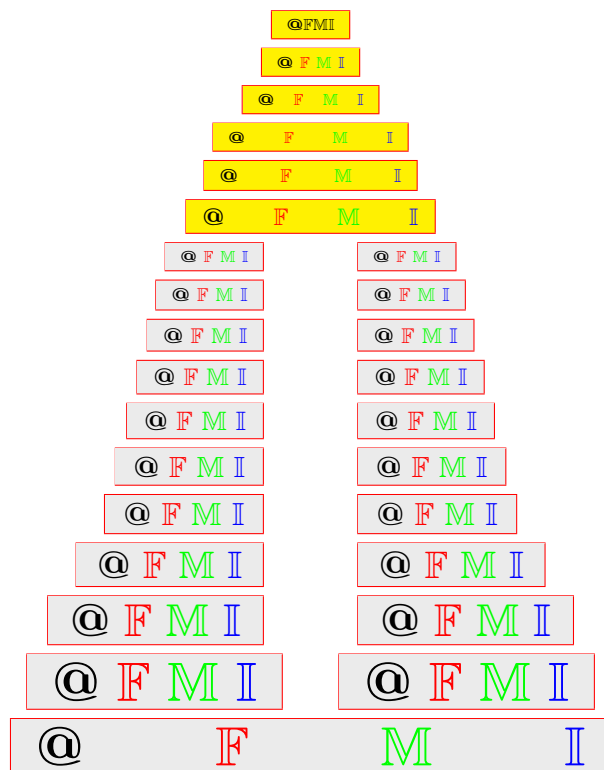


## Fuzzy ideals in demi-pseudocomplemented MS-algebras

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**ABSTRACT.** In this paper, we study fuzzy congruences, kernel fuzzy ideals and  $(\circ, *)$ -fuzzy ideals of a demi-pseudocomplemented MS-algebra  $L$ . In particular, we study the fuzzy congruence relation generated by fuzzy relation and its properties. Also we prove that  $(\circ, *)$ -fuzzy ideals form a sublattice of fuzzy ideals of  $L$ , and the set of these fuzzy ideals is isomorphic to the closed interval  $G_F$  and  $\chi_\iota$  of the fuzzy congruence lattice of  $L$  where  $G_F$  is Gliivenko fuzzy congruence and  $\chi_\iota$  is the universal fuzzy congruence.

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**Keywords:** Demi-pseudocomplemented MS-algebra, Fuzzy ideals,  $(\circ, *)$ -fuzzy ideals, Fuzzy congruences.

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### 1. INTRODUCTION

**Z**adeh [16] introduced the concepts of a fuzzy set. Fuzzy sets have been found to be very useful in diversely applied areas of science and technology. Extensive applications of the fuzzy set theory have been found in various fields such as logic programming, medical diagnosis, decision making problems and microelectronic fault analysis. Also fuzzy set theory is conveniently and successfully applied in Abstract Algebra. Rosenfeld [11] defined the notion of a fuzzy subgroup of a groups. Then many algebraists took interest to introduce fuzzy theory in various algebraic structures by fuzzyfying the formal theory. [1, 13, 15] introduced fuzzy ideals of distributive lattice. Recently, Alaba and Alemayehu [2] introduced the notion of clouser fuzzy ideals of MS-algebras. Also Alaba, Taye and Alemayehu [3] introduced the concept of  $\delta$ -fuzzy ideals in MS-algebras.

On the other hand, Blyth and Varlet [6] introduced the notion of MS-algebras as a common abstraction of de Morgan algebras and Stone algebras. Sankappanavar [12]

introduced the notion of demi-pseudocomplemented algebras, and Blyth, Fang and Wang [4] studied on ideals and congruences of distributive demi-pseudocomplemented algebras. More recently, Fang and Tan [7] characterized kernel ideals and  $(\circ, *)$ -ideals in demi-pseudocomplemented MS-algebras.

These studies motivated us to study fuzzy congruences, kernel fuzzy ideals and  $(\circ, *)$ -fuzzy ideals in demi-pseudocomplemented MS-algebras  $L$ . In particular, we study the fuzzy congruence relation generated by fuzzy relation and its properties. Also we prove that  $(\circ, *)$ -fuzzy ideals form a sublattice of fuzzy ideals of  $L$ , and the set of these fuzzy ideals is isomorphic to the interval  $[G_F, \chi_\iota]$  of the fuzzy congruence lattice of  $L$  where  $G_F$  is Glivenko fuzzy congruence, and  $\chi_\iota$  is the universal fuzzy congruence.

## 2. PRELIMINARIES

In this section, we recall basic concepts frequently used in this article.

**Definition 2.1** ([12]). A demi-pseudocomplemented algebra is an algebra  $(L, \wedge, \vee, *, 0, 1)$  in which  $(L, \wedge, \vee, 0, 1)$  is a bounded lattice and a unary operation  $x \longrightarrow x^*$  satisfying the following properties: for any  $x, y \in L$ ,

- (i)  $(x \vee y)^* = x^* \wedge y^*$ ,
- (ii)  $(x \wedge y)^{**} = x^{**} \wedge y^{**}$ ,
- (iii)  $0^* = 1$  and  $1^* = 0$ ,
- (iv)  $x^{***} = x^*$ ,
- (v)  $x^* \wedge x^{**} = 0$ .

As shown by Sankappanavara [12], in a demi-pseudocomplemented algebra, the following property holds:

- (vi)  $x^* \wedge (x^* \wedge y)^* = x^* \wedge y^*$ , for any  $x, y \in L$ .

**Definition 2.2** ([6]). An Ockham algebra is a bounded distributive lattice  $L$  together with a dual endomorphism  $f : L \longrightarrow L$ . An MS-algebra is an Ockham algebra in which dual endomorphism  $x \longrightarrow f(x)$  is determined by the inequality  $x \leq f^2(x)$ . As usual, we shall write  $x^\circ$  for  $f(x)$ .

**Definition 2.3** ([8]). A demi-pseudocomplemented Ockham algebra is an algebra  $(L, \wedge, \vee, f, *, 0, 1)$  of type  $(2, 2, 1, 1, 0, 0)$  where  $(L, \wedge, \vee, f, 0, 1)$  is an Ockham algebra,  $(L, \wedge, \vee, *, 0, 1)$  is a demi-p-lattice and the operations  $x \longrightarrow f(x)$  and  $x \longrightarrow x^*$  are linked by the identity  $f(x^*) = (f(x))^*$ .

Specially, if  $(L, \circ)$  is an MS-algebra, then  $(L, \wedge, \vee, \circ, *, 0, 1)$  is called demi-pseudocomplemented MS-algebra. We shall denote the class of demi-pseudocomplemented MS-algebra by dpMS-algebra.

**Theorem 2.4** ([8]). If  $(L, \circ, *)$  is a dpMS-algebra then the flowing statements hold:

- (1)  $x^* = x^{\circ\circ*} \forall x \in L$ ,
- (2)  $x^{\circ\circ} = x^{**} = x \forall x \in L^*$ ,
- (3)  $(x \wedge y)^* = x^* \vee y^* \forall x, y \in L$ ,
- (4)  $x^* \vee x^{**} = 1 \forall x \in L$ ,
- (5)  $x^* \wedge y = 0 \Rightarrow x^* \leq y^* \forall x, y \in L$ ,
- (6)  $(L^*, *)$  is a boolean algebra.

**Definition 2.5** ([9]). Let  $L$  be a lattice and let  $H \subseteq L \times L$ . We denote by  $\Theta(H)$  the smallest congruence relation containing  $H$ , and call it the congruence relation generated by  $H$ . If  $H = I \times I$ , where  $I$  is an ideal, we write  $\Theta[I]$  for  $\Theta(H)$ .

