Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 18, No. 2, (October 2019) pp. 161–171
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2019.18.2.161

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 18, No. 2, October 2019

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Received 1 June 2019; Accepted 23 June 2019

Abstract. This paper focuses on solving fuzzy variables linear programming problems with rough intervals coefficients (FVRILP). First, FVRILP problems can be decomposed into three linear programming problems with rough intervals coefficients (RILP) using a crisp linear technique. Then, interval method is utilized to convert each RILP problem into two linear programming problems with interval coefficients (ICLP). Second, the ensuing final, crisp linear programming problems are constructed for each ICLP problem and finally, they are solved to obtain a fuzzy rough solution for FVRILP problems in general. A numerical example is given to illustrate the applicability of the proposed method.

2010 AMS Classification: 90C05, 90C70, 90C99

Keywords: Fuzzy linear programming, Fuzzy variables, Interval coefficients, Rough interval coefficients.

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1. Introduction

Fuzzy linear programming (FLP) can be separated into two models. FLP with fuzzy coefficients and FLP with fuzzy constraints while the decision variables in these types are considered as crisp. In the fuzzy variables linear programming (FVLP), the decision-making variables are fuzzy while the coefficients of the objective function and the right-hand side coefficients are not fuzzy. Xiaozhong et al. [17] discussed the fuzzy linear programming problems with fuzzy variables and fuzzy coefficients. Nasseri and Ardil [13] proposed a method depends on linear ranking functions and simplex method to find the fuzzy basic feasible solution of the (FVLP). Pandian and Jayalakskmi [15] suggested a decomposition method to find the optimal solution of the integer linear programming problems with fuzzy variables. Some authors of [1, 4, 6, 10, 11] have presented several approaches based on ranking functions to transform the fuzzy linear programming problems into the corresponding deterministic linear programming problems.

Rough Intervals are helpful tools to treat the uncertainty in the decision-making problems. The linear programming with interval coefficients in the objective functions and/or in the constraints has been studied by several authors, [3, 7, 16]. Xu and Yao [18] have presented two approaches for solving random rough multi-objective programming problems. One is the interactive satisfying method which is used to find the satisfying solution of the decision maker, and the other applies the technique of random rough simulation-based genetic algorithm using compromise approach is obtained for solving the general random rough multi-objective programming problems. The basic knowledge of possibly optimal range, surely optimal range, rather satisfactory solutions, completely satisfactory solutions and rough optimal range of the linear programming with rough interval coefficients are discussed by Hamzehee et al. [5]. They transform the rough intervals linear programming problem into two linear programming problems with interval coefficients.

This paper is organized into five sections. In the next section, definitions of arithmetic operations on fuzzy numbers and on rough intervals that are necessary for the discussion here are reviewed. In Section 3, fuzzy variables linear programming problems with rough interval coefficients (FVRILP) are formulated, and a methodology is given to solve this type of problem. An illustrative numerical example for the proposed method is presented in Section 4. Finally, conclusions are drawn in Section 5.

2. Preliminaries

In this section, basic notation related to the fuzzy numbers and rough intervals used in this paper are given.

Definition 2.1 ([2])**.** A fuzzy number $\tilde{a} = (a_1, a_2, a_3)$ is a triangular fuzzy number, if its membership function is given by:

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\ 1, & x = a_2, \\ \frac{x-a_3}{a_2-a_3}, & a_2 \leq x \leq a_3, \\ 0, & otherwise. \end{cases}$$

Definition 2.2 ([9])**.** Let $\tilde{a} = (a_1, a_2, a_3)$ and $\tilde{b} = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. The basic arithmetic operations are defined as follows:

(i) Addition: $\tilde{a} + \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$,

(ii) Subtraction: $\tilde{a} - \tilde{b} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$,

(iii) Scalar multiplication: $k\tilde{a} = (ka_1, ka_2, ka_3)$ if $k \geq 0$,
$k\tilde{a} = (ka_3, ka_2, ka_1)$ if $k < 0$,

(iv) Multiplication: $\tilde{a} \times \tilde{b} = (a_1b_1, a_2b_2, a_3b_3)\,,\ a_1 \geq 0,\ ,$

$$\tilde{a} \times \tilde{b} = (a_1b_3, a_2b_2, a_3b_3)\,,\ a_1 < 0,\ a_3 \geq 0,$$

$$\tilde{a} \times \tilde{b} = (a_1b_3, a_2b_2, a_3b_1)\,,\ a_3 < 0.$$

Definition 2.3 ([12])**.** Let $A = \left(\left[\underline{a}^L, \underline{a}^U\right], \left[\overline{a}^L, \overline{a}^U\right]\right)$ and $B = \left(\left[\underline{b}^L, \underline{b}^U\right], \left[\overline{b}^L, \overline{b}^U\right]\right)$ be two rough intervals. The basic arithmetic operations are defined as follows:

(i) Addition: $A + B = \left(\left[\underline{a}^L + \underline{b}^L, \underline{a}^U + \underline{b}^U\right], \left[\overline{a}^L + \overline{b}^L, \overline{a}^U + \overline{b}^U\right]\right)$,

(ii) Subtraction: $A - B = \left(\left[\underline{a}^L - \underline{b}^U, \underline{a}^U - \underline{b}^L\right], \left[\overline{a}^L - \overline{b}^U, \overline{a}^U - \overline{b}^L\right]\right)$,

(iii) Negation: $-A = \left(\left[-\underline{a}^U, -\underline{a}^L\right], \left[-\overline{a}^U, -\overline{a}^L\right]\right)$,

(iv) Intersection: $A \cap B$

$$= \left(\left[max\left\{\underline{a}^L, \underline{b}^L\right\}, min\left\{\underline{a}^U, \underline{b}^U\right\}\right], \left[max\left\{\overline{a}^L, \overline{b}^L\right\}, min\left\{\overline{a}^U, \overline{b}^U\right\}\right]\right),$$

(v) Union: $A \cup B$

$$= \left(\left[min\left\{\underline{a}^L, \underline{b}^L\right\}, max\left\{\underline{a}^U, \underline{b}^U\right\}\right], \left[min\left\{\overline{a}^L, \overline{b}^L\right\}, max\left\{\overline{a}^U, \overline{b}^U\right\}\right]\right).$$

3. Methodology

3.1. **Formulation of the fuzzy variables linear programming problems with rough interval coefficients (FVRILP).** Let us consider the following FVRILP problems:

$$(FVRILP) : max\tilde{F} = \sum_{j=1}^{n} \left(\left[\underline{c}_j^L, \underline{c}_j^U\right], \left[\overline{c}_j^L, \overline{c}_j^U\right]\right) \otimes \tilde{x}_j, \tag{3.1}$$

subject to

$$\sum_{j=1}^{n} \left(\left[\underline{a}_{ij}^L, \underline{a}_{ij}^U\right], \left[\overline{a}_{ij}^L, \overline{a}_{ij}^U\right]\right) \otimes \tilde{x}_j \leq \tilde{b}_i \ \ (i = 1, 2, \ldots, m)$$

$$\tilde{x}_j \geq 0 \ \ (j = 1, 2, \ldots, n),$$

where $\tilde{x}_j \ \ (j = 1, 2, \ldots, n)$ is a vector of fuzzy decision variables, $\tilde{b}_i \ \ (i = 1, 2, \ldots, m)$ is a vector of fuzzy constants, and $\left(\left[\underline{c}_j^L, \underline{c}_j^U\right], \left[\overline{c}_j^L, \overline{c}_j^U\right]\right)$ and $\left(\left[\underline{a}_{ij}^L, \underline{a}_{ij}^U\right], \left[\overline{a}_{ij}^L, \overline{a}_{ij}^U\right]\right)$ $(i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n)$ are rough interval coefficients of objective function and constraints respectively.

3.2. **Defuzzification process.** Let all the fuzzy parameters and fuzzy variables for both objective function and constraints in Problem (3.1) be represented by triangular fuzzy numbers, each of which has values as follows:

$\tilde{F} = (F_1, F_2, F_3)$, $\tilde{x}_j = (t_j, y_j, z_j)$ and $\tilde{b}_i = \left(b_i^1, b_i^2, b_i^3\right)$ $(i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n)$.

Hence, Problem (3.1) can be reformulated as follows:

$$max\,(F_1, F_2, F_3) = \sum_{j=1}^{n} \left(\left[\underline{c}_j^L, \underline{c}_j^U\right], \left[\overline{c}_j^L, \overline{c}_j^U\right]\right) \otimes (t_j, y_j, z_j), \tag{3.2}$$

subject to

$$\sum_{j=1}^{n} \left(\left[\underline{a}_{ij}^L, \underline{a}_{ij}^U\right], \left[\overline{a}_{ij}^L, \overline{a}_{ij}^U\right]\right) \otimes (t_j, y_j, z_j) \leq \left(b_i^1, b_i^2, b_i^3\right) \ \ (i = 1, 2, \ldots, m)$$

$$t_j, y_j, z_j \geq 0 \ \ (j = 1, 2, \ldots, n).$$

Jayalakshmi and Pandian [8] presented a crisp technique, namely, the bounded and decomposition method, to find an optimal solution for fully fuzzy linear programming problems. Using this method, FVRILP problems can be converted into three rough interval linear programming (RILP) problems.

