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Tri-ideals and fuzzy tri-ideals of Γ-semirings

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Abstract. In this paper, we introduce the notion of tri-ideal and fuzzy tri-ideal of Γ-semirings, characterize the regular Γ-semiring in terms of fuzzy tri-ideals of Γ-semiring and study some of their properties.

2010 AMS Classification: 16Y60,16Y99, 03E72

Keywords: semiring, Γ-semiring, Γ- regular semiring, bi- quasi ideal, bi-interior ideal, fuzzy bi- quasi ideal,fuzzy bi-interior ideal, tri- ideal and fuzzy tri- ideal.

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1. Introduction

Semiring is an algebraic structure which is a common generalization of rings and distributive lattices, was first introduced by Vandiver [35] in 1934 but non-trivial examples of semirings had appeared in the earlier studies on the theory of commutative ideals of rings by Dedekind in 19th century. Semiring is a universal algebra with two binary operations called addition and multiplication, where one of them distributive over the other. In 1995, Murali Krishna Rao [13] introduced the notion of a Γ-semiring as a generalization of a Γ-ring, ring, ternary semiring and semiring. In structure, semirings lie between semigroups and rings.

We know that the notion of a one sided ideal of any algebraic structure is a generalization of notion of an ideal. The quasi ideals are generalization of left ideal and right ideal whereas the bi-ideals are generalization of quasi ideals. In 1952, the concept of bi-ideals was introduced by Good and Hughes [2] for semigroups.In 1976, the concept of interior-ideals was introduced by Lajos [9] for semigroups. The notion of bi-ideals in rings and semigroups was introduced by Lajos and Szasz [10]. Bi-ideal is a special case of (m-n) ideal. Steinfeld [33] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki [4, 5, 6] introduced the concept of quasi ideals for a semiring. Quasi ideals and bi-ideals in Γ-semirings studied by Jagtap and Pawar [7]. Rao [14, 15, 18, 21, 22, 23, 24, 25, 27], and Rao et al. [22, 28] studied ideals in ordered Γ-semiring, introduced the notion of bi-interior ideal, bi- quasi-interior ideal and left (right) bi-quasi ideal of semiring, Γ-semiring, Γ-semigroup and studied their properties and characterized simple Γ-semiring and regular Γ-semiring using these ideals. The fuzzy set theory was developed by Zadeh [36] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. The fuzzification of algebraic structure was introduced by Rosenfeld [32] and he introduced the notion of fuzzy subgroups in 1971. K. L. N. Swamy and U. M. Swamy [34] studied fuzzy prime ideals in rings in 1988. In 1982, Liu [28] defined and studied fuzzy subrings as well as fuzzy ideals in rings. Mandal [12] studied fuzzy ideals and fuzzy interior ideals in an ordered semiring. Rao [16, 17, 19, 20, 26, 29, 31] studied fuzzy $k-$ideals of Γ-semirings, $r-$fuzzy ideals of Γ-incline and $T-$fuzzy ideals of ordered Γ-semirings. Kuroki [8] studied fuzzy interior ideals in semigroups. In this paper,as a further generalization of ideals, we introduce the notion of tri-ideal of Γ-semiring as a generalization of ideal, left ideal, right ideal, bi-ideal, quasi ideal and interior ideal of Γ-semiring and the notion of fuzzy tri-ideal of Γ-semiring. We characterize the regular Γ-semiring in terms of fuzzy tri-ideal of Γ-semiring and studied some of their properties.

2. Preliminaries

In this section, we recall some of the fundamental concepts and definitions which are necessary for this paper.

Definition 2.1 ([1])**.** A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) is called a semiring, if it satisfies the following conditions:
(i) addition is a commutative operation,
(ii) multiplication distributes over addition both from the left and from the right,
(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$, for all $x \in S$.

Definition 2.2. Let M and Γ be two non-empty sets. Then M is called a Γ-semigroup, if it satisfies
(i) $x\alpha y \in M$,
(ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Definition 2.3. Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. A Γ-semigroup M is said to be Γ-semiring, if it satisfies the following axioms: for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,
(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$,
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$.

Every semiring M is a Γ-semiring with $\Gamma = M$ and ternary operation as the usual semiring multiplication

Definition 2.4. A Γ-semiring M is said to have zero element, if there exists an element $0 \in M$ such that $0 + x = x$ and $0\alpha x = x\alpha 0 = 0,$ for all $x \in M, \alpha \in \Gamma$.

Definition 2.5. Let M be a Γ-semiring. An element $a \in M$ is said to be regular element of M, if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

Definition 2.6. Let M be a Γ-semiring. If every element of M is a regular, then M is said to be regular Γ-semiring.

