Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 18, No. 2, (October 2019) pp. 173–180
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2019.18.2.173

@FMI

Fuzzy sublattices of lattices

Ch. Santhi Sundar Raj, K. Rama Prasad, K. Ramanuja Rao, Natnael Teshale A.

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 18, No. 2, October 2019

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 18, No. 2, (October 2019) pp. 173–180
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2019.18.2.173

@FMI

Fuzzy sublattices of lattices

Ch. Santhi Sundar Raj, K. Rama Prasad, K. Ramanuja Rao, Natnael Teshale A.

Received 13 June 2019; Revised 26 June 2019; Accepted 4 July 2019

Abstract. In this paper, we introduce the notion of L-fuzzy sublattice of a bounded lattice with truth values in a complete lattice satisfying the infinite meet distributive law and prove certain general properties of these, by observing that these form an algebraic fuzzy set system.

2010 AMS Classification: 06D72, 06F15, 08A72.

Keywords: L- fuzzy sublattice, Frame, Algebraic fuzzy set system.

Corresponding Author: Ch.Santhi Sundar Raj (santhisundarraj@yahoo.com)

1. Introduction

Ever since Zadeh [17] introduced the notion of a fuzzy subset of a non empty set X as a function from X into the unit interval $[0, 1]$, several algebraists studied fuzzy subalgebras of various algebraic systems such as groups, rings, modules, lattices with membership function assuming truth values in the unit intervel $[0, 1]$ of real numbers. Rosenfeld [8] defined the notion of a fuzzy subgroup of a group and since then several researchers worked on fuzzy subrings and ideals of rings [6, 7], fuzzy ideals of lattices [1, 5], fuzzy subspaces of a vecter space [4] and so on. In the above mentioned, the fuzzy statements take truth values in the interval $[0, 1]$ of real numbers, while crisp statements take truth values in the two-element set $\{0, 1\}$. However, Gougen [3] realised that the interval $[0, 1]$ is insufficient to have the truth values of general fuzzy statements and it is necessary to consider a more general class of lattices in place of $[0, 1]$. Swamy and other researchers in [9, 10, 11, 12, 13, 14, 15, 16][9-16] used a complete lattice satisfying the infinite meet distributivity, which are called frames, to have the truth values of general fuzzy statements.

In this paper, we introduce the notion of an L-fuzzy sublattice of a bounded lattice $(A, \wedge, \vee, 0, 1)$, having truth values (as suggested by Gougen [3]) in a complete lattice L satisfying the infinite meet distributive law and we prove certain important structural properties of these. Mainly, we observe that the set $\mathcal{FS}_L(A)$ of all L-fuzzy sublattice of A is an algebric fuzzy set system.

Throughout this paper, $L = (L, \wedge, \vee, 0, 1)$ stands for a non-trivial complete lattice in which the infinite meet distributive law is satsified. That is;

$$a \wedge \big(\bigvee_{s\in S} s\big) = \bigvee_{s\in S} (a \wedge s)$$

for any $S \subseteq L$ and $a \in L$ and A stands for a bounded lattice $(A, \wedge, \vee, 0, 1)$. We consider L-fuzzy subsets (for simplicity, fuzzy subsets) of A in the sense of Gougen [3]. Accordingly, an L-fuzzy subset of A is a mapping of A into L. If L is the unit interval $[0, 1]$ of real numbers, these are the usual fuzzy subsets of A.

2. L-fuzzy sublattices

A non-empty subset S of a bounded lattice $(A, \wedge, \vee, 0, 1)$ is called a sublattice of A if it is closed under the binary operations $\wedge$ and $\vee$ and containing smallest element 0 and largest element 1 of A. Let $\mathcal{S}(A)$ be the set of all sublattices of A. Then it is well known that $\mathcal{S}(A)$ is an algebraic closure set system on A (i.e, closed under arbitrary intersections and unions of upward directed subclasses of $\mathcal{S}(A)$). In the following, we introduce the notion of an L-fuzzy sublattice of a given bounded lattice $(A, \wedge, \vee, 0, 1)$ and prove certain properties of these.