**Definition 2.6** ([7]). An equivalence relation  $\theta$  is a congruence relation in dpMS-algebra  $L$ , if it is a lattice congruence and  $(a, b) \in \theta$  implies  $(a^\circ, b^\circ) \in \theta$  and  $(a^*, b^*) \in \theta$  for all  $a, b \in L$ . To distinguish lattice congruence of dpMS-algebra  $L$  from congruence of dpMS-algebra  $L$ , we shall use the subscript 'lat' to denote lattice congruence.

As shown [9], if  $I$  is an ideal of a distributive lattice  $L$ , then

$$(2.1) \quad (x, y) \in \Theta[I] \Leftrightarrow (\exists i \in I) x \vee i = y \vee i.$$

Dully, if  $F$  is a filter of a distributive lattice  $L$ , then

$$(2.2) \quad (x, y) \in \Theta[F] \Leftrightarrow (\exists j \in F) x \wedge j = y \wedge j.$$

For an ideal  $I$  of dpMS-algebra  $(L, ^\circ, *)$ , We shall write

$$I_\circ^\geq = \{x \in L : (\exists i \in I) i^\circ \leq x\}$$

$$I_{\circ\circ} = \{x \in L : (\exists i \in I) x \leq i^{\circ\circ}\}$$

and

$$I^\circ = \{y \in L, x^\circ = y : \exists x \in I\}.$$

Clearly,  $I^\circ \subseteq I_\circ^\geq$ ,  $I_\circ^\geq$  is a filter of  $L$  and  $I_{\circ\circ}$  is ideal of  $L$ . By Definition of  $I_\circ^\geq$  and  $I^\circ$  and by equation (2.2) Fang and Tan [7], characterize as

$$(2.3) \quad (x, y) \in \Theta_{lat}[I_\circ^\geq] \Leftrightarrow (\exists i \in I) x \wedge i^\circ = y \wedge i^\circ.$$

**Theorem 2.7** ([6]). Let  $I$  be an ideal of the dpMS-algebra  $L$ . Then

$$\Theta[I] = \Theta_{lat}[I_\circ^\geq] \vee \Theta_{lat}[I_{\circ\circ}].$$

We recall that for any nonempty set  $S$ , the characteristic function of  $S$ ,

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

**Definition 2.8.** [16] Let  $\mu$  be a fuzzy subset of  $S$  and let  $\alpha \in [0, 1]$ . Then the set

$$\mu_\alpha = \{x \in L : \alpha \leq \mu(x)\}$$

is called a level subset of  $\mu$ .

**Definition 2.9** ([13]). A fuzzy subset  $\mu$  of a bounded lattice  $L$  is said to be a fuzzy ideal of  $L$ , if for all  $x, y \in L$ ,

- (i)  $\mu(0) = 1$ ,
- (ii)  $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$ ,
- (iii)  $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$ .

In [13], Swamy and Raju observed that, a fuzzy subset  $\mu$  of a bounded lattice  $L$  is a fuzzy ideal of  $L$  if and only if  $\mu(0) = 1$  and  $\mu(x \vee y) = \mu(x) \wedge \mu(y)$  for all  $x, y \in L$ .

A fuzzy relation  $\theta$  on a set  $X$  is map  $\theta : X \times X \rightarrow [0, 1]$ . For any  $x, y \in X$  and fuzzy relations  $\theta$  and  $\phi$  on  $X$ ,  $(\theta \cap \phi)(x, y) = \min\{\theta(x, y), \phi(x, y)\}$ ,  $(\theta \cup \phi)(x, y) = \max\{\theta(x, y), \phi(x, y)\}$ ,  $\theta \subseteq \phi$  means  $\theta(x, y) \leq \phi(x, y)$ .

**Definition 2.10** ([10]). Suppose that  $\theta$  and  $\phi$  are two fuzzy relations on a set  $X$ . Then  $(\theta \circ \phi)(x, y) = \sup_{z \in X} (\theta(x, z) \wedge \phi(z, y))$ .

**Definition 2.11** ([10]). A fuzzy relation  $\phi$  on  $X$  is said to be a fuzzy equivalence relation on  $X$ , if

- (i)  $\phi(x, x) = 1$  for all  $x \in X$  (reflexive),
- (ii)  $\phi(x, y) = \phi(y, x)$  for all  $x, y \in L$  (symmetric),
- (iii)  $\phi(x, z) \geq \phi(x, y) \wedge \phi(y, z)$  for all  $x, y, z \in L$  (transitive).

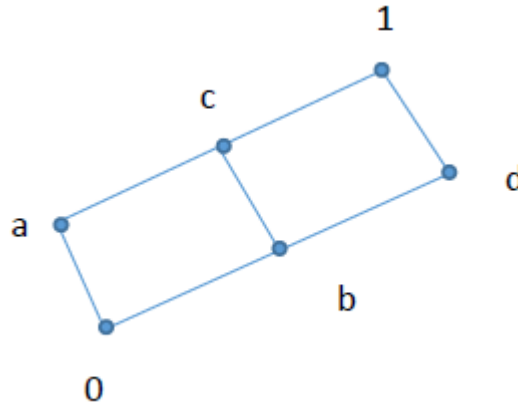
Through out the next sections,  $L$  stands for demi-pseudocomplemented MS-algebras unless otherwise mentioned.

### 3. FUZZY CONGRUENCES IN DEMI-PSEUDOCOMPLEMENTED MS-ALGEBRAS

**Definition 3.1.** A fuzzy relation  $\phi$  on a demi-pseudocomplemented MS-algebra  $(L, \circ, *)$  is called fuzzy congruence relation on  $(L, \circ, *)$ , if the following are satisfied:

- (i)  $\phi(x, x) = 1$  for all  $x \in L$ ,
- (ii)  $\phi(x, y) = \phi(y, x)$  for all  $x, y \in L$ ,
- (iii)  $\phi(x, z) \geq \phi(x, y) \wedge \phi(y, z)$  for all  $x, y, z \in L$ ,
- (iv)  $\phi(x \wedge z, y \wedge w) \wedge \phi(x \vee z, y \vee w) \geq \phi(x, y) \wedge \phi(z, w)$  for all  $x, y, z, w \in L$ ,
- (v)  $\phi(x^\circ, y^\circ) \wedge \phi(x^*, y^*) \geq \phi(x, y)$  for all  $x, y \in L$ .

**Example 3.2.** Consider the dpMS-algebra  $(L, \circ, *)$  given in Hasse diagram 1 below, unary operations  $\circ$  and  $*$  define in the table.



|         |   |   |   |   |   |   |
|---------|---|---|---|---|---|---|
|         | 0 | a | b | c | d | 1 |
| $\circ$ | 1 | d | a | 0 | a | 0 |
| $*$     | 1 | 0 | 1 | 0 | 1 | 0 |

Diagram 1

Define a fuzzy relation  $\varphi : L \times L \longrightarrow [0, 1]$  in Table 1

|              |       |       |       |       |       |       |       |       |       |       |       |
|--------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|              | (0,0) | (a,a) | (b,b) | (c,c) | (d,d) | (1,1) | (0,a) | (a,0) | (0,b) | (b,0) | (0,c) |
| $\varphi(,)$ | 1     | 1     | 1     | 1     | 1     | 1     | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   |

|       |       |       |       |       |       |       |       |       |       |       |       |       |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| (c,0) | (0,d) | (d,0) | (0,1) | (1,0) | (a,b) | (b,a) | (a,c) | (c,a) | (a,d) | (d,a) | (a,1) | (1,a) |
| 0.8   | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   |

|       |       |       |       |       |       |       |       |       |       |       |       |       |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| (b,c) | (c,b) | (b,d) | (d,b) | (b,1) | (1,b) | (c,d) | (c,d) | (d,c) | (c,1) | (1,c) | (1,d) | (d,1) |
| 0.8   | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   | 0.8   |

Table 1

We can be easily verified that  $\varphi$  is a fuzzy congruence on  $L$ .