Definition 2.7. A non-empty subset A of a Γ-semiring M is called

(i) a Γ-subsemiring of M if $(A, +)$ is a subsemigroup of $(M, +)$ and $A\Gamma A \subseteq A$.
(ii) a quasi ideal of M if A is a Γ-subsemiring of M and $A\Gamma M \cap M\Gamma A \subseteq A$.
(iii) a bi-ideal of M if A is a Γ-subsemiring of M and $A\Gamma M\Gamma A \subseteq A$.
(iv) an interior ideal of M if A is a Γ-subsemiring of M and $M\Gamma A\Gamma M \subseteq A$.
(v) a left (right) ideal of M if A is a Γ-subsemiring of M and $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$.
(vi) an ideal if A is a Γ-subsemiring of $M, A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.
(vii) a $k-$ideal if A is a Γ-subsemiring of $M, A\Gamma M \subseteq A, M\Gamma A \subseteq A$ and $x \in M,\ x + y \in A, y \in A$ then $x \in A$.
(viii) a bi-interior ideal of M if A is a Γ-subsemiring of M and $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$.
(ix) a left bi-quasi ideal (right bi-quasi ideal) of M if A is a subsemigroup of $(M, +)$ and $M\Gamma A \cap A\Gamma M\Gamma A \subseteq A$ $(A\Gamma M \cap A\Gamma M\Gamma A \subseteq A)$.
(x) a bi-quasi ideal of M if B is a $\Gamma-$subsemiring of M and $B$is a left bi-quasi ideal and a right bi-quasi ideal of M.
(xi) a left quasi-interior ideal (right quasi-interior ideal) of M if A is a Γ-subsemiring of M and $M\Gamma A\Gamma M\Gamma A \subseteq A$ $(A\Gamma M\Gamma A\Gamma M \subseteq A)$.
(xii) a quasi-interior of M if B is a $\Gamma-$subsemiring of M and $B$is a left quasi-interior ideal and a right quasi-interior ideal of M.
(xiii) a bi-quasi-interior ideal of M if A is a Γ-subsemiring of M and $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$.
(xiv) a left(right) weak-interior ideal of M if B is a $\Gamma-$subsemiring of M and $M\Gamma B\Gamma B \subseteq B(\ B\Gamma B\Gamma M \subseteq B)$.
(xv) a weak-interior ideal of M if B is a $\Gamma-$subsemiring of M and $B$is a left weak-interior ideal and a right weak-interior ideal)of M.

Definition 2.8. A Γ-semiring M is a left (right) simple Γ-semiring, if M has no proper left (right) ideal of M

Definition 2.9. A Γ-semiring M is said to be simple Γ-semring, if M has no proper ideals.

Definition 2.10. Let M be a non-empty set. A mapping $f : M \to [0, 1]$ is called a fuzzy subset of a Γ-semiring M.

Definition 2.11. Let f be a fuzzy subset of a non-empty set M, for $t \in [0, 1]$ the set $f_t = \{x \in M \mid f(x) \geq t\}$ is called a level subset of M with respect to f.

Definition 2.12. Let M be a Γ-semiring. A fuzzy subset μ of M is said to be fuzzy Γ-subsemiring of M, if it satisfies the following conditions:
(i) $\mu(x + y) \geq min\ \{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \geq min\ \{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.13. A fuzzy subset μ of a Γ-semiring M is called a fuzzy left (right) ideal of M, if it satisfies the following conditions:
(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$,

(ii) $\mu(x\alpha y) \geq \mu(y)$ $(\mu(x))$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.14. A fuzzy subset μ of a Γ-semiring M is called a fuzzy ideal of M, if it satisfies the following conditions:
(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \geq max$ $\{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.15. For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$ for all $x \in M$.

Definition 2.16. [6] Let f and g be fuzzy subsets of a $\Gamma-$semiring M. Then $f \circ g, f + g, f \cup g, f \cap g$, are defined by

$$f \circ g(z) = \begin{cases} \underset{z=x\alpha y}{sup}\ \{min\{f(x), g(y)\}\}, \\ 0, \quad \text{otherwise} \end{cases} \qquad f + g(z) = \begin{cases} \underset{z=x+y}{sup}\ \{min\{f(x), g(y)\}\}, \\ 0, \quad \text{otherwise} \end{cases}$$

$$f \cup g(z) = max\{f(z), g(z)\} \ \ ; \ f \cap g(z) = min\{f(z), g(z)\}$$

for all $x, y, z \in M, \alpha \in \Gamma$.

Definition 2.17. Let A be a non-empty subset of M. The characteristic function of A is a fuzzy subset of M, defined by

$$\chi_{_A}(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

3. Tri- ideals of Γ-semirings

In this section, we introduce the notion of tri-ideal as a generalization of bi-ideal, quasi-ideal and interior ideal of a Γ-semiring and study the properties of tri-ideal of a Γ-semiring.Throughout this paper$M$ is a Γ-semiring with unity element.