Now, using the arithmetic operations of Definition 2.2 and the method described in [8], Problem (3.2) can be decomposed into three RILP problems:

(1) **Middle level rough interval linear programming (MRILP) problems:**

(3.3)
$$(MRILP): \; maxF_2 = \sum_{j=1}^{n} middle\ value\ of\ \left(\left[\underline{c}_j^L, \underline{c}_j^U\right], \left[\overline{c}_j^L, \overline{c}_j^U\right]\right) \otimes (t_j, y_j, z_j),$$

subject to

$$\sum_{j=1}^{n} middle\ value\ of\ \left(\left[\underline{a}_{ij}^L, \underline{a}_{ij}^U\right], \left[\overline{a}_{ij}^L, \overline{a}_{ij}^U\right]\right) \otimes (t_j, y_j, z_j) \leq \left(b_i^1, b_i^2, b_i^3\right) (i = 1, 2, \ldots, m)$$

$$t_j, y_j, z_j \geq 0 \; (j = 1, 2, \ldots, n).$$

(2) **Upper level rough interval linear programming (URILP) problems:**

(3.4)
$$(URILP): \; maxF_3 = \sum_{j=1}^{n} upper\ value\ of\ \left(\left[\underline{c}_j^L, \underline{c}_j^U\right], \left[\overline{c}_j^L, \overline{c}_j^U\right]\right) \otimes (t_j, y_j, z_j),$$

subject to

$$\sum_{j=1}^{n} upper\ value\ of\ \left(\left[\underline{a}_{ij}^L, \underline{a}_{ij}^U\right], \left[\overline{a}_{ij}^L, \overline{a}_{ij}^U\right]\right) \otimes (t_j, y_j, z_j) \leq \left(b_i^1, b_i^2, b_i^3\right) (i = 1, 2, \ldots, m)$$

$$\sum_{j=1}^{n} upper\ value\ of\ \left(\left[\underline{c}_j^L, \underline{c}_j^U\right], \left[\overline{c}_j^L, \overline{c}_j^U\right]\right) \otimes (t_j, y_j, z_j) \geq \left(\left[F_{*2}^L, F_{*2}^U\right], \left[F_2^{*L}, F_2^{*U}\right]\right),$$

$$z_j \geq y_j \; (j = 1, 2, \ldots, n),$$

$$t_j, y_j, z_j \geq 0 \; (j = 1, 2, \ldots, n),$$

where $\left(\left[F_{*2}^L, F_{*2}^U\right], \left[F_2^{*L}, F_2^{*U}\right]\right)$ is the rough optimal range of the objective function F_2, it will be illustrated later.

(3) **Lower level rough interval linear programming (LRILP) problems:**

(3.5)
$$(LRILP): \; maxF_1 = \sum_{j=1}^{n} lower\ value\ of\ \left(\left[\underline{c}_j^L, \underline{c}_j^U\right], \left[\overline{c}_j^L, \overline{c}_j^U\right]\right) \otimes (t_j, y_j, z_j),$$

subject to

$$\sum_{j=1}^{n} lower\ value\ of\ \left(\left[\underline{a}_{ij}^{L}, \underline{a}_{ij}^{U}\right],\ \left[\overline{a}_{ij}^{L},\ \overline{a}_{ij}^{U}\right]\right) \otimes (t_j, y_j\ , z_j) \leq \left(b_i^1, b_i^2, b_i^3\right)\ (i = 1, 2, \ldots, m)$$

$$\sum_{j=1}^{n} lower\ value\ of\ \left(\left[\underline{c}_{j}^{L}, \underline{c}_{j}^{U}\right],\ \left[\overline{c}_{j}^{L},\ \overline{c}_{j}^{U}\right]\right) \otimes (t_j, y_j\ , z_j) \leq \left(\left[F_{*2}^{L}, F_{*2}^{U}\right],\ \left[F_2^{*L},\ F_2^{*U}\right]\right),$$

$$t_j \leq y_j\ \ (j = 1, 2, \ldots, n),$$

$$t_j, y_j\ , z_j \geq 0\ \ (j = 1, 2, \ldots, n).$$

3.3. **Crisp linear programming problems.** Before we go any further, the following definitions and theorems are needed:

Theorem 3.1 ([3]). *Consider the inequality* $\sum_{j=1}^{n}\left[\underline{a}_{ij},\ \overline{a}_{ij}\right]x_j \leq \left[\underline{b}_i,\ \overline{b}_i\right]$, *where* $x_j \geq 0\ \ (\ j = 1,\ 2,\ \ldots,\ n)$. *Then,* $\sum_{j=1}^{n}\overline{a}_{ij}x_j \leq \underline{b}_i$ *and* $\sum_{j=1}^{n}\underline{a}_{ij}x_j \leq \overline{b}_i$ *are the minimum value range and maximum value range inequalities, respectively.*

Theorem 3.2. *Consider the inequality* $\sum_{j=1}^{n}\left[\underline{a}_{ij},\ \overline{a}_{ij}\right]x_j \geq \left[\underline{b}_i,\ \overline{b}_i\right]$, *where* $x_j \geq 0\ \ (\ j = 1,\ 2,\ \ldots,\ n)$. *Then,* $\sum_{j=1}^{n}\overline{a}_{ij}x_j \geq \underline{b}_i$ *and* $\sum_{j=1}^{n}\underline{a}_{ij}x_j \leq \overline{b}_i$ *are the minimum value range and maximum value range inequalities, respectively.*

Let $F_r = \left(\underline{F}_r,\ \overline{F}_r\right) = \left(\left[\underline{F}_r^{L}, \underline{F}_r^{U}\right],\ \left[\overline{F}_r^{L},\ \overline{F}_r^{U}\right]\right)\ (r = 1, 2, 3)$.