**Theorem 3.3.** *A fuzzy equivalence relation is a fuzzy congruence on  $(L, ^\circ, ^*)$  if and only if  $\phi(x, y) \leq \phi(x \wedge z, y \wedge z) \wedge \phi(x \vee z, y \vee z) \wedge \phi(x^\circ, y^\circ) \wedge \phi(x^*, y^*)$ , for all  $x, y, z \in L$ .*

*Proof.* The forward proof is clear. Conversely, let  $\phi$  be a fuzzy equivalence relation satisfying the following: for all  $x, y, z \in L$ ,

$$\phi(x, y) \leq \phi(x \wedge z, y \wedge z) \wedge \phi(x \vee z, y \vee z) \wedge \phi(x^\circ, y^\circ) \wedge \phi(x^*, y^*).$$

This implies

$$\begin{aligned} \phi(x, y) &\leq \phi(x \wedge z, y \wedge z), \quad \phi(x, y) \leq \phi(x \vee z, y \vee z), \\ \phi(x, y) &\leq \phi(x^\circ, y^\circ), \quad \phi(x, y) \leq \phi(x^*, y^*). \end{aligned}$$

Thus for all  $x, y, z, w \in L$ ,

$$\phi(x, y) \wedge \phi(z, w) \leq \phi(x \wedge z, y \wedge z) \wedge \phi(y \wedge z, y \wedge w) \leq \phi(x \wedge z, y \wedge w).$$

Similarly,  $\phi(x, y) \wedge \phi(z, w) \leq \phi(x \vee z, y \vee w)$ . So  $\phi$  is a fuzzy congruence relation of  $(L, ^\circ, ^*)$ .  $\square$

**Example 3.4.** Let  $(L, ^\circ, ^*)$  be an dpMS-algebra. Define the fuzzy relations  $\phi$  and  $G_F$  on  $L$  by:

$$\phi(x, y) = \begin{cases} 1 & \text{if } x^\circ = y^\circ \\ 0 & \text{otherwise} \end{cases}$$

and

$$G_F(x, y) = \begin{cases} 1 & \text{if } x^* = y^* \\ 0 & \text{otherwise,} \end{cases}$$

for any  $x, y \in L$ . Then

- (1)  $\phi$  and  $G_F$  are fuzzy congruences of  $(L, ^\circ, ^*)$ ,
- (2)  $\phi \subseteq \Theta$ .

We call the fuzzy congruence  $G_F$  is Gliivenko fuzzy congruence.

**Theorem 3.5.** *A fuzzy relation  $\phi$  on  $(L, ^\circ, ^*)$  is a fuzzy congruence relation if and only if every level subsets  $\phi_\alpha$  of  $(L, ^\circ, ^*)$ ,  $\alpha \in [0, 1]$  is a congruence relation on  $(L, ^\circ, ^*)$ .*

**Corollary 3.6.** *An equivalence relation  $\phi$  is a congruence relation on  $L$  if and only if its characteristic function  $\chi_\phi$  is a fuzzy congruence on  $L$ .*

**Lemma 3.7.** *If  $\{\phi_i : i \in \Delta\}$  is a family of fuzzy congruences of  $(L, \circ, *)$ , then  $\cap_{i \in \Delta} \phi_i$  is a fuzzy congruence on  $(L, \circ, *)$ .*

The set of all fuzzy congruences of  $L$  is denoted by  $\mathcal{FC}(L)$  and the set of all congruences of  $L$  is denoted by  $\mathcal{C}(L)$ .  $\omega = \{(x, y) \in L \times L : x = y\}$  and  $\iota = L \times L$  are the smallest and the largest elements of  $\mathcal{C}(L)$  respectively and

$$\chi_\omega(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \omega \\ 0 & \text{otherwise,} \end{cases}$$

and  $\chi_\iota(x, y) = 1$  for all  $x, y \in L$  are the smallest and the largest elements of  $\mathcal{FC}(L)$  respectively.

**Theorem 3.8.**  *$(\mathcal{FC}(L), \subseteq)$  is a complete lattice.*

*Proof.* Clearly, we note that both fuzzy congruence relations  $\chi_\omega$  and  $\chi_\iota$  are the least and the greatest elements of  $\mathcal{FC}(L)$ , respectively. Then clearly,  $\cap_{i \in \Delta} \phi_i$  is a lower bound of any family  $\{\phi_i : i \in \Delta\}$  of fuzzy congruences of  $L$  and  $(\mathcal{FC}(L), \subseteq)$  is poset. Let  $\Theta$  be any lower bound of  $\{\phi_i : i \in \Delta\}$ . Then  $\Theta \subseteq \phi_i$ , for all  $i \in \Delta$ . Thus  $\Theta \subseteq \cap_{i \in \Delta} \phi_i$ . So  $\cap_{i \in \Delta} \phi_i$  is a greatest lower bound of  $\{\phi_i : i \in \Delta\}$ . Hence  $(\mathcal{FC}(L), \subseteq)$  is a complete lattice.  $\square$

Next we define the fuzzy quotient demi-pseudocomplemented MS-algebra induced by fuzzy congruence relation.

**Definition 3.9.** Let  $(L, \wedge, \vee, \circ, *, 0, 1)$  be a demi-pseudocomplemented MS-algebra,  $x \in L$  and  $\theta$  be a fuzzy congruence on  $L$ . The fuzzy congruence determined by  $x$  and  $\theta$ , denoted by  $\theta_x$ , is the fuzzy subset of  $L$  defined by  $\theta_x(y) = \theta(x, y)$ ,  $\forall y \in L$ .

Let  $L/\theta$  denote the set of all fuzzy congruence class, that is  $L/\theta = \{\theta_x : x \in L\}$ .

**Remark 3.10.** If  $\theta$  is a fuzzy congruence of  $L$  and  $x, y \in L$ , then  $\theta_x = \theta_y \Leftrightarrow \theta(x, y) = 1$ .

**Theorem 3.11.** *Let  $\theta$  be a congruence of a demi-pseudocomplemented MS-algebra  $(L, \wedge, \vee, \circ, *, 0, 1)$ . For any  $\theta_x, \theta_y \in L/\theta$ , define*

$$\theta_x \wedge \theta_y = \theta_{x \wedge y}, \theta_x \vee \theta_y = \theta_{x \vee y}, (\theta_x)^\circ = \theta_{x^\circ}, \text{ and } (\theta_x)^* = \theta_{x^*}.$$

*Then  $(L/\theta, \wedge, \vee, \circ, *, \theta_0, \theta_1)$  is a demi-pseudocomplemented MS-algebra, where  $\theta_0$ , and  $\theta_1$  are the smallest and largest elements of  $L/\theta$  respectively.*

*Proof.* we should first prove that the operations on  $L/\theta$  are well defined. To do this, for any any  $x, y, w$  and  $z \in L$ , suppose  $\theta_x = \theta_w$  and  $\theta_y = \theta_z$ . Then

$$\begin{aligned} & \theta(x, w) = 1 \text{ and } \theta(y, z) = 1 \\ \implies & \theta(x \wedge y, w \wedge z) \geq \theta(x, w) \wedge \theta(y, z) = 1 \\ \implies & \theta_{x \wedge y} = \theta_{w \wedge z}. \end{aligned}$$

Similarly,  $\theta_{x \vee y} = \theta_{w \vee z}$ . Thus the operations of  $L/\theta$  are well defined.