Definition 3.1. A non-empty subset B of a Γ-semiring M is said to be right tri-ideal of M, if B is a Γ-subsemiring of M and $B\Gamma B\Gamma M\Gamma B \subseteq B$.

Definition 3.2. A non-empty subset B of a Γ-semiring M is said to be left tri-ideal of M, if B is a Γ-subsemiring of M and $B\Gamma M\Gamma B\Gamma B \subseteq B$.

Definition 3.3. A non-empty subset B of a Γ-semiring M is said to be tri-ideal of M, if B is a Γ-subsemiring of M and B is a left tri-ideal and a right tri-ideal of M.

Remark 3.4. A tri-ideal of a Γ-semiring M need not be quasi-ideal, interior ideal, bi-interior ideal and bi-quasi ideal of a Γ-semiring M.

Example 3.5. Let I be the set of all rational numbers and $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in I \right\}$
Then M is a Γ-semiring with respect to usual addition of matrices and ternary operation is defined as the usual matrix multiplication. If $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, 0 \neq b \in Q \right\}$, then A is a right tri-ideal but not a bi-ideal of the Γ-semiring M.

Example 3.6.

(i) Let Q be the set of all rational numbers, $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Q \right\}$ be the additive semigroup of M matrices and $\Gamma = M$. A ternary operation $A\alpha B$ is defined as usual matrix multiplication of A, α, B, for all $A, \alpha, B \in M$. Then M is a Γ-semiring

(a) If $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$, then R is a quasi ideal of the Γ-semiring M and R is neither a left ideal nor a right ideal.

(b) If $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid 0 \neq a \in Q \right\}$, then S is a bi-ideal of the Γ-semiring M.

(ii) If $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\}$ and $\Gamma = M$ then M is a Γ-semiring with respect to usual addition of matrices and ternary operation is defined as usual matrix multiplication and $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$. Thus A is not a bi-ideal of the Γ-semiring M.

(iii) If $M = \left\{ \begin{pmatrix} a & c \\ b & 0 \end{pmatrix} \mid a, b, c \in Q \right\}$ and $\Gamma = M$, then M is a Γ-semiring with respect to usual addition of matrices and ternary operation is defined as usual matrix multiplication and $A = \left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$. Thus A is a tri- ideal of the Γ-semiring M.

Example 3.7. Let N be a the set of all even natural numbers and $\Gamma = N$ be additive abelian semigroups.A ternary operation is defined as $(x, \alpha, y) \to x + \alpha + y$, where $+$ is the usual addition of integers. Then N is a $\Gamma-$semiring. A subset $I = 4N$ of N is a tri-ideal of N but not bi-ideal,quasi-ideal, interior ideal, bi-interior ideal and bi-quasi ideal of Γ-semiring N.

In the following theorem, we mention some important properties and we omit the proofs since they are straightforward.

Theorem 3.8. *Let M be a Γ-semring. Then the following are hold.*

(1) *Every left ideal is a tri-ideal of M.*
(2) *Every right ideal is a tri-ideal of M.*
(3) *Every quasi ideal is a tri-ideal of M.*
(4) *Every ideal is a tri-ideal of M.*
(5) *Intersection of a right ideal and a left ideal of M is a tri-ideal of M.*
(6) *If L is a left ideal and R is a right ideal of a Γ-semiring M then $B = R\Gamma L$ is a tri-ideal of M.*
(7) *If B is a tri-ideal and T is an interior ideal of M then $B \cap T$ is a tri-ideal of M.*
(8) *Let M be a Γ-semiring and B be a Γ-subsemiring of M. If $M\Gamma M\Gamma M\Gamma B \subseteq B$ then B is a left tri-ideal of M.*
(9) *Let M be a Γ-semiring and B be a Γ-subsemiring of M. If $M\Gamma M\Gamma M\Gamma B \subseteq B$ and $B\Gamma M\Gamma M\Gamma M \subseteq B$ then B is a tri-ideal of M.*
(10) *Intersection of a right tri-ideal and a left tri-ideal of M is a tri-ideal of M.*

(11) *If L is a left ideal and R is a right ideal of M then $B = R \cap L$ is a tri-ideal of M.*

Theorem 3.9. *If $B\Gamma M\Gamma B\Gamma B = B$, for all left tri- ideals B of ar Γ-semiring M, then M is a regular Γ-semiring.*

