = \{(e^2 = \{\frac{x^1}{1}\}, 1)\},\ G^c_\delta = \{(e^1 = \{\frac{x^1}{1}\}, 1)\}.$$

For the GFS point $(e^1_\lambda, x^1_\alpha) \sqcap F^c_\mu = \widetilde{0}_\theta$, there exist GFS open sets F_μ and G_δ such that $(e^1_\lambda, x^1_\alpha) \widetilde{\in} F_\mu$, $F^c_\mu \sqsubseteq G_\delta$ and $F_\mu \sqcap G_\delta = \widetilde{0}_\theta$.

For the GFS point $(e^2_\lambda, x^1_\alpha) \sqcap G^c_\delta = \widetilde{0}_\theta$, there exist GFS open sets G_δ and F_μ such that $(e^1_\lambda, x^1_\alpha) \widetilde{\in} G_\delta$, $G^c_\delta \sqsubseteq F_\mu$ and $G_\delta \sqcap F_\mu = \widetilde{0}_\theta$. Thus (X, T, E) is a $GFST_3-$space.

Theorem 3.26. *If (X, T, E) is a GFS regular space, then for any GFS open set G_δ and a GFS point (e_λ, x_α) in (X, E) such that $(e_\lambda, x_\alpha) \sqcap G^c_\delta = \widetilde{0}_\theta$, there exists a GFS open set M_ψ such that $(e_\lambda, x_\alpha) \widetilde{\in} M_\psi \sqsubseteq cl(M_\psi) \sqsubseteq G_\delta$.*

Proof. Suppose that (X, T, E) is a GFS regular space. Let G_δ be a GFS open set over (X, E) such that $(e_\lambda, x_\alpha) \sqcap G^c_\delta = \widetilde{0}$. Then clearly, G^c_δ is a GFS closed set over (X, E) such that $(e_\lambda, x_\alpha) \sqcap G^c_\delta = \widetilde{0}$. Then by the hypothesis, there exist two disjoint GFS open sets M_ψ and N_η such that $(e_\lambda, x_\alpha) \widetilde{\in} M_\psi$ and $G^c_\delta \sqsubseteq N_\eta$. Now, N^c_η is GFS closed set of (X, E) such that $M_\psi \sqsubseteq N^c_\eta \sqsubseteq G_\delta$. Thus $(e_\lambda, x_\alpha) \widetilde{\in} M_\psi \sqsubseteq cl(M_\psi)$ and $M_\psi \sqsubseteq N^c_\eta \sqsubseteq G_\delta$. So $cl(M_\psi) \sqsubseteq G_\delta$. Hence $(e_\lambda, x_\alpha) \widetilde{\in} M_\psi \sqsubseteq cl(M_\psi) \sqsubseteq G_\delta$. □

Theorem 3.27. *Every GFS regular space, in which every GFS point is GFS closed, is a $GFST_2$-space.*

Proof. Let (e_λ, x_α), (e'_γ, y_β) be two distinct GFS points of a GFS regular space (X, T, E). Then by hypothesis, (e'_γ, y_β) is GFS closed set and $(e_\lambda, x_\alpha) \sqcap (e'_\gamma, y_\beta) = \widetilde{0}_\theta$. From the GFS regularity, there exist disjoint GFS open sets M_ψ and N_η such that $(e_\lambda, x_\alpha) \widetilde{\in} M_\psi$ and $(e'_\gamma, y_\beta) \sqsubseteq N_\eta$. Thus $(e_\lambda, x_\alpha) \widetilde{\in} M_\psi$ and $(e'_\gamma, y_\beta) \widetilde{\in} N_\eta$. So (X, T, E) is a $GFST_2-$space. □

Remark 3.28. Every $GFST_3-$space, in which every GFS point is GFS closed, is $GFST_2$.

Theorem 3.29. *The GFS set K_γ is GFS closed in the subspace (Y, T_Y, E) of (X, T, E) if and only if $K_\gamma = H^Y_\nu \sqcap J_\sigma$, for some GFS closed set J_σ in T (for H^Y_ν see Definition 2.12).*