For any  $x \in L$ ,  $\theta_x \wedge \theta_0 = \theta_{x \wedge 0} = \theta_0$ . Then  $\theta_0 \leq \theta_x$ , for all  $\theta_x \in L/\theta$ . Thus  $\theta_0$  is the smallest element of  $L/\theta$ . Similarly,  $\theta_1$  is the largest element of  $L/\theta$ .

Now let  $\theta_x, \theta_y \in L/\theta$ . Then

$$(\theta_x \vee \theta_y)^* = (\theta_{x \vee y})^* = \theta_{(x \vee y)^*} = \theta_{x^* \wedge y^*} = \theta_{x^*} \wedge \theta_{y^*} = (\theta_x)^* \wedge (\theta_y)^*.$$

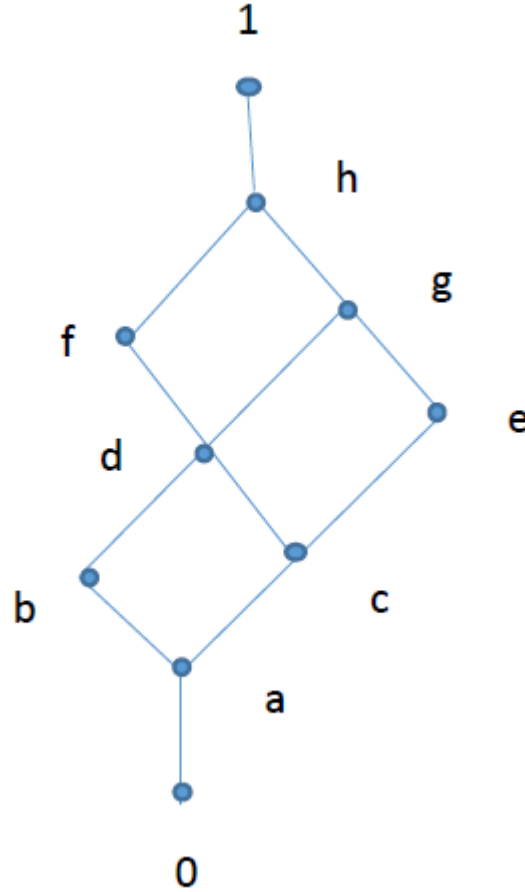
Thus

$$\begin{aligned}(\theta_x \wedge \theta_y)^{**} &= (\theta_{x \wedge y})^{**} = \theta_{(x \wedge y)^{**}} = \theta_{x^{**} \wedge y^{**}} \\ &= \theta_{x^{**}} \wedge \theta_{y^{**}} = (\theta_x)^{**} \wedge (\theta_y)^{**}.\end{aligned}$$

Similarly, we can easily show that  $(\theta_0)^* = \theta_1$ ,  $(\theta_1)^* = \theta_0$ ,  $(\theta_x)^{***} = (\theta_x)^*$  and  $(\theta_x)^* \wedge (\theta_x)^{**} = \theta_0$  and  $(\theta_x)^* \wedge ((\theta_x)^* \wedge \theta_y)^* = (\theta_x)^* \wedge (\theta_y)^*$ . Hence  $(L/\theta, \wedge, \vee, *, \theta_0, \theta_1)$  is a demi-pseudocomplemented algebra. Similarly, we can see that  $(L/\theta, \wedge, \vee, \circ, \theta_0, \theta_1)$  is an MS-algebra. Hence  $(L/\theta, \wedge, \vee, \circ, *, \theta_0, \theta_1)$  is a demi-pseudocomplemented MS-algebra.  $\square$

$L/\theta$  is called the fuzzy quotient a demi-pseudocomplemented MS-algebra  $L$  induced by  $\theta$ . It is clear that the map  $x \rightarrow L/\theta$  is a homomorphism from  $L$  onto  $L/\theta$ .

**Example 3.12.** Consider the dpMS-algebra  $(L, \circ, *)$  given in Hasse diagram 2, unary operations  $\circ$  and  $*$  define in the table.



|         | 0 | a | b | c | d | e | f | g | h | 1 |
|---------|---|---|---|---|---|---|---|---|---|---|
| $\circ$ | 1 | h | h | h | h | f | g | f | d | 0 |
| $*$     | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |

diagram 2

Define a fuzzy relation  $\varphi : L \times L \longrightarrow [0, 1]$  in the Table 2

|              |       |       |       |       |       |       |       |       |       |       |       |       |
|--------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|              | (0,0) | (a,a) | (b,b) | (c,c) | (d,d) | (e,e) | (f,f) | (g,g) | (h,h) | (1,1) | (0,a) | (a,0) |
| $\varphi(,)$ | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
| (0,b)        | (b,0) | (0,c) | (c,0) | (0,d) | (d,0) | (a,b) | (b,a) | (a,c) | (c,a) | (a,d) | (d,a) | (c,b) |
| 1            | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
| (b,c)        | (b,d) | (d,b) | (c,d) | (d,c) | (e,g) | (g,e) | (h,1) | (1,h) | (0,e) | (e,0) | (0,f) | (f,0) |
| 1            | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 0.7   | 0.7   | 0.7   | 0.7   |
| (0,g)        | (g,0) | (0,h) | (h,0) | (0,1) | (1,0) | (a,e) | (e,a) | (a,f) | (f,a) | (a,g) | (g,a) | (a,h) |
| 0.7          | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   |
| (h,a)        | (a,1) | (1,a) | (e,f) | (f,e) | (b,e) | (e,b) | (b,f) | (f,b) | (b,g) | (g,b) | (b,h) | (h,b) |
| 0.7          | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   |
| (b,1)        | (1,b) | (c,e) | (e,c) | (c,f) | (f,c) | (c,g) | (g,c) | (c,h) | (h,c) | (c,1) | (1,c) | (d,e) |
| 0.7          | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   |
| (e,d)        | (d,f) | (f,d) | (d,g) | (g,d) | (d,h) | (h,d) | (d,1) | (1,d) | (e,h) | (h,e) | (e,1) | (1,e) |
| 0.7          | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   |
|              | (f,g) | (g,f) | (f,h) | (h,f) | (f,1) | (1,f) | (g,h) | (h,g) | (g,1) | (1,g) |       |       |
|              | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   | 0.7   |       |       |

Table 2

we can easily verified that  $\varphi$  is a fuzzy congruence on  $L$ ,  $\varphi_0 = \varphi_a = \varphi_b = \varphi_c = \varphi_d$ ,  $\varphi_e = \varphi_g$  and  $\varphi_h = \varphi_1$ . This implies  $L/\varphi = \{\varphi_0, \varphi_e, \varphi_f, \varphi_1\}$  and  $(L/\varphi, \vee, \wedge, \circ, *, 0, 1)$  is a dpMS-algebra, where  $\varphi_0$ , and  $\varphi_1$  are the smallest and largest elements of  $L/\varphi$  respectively.

Now we define the join of two fuzzy congruence of  $L$ .

**Definition 3.13.** Let  $\phi$  and  $\varphi$  be any two fuzzy congruence relations of a dpMS-algebra  $(L, \circ, *)$ . Then define  $\phi \vee \varphi = \cap \{\Theta \in \mathcal{FC}(L) : \phi \subseteq \Theta \text{ and } \varphi \subseteq \Theta\}$ , i.e.,  $\phi \vee \varphi$  is the smallest fuzzy congruence containing  $\phi \cup \varphi$ .