Proof. Suppose $B\Gamma M\Gamma B\Gamma B = B$, for all tri- ideals B of M. Let $B = R \cap L$, where R and L are ideals of M.
Then B is a tri-ideal of M. Thus

$$\begin{aligned}(R \cap L)\Gamma M\Gamma(R \cap L)\Gamma(R \cap L) &= R \cap L \\ &= (R \cap L)\Gamma M\Gamma(R \cap L)\Gamma(R \cap L) \\ &\subseteq R\Gamma M\Gamma L\Gamma L \\ &\subseteq R\Gamma L \\ &\subseteq R \cap L. \quad [\text{Since } R\Gamma L \subseteq L \text{ and } R\Gamma L \subseteq R)]\end{aligned}$$

So $R \cap L = R\Gamma L$. Hence M is a regular Γ-semiring. □

Theorem 3.10. *Let M be a commutative regular Γ-semiring. Then $B\Gamma M\Gamma B\Gamma B = B$, for all left tri- ideals B of M.*

Proof. Suppose M is a regular commutative Γ- semiring, B is a tri-ideal of M and $x \in B$. Then $B\Gamma M\Gamma B\Gamma B \subseteq B$ and there exist $y \in M, \alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x$. Thus $x = x\alpha y\beta y\alpha x\beta x \in B\Gamma M\Gamma B\Gamma B$. So $x \in B\Gamma M\Gamma B\Gamma B$. Hence $B\Gamma M\Gamma B\Gamma B = B$. □

4. Fuzzy tri- ideals of Γ-semirings

In this section, we introduce the notion of fuzzy right(left) tri interior ideal as a generalization of fuzzy bi-ideal of a Γ-semiring and study the properties of fuzzy right(left) tri interior ideals.

Definition 4.1. A fuzzy subset μ of a $\Gamma-$ semiring M is called a fuzzy left (right) tri interior, if

(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in M$,
(ii) $\mu \circ \chi_M \circ \mu \circ \mu \subseteq \mu\big(\mu \circ \mu \circ \chi_M \circ \mu \subseteq \mu\big)$.

A fuzzy subset μ of a Γ-semiring M is called a fuzzy tri-ideal, if it is a left tri-ideal and a right tri-ideal of M.

Example 4.2. Let Q be the set of all rational numbers and $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\}$ and $\Gamma = M$. Then M is a Γ-semiring with respect to usual addition of matrices and a ternary operation is defined as the usual matrix multiplication.
If $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, 0 \neq b \in Q \right\}$, then A is a right tri-ideal but not a bi-ideal of Γ-semiring M. Define $\mu : M \to [0, 1]$ such that $\mu(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$
Then μ is a fuzzy right tri-ideal of M.

Theorem 4.3. *A non-empty fuzzy subset μ of a Γ-semiring M is a fuzzy Γ-subsemiring of M if and only if $\mu \circ \mu \subseteq \mu$.*

Proof. Suppose μ is a fuzzy Γ-subsemiring of M and $x \in M$. Then

$$\begin{aligned}\mu \circ \mu(x) &= \sup_{x=a\alpha b} \min\{\mu(a), \mu(b)\} \\ &\leq \sup_{x=a\alpha b} \mu(a\alpha b), \text{ since } \mu(a\alpha b) \geq \min\{\mu(a), \mu(b)\} \\ &= \mu(x).\end{aligned}$$

If $a, b \in M$ does not exist such that $x = a\alpha b$, then $\mu \circ \mu(x) = 0 \leq \mu(x)$, for all $x \in M$. Thus $\mu \circ \mu \subseteq \mu$.

Conversely, suppose that $\mu \circ \mu \subseteq \mu$ and $x, y \in M, \alpha \in \Gamma$. Then

$$\begin{aligned}\mu(x\gamma y) &\geq \mu \circ \mu(x\gamma y) \\ &= \sup \min\{\mu(x), \mu(y)\} \\ &\geq \min\{\mu(x), \mu(y)\},\end{aligned}$$

since μ is a fuzzy Γ-subsemiring of M. □

Theorem 4.4. *Every fuzzy right ideal of a Γ-semiring M is a fuzzy left tri-ideal of M.*

Proof. Let μ be a fuzzy right ideal of the $\Gamma-$ semiring M and $x \in M$. Then

$$\begin{aligned}\mu \circ \chi_M(x) &= \sup_{x=a\alpha b} \min\{\mu(a), \chi_M(b)\}, \ a, \ b \in M, \alpha \in \Gamma \\ &= \sup_{x=a\alpha b} \mu(a) \\ &\leq \sup_{x=a\alpha b} \mu(a\alpha b) \\ &= \mu(x).\end{aligned}$$

Thus $\mu \circ \chi_M(x) \leq \mu(x)$.