Proof. Let K_γ be GFS closed set in (Y, E). Then $(K_\gamma)^c_{T_Y}$ is GFS open in T_Y, where $(K_\gamma)^c_{T_Y}$ is the closure of K_γ in the subspace (Y, T_Y, E). Since $(K_\gamma)^c_{T_Y} \in T_Y$, there exist $G_\delta \in T$ such that $(K_\gamma)^c_{T_Y} = H^Y_\nu \sqcap G_\delta$. Thus

$$\begin{aligned} K_\gamma &= (H^Y_\nu \sqcap G_\delta)^c_{T_Y} = H^Y_\nu \sqcap (H^Y_\nu \sqcap G_\delta) \\ &= (H^Y_\nu \sqcap (H^Y_\nu \sqcap G_\delta)^c \\ &= H^Y_\nu \sqcap ((H^Y_\nu)^c \sqcup G^c_\delta) \\ &= (H^Y_\nu \sqcap (H^Y_\nu)^c) \sqcup (H^Y_\nu \sqcap G^c_\delta) \\ &= H^Y_\nu \sqcap G^c_\delta. \end{aligned}$$

Let $G^c_\delta = J_\sigma$. Then J_σ is GFS closed set in (X, E) and $K_\gamma = H^Y_\nu Y \sqcap J_\sigma$. Conversely, let J_σ be GFS closed set in (X, E) and $K_\gamma = H^Y_\nu \sqcap J_\sigma$. We need to show that$K_\gamma$ is GFS closed set in T_Y, that is,

$$(K_\gamma)^c_{T_Y} = H^Y_\nu - (H^Y_\nu \sqcap J_\sigma) = H^Y_\nu \sqcap (H^Y_\nu \sqcap J_\sigma)^c = H^Y_\nu \sqcap = H^Y_\nu \sqcap J^c_\sigma.$$

Since J_σ is GFS closed set in (X, E), $J^c_\sigma \in T$. Then $(K_\gamma)^c_{T_Y} \in T_Y$ i.e, K_γ is GFS closed set in (Y, E). □

Theorem 3.30. *A GFS subspace (Y, T_Y, E) of a $GFST_3-$space (X, T, E) is $GFST_3$.*

Proof. By theorem 3.11, (Y, T_Y, E) is $GFST_1-$space. Now, we want to prove that (X, T_Y, E) is a GFS regular space. Let K_γ be a GFS closed set in (Y, E) and (e'_γ, y_β) be a GFS point in (Y, E) such that $(e'_\gamma, y_\beta) \sqcap K_\gamma = \widetilde{0}_\theta$. Then $K_\gamma = H^Y_\nu \sqcap G_\delta$, for some GFS closed set in (X, E). Thus $(e'_\gamma, y_\beta) \sqcap (H^Y_\nu \sqcap G_\delta) = \widetilde{0}_\theta$. But $(e'_\gamma, y_\beta) \widetilde{\in} H^Y_\nu$. So $(e'_\gamma, y_\beta) \sqcap G_\delta = \widetilde{0}_\theta$. Since (X, T, E) is GFS regular, there exist disjoint GFS open sets M_ψ and N_η in T such that $(e'_\gamma, y_\beta) \widetilde{\in} M_\psi$ and $G_\delta \sqsubseteq N_\eta$. It follows that, $H^Y_\nu \sqcap M_\psi$ and $H^Y_\nu \sqcap N_\eta$ are disjoint GFS open sets in T_Y such that $(e'_\gamma, y_\beta) \widetilde{\in} H^Y_\nu \sqcap M_\psi$ and $(e'_\gamma, y_\beta) \widetilde{\in} H^\nu_\nu \sqcap N_\eta$. So (Y, T_Y, E) is GFS regular and hence $GFST_3$. □

Definition 3.31. Let (X, T, E) be a $GFST-$space. If for every disjoint GFS closed sets H_ν, K_γ there exist disjoint GFS open sets M_ψ and N_η such that $H_\nu \sqsubseteq M_\psi$, $K_\gamma \sqsubseteq N_\eta$. Then (X, T, E) is called generalized fuzzy soft normal space (GFS normal space for short). (X, T, E) is called a generalized fuzzy soft T_4-space ($GFST_4-$space, in short), if it is GFS normal and $GFST_1-$space.

Example 3.32. Let $X = \{x^1, x^2\}$, $E = \{e^1, e^2\}$ and $T = \{\widetilde{0}_\theta, \widetilde{1}_\Delta, F_\mu, G_\delta, H_\nu, K_\gamma\}$, where

$$F_\mu = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{0}\}, 1), (e^2 = \{\frac{x^1}{0}, \frac{x^2}{0}\}, 0)\},$$

$$G_\delta = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{0}\}, 1), (e^2 = \{\frac{x^1}{0}, \frac{x^2}{1}\}, 1)\},$$

$$H_\nu = \{(e^1 = \{\frac{x^1}{0}, \frac{x^2}{1}\}, 0), (e^2 = \{\frac{x^1}{1}, \frac{x^2}{0}\}, 0)\},$$

$$K_\gamma = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1), (e^2 = \{\frac{x^1}{1}, \frac{x^2}{0}\}, 0)\}.$$