**Theorem 3.14.** Let  $\phi$  and  $\varphi$  be any fuzzy congruence relations on a dpMS-algebra  $(L, \circ, *)$ . Then  $\phi \vee \varphi = \cup_{n=1}^{\infty} \Theta_n$ , where  $\Theta_1 = \phi \circ \varphi \circ \phi$ ,  $\Theta_2 = \phi \circ \varphi \circ \phi \circ \varphi \circ \phi$ ,  $\Theta_3 = \phi \circ \varphi \circ \phi \circ \varphi \circ \phi \circ \varphi \circ \phi, \dots$ .

*Proof.* Let  $\kappa = \cup_{n=0}^{\infty} \Theta_n$ . We prove that  $\kappa$  is the smallest fuzzy congruence relation in a dpMS-algebra  $(L, \circ, *)$  containing  $\phi$  and  $\varphi$ . It can be easily verified that  $\phi \subseteq \Theta_1 \subseteq \Theta_2 \subseteq, \dots$  and  $\varphi \subseteq \Theta_1 \subseteq \Theta_2 \subseteq, \dots$  and so  $\Theta_n \subseteq \phi \vee \varphi$ .

Now we see that  $\kappa$  is a fuzzy congruence relation in dpMS-algebra  $(L, \circ, *)$ .

(1)  $1 = \phi(x, x) \leq \Theta_1(x, x) \leq \cup_{n=0}^{\infty} \Theta_n(x, x) = \kappa(x, x)$ . Then  $\kappa(x, x) = 1$ .

(2) Symmetric is straightforward.

(3)  $\kappa(x, y) \wedge \kappa(y, z) = \cup_{n=1}^{\infty} \Theta_n(x, y) \wedge \cup_{n=1}^{\infty} \Theta_n(y, z) = \sup_n \Theta_n(x, y) \wedge \sup_n \Theta_n(y, z) \leq \cup_{n=1}^{\infty} \Theta_n(x, z)$  since  $\Theta_n(x, y) \wedge \Theta_m(y, z) \leq \Theta_{n+m}(x, z)$ , for any real number  $n$  and  $m$ .

$$\begin{aligned}
 (4) \quad \kappa(x, y) &= \bigcup_{n=1}^{\infty} \Theta_n(x, y) \\
 &= \sup_n \left( \sup_{z_1, z_2, \dots, z_{2n}} (\phi(x, z_1) \wedge \varphi(z_1, z_2) \wedge \phi(z_2, z_3) \wedge \dots \wedge \phi(z_{2n}, y)) \right) \\
 &\leq \sup_n \left( \sup_{z_1 \wedge c, z_2 \wedge c, \dots, z_{2n} \wedge c} (\phi(x \wedge c, z_1 \wedge c) \wedge \varphi(z_1 \wedge c, z_2 \wedge c) \right. \\
 &\quad \left. \wedge \phi(z_2 \wedge c, z_3 \wedge c) \dots \wedge \phi(z_{2n} \wedge c, y \wedge c)) \right) \\
 &= \bigcup_{n=1}^{\infty} \Theta_n(x \wedge c, y \wedge c) = \kappa(x \wedge c, y \wedge c).
 \end{aligned}$$

Similarly, we can show that  $\kappa(x, y) \leq \kappa(x \vee c, y \vee c)$ .

$$\begin{aligned}
 (5) \quad \kappa(x, y) &= \bigcup_{n=1}^{\infty} \Theta_n(x, y) \\
 &= \sup_n \left( \sup_{z_1, z_2, \dots, z_{2n}} (\phi(x, z_1) \wedge \varphi(z_1, z_2) \wedge \phi(z_2, z_3) \wedge \dots \wedge \phi(z_{2n}, y)) \right) \\
 &\leq \sup_n \left( \sup_{z_1^\circ, z_2^\circ, \dots, z_{2n}^\circ} (\phi(x^\circ, z_1^\circ) \wedge \varphi(z_1^\circ, z_2^\circ) \wedge \phi(z_2^\circ, z_3^\circ) \dots \wedge \phi(z_{2n}^\circ, y^\circ)) \right) \\
 &= \bigcup_{n=1}^{\infty} \Theta_n(x^\circ, y^\circ) = \kappa(x^\circ, y^\circ). \\
 (6) \quad \kappa(x, y) &= \bigcup_{n=1}^{\infty} \Theta_n(x, y) \\
 &= \sup_n \left( \sup_{z_1, z_2, \dots, z_{2n}} (\phi(x, z_1) \wedge \varphi(z_1, z_2) \wedge \phi(z_2, z_3) \wedge \dots \wedge \phi(z_{2n}, y)) \right) \\
 &\leq \sup_n \left( \sup_{z_1^*, z_2^*, \dots, z_{2n}^*} (\phi(x^*, z_1^*) \wedge \varphi(z_1^*, z_2^*) \wedge \phi(z_2^*, z_3^*) \dots \wedge \phi(z_{2n}^*, y^*)) \right) \\
 &= \bigcup_{n=1}^{\infty} \Theta_n(x^*, y^*) = \kappa(x^*, y^*).
 \end{aligned}$$

This implies  $\kappa$  is fuzzy congruence of a dpMS-algebra  $(L, ^\circ, *)$ .

Finally, let  $\tau$  be any fuzzy congruence relation such that  $\phi \subseteq \tau$  and  $\varphi \subseteq \tau$ . Then we prove that  $\kappa \subseteq \tau$ .

$$\begin{aligned}
 \kappa(x, y) &= \bigcup_{n=1}^{\infty} \Theta_n(x, y) \\
 &= \sup_n \left( \sup_{z_1, z_2, \dots, z_{2n}} (\phi(x, z_1) \wedge \varphi(z_1, z_2) \wedge \phi(z_2, z_3) \dots \wedge \phi(z_{2n}, y)) \right) \\
 &\leq \sup_n \left( \sup_{z_1, z_2, \dots, z_{2n}} (\tau(x, z_1) \wedge \tau(z_1, z_2) \wedge \dots \wedge \tau(z_{2n}, y)) \right) \\
 &= \sup_n \tau(x, y) = \tau(x, y).
 \end{aligned}$$

Thus  $\kappa$  is the smallest fuzzy congruence such that  $\phi \subseteq \tau$  and  $\varphi \subseteq \tau$ . So  $\phi \vee \varphi = \bigcup_{n=0}^{\infty} \Theta_n$ .  $\square$

**Definition 3.15.** Let  $L$  be an dpMS-algebra. The fuzzy congruence generated by the fuzzy relation  $\phi$  of  $L$  is defined by  $\bar{\Theta}(\phi) = \bigcap \{\vartheta \in FC(L) : \phi \subseteq \vartheta\}$ . If  $\phi = \mu \times \mu$  is the product of fuzzy ideal  $\mu$  by itself, where  $(\mu \times \mu)(x, y) = \mu(x) \wedge \mu(y)$  for all  $(x, y) \in L \times L$ . We write  $\bar{\Theta}[\mu]$  instead of  $\bar{\Theta}(\phi)$  i.e  $\bar{\Theta}[\mu]$  is the smallest fuzzy congruence containing  $\mu \times \mu$ .

**Theorem 3.16.** Let  $\phi$  be a fuzzy relation of dpMS-algebra  $(L, ^\circ, *)$ . Then  $\bar{\Theta}(\phi)(x, y) = \sup\{\alpha : (x, y) \in \Theta(\phi_\alpha)\}$ , for any  $(x, y) \in L \times L$  and  $\alpha \in [0, 1]$ .

*Proof.* Let  $\varphi(x, y) = \sup\{\alpha : (x, y) \in \Theta(\phi_\alpha)\}$ , for any  $(x, y) \in L \times L$  and  $\alpha \in [0, 1]$ . Then we prove that  $\varphi = \bar{\Theta}(\phi)$ . First we see that  $\varphi$  is a fuzzy congruence of  $(L, ^\circ, *)$ .