Now
$$\begin{aligned}\mu \circ \chi_M \circ \mu \circ \mu(x) &= \sup_{x=u\alpha v\beta s} \min\{\mu \circ \chi_M(u\alpha v), \mu \circ \mu(s)\} \\ &\leq \sup_{x=u\alpha v\beta s} \min\{\mu(u\alpha v), \mu(s)\} \\ &= \mu(x).\end{aligned}$$

So μ is a fuzzy left tri-ideal of the Γ-semiring M. □

Corollary 4.5. *Every fuzzy left ideal of a Γ-semiring M is a fuzzy right tri-ideal of M.*

Corollary 4.6. *Every fuzzy ideal of a Γ-semiring M is a fuzzy tri-ideal of M.*

Theorem 4.7. *Let M be a Γ-semiring and μ be a non-empty fuzzy subset of M. A fuzzy subset μ is a fuzzy left tri-ideal of a Γ-semiring M if and only if the level subset μ_t of μ is a left tri-ideal of a Γ-semiring M, for every $t \in [0, 1]$, where $\mu_t \neq \phi$.*

Proof. Let M be a Γ-semiring and μ be a non-empty fuzzy subset of M. Suppose μ is a fuzzy left tri-ideal of the Γ-semiring M, $\mu_t \neq \phi, t \in [0, 1]$ and $a, b \in \mu_t$. Then

$$\begin{aligned}&\mu(a) \geq t, \mu(b) \geq t \\ \Rightarrow &\mu(a + b) \geq \min\{\mu(a), \mu(b)\} \geq t \\ \Rightarrow &a + b \in \mu_t.\end{aligned}$$

Let $x \in \mu_t\Gamma M\Gamma\mu_t\Gamma\mu_t$. Then $x = b\alpha a\beta d\gamma c$, where $a \in M, b, c, d \in \mu_t, \alpha, \beta$ and $\gamma \in \Gamma$. Thus

$$\mu \circ \chi_M \circ \mu \circ \mu(x) \geq t$$
$$\Rightarrow \mu(x) \geq \mu \circ \chi_M \circ \mu \circ \mu(x) \geq t.$$

So $x \in \mu_t$. Hence μ_t is a left tri-ideal of the Γ-semiring M.

Conversely, suppose that μ_t is a left tri-ideal of the Γ-semiring M, for all $t \in Im(\mu)$. Let $x, y \in M, \alpha \in \Gamma, \mu(x) = t_1, \mu(y) = t_2$ and $t_1 \geq t_2$. Then $x, y \in \mu_{t_2}$.

$$\Rightarrow x + y \in \mu_{t_2} \text{ and } x\alpha y \in \mu_{t_2}$$
$$\Rightarrow \mu(x + y) \geq t_2 = \min\{t_1, t_2\} = \min\{\mu(x), \mu(y)\}.$$

Thus $\mu(x+y) \geq t_2 = \min\{\mu(x), \mu(y)\}$. We have $\mu_l\Gamma M\Gamma\mu_l\Gamma\mu_l \subseteq \mu_l$, for all $l \in Im(\mu)$. Suppose $t = \min\{Im(\mu)\}$. Then $\mu_t\Gamma M\Gamma\mu_t\Gamma\mu_t \subseteq \mu_t$. Thus $\mu \circ \chi_M \circ \mu \circ \mu \subseteq \mu$. So μ is a fuzzy left tri-ideal of the Γ-semiring M. □

Corollary 4.8. *Let M be a Γ-semiring and μ be a non-empty fuzzy subset of M. A fuzzy subset μ is a fuzzy right tri-ideal of a Γ-semiring if and only if the level subset μ_t of μ is a right tri-ideal of a Γ-semiring M for every $t \in [0, 1]$, where $\mu_t \neq \phi$.*

Theorem 4.9. *Let I be a non-empty subset of a Γ-semiring M and χ_I be the characteristic function of I. Then I is a right tri-ideal of a Γ-semiring M if and only if χ_I is a fuzzy right tri-ideal of a $\Gamma-$ semiring M.*

Proof. Let I be a non-empty subset of the Γ-semiring M and χ_I be the characteristic function of I. Suppose I is a right tri-ideal of the Γ-semiring M. Obviously χ_I is a fuzzy Γ-subsemiring of M. We have $I\Gamma I\Gamma M\Gamma I \subseteq I$. Then

$$\begin{aligned}\chi_I \circ \chi_I \circ \chi_M \circ \chi_I &= \chi_{I\Gamma I\Gamma M\Gamma I}\\ &= \chi_{I\Gamma I\Gamma M\Gamma I}\\ &\subseteq \chi_I.\end{aligned}$$

Thus χ_I is a fuzzy right tri-ideal of the $\Gamma-$semiring M.