Definition 2.1. An L-fuzzy subset λ of A is said to be an L-fuzzy sublattice of A if and only if it satisfies the following conditions:

$$\lambda(0) = 1 = \lambda(1) \text{ and}$$
$$\lambda(x) \wedge \lambda(y) \leq \lambda(x \wedge y) \wedge \lambda(x \vee y), \text{ for all } x, y \in A.$$

We can identify any sublattice of A with an L-fuzzy sublattice of A by the following theorem. For any subset S of A the characterstic map $\chi_S : A \to L$ is defined by

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

Theorem 2.2. *For any subset S of A, S is a sublattice of A if and only if χ_S is an L-fuzzy sublattice of A.*

Next we characterize L-fuzzy sublattices by their α-cuts. First let us recall that, for any L-fuzzy subset λ of A and $\alpha \in L$, the α-cut of λ is defined by

$$\lambda^{-1}[\alpha, 1] = \{x \in A : \alpha \leq \lambda(x)\}.$$

Note that the α-cut of χ_S is given by

$$\chi_S^{-1}[\alpha, 1] = \begin{cases} A & \text{if } \alpha = 0 \\ S & \text{if } \alpha \neq 0. \end{cases}$$

The above theorem can rephrased as χ_S is an L-fuzzy sublattice of A if and only if $\chi_S^{-1}[\alpha, 1]$ is a sublattice of A, for each $\alpha \in L$.

Theorem 2.3. *An L-fuzzy subset λ of A is an L-fuzzy sublattice of A if and only if $\lambda^{-1}[\alpha, 1]$ is a sublattice of A, for each $\alpha \in L$.*

Proof. Suppose that λ is an L-fuzzy sublattice of A and $\alpha \in L$. Then $\lambda(0) = 1 = \lambda(1)$. Thus $0,\ 1 \in \lambda^{-1}[\alpha, 1]$. So $\lambda^{-1}[\alpha, 1] \neq \varnothing$.

Let x and $y \in \lambda^{-1}[\alpha, 1]$. Then $\alpha \leq \lambda(x)$ and $\alpha \leq \lambda(y)$. Thus

$$\alpha \leq \lambda(x) \wedge \lambda(y) \leq \lambda(x \wedge y) \text{ and } \alpha \leq \lambda(x) \vee \lambda(y) \leq \lambda(x \vee y).$$

So $x \wedge y,\ x \vee y \in \lambda^{-1}[\alpha, 1]$. Hence $\lambda^{-1}[\alpha, 1]$ is a sublattice of A.

Conversely, suppose that for each $\alpha \in L, \lambda^{-1}[\alpha, 1]$ is a sublattice of A. In particular, $\lambda^{-1}[1, 1]$ is a sublattice of A and then $0, 1 \in \lambda^{-1}[1, 1]$, it follows that $\lambda(0) = 1 = \lambda(1)$. Let x and $y \in A$. Put $\alpha = \lambda(x) \wedge \lambda(y)$. Then $\alpha \leq \lambda(x)$ and $\alpha \leq \lambda(y)$. Thus $x,\ y \in \lambda^{-1}[\alpha, 1]$. Since $\lambda^{-1}[\alpha, 1]$ is a sublattice of A, $x \vee y,\ x \wedge y \in \lambda^{-1}[\alpha, 1]$. Thus $\lambda(x) \wedge \lambda(y) \leq \lambda(x \wedge y)$ and $\lambda(x) \vee \lambda(y) \leq \lambda(x \vee y)$. So λ is an L-fuzzy sublattice of A. □

Let $\mathcal{FS}_L(A)$ denote the set of all L-fuzzy sublattices of A. We have the point-wise ordering on $\mathcal{FS}_L(A)$ which is defined by

$$\lambda \leq \mu \text{ if and only if } \lambda(x) \leq \mu(x)$$

for all $x \in A$ and for any $\lambda, \mu \in \mathcal{FS}_L(A)$. The following is straightforward verification.