Definition 3.3. If $F_{*r} = \left(\left[F_{*r}^{L}, F_{*r}^{U}\right],\ \left[F_r^{*L},\ F_r^{*U}\right]\right)\ (r = 1, 2, 3)$, is the rough optimal range of Problems (3.3)- (3.5), then the rough optimal range of Problem (3.2) is defined as:

$$\left(\left[\sum_{r=1}^{3} F_{*r}^{L}, \sum_{r=1}^{3} F_{*r}^{U}\right],\ \left[\sum_{r=1}^{3} F_r^{*L},\ \sum_{r=1}^{3} F_r^{*U}\right]\right).$$

Definition 3.4. The optimal solution of Problem (3.2) which has optimal value in $\left[\sum_{r=1}^{3} F_{*r}^{L}, \sum_{r=1}^{3} F_{*r}^{U}\right]$ $\left(\left[\sum_{r=1}^{3} F_r^{*L},\ \sum_{r=1}^{3} F_r^{*U}\right]\right)$ is called a completely (rather) satisfactory solution.

(1) **Crisp middle level linear programming (CMLP) problems:**

Using the method suggested in [5], MRILP problems are converted into two linear programming problems with interval coefficients (ICLP), as follows:

$max\overline{F}_2 = \sum_{j=1}^{n}\left[\overline{c}_j^{L},\ \overline{c}_j^{U}\right]y_j$ (1.a)	$max\underline{F}_2 = \sum_{j=1}^{n}\left[\underline{c}_j^{L}, \underline{c}_j^{U}\right]y_j$ (1.b)
subject to	subject to
$\sum_{j=1}^{n}\left[\overline{a}_{ij}^{L},\ \overline{a}_{ij}^{U}\right]y_j \leq b_i^2\ \ (i = 1, 2, \ldots, m)$	$\sum_{j=1}^{n}\left[\underline{a}_{ij}^{L}, \underline{a}_{ij}^{U}\right]y_j \leq b_i^2\ \ (i = 1, 2, \ldots, m)$
$y_j \geq 0\ \ (j = 1, 2, \ldots, n)$.	$y_j \geq 0\ \ (j = 1, 2, \ldots, n)$.

The ICLP Problems (1.a) and (1.b) are transformed to four crisp linear programming (LP) problems, as follows:

$max\overline{F}_2^U = \sum_{j=1}^n \overline{c}_j^U y_j$ (2.a)
subject to
$\sum_{j=1}^n \overline{a}_{ij}^L y_j \leq b_i^2 \ (i = 1, 2, \ldots, m)$
$y_j \geq 0 \ (j = 1, 2, \ldots, n).$

$max\overline{F}_2^L = \sum_{j=1}^n \overline{c}_j^L y_j$ (2.b)
subject to
$\sum_{j=1}^n \overline{a}_{ij}^U y_j \leq b_i^2 \ (i = 1, 2, \ldots, m)$
$y_j \geq 0 \ (j = 1, 2, \ldots, n).$

$max\underline{F}_2^U = \sum_{j=1}^n \underline{c}_j^U y_j$ (2.c)
subject to
$\sum_{j=1}^n \underline{a}_{ij}^L y_j \leq b_i^2 \ (i = 1, 2, \ldots, m)$
$y_j \geq 0 \ (j = 1, 2, \ldots, n).$

$max\underline{F}_2^L = \sum_{j=1}^n \underline{c}_j^L y_j$ (2.d)
subject to
$\sum_{j=1}^n \underline{a}_{ij}^U y_j \leq b_i^2 \ (i = 1, 2, \ldots, m)$
$y_j \geq 0 \ (j = 1, 2, \ldots, n).$

Solving the problems (2.a)–(2.d), the rough optimal range to MRILP Problem (3.3) is $\left(\left[F_{*2}^L, F_{*2}^U\right], \ \left[F_2^{*L}, \ F_2^{*U}\right]\right)$ with completely satisfactory solutions $\left(y_{*j}^L, \ y_{*j}^U\right)$ and rather satisfactory solutions $\left(y_j^{*L}, \ y_j^{*U}\right)$.