- (1)  $\varphi(x, x) = \sup\{\alpha : (x, x) \in \Theta(\phi_\alpha)\} = 1$ .
- (2) Symmetric is straightforward.
- (3)  $\varphi(x, y) \wedge \varphi(y, z) = \sup\{\alpha : (x, y) \in \Theta(\phi_\alpha)\} \wedge \sup\{\lambda : (y, z) \in \Theta(\phi_\lambda)\}$   
 $= \sup\{\alpha \wedge \lambda : (x, y) \in \Theta(\phi_\alpha), (y, z) \in \Theta(\phi_\lambda)\}$   
 $\leq \sup\{\alpha \wedge \lambda : (x, y) \in \Theta(\phi_{\alpha \wedge \lambda}), (y, z) \in \Theta(\phi_{\alpha \wedge \lambda})\}$

$$\begin{aligned}
 &\leq \sup\{\alpha \wedge \lambda : (x, z) \in \Theta(\phi_{\alpha \wedge \lambda})\} \\
 &= \varphi(x, z). \\
 (4) \quad \varphi(x, y) \wedge \varphi(w, z) &= \sup\{\alpha : (x, y) \in \Theta(\phi_\alpha)\} \wedge \sup\{\lambda : (w, z) \in \Theta(\phi_\lambda)\} \\
 &= \sup\{\alpha \wedge \lambda : (x, y) \in \Theta(\phi_\alpha), (w, z) \in \Theta(\phi_\lambda)\} \\
 &\leq \sup\{\alpha \wedge \lambda : (x, y) \in \Theta(\phi_{\alpha \wedge \lambda}), (w, z) \in \Theta(\phi_{\alpha \wedge \lambda})\} \\
 &\leq \sup\{\alpha \wedge \lambda : (x \wedge w, y \wedge z) \in \Theta(\phi_{\alpha \wedge \lambda})\} \\
 &= \varphi(x \wedge w, y \wedge z).
 \end{aligned}$$

Similarly, we can prove that  $\varphi(x, y) \wedge \varphi(y, z) \leq \varphi(x \vee w, y \vee z)$ .

(5)  $\varphi(x, y) = \sup\{\alpha : (x, y) \in \Theta(\phi_\alpha)\} \leq \sup\{\alpha : (x^\circ, y^\circ) \in \Theta(\phi_\alpha)\} = \varphi(x^\circ, y^\circ)$ . Similarly,  $\varphi(x, y) \leq \varphi(x^*, y^*)$ . Then  $\varphi$  is a fuzzy congruence of a dpMS-algebra  $(L, ^\circ, *)$ . Now  $\phi(x, y) = \sup\{\alpha : (x, y) \in \phi_\alpha\} \leq \sup\{\alpha : (x, y) \in \Theta(\phi_\alpha)\} = \varphi(x, y)$ , for any  $x, y \in L$ . Thus  $\phi \subseteq \varphi$ . Finally, let  $\tau$  be any fuzzy congruence of a dpMS-algebra  $(L, ^\circ, *)$  such that  $\phi \subseteq \tau$ . We see that  $\varphi \subseteq \tau$ . If  $\phi \subseteq \tau$ , then  $\phi_\alpha \subseteq \tau_\alpha$  and thus  $\Theta(\phi_\alpha) \subseteq \Theta(\tau_\alpha) = \tau_\alpha$ . Now,

$$\varphi(x, y) = \sup\{\alpha : (x, y) \in \Theta(\phi_\alpha)\} \leq \sup\{\alpha : (x, y) \in \tau_\alpha\} = \tau(x, y).$$

So  $\varphi \subseteq \tau$ . Hence  $\varphi = \overline{\Theta}(\phi)$ .  $\square$

#### 4. KERNEL FUZZY IDEALS AND FUZZY CONGRUENCES

**Definition 4.1.** Kernel fuzzy ideal of a dpMS-algebra  $(L, ^\circ, *)$  is a fuzzy ideal  $\mu$  of  $L$  for which there exists a fuzzy congruence  $\varphi$  on  $L$  such that  $\mu = \text{Ker}\varphi$  i.e.,  $\mu$  is the kernel fuzzy congruence  $\varphi$ , where  $\text{Ker}\varphi(x) = \varphi(x, 0)$  for all  $x \in L$ .

**Definition 4.2.** Cokernel fuzzy filter of a dpMS-algebra  $(L, ^\circ, *)$  is a fuzzy filter  $\eta$  of  $L$  for which there exists a fuzzy congruence  $\psi$  on  $L$  such that  $\eta = \text{CoKer}\psi$  i.e.  $\eta$  is the Cokernel fuzzy congruence  $\psi$ , where  $\text{CoKer}\psi(x) = \psi(x, 1)$  for all  $x \in L$ .

**Example 4.3.** Consider the dpMS-algebra  $(L, ^\circ, *)$  and the fuzzy congruence  $\varphi$  in Example 3.2. Define fuzzy subsets  $\mu$  and  $\eta$  of  $(L, ^\circ, *)$  as follows:

$$\mu(0) = 1, \mu(a) = \mu(b) = \mu(c) = \mu(d) = \mu(1) = 0.8$$

and

$$\eta(1) = 1, \eta(a) = \eta(b) = \eta(c) = \eta(d) = \eta(0) = 0.8.$$

Then  $\mu(x) = \varphi(0, x)$  for all  $x \in L$  and  $\eta(x) = \varphi(1, x)$ , for all  $x \in L$ . Thus  $\mu$  is a kernel fuzzy ideal and  $\eta$  is a cokernel fuzzy filter of a dpMS-algebra  $(L, ^\circ, *)$ .

In the the following Theorem, we characterized a kernel fuzzy ideal of a dpMS-algebra.

**Theorem 4.4.** Let  $(L, ^\circ, *)$  be a dpMS-algebra. If  $\mu$  be a kernel fuzzy ideal, then  $\mu(x) \leq \mu(x^{\circ\circ}) \wedge \mu(x^{\circ*}) \wedge \mu(x^{**}), \forall x \in L$ .

*Proof.* The proof follows from the Definition 4.1.  $\square$

For a fuzzy ideal  $\mu$  of a dpMS-algebra  $(L, ^\circ, *)$ , define

$$\begin{aligned}
 \mu_{\circ}^{\geq}(x) &= \sup\{\mu(i) : i^{\circ} \leq x, i \in L\} \text{ for } \forall x \in L, \\
 \mu^{\circ}(x) &= \sup\{\mu(i) : i^{\circ} = x, i \in L\} \text{ for } \forall x \in L, \\
 \mu_{\circ\circ}(x) &= \sup\{\mu(i) : x \leq i^{\circ\circ}, i \in L\} \text{ for } \forall x \in L.
 \end{aligned}$$

Then it can be easily verified that  $\mu_{\circ}^{\geq}$  is a fuzzy filter of  $L$ ,  $\mu_{\circ\circ}$  is a fuzzy ideal of  $L$  and the  $\alpha$ -level subsets of  $\mu_{\circ}^{\geq}$ ,  $\mu^{\circ}$ , and  $\mu_{\circ\circ}$  are

$$(\mu_{\circ}^{\geq})_{\alpha} = \{x \in L : \exists i \in \mu_{\alpha} \ i^{\circ} \leq x\},$$

$$\mu_{\alpha}^{\circ} = \{y \in L, x^{\circ} = y : \exists x \in \mu_{\alpha}\}$$

and

$$(\mu_{\circ\circ})_{\alpha} = \{x \in L : (\exists i \in \mu_{\alpha}) \ x \leq i^{\circ\circ}\}, \text{ respectively.}$$