Theorem 2.4. *$\big(\mathcal{FS}_L(A), \leq\big)$ is a complete lattice in which, for any $\{\lambda_i\}_{i \in \Delta} \subseteq \mathcal{FS}_L(A)$, the infimum and supremum are given by*

$$\bigwedge_{i \in \Delta} \lambda_i = \textit{ The point-wise infimum of } \{\lambda_i\}_{i \in \Delta}$$

and

$$\bigvee_{i \in \Delta} \lambda_i = \bigwedge \big\{\lambda \in \mathcal{FS}_L(A) : \lambda_i \leq \lambda \ \textit{ for all } \ i \in \Delta\big\}.$$

For any $S \subseteq A$, we denote the sublattice of A generated by S by $\langle S \rangle$ (i.e; the smallest sublattice containing S). It is well known that $S \mapsto \langle S \rangle$ is an algebraic closure operator on A and hence

$$\langle S \rangle = \bigcup \big\{\langle T \rangle : T \text{ is a finite subset of } S\big\}.$$

Note that the set $\{0, 1\}$ is the small sublattice of the bounded lattice $(A, \wedge, \vee, 0, 1)$. Now, we prove an important characterization of an L-fuzzy sublattice of A and this will be used repeatedly throughout this paper.

Theorem 2.5. *The following are equivalent to each other, for any L-fuzzy subset λ of A:*

(1) *λ is an L-fuzzy sublattice of A,*
(2) *for any $S \subseteq A$ and $a \in \langle S \rangle$, $\lambda(a) \geq \bigwedge\limits_{x \in S} \lambda(x)$,*
(3) *for any $S \subseteq A$, $\bigwedge\limits_{x \in S} \lambda(x) = \bigwedge\limits_{x \in \langle S \rangle} \lambda(x)$.*

Proof. (1)⇒(2): Suppose that λ is an L-fuzzy sublattice of A and $S \subseteq A$. Put $\alpha = \bigwedge\limits_{x \in S} \lambda(x)$. Then $\alpha \leq \lambda(x)$, for all $x \in S$. Thus $S \subseteq \lambda^{-1}[\alpha, 1]$. By Theorem 2.3, $\lambda^{-1}[\alpha, 1]$ is a sublattice of A containing S. So $\langle S \rangle \subseteq \lambda^{-1}[\alpha, 1]$. Hence $\alpha \leq \lambda(a)$, for all $a \in \langle S \rangle$. Therefore $\lambda(a) \geq \bigwedge\limits_{x \in S} \lambda(x)$, for all $a \in \langle S \rangle$.

(2)⇒(1): Suppose that the condition (2) holds. In particular, if $S = \varnothing$, then $\bigwedge_{x\in S} \lambda(x) = 1$. Thus $\lambda(a) \geq 1$, for all $a \in \langle\varnothing\rangle$. Since $\langle\varnothing\rangle = \{0,1\}$, $\lambda(0) \geq 1$ and $\lambda(1) \geq 1$. This implies $\lambda(0) = 1 = \lambda(1)$. If $S = \{x, y\}$, then $x \wedge y$ and $x \vee y \in \langle S\rangle$. Thus $\lambda(x) \wedge \lambda(y) \leq \lambda(x \wedge y) \wedge \lambda(x \vee y)$. So λ is an L-fuzzy sublattice A.

(2)⇒(3): Suppose that the condition (2) holds. Then we have

$$\bigwedge_{x\in S} \lambda(x) \leq \bigwedge_{a\in\langle S\rangle} \lambda(a) \leq \bigwedge_{x\in S} \lambda(x).$$

Thus $\bigwedge_{x\in S} \lambda(x) = \bigwedge_{x\in\langle S\rangle} \lambda(x)$.