(2) **Crisp upper level linear programming (CULP) problems:**

Using the method suggested in [5], URILP problems are converted into two ICLP problems, as follows:

$max\overline{F}_3 = \sum_{j=1}^n \left[\overline{c}_j^L, \ \overline{c}_j^U\right] z_j$ (3.a)
subject to
$\sum_{j=1}^n \left[\overline{a}_{ij}^L, \ \overline{a}_{ij}^U\right] z_j \leq b_i^3, \ (i = 1, 2, \ldots, m)$
$\sum_{j=1}^n \left[\overline{c}_j^L, \ \overline{c}_j^U\right] z_j \geq \left[F_2^{*L}, \ F_2^{*U}\right],$
$z_j \geq y_j, \ (j = 1, 2, \ldots, n)$
$z_j \geq 0, \ (j = 1, 2, \ldots, n).$

$max\underline{F}_3 = \sum_{j=1}^n \left[\underline{c}_j^L, \underline{c}_j^U\right] z_j$ (3.b)
subject to
$\sum_{j=1}^n \left[\underline{a}_{ij}^L, \underline{a}_{ij}^U\right] z_j \leq b_i^3, \ (i = 1, 2, \ldots, m)$
$\sum_{j=1}^n \left[\underline{c}_j^L, \underline{c}_j^U\right] z_j \geq \left[F_{*2}^L, F_{*2}^U\right],$
$z_j \geq y_j, \ (j = 1, 2, \ldots, n)$
$z_j \geq 0, \ (j = 1, 2, \ldots, n).$

The ICLP Problems (3.a) and (3.b) are transformed to four LP problems, as follows:

$max\overline{F}_3^U = \sum_{j=1}^n \overline{c}_j^U z_j$ (4.a)
subject to
$\sum_{j=1}^n \overline{a}_{ij}^L z_j \leq b_i^3 \ (i = 1, 2, \ldots, m)$
$\sum_{j=1}^n \overline{c}_j^L z_j \leq F_2^{*U},$
$z_j \geq y_j \ (j = 1, 2, \ldots, n),$
$z_j \geq 0 \ (j = 1, 2, \ldots, n).$

$max\overline{F}_3^L = \sum_{j=1}^n \overline{c}_j^L z_j$ (4.b)
subject to
$\sum_{j=1}^n \overline{a}_{ij}^U z_j \leq b_i^3 \ (i = 1, 2, \ldots, m)$
$\sum_{j=1}^n \overline{c}_j^U z_j \geq F_2^{*L},$
$z_j \geq y_j \ (j = 1, 2, \ldots, n),$
$z_j \geq 0 \ (j = 1, 2, \ldots, n).$

$max\underline{F}_3^U = \sum_{j=1}^n \underline{c}_j^U z_j$ (4.c)
subject to
$\sum_{j=1}^n \underline{a}_{ij}^L z_j \leq b_i^3 \ (i = 1, 2, \ldots, m)$
$\sum_{j=1}^n \underline{c}_j^L z_j \leq F_{*2}^U,$
$z_j \geq y_j \ (j = 1, 2, \ldots, n),$
$z_j \geq 0 \ (j = 1, 2, \ldots, n).$

$max\underline{F}_3^L = \sum_{j=1}^n \underline{c}_j^L z_j$ (4.d)
subject to
$\sum_{j=1}^n \underline{a}_{ij}^U z_j \leq b_i^3 \ (i = 1, 2, \ldots, m)$
$\sum_{j=1}^n \underline{c}_j^U z_j \geq F_2^{*L},$
$z_j \geq y_j \ (j = 1, 2, \ldots, n),$
$z_j \geq 0 \ (j = 1, 2, \ldots, n).$

Solving the problems (4.a)–(4.d), the rough optimal range to URILP Problem (3.4) is $\left(\left[F_{*3}^L, F_{*3}^U\right], \ \left[F_3^{*L}, \ F_3^{*U}\right]\right)$ with completely satisfactory solutions $\left(z_{*j}^L, \ z_{*j}^U\right)$ and rather satisfactory solutions $\left(z_j^{*L}, \ z_j^{*U}\right)$.

(3) **Crisp lower level linear programming (CLLP) problems:**

Using the method suggested in [5], LRILP problems are converted into two ICLP problems, as follows:

$max\overline{F}_1 = \sum_{j=1}^{n} \left[\overline{c}_j^L,\ \overline{c}_j^U\right] t_j$ (5.a)
subject to
$\sum_{j=1}^{n} \left[\overline{a}_{ij}^L,\ \overline{a}_{ij}^U\right] t_j \leq b_i^1$ $(i = 1, 2, \dots, m)$
$\sum_{j=1}^{n} \left[\overline{c}_j^L,\ \overline{c}_j^U\right] t_j \leq \left[F_2^{*L},\ F_2^{*U}\right]$,
$t_j \leq y_j$ $(j = 1, 2, \dots, n)$
$t_j \geq 0$ $(j = 1, 2, \dots, n)$.

$max\underline{F}_1 = \sum_{j=1}^{n} \left[\underline{c}_j^L, \underline{c}_j^U\ \right] t_j$ (5.b)
subject to
$\sum_{j=1}^{n} \left[\underline{a}_{ij}^L, \underline{a}_{ij}^U\ \right] t_j \leq b_i^1$ $(i = 1, 2, \dots, m)$
$\sum_{j=1}^{n} \left[\underline{c}_j^L, \underline{c}_j^U\ \right] t_j \leq \left[F_{*2}^L, F_{*2}^U\ \right]$,
$t_j \leq y_j$ $(j = 1, 2, \dots, n)$
$t_j \geq 0$ $(j = 1, 2, \dots, n)$.