By Definition 3.15, Theorem 3.16 and equations (2.1), (2.2), we can write

$$\begin{aligned} \bar{\Theta}_{lat}[\mu](x, y) &= \sup\{\alpha : (x, y) \in \Theta_{lat}[\mu_{\alpha}]\}, \\ \bar{\Theta}_{lat}[\mu_{\circ\circ}](x, y) &= \sup\{\alpha : (x, y) \in \Theta_{lat}[(\mu_{\circ\circ})_{\alpha}]\} \\ &= \sup\{\alpha : x \vee i = y \vee i, \ i \in [(\mu_{\circ\circ})_{\alpha}]\} \\ &= \sup\{\alpha : x \vee i = y \vee i, \ i \leq j^{\circ\circ} \text{ for some } j \in \mu_{\alpha}\} \end{aligned}$$

and

$$\begin{aligned} \bar{\Theta}_{lat}[\mu_{\circ}^{\geq}](x, y) &= \sup\{\alpha : (x, y) \in \Theta_{lat}[(\mu_{\circ}^{\geq})_{\alpha}]\} \\ &= \sup\{\alpha : x \wedge i = y \wedge i, \ i \in (\mu_{\circ}^{\geq})_{\alpha}\} \\ &= \sup\{\alpha : x \wedge i = y \wedge i, \ j^{\circ} \leq i \text{ for some } j \in \mu_{\alpha}\}. \end{aligned}$$

Also by equation (2.3), we have  $\bar{\Theta}_{lat}[\mu_{\circ}^{\geq}](x, y) = \sup\{\alpha : x \wedge i^{\circ} = y \wedge i^{\circ}, \ i \in \mu_{\alpha}\}$ .

**Corollary 4.5.** *If  $\mu$  is a kernel fuzzy ideal of a dpMS-algebra  $(L, ^{\circ}, *)$ , then  $\mu = \mu_{\circ\circ}$  i.e  $\mu(x^{\circ\circ}) \geq \mu(x)$  for  $\forall x \in L$ .*

*Proof.*  $\mu_{\circ\circ}(x) = \sup\{\mu(i) : x \leq i^{\circ\circ}\} \leq \sup\{\mu(i^{\circ\circ}) : x \leq i^{\circ\circ}\} \leq \mu(x)$ . It is obvious that  $\mu \subseteq \mu_{\circ\circ}$ . This implies  $\mu = \mu_{\circ\circ}$ .  $\square$

The description of  $\bar{\Theta}_{lat}[\mu_{\circ}^{\geq}]$  can be characterized as follows.

**Lemma 4.6.** *Let  $(L, ^{\circ}, *)$  be a dpMS-algebra and  $\mu$  be a kernel fuzzy ideal of  $L$ . Then following conditions hold.*

- (1)  $\bar{\Theta}_{lat}[\mu_{\circ}^{\geq}](x, y) \leq \bar{\Theta}_{lat}[\mu](x^{\circ}, y^{\circ}) \wedge \bar{\Theta}_{lat}[\mu](x^*, y^*),$
- (2)  $\bar{\Theta}_{lat}[\mu](x, y) \leq \bar{\Theta}_{lat}[\mu_{\circ}^{\geq}](x^{\circ}, y^{\circ}) \wedge \bar{\Theta}_{lat}[\mu_{\circ}^{\geq}](x^*, y^*).$

*Proof.* (1)  $\bar{\Theta}_{lat}[\mu_{\circ}^{\geq}](x, y) = \sup\{\alpha : (x, y) \in \Theta_{lat}[(\mu_{\circ}^{\geq})_{\alpha}]\}$   
 $= \sup\{\alpha : x \wedge i = y \wedge i, \ i \in (\mu_{\circ}^{\geq})_{\alpha}\}$   
 $= \sup\{\alpha : x \wedge i = y \wedge i, \ j^{\circ} \leq i \text{ for some } j \in \mu_{\alpha}\}$   
 $\leq \sup\{\alpha : x^{\circ} \vee i^{\circ} = y^{\circ} \vee i^{\circ}, \ i^{\circ} \leq j^{\circ\circ} \text{ for some } j \in \mu_{\alpha}\}$   
 $= \bar{\Theta}_{lat}[\mu_{\circ\circ}](x^{\circ}, y^{\circ}) = \bar{\Theta}_{lat}[\mu](x^{\circ}, y^{\circ}),$

since  $\mu$  is a kernel fuzzy ideal of  $L$ . Also

$$\begin{aligned} \bar{\Theta}_{lat}[\mu_{\circ}^{\geq}](x, y) &= \sup\{\alpha : (x, y) \in \Theta_{lat}[(\mu_{\circ}^{\geq})_{\alpha}]\} \\ &= \sup\{\alpha : x \wedge i = y \wedge i, \ i \in (\mu_{\circ}^{\geq})_{\alpha}\} \\ &\leq \sup\{\alpha : x^* \vee i^* = y^* \vee i^*, \ i^* \leq j^{\circ*} = j^{\circ\circ\circ} \text{ for some } j^{\circ*} \in \mu_{\alpha}\} \\ &= \bar{\Theta}_{lat}[\mu_{\circ\circ}](x^*, y^*) = \bar{\Theta}_{lat}[\mu](x^*, y^*). \end{aligned}$$

Thus  $\bar{\Theta}_{lat}[\mu_{\circ}^{\geq}](x, y) \leq \bar{\Theta}_{lat}[\mu](x^{\circ}, y^{\circ}) \wedge \bar{\Theta}_{lat}[\mu](x^*, y^*)$ .

$$\begin{aligned} (2) \ \bar{\Theta}_{lat}[\mu](x, y) &= \sup\{\alpha : (x, y) \in \Theta_{lat}[\mu_{\alpha}]\} \\ &= \sup\{\alpha : x \vee i = y \vee i, \ i \in \mu_{\alpha}\} \\ &= \sup\{\alpha : x \vee i = y \vee i, \ i \in (\mu_{\circ\circ})_{\alpha}\} \\ &= \sup\{\alpha : x \vee i = y \vee i, \ i \leq j^{\circ\circ}, \ j \in \mu_{\alpha}\} \end{aligned}$$

$$\begin{aligned}
 &\leq \sup\{\alpha : x^\circ \wedge i^\circ = y^\circ \wedge i^\circ, j^\circ \leq i^\circ, j \in \mu_\alpha\} \\
 &= \bar{\Theta}_{lat}[\mu_\alpha^\circ](x^\circ, y^\circ), \\
 \bar{\Theta}_{lat}[\mu](x, y) &= \sup\{\alpha : (x, y) \in \bar{\Theta}_{lat}[\mu_\alpha]\} \\
 &= \sup\{\alpha : x \vee i = y \vee i, i \in \mu_\alpha\} \\
 &= \sup\{\alpha : x \vee i = y \vee i, i \in (\mu_{\circ\circ})_\alpha\} \\
 &= \sup\{\alpha : x \vee i = y \vee i, i \leq j^{\circ\circ}, j \in \mu_\alpha\} \\
 &\leq \sup\{\alpha : x^* \wedge i^* = y^* \wedge i^*, j^{\circ\circ*} \leq i^* = i^{*\circ\circ}, j \in \mu_\alpha\} \\
 &= \bar{\Theta}_{lat}(\mu_\alpha^\circ)(x^*, y^*).
 \end{aligned}$$

Thus  $\bar{\Theta}_{lat}[\mu](x, y) \leq \bar{\Theta}_{lat}[\mu_\alpha^\circ](x^\circ, y^\circ) \wedge \bar{\Theta}_{lat}[\mu_\alpha^\circ](x^*, y^*)$ .  $\square$

**Theorem 4.7.** Let  $(L, \circ, *)$  be a dpMS-algebra, and  $\mu$  be a kernel fuzzy ideal of  $L$ . Then we have  $\bar{\Theta}[\mu] = \bar{\Theta}_{lat}[\mu] \vee \bar{\Theta}_{lat}[\mu_\alpha^\circ]$ .