Conversely, suppose that χ_I is a fuzzy right tri-ideal of M. Then I is a Γ-subsemiring of M. We have

$$\chi_I \circ \chi_I \circ \chi_M \circ \chi_I \subseteq \chi_I$$
$$\Rightarrow \chi_{I\Gamma I\Gamma M\Gamma I} \subseteq \chi_I.$$

Thus $I\Gamma I\Gamma M\Gamma I \subseteq I$. So I is a right tri-ideal of the Γ-semiring M. □

Theorem 4.10. *Let M be a regular $\Gamma-$ semiring. Then μ is a fuzzy left tri-ideal of M if and only if μ is a fuzzy right left ideal of M.*

Proof. Let μ be a fuzzy left tri-ideal of the $\Gamma-$ semiring M and $x \in M$. Then

$$\Rightarrow \mu \circ \chi_M \circ \mu \circ \mu \subseteq \mu$$

Since M is a regular, there exist $y \in M, \alpha, \beta \in \Gamma$, such that $x = x\alpha y\beta x$.

Then $\mu \circ \chi_M \circ \mu \circ \mu(x) = \sup_{x = x\alpha y\beta x} \min\{\mu \circ \chi_M(x\alpha y), \mu(x)\}) \leq \sup_{x = x\alpha y\beta x} \min\{\mu(x\alpha y), \mu(x)\} = \mu(x)$

$$\Rightarrow \mu \circ \chi_M(x) \leq \mu(x)$$

Hence μ is a fuzzy right ideal of M. By Theorem 4.4, Converse is obvious. □

Corollary 4.11. *Let M be a regular $\Gamma-$ semiring. Then μ is a fuzzy right tri-ideal of M if and only if μ is a fuzzy left ideal of M.*

Theorem 4.12. *If μ and λ are fuzzy right tri- ideals of a Γ-semiring M, then $\mu \cap \lambda$ is a fuzzy right tri-ideal of a $\Gamma -$ semiring M.*

Proof. Let μ and λ be fuzzy right-tri ideals of the Γ-semiring $M, x, y \in M$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned}
\mu \cap \lambda(x+y) &= \min\{\mu(x+y), \lambda(x+y)\} \\
&\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\} \\
&= \min\{\min\{\mu(x), \lambda(x)\}, \min\{\mu(y), \lambda(y)\}\} \\
&= \min\{\mu \cap \lambda(x), \mu \cap \lambda(y)\} \\
\chi_M \circ \mu \cap \lambda(x) &= \sup_{x=a\alpha b} \min\{\chi_M(a), \mu \cap \lambda(b)\} \\
&= \sup_{x=a\alpha b} \min\{\chi_M(a), \min\{\mu(b), \lambda(b)\}\} \\
&= \sup_{x=a\alpha b} \min\{\min\{\chi_M(a), \mu(b)\}, \min\{\chi_M(a), \lambda(b)\}\} \\
&= \min\{\sup_{x=a\alpha b} \min\{\chi_M(a), \mu(b)\}, \sup_{x=a\alpha b} \min\{\chi_M(a), \lambda(b)\}\} \\
&= \min\{\chi_M \circ \mu(x), \chi_M \circ \lambda(x)\} \\
&= \chi_M \circ \mu \cap \chi_M \circ \lambda(x).
\end{aligned}$$

Thus $\chi_M \circ \mu \cap \chi_M \circ \lambda = \chi_M \circ \mu \cap \lambda$. On the other hand,

$$\begin{aligned}
&\mu \cap \lambda \circ \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda(x) \\
&= \sup_{x=a\alpha b\beta c} \min\{\mu \circ \mu \cap \lambda \circ \lambda(a), \chi_M \circ \mu \cap \lambda(b\beta c)\} \\
&= \sup_{x=a\alpha b\beta c} \min\{\mu \circ \mu(a), \lambda \circ \lambda(a)\}\}, \min\{\chi_M \circ \mu(b\beta c), \chi_M \circ \lambda(b\beta c)\}\} \\
&= \sup_{x=a\alpha \beta bc} \min\{\min\{\mu \circ \mu(a), \chi_M \circ \mu(b\beta c)\}, \min\{\lambda \circ \lambda(a), \chi_M \circ \lambda(b\beta c)\}\} \\
&= \min\{\sup_{x=a\alpha b\beta c} \min\{\mu \circ \mu(a), \chi_M \circ \mu(b\beta c)\}, \sup_{x=a\alpha b\beta c} \min\{\lambda \circ \lambda(a), \chi_M \circ \lambda(b\beta c)\}\} \\
&= \min\{\mu \circ \mu \circ \chi_M \circ \mu(x), \lambda \circ \lambda \circ \chi_M \circ \lambda(x)\} \\
&= \mu \circ \mu \circ \chi_M \circ \mu \cap \lambda \circ \lambda \circ \chi_M \circ \lambda(x).
\end{aligned}$$