The ICLP Problems (5.a) and (5.b) are transformed to four LP problems, as follows:

$max\overline{F}_1^U = \sum_{j=1}^{n} \overline{c}_j^U t_j$ (6.a)
subject to
$\sum_{j=1}^{n} \overline{a}_{ij}^L t_j \leq b_i^1$ $(i = 1, 2, \dots, m)$
$\sum_{j=1}^{n} \overline{c}_j^L t_j \leq F_2^{*U}$,
$t_j \leq y_j$ $(j = 1, 2, \dots, n)$,
$t_j \geq 0$ $(j = 1, 2, \dots, n)$.

$max\overline{F}_1^L = \sum_{j=1}^{n} \overline{c}_j^L t_j$ (6.b)
subject to
$\sum_{j=1}^{n} \overline{a}_{ij}^U t_j \leq b_i^1$ $(i = 1, 2, \dots, m)$
$\sum_{j=1}^{n} \overline{c}_j^U t_j \leq F_2^{*L}$,
$t_j \leq y_j$ $(j = 1, 2, \dots, n)$,
$t_j \geq 0$ $(j = 1, 2, \dots, n)$.

$max\underline{F}_1^U = \sum_{j=1}^{n} \underline{c}_j^U t_j$ (6.c)
subject to
$\sum_{j=1}^{n} \underline{a}_{ij}^L t_j \leq b_i^1$ $(i = 1, 2, \dots, m)$
$\sum_{j=1}^{n} \underline{c}_j^L t_j \leq F_{*2}^U$,
$t_j \leq y_j$ $(j = 1, 2, \dots, n)$,
$t_j \geq 0$ $(j = 1, 2, \dots, n)$.

$max\underline{F}_1^L = \sum_{j=1}^{n} \underline{c}_j^L t_j$ (6.d)
subject to
$\sum_{j=1}^{n} \underline{a}_{ij}^U t_j \leq b_i^1$ $(i = 1, 2, \dots, m)$
$\sum_{j=1}^{n} \underline{c}_j^U t_j \leq F_2^{*L}$,
$t_j \leq y_j$ $(j = 1, 2, \dots, n)$,
$t_j \geq 0$ $(j = 1, 2, \dots, n)$.

Solving the problems (6.a)–(6.d), the rough optimal range to LRILP Problem (3.5) is $\left(\left[F_{*1}^L, F_{*1}^U\ \right],\ \left[F_1^{*L},\ F_1^{*U}\right]\right)$ with completely satisfactory solutions $\left(t_{*j}^L,\ t_{*j}^U\right)$ and rather satisfactory solutions $\left(t_j^{*L},\ t_j^{*U}\right)$.

Hence, the rough optimal range of FVRILP problems (3.1) is $\left(\left[\sum_{r=1}^{3} F_{*r}^L, \sum_{r=1}^{3} F_{*r}^U\ \right],\ \left[\sum_{r=1}^{3} F_r^{*L},\ \sum_{r=1}^{3} F_r^{*U}\right]\right)$ with completely satisfactory fuzzy solution is $\left(\tilde{x}_{*j}^L,\ \tilde{x}_{*j}^U\right)$ and rather satisfactory fuzzy solution is $(\tilde{x}_j^{*L}$ and $\tilde{x}_j^{*U})$.

4. Numerical example

The following example demonstrates the computational procedure for an FVRILP problem:

$$\max\tilde{F} = ([1, 4],\ [0, 6])\ \tilde{x}_1 + ([2, 3],\ [1, 4])\ \tilde{x}_2,$$

subject to

$$([2, 4],\ [1, 4])\ \tilde{x}_1 + ([3, 5],\ [2, 6])\ \tilde{x}_2 \leq \tilde{b}_1,$$

$$([3, 4],\ [2, 6])\ \tilde{x}_1 + ([1, 3],\ [0, 5])\ \tilde{x}_2 \leq \tilde{b}_2,$$

$$\tilde{x}_1,\ \tilde{x}_2 \geq 0.$$

The first step in the procedure is to use arithmetical operations [9] and the bound and decomposition method [8] to transform an FVRILP problem into RILP problem.

Let $\tilde{F} = (F_1, F_2\ , F_3)$, $\tilde{x}_j = (t_j, y_j\ , z_j)$, $(j = 1,\ 2)$, $\tilde{b}_1 = (9, 6\ , 11)$ and $\tilde{b}_2 = (4, 8\ , 10)$, then an RILP problem can be formulated, as follows:

$$\max F_1 = ([1, 4],\ [0, 6])\ t_1 + ([2, 3],\ [1, 4])\ t_2,$$

$$\max F_2 = ([1,4]\,,\ [0,6])\ y_1 + ([2,3]\,,\ [1,4])\, y_2,$$

$$\max F_3 = ([1,4]\,,\ [0,6])\ z_1 + ([2,3]\,,\ [1,4])\, z_2,$$

subject to

$$([2,4]\,,\ [1,4])\, t_1 + ([3,5]\,,\ [2,6])\, t_2 \leq 6,$$

$$([2,4]\,,\ [1,4])\, y_1 + ([3,5]\,,\ [2,6])\, y_2 \leq 9,$$

$$([2,4]\,,\ [1,4])\, z_1 + ([3,5]\,,\ [2,6])\, z_2 \leq 11,$$

$$([3,4]\,,\ [2,6])\, t_1 + ([1,3]\,,\ [0,5])\, t_2 \leq 4,$$

$$([3,4]\,,\ [2,6])\, y_1 + ([1,3]\,,\ [0,5])\, y_2 \leq 8,$$

$$([3,4]\,,\ [2,6])\, z_1 + ([1,3]\,,\ [0,5])\, z_2 \leq 10,$$

$$t_1, y_1\ , z_1, t_2, y_2\ , z_2 \geq 0.$$

Using bound and decomposition method [8], an RILP problem can be decomposed into three RILP problems, namely, MRILP, URILP and LRILP problems, as follows:

(1) **MRILP Problem:**

$$\text{MRILP}:\ \max F_2 = ([1,4]\,,\ [0,6])\ y_1 + ([2,3]\,,\ [1,4])\, y_2,$$

subject to

$$([2,4]\,,\ [1,4])\, y_1 + ([3,5]\,,\ [2,6])\, y_2 \leq 9,$$

$$([3,4]\,,\ [2,6])\, y_1 + ([1,3]\,,\ [0,5])\, y_2 \leq 8,$$

$$y_1,\ \ y_2 \geq 0.$$

Using the interval method [5], MRILP problem is transformed to LP problems LP1, LP2, LP3 and LP4.

LP1 : $\max \overline{F}_2^U = 6\ y_1 + 4y_2,$
subject to
$y_1 + 2y_2 \leq 9,$
$2y_1 \qquad \leq 8,$
$y_1,\ \ y_2 \geq 0.$

LP2 : $\max \overline{F}_2^L = y_2,$
subject to
$4y_1 + 6y_2 \leq 9,$
$6y_1 + 5y_2 \leq 8,$
$y_1,\ \ y_2 \geq 0.$

LP3 : $\max \underline{F}_2^U = 4\ y_1 + 3y_2,$
subject to
$2y_1 + 3y_2 \leq 9,$
$3y_1 + \ \ y_2 \leq 8,$
$y_1,\ \ y_2 \geq 0.$

LP4 : $\max \underline{F}_2^L = y_1 + 2y_2,$
subject to
$4y_1 + 5y_2 \leq 9,$
$4y_1 + 3y_2 \leq 8,$
$y_1,\ \ y_2 \geq 0.$

The rough optimal range to MRILP problem is $F_2^* = \left(\left[F_{*2}^L, F_{*2}^U\right],\ \left[F_2^{*L},\ F_2^{*U}\right]\right) = ([3.6,\ 13.3],\ \ [1.5,\ 34])$, completely satisfactory solution is $\left(y_{*j}^L,\ y_{*j}^U\right) = ((0,\ 1.8)\,,\ (2.1,\ 1.6))$ and rather satisfactory solution is $\left(y_j^{*L},\ y_j^{*U}\right) = ((0,\ 1.5)\,,\ (4,\ 2.5))$.

(1) **URILP Problem:**

$$\text{URILP}: \max F_3 = ([1,4]\,,\ [0,6])\ z_1 + ([2,3]\,,\ [1,4])\, z_2,$$

subject to

$$([2,4]\,,\ [1,4])\, z_1 + ([3,5]\,,\ [2,6])\, z_2 \leq 11,$$

$$([3,4]\,,\ [2,6])\, z_1 + ([1,3]\,,\ [0,5])\, z_2 \leq 10,$$

$$([1,4]\,,\ [0,6])\, z_1 + ([2,3]\,,\ [1,4])\, z_2 \geq ([3.6,\ 13.3]\,,\ [1.5,\ 34])\,,$$

$$z_1 \geq y_1,\ z_2 \geq y_2,$$

$$z_1,\ z_2 \geq 0.$$

Using the interval method [5], URILP problem is transformed to LP problems LP5, LP6, LP7 and LP8.

LP5 : $\max \overline{F}_3^U = 6\ z_1 + 4z_2,$
subject to
$z_1 + 2z_2 \leq 11,$
$2z_1 \leq 10,$
$z_2 \leq 34,$
$z_1,\ \ z_2 \geq 0.$

LP6 : $\max \overline{F}_3^L = z_2,$
subject to
$4z_1 + 6z_2 \leq 11,$
$6z_1 + 5z_2 \leq 10,$
$6z_1 + 4z_2 \geq 1.5,$
$z_1,\ \ z_2 \geq 0.$

LP7 : $\max \underline{F}_3^U = 4\ z_1 + 3z_2,$
subject to
$2z_1 + 3z_2 \leq 11,$
$3z_1 + \ z_2 \leq 10,$
$z_1 + 2z_2 \leq 13.3,$
$z_1,\ \ z_2 \geq 0.$