Then T is GFS topology over (X, E). Now, $T^c = \{\widetilde{0}_\theta, \widetilde{1}_\Delta, F^c_\mu, G^c_\delta, H^c_\nu, K^c_\gamma\}$ where

$$F^c_\mu = \{(e^1 = \{\frac{x^1}{0}, \frac{x^2}{1}\}, 0), (e^2 = \{\frac{x^1}{1}, \frac{x^2}{1}\}, 1)\},$$

$$G^c_\delta = \{(e^1 = \{\frac{x^1}{0}, \frac{x^2}{1}\}, 0), (e^2 = \{\frac{x^1}{1}, \frac{x^2}{0}\}, 0)\},$$

$$H^c_\nu = \{(e^1 = \{\frac{x^1}{1}, \frac{x^2}{0}\}, 1), (e^2 = \{\frac{x^1}{0}, \frac{x^2}{1}\}, 1)\},$$

$$K^c_\gamma = \{(e^1 = \{\frac{x^1}{0}, \frac{x^2}{0}\}, 0), (e^2 = \{\frac{x^1}{0}, \frac{x^2}{1}\}, 1)\}.$$

For the two disjoint GFS closed sets G^c_δ and H^c_ν there exist $G_\delta, H_\nu \in T$ such that $H^c_\nu \sqsubseteq G_\delta, G^c_\delta \sqsubseteq H_\nu$ and $G_\delta \sqcap H_\nu = \widetilde{0}_\theta$. For the two disjoint GFS closed sets G^c_δ and K^c_γ there exist $G_\delta, H_\nu \in T$ such that $G^c_\delta \sqsubseteq H_\nu, K^c_\gamma \sqsubseteq G_\delta$ and $H_\nu \sqcap G_\delta = \widetilde{0}_\theta$. Then (X, T, E) is a GFS normal space, but not $GFST_1-$ space and thus not $GFST_4-$ space.

Theorem 3.33. *If (X, T, E) is a GFS normal space, then for each GFS closed set K_γ in (X, E) and any GFS open set G_δ in (X, E) such that $K_\gamma \sqcap G^c_\delta = \widetilde{0}_\theta$, then there exists a GFS open set H_ν such that $K_\gamma \sqsubseteq M_\psi \sqsubseteq cl(M_\psi) \sqsubseteq G_\delta$.*

Proof. Suppose (X, T, E) is a GFS normal space. Let K_γ be a GFS closed set in (X, E) and G_δ be a GFS open set in (X, E) such that $K_\gamma \sqcap G^c_\delta = \widetilde{0}_\theta$. Then $K_\gamma \sqsubseteq G_\delta$. Now, K_γ and G^c_δ are two disjoint GFS closed sets in (X, E). Since (X, T, E) is GFS normal, there exist two disjoint GFS open sets M_ψ and N_η such that $K_\gamma \sqsubseteq M_\psi$, $G^c_\delta \sqsubseteq N_\eta$ and $M_\psi \sqcap N_\eta = \widetilde{0}_\theta$. Thus $M_\psi \sqsubseteq N^c_\eta$, but N^c_η is a GFS closed set. So $cl(M_\psi) \sqsubseteq N^c_\eta$. Hence we have $K_\gamma \sqsubseteq M_\psi \sqsubseteq cl(M_\psi) \sqsubseteq G_\delta$. □

Theorem 3.34. *GFS closed subspace (Y, T_Y, E) of a GFS normal-space (X, T, E) is GFS normal.*

Proof. Let $(K_\nu)_1$ and $(K_\nu)_2$ be disjoint GFS closed sets in (Y, E). Then $(K_\nu)_1 = H^Y_\nu \sqcap (F_\mu)_1$ and $(K_\nu)_2 = H^Y_\nu \sqcap (F_\mu)_2$, for some GFS closed sets $(F_\mu)_1$, $(F_\mu)_2$ in (X, E). Since H^Y_ν is GFS closed set in (X, E), $(K_\nu)_1$ and $(K_\nu)_2$ are disjoint GFS closed sets in (X, E). Since (X, T, E) is GFS normal, there exist disjoint GFS open sets $(G_\delta)_1$ and $(G_\delta)_2$ in(X, E) such that $(K_\nu)_1 \sqsubseteq (G_\delta)_1$, $(K_\nu)_2 \sqsubseteq (G_\delta)_2$. Thus

$$(K_\nu)_1 \sqsubseteq (M_\psi)_1 = H^Y_\nu \sqcap (G_\delta)_1,\ (K_\nu)_1 \sqsubseteq (M_\psi)_2 = H^Y_\nu \sqcap (G_\delta)_2.$$