(3)⇒(2): The proof is trivial. □

If $\{\lambda_i : i \in\triangle\}$ is a family of L-fuzzy sublattices of A, then $\bigwedge_{i\in\triangle} \lambda_i$ (the point-wise infimum of $\lambda_i's$) is an L-fuzzy sublattice of A (by Theorem 2.4) and in particular, if λ is any L-fuzzy subset of A then the point-wise infimum of all L-fuzzy sublattices of A containing λ is an L-fuzzy sublattice of A and which becomes the L-fuzzy sublattice $\overline{\lambda}$ generated by λ. In the following we give a precise discription of the L-fuzzy sublattice $\overline{\lambda}$ generated by a given L-fuzzy subset λ of A. We write $S \Subset A$ to mean that S is a finite subset of A.

Theorem 2.6. *Let λ be an L-fuzzy subset of A. Then, the L-fuzzy sublattice $\overline{\lambda}$ generated by λ is given by*

$$\overline{\lambda}(x) = \bigvee \Big\{ \bigwedge_{a\in S} \lambda(a) : S \Subset A \text{ and } x \in \langle S\rangle \Big\},$$

for any $x \in A$.

Proof. Let $\nu(x) = \bigvee \big\{ \bigwedge_{a\in S} \lambda(a) : S \Subset A \text{ and } x \in \langle S\rangle \big\}$. We shall prove that ν is the smallest L-fuzzy sublattice of A containing λ. If $S = \{x\}$, then $x \in \langle S\rangle$, for any $x \in A$. Thus $\lambda(x) \leq \nu(x)$, for all $x \in A$. So $\lambda \leq \nu$. Since the set $\{0, 1\}$ is the smallest sublattice of A containing the empty set $\varnothing$, $0,\ 1 \in \langle\varnothing\rangle$ and $\bigwedge_{a\in\varnothing} \lambda(a) = 1$. This implies that $1 \leq \nu(0)$ and $1 \leq \nu(1)$. Hence $\nu(0) = 1 = \nu(1)$.

Now, using the infinite meet distributivity in L, we have

$$\nu(x) \wedge \nu(y) = \bigvee \Big\{ \big(\bigwedge_{a\in S} \lambda(a)\big) \wedge \big(\bigwedge_{b\in T} \lambda(b)\big) : S, T \Subset A \text{ and } x \in \langle S\rangle, y \in \langle T\rangle \Big\}.$$

Let S and $T \Subset A$ and $x \in \langle S\rangle, y \in \langle T\rangle$ and $F = S \cup T$. Then $F \Subset A$ and $x, y \in \langle F\rangle$. Thus $x \wedge y,\ x \vee y \in \langle F\rangle$. So

$$\big(\bigwedge_{a\in S} \lambda(a)\big) \wedge \big(\bigwedge_{b\in T} \lambda(b)\big) = \bigwedge_{a,b\in F} \big(\lambda(a) \wedge \lambda(b)\big) \leq \nu(x \wedge y) \wedge \nu(x \vee y).$$

It follows that, $\nu(x) \wedge \nu(y) \leq \nu(x \wedge y) \wedge \nu(x \vee y)$. Hence ν is an L-fuzzy sublattice of A.

If μ is any other L-fuzzy sublattice of A containing λ, then for any finite subset S in A and $x \in \langle S\rangle$, we have,

$$\bigwedge_{a\in S} \lambda(a) \leq \bigwedge_{a\in S} \mu(a) \leq \mu(x). \quad \text{(by Theorem 2.5)}$$

Thus $\nu(x) \leq \mu(x)$ .for all $x \in A$. So $\nu \leq \mu$. Hence $\overline{\lambda} = \nu$. □