LP8 : $\max \underline{F}_3^L = z_1 + 2z_2,$
subject to
$4z_1 + 5z_2 \leq 11,$
$4z_1 + 3z_2 \leq 10,$
$4z_1 + 3z_2 \geq 3.6,$
$z_1,\ \ z_2 \geq 0.$

The rough optimal range to URILP problem is $F_3^* = \left(\left[F_{*3}^L, F_{*3}^U\right],\ \left[F_3^{*L},\ F_3^{*U}\right]\right) = ([4.4,\ 16.4],\ \ [1.8,\ 42])$, completely satisfactory solution is $\left(z_{*j}^L,\ z_{*j}^U\right) = ((0,\ 2.2)\,,\ (2.7,\ 1.9))$ and rather satisfactory solution is $\left(z_j^{*L},\ z_j^{*U}\right) = ((0,\ 1.8)\,,\ (5,\ 3))$.

(1) **LRILP Problem:**

$$\text{LRILP}: \max F_1 = ([1,4]\,,\ [0,6])\ t_1 + ([2,3]\,,\ [1,4])\, t_2,$$

subject to

$$([2,4]\,,\ [1,4])\, t_1 + ([3,5]\,,\ [2,6])\, t_2 \leq 6,$$

$$([3,4]\,,\ [2,6])\, t_1 + ([1,3]\,,\ [0,5])\, t_2 \leq 4,$$

$$([1,4]\,,\ [0,6])\, t_1 + ([2,3]\,,\ [1,4])\, t_2 \leq ([3.6,\ 13.3]\,,\ [1.5,\ 34])\,,$$

$$t_1 \leq y_1,\ t_2 \leq y_2,$$

$$t_1, t_2 \geq 0.$$

Using the interval method [5], LRILP problem is transformed to LP problems LP9, LP10, LP11 and LP12.

LP9 : $\max \overline{F}_1^U = 6\ t_1 + 4t_2,$
subject to
$t_1 + 2t_2 \leq 6,$
$2t_1 \quad\quad \leq 4,$
$t_2 \leq 34,$
$t_1,\ \ t_2 \geq 0.$

LP10 : $\max \overline{F}_1^L = t_2,$
subject to
$4t_1 + 6t_2 \leq 6,$
$6t_1 + 5t_2 \leq 4,$
$6t_1 + 4t_2 \leq 1.5,$
$t_1,\ \ t_2 \geq 0.$

LP11 : $\max \underline{F}_1^U = 4\ t_1 + 3t_2,$
subject to
$2t_1 + 3t_2 \leq 6,$
$3t_1 + \ \ t_2 \leq 4,$
$t_1 + 2t_2 \leq 13.3,$
$t_1,\ \ t_2 \geq 0.$

LP12 : $\max \underline{F}_3^L = t_1 + 2t_2,$
subject to
$4t_1 + 5t_2 \leq 6,$
$4t_1 + 3t_2 \leq 4,$
$4t_1 + 3t_2 \leq 3.6,$
$t_1,\ \ t_2 \geq 0.$

The rough optimal range to LRILP problem is
$F_1^* = \left(\left[F_{*1}^L, F_{*1}^U\right], \left[F_1^{*L},\ F_1^{*U}\right]\right) = ([2.4,\ 7.7],\ \ [0.4,\ 20])$, completely satisfactory solution is $\left(t_{*j}^L,\ t_{*j}^U\right) = ((0,\ 1.2),\ (0.8,\ 1.4))$ and rather satisfactory solution is $\left(t_j^{*L},\ t_j^{*U}\right) = ((0,\ 0.4),\ (2,\ 2))$.

Finally, the rough optimal range for an FVRILP problem is $([10.4,\ 37.4],\ [3.7,\ 96])$, while the completely satisfactory fuzzy solutions are $(0,\ 1.2,\ 0,\ 1.8,\ 0,\ 2.2)$ and $(0.9,\ 1.4,\ 2.1,\ 1.6,\ 2.7,\ 1.9)$ and the rather satisfactory fuzzy solutions are found $(0,\ 0.4,\ 0,\ 1.5,\ 0,\ 1.8)$ and $(2,\ 2,\ 4,\ 2.5,\ 5,\ 3)$.

5. Conclusions

This study presents a method to tackle linear programming problems with rough interval coefficients in a fuzzy environment (FVRILP). This method uses a crisp linear technique to convert FVRILP problems into three rough interval linear programming (RILP) problems. The interval method can be used to convert each RILP problem into four crisp linear programming (LP) problems. Then, each LP problem can be solved independently to find the final solution. To test the validity of this method, a numerical example is provided. The results obtained show the applicability and accuracy of the proposed method.

However, there is another interesting area of research in solving linear programming problems with rough interval coefficients in a fuzzy environment. The level-sum method of Pandian proposed in [14] is a promising point in future to be used to treat such formulated problems in this paper.

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