From definition T_Y, we have $(M_\psi)_1, (M_\psi)_2 \in T_Y$ and $(M_\psi)_1 \sqcap (M_\psi)_2 = H_\nu^Y \sqcap (G_\delta)_1 \sqcap H_\nu^Y \sqcap (G_\delta)_2 = H_\nu^Y \sqcap [(G_\delta)_1 \sqcap (G_\delta)_2] = H_\nu^Y \sqcap \widetilde{0}_\theta = \widetilde{0}_\theta$. Thus (Y, T_Y, E) is a GFS normal space. □

Theorem 3.35. *Every GFS normal space, in which every GFS point (e_λ, x_α) is GFS closed, is a GFS regular space.*

Proof. Let (X, T, E) be a GFS normal space. Let (e_λ, x_α) be a GFS point. Let K_γ be a GFS closed set such that $(e_\lambda, x_\alpha) \sqcap K_\gamma = \widehat{0}_\theta$. Since (e_λ, x_α) is a GFS closed set in (X, E), there exist disjoint GFS open sets M_ψ and N_η such that $(e_\lambda, x_\alpha) \sqsubseteq M_\psi, K_\gamma \sqsubseteq N_\eta$. Then $(e_\lambda, x_\alpha) \widetilde{\in} M_\psi, K_\gamma \sqsubseteq N_\eta$. Thus (X, T, E) is a GFS regular space. □

Remark 3.36. Every $GFST_4-$space, in which every GFS point is GFS closed, is $GFST_3$.

Theorem 3.37. *Let $f_{up} : (X, T_1, E_1) \longrightarrow (Y, T_2, E_2)$ be a GFS bijective GFS open mapping. If (X, T_1, E_1) is a $GFST_i-$space, then (Y, T_2, E_2) is a $GFST_i -$ space., $i = 0,\ 1,\ 2$*

Proof. We prove the theorem for ($i = 2$, for example), the other cases are similar. Let (X, T_1, E_1) is a $GFST_2-$space and $f_{up} : (X, T_1, E_1) \longrightarrow (Y, T_2, E_2)$ be a GFS bijective GFS open mapping. We want to show that (Y, T_2, E_2) is a $GFST_2-$space. Let (e'_λ, x_α),(s'_γ, y_β) be two distinct GFS points in (Y, E_2). Since f_{up} is GFS bijection, there exist two distinct GFS points (e_λ, a_α), (s'_γ, b_β) in (X, E_1) such that $f_{up}(e_\lambda, a_\alpha) = (e'_\lambda, x_\alpha) f_{up}(s_\gamma, b_\beta) = (s'_\gamma, y_\beta)$. But (X, T_1, E_1) is a $GFST_2-$space. Then there exist disjoint GFS open sets F_μ and G_δ in (X, E_1) such that $(e_\lambda, a_\alpha) \widetilde{\in} F_\mu, (e'_\lambda, x_\alpha) \widetilde{\in} G_\delta$. It follows that, $f_{up}(e_\lambda, a_\alpha) = (e'_\lambda, x_\alpha) \widetilde{\in} f_{up}(F_\mu)$, $f_{up}(s_\gamma, b_\beta) = (s'_\gamma, y_\beta) \widetilde{\in} f_{up}(G_\delta)$ and $f_{up}(F_\mu) \sqcap f_{up}(G_\delta) = f_{up}(F_\mu \sqcap G_\delta) = f_{up}(\widetilde{0}_{\theta_X}) = \widetilde{0}_{\theta_Y}$ (from Theorem 2.20).

Since $F_\mu, G_\delta \in T_1$ and f_{up} is GFS open and $f_{up}(F_\mu), f_{up}(G_\delta) \in T_2$, there exist disjoint GFS open sets $f_{up}(F_\mu)$ and $f_{up}(G_\delta)$ in (Y, E_2) such that $(e'_\lambda, x_\alpha) \widetilde{\in} f_{up}(F_\mu)$ and $(s'_\gamma, y_\beta) \widetilde{\in} f_{up}(G_\delta)$. Thus (Y, T_2, E_2) is a $GFST_2-$ space. □

Definition 3.38. The property P is called a generalized fuzzy soft topological property (GFS topological property, in short), if it is preserved under a GFS homeomorphism mapping.

Remark 3.39. The property of being $GFST_i-$space ($i = 0,\ 1,\ 2$) is a GFS topological property.