It can be easily proved that the supremum of a class $\{\lambda_i\}_{i\in\Delta}$ of L-fuzzy sublattices of A is given by

$$\bigvee_{i\in\Delta} \lambda_i = \overline{\mu}, \text{ where } \mu(x) = \bigvee\{\lambda_i(x) : i \in \Delta\}.$$

and then by Theorem 2.6,

$$(\bigvee_{i\in\Delta} \lambda_i)(x) = \bigvee\Big\{ \bigwedge_{a\in S} \mu(a) : S \Subset A \text{ and } x \in \langle S\rangle\Big\}.$$

Since $\langle S\rangle = \bigcup\big\{\langle T\rangle : T \Subset S\big\}$ for any $S \subseteq A$, it can be easily verified that

$$\overline{\lambda}(x) = \bigvee\Big\{ \bigwedge_{a\in S} \lambda(a) : S \subseteq A \text{ and } x \in \langle S\rangle\Big\}.$$

Let us recall that the constant maps $\overline{0}$ and $\overline{1}$ of A are defined by $\overline{0}(x) = 0$, the smallest element in L and $\overline{1}(x) = 1$, the largest element in L. It can be easily verified that the L-fuzzy sublattice generated by $\overline{0}$ is the smallest L-fuzzy sublattice of A and $\overline{1}$ is the largest L-fuzzy sublattice of A. These ideas are generalized in the following.

Theorem 2.7. *For any sublattice S of A, $\overline{\chi}_S = \chi_{\langle S\rangle}$.*

Proof. For any $S \subseteq A$, we have

$$\overline{\chi}_S(x) = \bigvee\Big\{ \bigwedge_{a\in T} \chi_S(a) : T \subseteq A \text{ and } x \in \langle T\rangle\Big\}.$$

Since $S \subseteq \langle S\rangle, \chi_S \leq \chi_{\langle S\rangle}$. Then $\overline{\chi_S} \leq \chi_{\langle S\rangle}$. On the other hand, if $x \notin \langle S\rangle$, then $\chi_{\langle S\rangle}(x) = 0 \leq \overline{\chi_S}(x)$. If $x \in \langle S\rangle$, then $\overline{\chi_S}(x) \geq \bigwedge_{a\in S} \chi_S(a) = 1 = \chi_{\langle S\rangle}(x)$. Thus $\overline{\chi}_S = \chi_{\langle S\rangle}$. □

Next, we describe the α-cut of $\overline{\lambda}$. First let us recall that, for any $\alpha \in L$ and $C \subseteq L$, C is called a cover of α if $\alpha \leq Sup\ C$.

Theorem 2.8. *Let λ be an L-fuzzy subset of A and $\overline{\lambda}$ is the L-fuzzy sublattice of A generated by λ. Then for any $\alpha \in L$,*

$$\overline{\lambda}^{-1}[\alpha, 1] = \bigcup\Big\{ \bigcap_{\beta\in C} \langle\lambda^{-1}[\beta, 1]\rangle : C \textit{ is a cover of } \alpha \textit{ in } L\Big\}.$$

Proof. For any $x \in A$, we have

$$\overline{\lambda}(x) = \bigvee\Big\{ \bigwedge_{a\in S} \lambda(x) : S \subseteq A \quad \text{and} \quad x \in \langle S\rangle\Big\}.$$

Let $\alpha \in L$. Suppose there exists a cover C of α in L such that $x \in \langle\lambda^{-1}[\beta, 1]\rangle$, for all $\beta \in C$. Then for any $a \in \lambda^{-1}[\beta, 1]$, we have $\beta \leq \lambda(a)$. Thus

$$\beta \leq \bigwedge_{a\in\lambda^{-1}[\beta,1]} \lambda(a) \leq \overline{\lambda}(x).$$

So $\alpha \leq \text{Sup } C \leq \overline{\lambda}(x)$. Hence $x \in \overline{\lambda}^{-1}[\alpha, 1]$.