So $\mu \cap \lambda \circ \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = \mu \circ \mu \circ \chi_M \circ \mu \cap \lambda \circ \lambda \circ \chi_M \circ \lambda$. Hence

$$\mu \cap \lambda \circ \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = \mu \circ \mu \circ \chi_M \circ \mu \cap \lambda \circ \lambda \circ \chi_M \circ \lambda \subseteq \mu \cap \lambda.$$

Therefore $\mu \cap \lambda$ is a fuzzy right tri-ideal of the $\Gamma-$ semiring M. This completes the proof. □

Corollary 4.13. *If μ and λ are fuzzy left tri- ideals of a Γ-semiring M, then $\mu \cap \lambda$ is a fuzzy left tri-ideal of a Γ-semiring M.*

Corollary 4.14. *Let μ and λ be fuzzy tri- ideals of a Γ-semiring M. Then $\mu \cap \lambda$ is a fuzzy tri-ideal of a Γ-semiring M.*

Theorem 4.15. *If μ and λ are fuzzy right tri- ideals of a Γ-semiring M, then $\mu \cup \lambda$ is a fuzzy right tri-ideal of a Γ-semiring M.*

Proof. Let μ and λ be fuzzy right- tri ideals of the Γ-semiring $M, x, y \in M$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned}
\mu \cup \lambda(x+y) &= \max\{\mu(x+y), \lambda(x+y)\} \\
&\geq \max\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\} \\
&= \min\{\max\{\mu(x), \lambda(x)\}, \max\{\mu(y), \lambda(y)\}\} \\
&= \min\{\mu \cup \lambda(x), \mu \cup \lambda(y)\} \\
\chi_M \circ \mu \cup \lambda(x) &= \sup_{x=a\alpha b} \min\{\chi_M(a), \mu \cup \lambda(b)\} \\
&= \sup_{x=a\alpha b} \min\{\chi_M(a), \max\{\mu(b), \lambda(b)\}\} \\
&= \sup_{x=a\alpha b} \max\{\min\{\chi_M(a), \mu(b)\}, \min\{\chi_M(a), \lambda(b)\}\} \\
&= \max\{\sup_{x=a\alpha b} \min\{\chi_M(a), \mu(b)\}, \sup_{x=a\alpha b} \min\{\chi_M(a), \lambda(b)\}\} \\
&= \max\{\chi_M \circ \mu(x), \chi_M \circ \lambda(x)\} \\
&= \chi_M \circ \mu \cup \chi_M \circ \lambda(x).
\end{aligned}$$

Thus $\chi_M \circ \mu \cup \chi_M \circ \lambda = \chi_M \circ \mu \cup \lambda$. On the other hand,

$$\begin{aligned}
&\mu \cup \lambda \circ \mu \cup \lambda \circ \chi_M \circ \mu \cup \lambda(x) \\
&= \sup_{x=a\alpha b\beta c} \min\{\mu \circ \mu \cup \lambda \circ \lambda(a), \chi_M \circ \mu \cup \lambda(b\beta c)\} \\
&= \sup_{x=a\alpha\beta bc} \max\{\min\{\mu \circ \mu(a), \chi_M \circ \mu(b\beta c)\}, \min\{\lambda \circ \lambda(a), \chi_M \circ \lambda(b\beta c)\}\} \\
&= \max\{\sup_{x=a\alpha b\beta c} \min\{\mu \circ \mu(a), \chi_M \circ \mu(b\beta c)\}, \sup_{x=a\alpha b\beta c} \min\{\lambda \circ \lambda(a), \chi_M \circ \lambda(b\beta c)\}\} \\
&= \max\{\mu \circ \mu \circ \chi_M \circ \mu(x), \lambda \circ \lambda \circ \chi_M \circ \lambda(x)\} \\
&= \mu \circ \mu \circ \chi_M \circ \mu \cup \lambda \circ \lambda \circ \chi_M \circ \lambda(x).
\end{aligned}$$

So $\mu \cup \lambda \circ \mu \cup \lambda \circ \chi_M \circ \mu \cup \lambda = \mu \circ \mu \circ \chi_M \circ \mu \cup \lambda \circ \lambda \circ \chi_M \circ \lambda$. Hence

$$\mu \cup \lambda \circ \mu \cup \lambda \circ \chi_M \circ \mu \cup \lambda = \mu \circ \mu \circ \chi_M \circ \mu \cup \lambda \circ \lambda \circ \chi_M \circ \lambda \subseteq \mu \cup \lambda.$$

Therefore $\mu \cup \lambda$ is a fuzzy right tri-ideal of the Γ-semiring M. This completes the proof. □