Theorem 3.40. *The property of being $GFST_i-$space ($i = 3,\ 4$) is a GFS topological property or it is preserved under a GFS homeomorphism mapping.*

Proof. We prove the theorem for ($i = 3$, for example), the other cases are similar.

Since the property of being $GFST_1-$ space is a GFS topological property, we only show that the property of GFS regularity is a GFS topological property. Let $f_{up} : (X, T_1, E_1) \longrightarrow (Y, T_2, E_2)$ be a GFS homeomorphism and (X, T_1, E_1) be a GFS regular space. Let J_σ be a GFS closed set in (Y, E_2) and let (e'_λ, y_α) be a GFS point in (Y, E_2) such that $(e'_\lambda, y_\alpha) \sqcap J_\sigma = \widetilde{0}_{\theta_Y}$. Since f_{up} is GFS surjective, there exists a GFS point (e_λ, x_α) in (X, E_1) such that $f_{up}(e_\lambda, x_\alpha) = (e'_\lambda, y_\alpha)$. Since f_{up} is GFS continuous and J_σ GFS closed in (Y, E_2), we have $f_{up}^{-1}(J_\sigma)$ a GFS closed set in (X, E_1) [from Theorem 2.19]. It follows that $f_{up}(e_\lambda, x_\alpha) = (e'_\lambda, y_\alpha)$. Then $(e_\lambda, x_\alpha) = f_{up}^{-1}(e'_\lambda, y_\alpha)$ [as f_{up} is GFS injective]. Since $(e'_\lambda, y_\alpha) \sqcap J_\sigma = \widetilde{0}_{\theta_Y}$, $f_{up}^{-1}((e'_\lambda, y_\alpha) \sqcap J_\sigma) = f_{up}^{-1}(\widetilde{0}_{\theta_Y}) = \widetilde{0}_{\theta_X}$. Thus $(e_\lambda, x_\alpha) \sqcap f_{up}^{-1}(J_\sigma) = \widetilde{0}_{\theta_X}$.

Now, $f_{up}^{-1}(J_\sigma)$ is GFS closed set over (X, E_1) and (e_λ, x_α) is a GFS point in (X, E_1) such that $(e_\lambda, x_\alpha) \sqcap f_{up}^{-1}(J_\sigma) = \widetilde{0}_{\theta_X}$. But (X, T_1, E_1) is a GFS regular space. So there exist disjoint GFS open sets G_δ and H_ν in (X, E_1) such that $(e_\lambda, x_\alpha) \widetilde{\in} G_\delta$, $f_{up}^{-1}(J_\sigma) \sqsubseteq H_\nu$. Hence $f_{up}(e_\lambda, x_\alpha) = (e'_\lambda, y_\alpha) \widetilde{\in} f_{up}(G_\delta)$, $f_{up}(f_{up}^{-1}(J_\sigma)) = J_\sigma \sqsubseteq f_{up}(H_\nu)$ [as f_{up} is GFS surjective] and $f_{up}(G_\delta) \sqcap f_{up}(H_\nu) = f_{up}(G_\delta \sqcap H_\nu) = f_{up}(\widetilde{0}_{\theta_X}) = \widetilde{0}_{\theta_Y}$ [from Theorem 2.20]. Since f_{up} is a GFS open mapping, $f_{up}(G_\delta), f_{up}(H_\nu) \in T_2$. Now, there exists disjoint GFS open sets $f_{up}(G_\delta)$ and $f_{up}(H_\nu)$ in (Y, E_2) such that $(e'_\lambda, y_\alpha) \widetilde{\in} f_{up}(G_\delta)$ and $J_\sigma \sqsubseteq f_{up}(H_\nu)$. Therefore (Y, T_2, E_2) is a GFS regular space. □

4. Conclusion

In the present work, we have continued to study the properties of generalized fuzzy soft topological spaces. We introduced some new concepts in generalized fuzzy soft topological spaces such as generalized fuzzy soft separation axioms T_i $(i = 0,\ 1,\ 2)$, generalized fuzzy soft regular and generalized fuzzy soft normal by used generalized fuzzy soft open sets and presented fundamentals properties such as hereditary properties and topological properties in generalized fuzzy soft topological spaces. Also, we discussed some of the characteristics that not hold of generalized fuzzy soft topological spaces in general which is a departure from general topology. For future works, we consider to study generalized fuzzy soft separation axioms T_i $(i = 0,\ 1,\ 2,\ 3,\ 4)$ by using generalized fuzzy soft quasi-coincident relation and generalized fuzzy soft neighborhood system.

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