On the other hand, let $x \in \overline{\lambda}^{-1}[\alpha, 1]$. Then $\alpha \leq \overline{\lambda}(x)$. For any $S \subseteq A$ with $x \in \langle S \rangle$, let $\beta_S = \bigwedge_{a \in C} \lambda(a)$ and $C = \{\beta_S : S \subseteq A \text{ and } x \in \langle S \rangle\}$. Then, clearly C is a cover of α in L. Let $\beta_S \in C$. Then for any $a \in S$, we have $\beta_S \leq \lambda(a)$. Thus $S \subseteq \lambda^{-1}[\beta_S, 1]$. So $x \in \langle S \rangle \subseteq \langle \lambda^{-1}[\beta_S, 1] \rangle$, for every $\beta_S \in C$. □

In the following we define α-level L-fuzzy sublattice of a bounded lattice $(A, \wedge, \vee, 0, 1)$, which slightly generalize the notion of the characterstic map χ_S corresponding to a sublattice S of A.

Theorem 2.9. *For any subset S of A and $\alpha \in L$, we define $\alpha_S : A \to L$ by*

$$\alpha_S(x) = \begin{cases} 1 & \text{if } x \in S \\ \alpha & \text{if } x \notin S. \end{cases}$$

Then $\overline{\alpha}_S = \alpha_{\langle S \rangle}$.

Corollary 2.10. *For any $S \subseteq A$ and $\alpha \in L$, α_S is an L-fuzzy sublattice of A if and only if S is a sublattice of A.*

It can be seen that the correspondence $S \mapsto \alpha_S$ establishes an isomorphism from the lattice $\mathcal{S}(A)$ of all sublattices of A onto the lattice of α-level L-fuzzy sublattices of A. Also, for any proper sublattice S of A, the mapping $\alpha \mapsto \alpha_S$ is an isomorphism of L onto a complete sublattice of the lattice of L-fuzzy sublattices of A.

Now the following theorem provides a method for constructing L-fuzzy sublattice of A from their α-cuts satisfying certain conditions.

Theorem 2.11. *Let $\{S_\alpha : \alpha \in L\}$ be a family of sublattices of A such that $\bigcap_{\alpha \in M} S_\alpha = S_{\bigwedge_{\alpha \in F} \alpha}$ for all $M \subseteq L$. Define $\lambda : A \to L$ by*

$$\lambda(x) = \bigvee \{\alpha \in L : x \in S_\alpha\}.$$

Then λ is an L-fuzzy sublattice of A such that S_α is precisely $\lambda^{-1}[\alpha, 1]$, for any $\alpha \in L$ and conversely every L-fuzzy sublattice of A can be defined as above.

Proof. By the definition of λ, for any $x \in A$ and $\alpha \in L$, we have

$$x \in S_\alpha \Rightarrow \alpha \leq \lambda(x) \Rightarrow x \in \lambda^{-1}[\alpha, 1].$$

Then $S_\beta \subseteq \lambda^{-1}[\beta, 1,]$ for all $\beta \in L$. Clearly, $\alpha \mapsto S_\alpha$ is an antitone.

Now,
$$\begin{aligned} x \in \lambda^{-1}[\beta, 1] &\Rightarrow \beta \leq \lambda(x) = \vee\{\alpha \in L : x \in S_\alpha\} \\ &\Rightarrow \beta = \beta \wedge \big(\vee \{\alpha \in L : x \in S_\alpha\}\big) \\ &\Rightarrow \beta = \vee\{\beta \wedge \alpha : x \in S_\alpha\} \\ &\Rightarrow S_\beta = \bigcap_{x \in S_\alpha} S_{\beta \wedge \alpha} \\ &\Rightarrow x \in \bigcap_{x \in S_\alpha} S_{\beta \wedge \alpha} = S_\beta \text{ (since } S_\alpha \subseteq S_{\beta \wedge \alpha}). \end{aligned}$$

Thus $\lambda^{-1}[\beta, 1] = S_\beta$, for all $\beta \in L$. So by Theorem 2.3, λ is an L-fuzzy sublattice of A.