Corollary 4.16. *If μ and λ are fuzzy left tri- ideals of a Γ-semiring M then $\mu \cup \lambda$ is a fuzzy left tri-ideal of a Γ-semiring M.*

Corollary 4.17. *Let μ and λ be fuzzy tri- ideals of a Γ-semiring M. Then $\mu \cup \lambda$ is a fuzzy tri-ideal of a Γ-semiring M.*

Theorem 4.18. *M is a regular Γ-semiring if and only if $\mu = \mu \circ \chi_M \circ \mu \circ \mu$, for any fuzzy left tri- ideal μ of a Γ-semiring M.*

Proof. Let μ be a fuzzy left tri-ideal of the regular Γ-semiring M and $x, y \in M, \alpha, \beta \in \Gamma$. Then $\mu \circ \chi_M \circ \mu \circ \mu \subseteq \mu$. Thus

$$\begin{aligned}
\mu \circ \chi_M \circ \mu \circ \mu(x) &= \sup_{x=x\alpha y\beta x} \{\min\{\mu \circ \chi_M(x), \mu \circ \mu(y\beta x)\}\} \\
&\geq \sup_{x=x\alpha y\beta x} \{\min\{\mu(x), \mu(y\beta x)\}\} \\
&= \mu(x).
\end{aligned}$$

So $\mu \subseteq \mu \circ \chi_M \circ \mu \circ \mu$. Hence $\mu \circ \chi_M \circ \mu \circ \mu = \mu$.

Conversely, suppose that $\mu = \mu \circ \chi_M \circ \mu \circ \mu$, for any fuzzy tri-ideal μ of the Γ-semiring M. Let B be a left tri-ideal of the Γ-semiring M. Then by Theorem 4.9, χ_B be a fuzzy left tri-ideal of the Γ-semiring M. Thus

$$\begin{aligned} \chi_B &= \chi_B \circ \chi_M \circ \chi_B \circ \chi_B \\ &= \chi_{B\Gamma M\Gamma B\Gamma B} \\ B &= B\Gamma M\Gamma B\Gamma B. \end{aligned}$$

So by Theorem 3.9, M is a regular Γ-semiring. □

Theorem 4.19. *Let M be a Γ-semiring. Then M is a regular if and only if $\mu \cap \gamma \subseteq \mu \circ \gamma \circ \mu \circ \mu$, for every fuzzy left tri-ideal μ and every fuzzy ideal γ of Γ-semiring M.*

Proof. Let M be a regular Γ-semiring and $x \in M$. Then there exist $y \in M, \alpha, \beta \in \Gamma$. such that $x = x\alpha y\beta x$. Thus

$$\begin{aligned} &\mu \circ \gamma \circ \mu \circ \mu(x) \\ =& \sup_{x = x\alpha y\beta x} \{\min\{\mu \circ \gamma(x\alpha y), \mu \circ \mu(x)\}\} \\ =& \min\{ \sup_{x\alpha y = x\alpha y\beta x\alpha y} \{\min\{\mu(x), \gamma(y\beta x\alpha y)\}, \sup_{x = x\alpha y\beta x} \{\min\{\mu(x), \mu(y\beta x)\}\}\} \\ \geq& \min\{\min\{\mu(x), \gamma(x)\}, min\{\mu(x), \gamma(x)\}\} \end{aligned}$$

So $\mu \cap \gamma \subseteq \mu \circ \gamma \circ \mu \circ \mu$.

Conversely, suppose that the condition holds. Let μ be a fuzzy left tri-ideal of the Γ-semiring M. Then $\mu \cap \chi_M \subseteq \mu \circ \chi_M \circ \mu \circ \mu$. thus $\mu \subseteq \mu \circ \chi_M \circ \mu \circ \mu$. So by Theorem 4.15, M is a regular semiring. □

Corollary 4.20. *Let M be a Γ-semiring. Then M is a regular if and only if $\mu \cap \gamma \subseteq \mu \circ \mu \circ \gamma \circ \mu$, for every fuzzy right tri-ideal μ and every fuzzy ideal γ of Γ-semiring M.*

5. Conclusion

As a further generalization of ideals, we introduced the notion of a tri-ideal of Γ-semiring as a generalization of ideal, left ideal, right ideal, bi-ideal,quasi ideal, bi-quasi ideal, bi-interior ideal, bi-quasi interior ideal and interior ideal of Γ-semiring and studied some of their properties. We also introduced the notion of fuzzy right (left) tri-ideal of a Γ-semiring and characterized the regular Γ-semiring in terms of fuzzy right(left) tri- ideals of a Γ-semiring and studied some of their algebraical properties. In continuity of this paper, we study prime tri- ideals, maximal and minimal tri- ideals of Γ-semirings.

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