The converse is clear, since for an L-fuzzy subset λ of A, $\bigcap_{\alpha \in M} \lambda^{-1}[\alpha, 1] = \lambda^{-1}\big[\bigvee_{\alpha \in M} \alpha, 1\big]$, for any $M \subseteq L$ and clearly, $\lambda(x) = \bigvee \{\alpha \in L : x \in \lambda^{-1}[\alpha, 1]\}$. □

Corollary 2.12. *Let $S_1 \subseteq S_2 \subseteq S_3 \subseteq \dots$ be an increasing sequence of sublattices of A such that $\bigcup_{n=1}^{\infty} S_n = A$ and $1 = \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$ be a decreasing sequence of elements in L. Then, the L-fuzzy subset λ defined by $\lambda(x) = \alpha_n$, where n is the least integer such that $x \in S_n$, is an L-fuzzy sublattice of A and the α-cut of λ is given by*

$$\lambda_\alpha = \begin{cases} A & \text{if } \alpha \leq \alpha_n \text{ for all } n \\ S_n & \text{if } n \text{ is the largest such that } \alpha_n \geq \alpha. \end{cases}$$

Corollary 2.13. *Let $S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots \subseteq S_n = A$ be finite increasing sequence of sublattices of A and $1 = \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots \geq \alpha_n$ be a finite decreasing sequence of elements in L. Then, the L-fuzzy subset λ defined by $\lambda(x) = \alpha_i$, if i is the least integer such that $x \in S_i$ is an L-fuzzy sublattice of A.*

A class $\mathscr{C}$ of L-fuzzy subsets of a non-empty set X is called directed above if, for any $\lambda, \mu \in \mathscr{C}$, there exists $\gamma \in \mathscr{C}$ such that $\lambda \leq \gamma$ and $\mu \leq \gamma$. $\mathscr{C}$ is said to be an algebraic fuzzy set system if, $\mathscr{C}$ is closed under point-wise infimums and closed under the point-wise supremums of directed above subclsses of $\mathscr{C}$.

Theorem 2.14. *The class $\mathcal{FS}_L(A)$ of all L-fuzzy sublattices of A is an algebraic fuzzy set system.*

Proof. By Theorem 2.4, we have $\mathcal{FS}_L(A)$ is closed under point-wise infimums. Let $\{\lambda_i : i \in \triangle\}$ be a directed above subclass of $\mathcal{FS}_L(A)$ and for any $x \in A$,

$$\mu(x) = \bigvee_{i \in \triangle} \lambda_i(x) = Sup\{\lambda_i(x) : i \in \triangle\}.$$

Then clearly, $\mu(0) = 1 = \mu(1)$, since each λ_i is an L-fuzzy sublattice of A. Now, let $x, y \in A$. Then by the infinite meet distributivity in L,

$$(*) \qquad \mu(x) \wedge \mu(y) = \big(\bigvee_{i \in \triangle} \lambda_i(x)\big) \wedge \big(\bigvee_{i \in \triangle} \lambda_i(y)\big) = \bigvee_{i,j \in \triangle} \big(\lambda_i(x) \wedge \lambda_j(y)\big).$$

Since for any $i, j \in \triangle$, there exists $k \in \triangle$ such that $\lambda_i \leq \lambda_k$ and $\lambda_j \leq \lambda_k$ and

$$\lambda_i(x) \wedge \lambda_j(y) \leq \lambda_k(x) \wedge \lambda_k(y) \leq \lambda_k(x \wedge y) \wedge \lambda_k(x \vee y).$$

Thus by ($\star$), it follows that $\mu(x) \wedge \mu(y) \leq \mu(x \wedge y) \wedge \mu(x \vee y)$. So μ is an L-fuzzy sublattice of A. □

3. Conclusions

In this paper, we have studied the structural theory of fuzzy sublattices of a bounded lattice with truth values in a complete lattice satisfying the infinite meet distributive law such a lattice is called a frame. Here, we have proved that the class of fuzzy sublattices of a bounded lattice is a complete lattice and form an algebraic fuzzy system. We want to know whether this class form an algebraic lattice or not and to investigate prime and maximal fuzzy sublattices and prime spectrum of fuzzy sublattices. We leave these concepts for future study.

Acknowledgements. Authors are very thankful to the anonymous reviewers for the valuable suggestions and comments to improve this paper.

References

[1] M. Attallah, Completely fuzzy prime ideals of distributie laattices, The Journal of Fuzzy Mathematics 8 (2000) 151–156.
[2] G. Birkhoff, Lattice Theory, Colloquium Publications, Amer. Math. Soc. Vol. 25 1967.
[3] J. A. Goguen, L-fuzzy sets, J. Math. Anal. Appl. 18 (1967) 145–174.
[4] A. K. Katsaras and D. B. Liu, Fuzzy vector spaces and fuzzy topological vector spaces, J. Math. Anal. Appl. 58 (1977) 135–146.
[5] S. Lehmke, Some properties of fuzzy ideals of lattices, Fuzzy-IEEE'97 (1997) 813–818.
[6] D. S. Malik and J. N. Mordeson, Fuzzy maximal, radical and primary ideals of rings, Inform. Sci. 53 (1991) 237–250.
[7] D. S. Malik and J. N. Mordeson, Extensions of fuzzy subrings and fuzzy ideals, Fuzzy sets and systems 45 (1992) 245–251.
[8] A. Rosenfeld, Fuzzy groups, J. Maths. Anal. Appl. 35 (1971) 512–517.
[9] Ch. Santhi Sundar Raj, A. Natnael Teshale and U. M. Swamy, Fuzzy Prime Ideals of ADLs, International Journal of Computing Science and Applied Mathematics 4 (2) (2018) 32–36.
[10] Ch. Santhi Sundar Raj, A. Natnael Teshale and U. M. Swamy, Fuzzy Maximal Ideals of ADLs, International Journal of Mathematics and Its application 6 (1-D) (2018) 819–824.
[11] U. M. Swamy, Ch. Santhi Sundar Raj and Natnael Teshale A. Fuzzy Ideals of Almost Distributive Lattices, Ann. Fuzzy Math. Inform. 14 (4) (2017) 371–379.
[12] U. M. Swamy, Ch. Santhi Sundar Raj and A. Natnael Teshale, L-Fuzzy Filters of Almost Distributive Lattices, International Journal of Mathematics and Soft Computing 8 (1) (2018) 35–43.
[13] U. M. Swamy and D. V. Raju, Algebraic Fuzzy systems, Fuzzy sets and systems 41 (1991) 187–194.
[14] U. M. Swamy and D. V. Raju, Irridducibility in Algebraic Fuzzy systems, Fuzzy sets and systems 41 (1991) 233–241.
[15] U. M. Swamy and D. V. Raju, Fuzzy ideals and congruences of lattices, Fuzzy sets and systems 95 (1998) 249–253.
[16] U. M. Swamy and K. L. N. Swamy, Fuzzy Prime Ideals of Rings, J. Math. Anal. Appl. 134 (1988) 94–103.
[17] L. A. Zadeh, Fuzzy set, Information and control 8 (1965) 338–353.

Ch. SANTHI SUNDAR RAJ (santhisundarraj@yahoo.com)
Department of Mathematics,Andhara University,Visakhapatnam 530003,A.P, India

K. RAMA PRASAD (ramaprasadkotni0@gmail.com)
Department of Mathematics,Andhara University,Visakhapatnam 530003,A.P, India

K.RAMANUJA RAO (ramanuja.kotti@fnu.ac.fj)
Department of Mathematics, Fiji National University Lautoka, FIJI

NATNAEL TESHALE A. (yenatnaelau@yahoo.com)
Department of Mathematics,Andhara University,Visakhapatnam 530003,A.P, India