*Proof.* Let  $\varphi = \bar{\Theta}_{lat}[\mu_\alpha^\circ] \vee \bar{\Theta}_{lat}[\mu]$ . We see that  $\varphi$  is the smallest fuzzy congruence containing  $\mu \times \mu$ . Clearly,  $\mu \times \mu \subseteq \varphi$ . Now we prove that  $\varphi$  is a fuzzy congruence of  $L$ . Since  $\bar{\Theta}_{lat}[\mu_\alpha^\circ]$  and  $\bar{\Theta}_{lat}[\mu]$  are lattice fuzzy congruences,  $\varphi$  is a lattice fuzzy congruence of  $L$ .

By Theorem 3.16 and Lemma 4.6 ,

$$\begin{aligned}
 &\varphi(x, y) \\
 &= \sup_n (\sup_{z_1, z_2, \dots, z_{2n}} (\bar{\Theta}_{lat}[\mu_\alpha^\circ](x, z_1) \wedge \bar{\Theta}_{lat}[\mu](z_1, z_2) \wedge \bar{\Theta}_{lat}[\mu_\alpha^\circ](z_2, z_3) \wedge \\
 &\quad \dots \wedge \bar{\Theta}_{lat}[\mu_\alpha^\circ](z_{2n}, y)) \\
 &\leq \sup_n (\sup_{z_1^\circ, z_2^\circ, \dots, z_{2n}^\circ} (\bar{\Theta}_{lat}[\mu](x^\circ, z_1^\circ) \wedge \bar{\Theta}_{lat}[\mu_\alpha^\circ](z_1^\circ, z_2^\circ) \wedge \bar{\Theta}_{lat}[\mu](z_2^\circ, z_3^\circ) \wedge \\
 &\quad \dots \wedge \bar{\Theta}_{lat}[\mu_\alpha^\circ](z_{2n}^\circ, y^\circ)) \\
 &= \varphi(x^\circ, y^\circ).
 \end{aligned}$$

Similarly,  $\varphi(x, y) \leq \varphi(x^*, y^*)$ . Thus  $\varphi$  is a fuzzy congruence of a dpMS-algebra  $L$ .

Finally, we see that  $\varphi$  is the smallest fuzzy congruence containing  $\mu \times \mu$ . Since  $\varphi$  is a fuzzy congruence containing  $\mu \times \mu$ ,  $\bar{\Theta}[\mu] \subseteq \varphi$ .

Clearly  $\bar{\Theta}_{lat}[\mu] \subseteq \bar{\Theta}[\mu]$  and also by Theorem 2.7 and Corollary 4.5,  $\bar{\Theta}_{lat}[(\mu_\alpha^\circ)](x, y) = \sup\{\alpha : (x, y) \in \bar{\Theta}_{lat}[(\mu_\alpha^\circ)_\alpha]\} \leq \sup\{\alpha : (x, y) \in \bar{\Theta}[\mu_\alpha]\} = \bar{\Theta}[\mu](x, y)$ . Thus  $\bar{\Theta}_{lat}[\mu_\alpha^\circ] \vee \bar{\Theta}_{lat}[\mu] \subseteq \bar{\Theta}[\mu]$ . So  $\varphi \subseteq \bar{\Theta}[\mu]$ . Hence  $\varphi = \bar{\Theta}[\mu]$ .  $\square$

## 5. $(\circ, *)$ -FUZZY IDEALS

**Definition 5.1.** A fuzzy ideal  $\mu$  of a dpMS-algebra  $(L, \circ, *)$  is said to be a  $(\circ, *)$ -fuzzy ideal if  $\mu(x) = \mu(x^{\circ*})$  for all  $x \in L$ .

**Example 5.2.** In Example 4.3, the kernel fuzzy ideal  $\mu$  is not  $(\circ, *)$ -fuzzy ideal. Because  $0.8 = \mu(b) \neq \mu(b^{\circ*}) = \mu(a^*) = \mu(0) = 1$ .

**Theorem 5.3.** Let  $(L, \circ, *)$  be a dpMS-algebra. If  $\mu$  is  $(\circ, *)$ -fuzzy ideal of  $L$ , then  $\mu(x^{**}) = \mu(x) \forall x \in L$ .

*Proof.* Suppose that  $\mu$  is a  $(\circ, *)$ -fuzzy ideal of  $L$ . Then  $\mu(x) = \mu(x^{\circ*}) = \mu(x^{(\circ*)^{\circ*}}) = \mu(x^{**})$ .  $\square$

**Theorem 5.4.**  $\mu$  is a  $(\circ, *)$ -fuzzy ideal of  $L$  if and only if  $\mu_\alpha, \forall \alpha \in [0, 1]$  is a  $(\circ, *)$ -ideal of  $L$ .

**Corollary 5.5.**  $I$  is a  $(\circ, *)$ -ideal of  $L$  if and only if  $\chi_I$  is  $(\circ, *)$ -fuzzy ideal of  $L$ .

**Example 5.6.** Consider the dpMS-algebra  $(L, \circ, *)$  describe in the Hasse Diagram 2.

Define a fuzzy subset  $\mu$  of  $L$  as follows:

$$\mu(0) = \mu(a) = \mu(b) = \mu(c) = \mu(d) = \mu(f) = 1$$

and

$$\mu(e) = \mu(g) = \mu(h) = \mu(1) = 0.7.$$

Then it can be easily verified that  $\mu$  is a  $(\circ, *)$ -fuzzy ideal.

As indicated in Example 5.6,  $\text{Ker}G_F = \mu$  is a  $(\circ, *)$ -fuzzy ideal. The following Theorem shows that it is true in general case.

**Theorem 5.7.** *If  $(L, \circ, *)$  is a dpMS-algebra, then  $\text{Ker}G_F$  is the smallest a  $(\circ, *)$ -fuzzy ideal of  $L$ .*

*Proof.* We prove that  $\text{Ker}G_F(x^{\circ}) = G_F(x^{\circ}, 0) = \text{Ker}G_F(x)$ , for any  $x \in L$ .

Suppose  $G_F(x^{\circ}, 0) = 1$ . Then  $x^{\circ*} = 0^*$ . This implies  $x^{\circ**} = 0^*$ . Thus we have  $x^{**} = x^{\circ**} = 0$ . So  $x^{***} = x^{\circ**} = 0^* = x^*$ . Hence  $G_F(x^{**}, 0) = G_F(x, 0) = 1$ . Therefore  $\text{Ker}G_F(x^{\circ}) = \text{Ker}G_F(x^{**}) = \text{Ker}G_F(x) = 1$ .

Suppose  $G_F(x^{\circ}, 0) = 0$ . Then  $x^{\circ*} \neq 0^*$ . We see that  $x^{***} \neq 0^*$  and  $x^* \neq 0^*$ . Now we prove by contradiction. Assume that  $x^* = 0^*$ . Then  $x^{\circ*} = 0^*$ . This contradicts the hypothesis. Also assume that  $x^{***} = 0^*$ . Then  $x^{\circ*} = x^{\circ**} = 0^{\circ*} = 0^*$ . It also contradicts the hypothesis. Thus

$$\text{Ker}G_F(x^{\circ}) = \text{Ker}G_F(x^{**}) = \text{Ker}G_F(x) = 0.$$

So in either cases, for any  $x \in L$ ,  $\text{Ker}G_F(x^{\circ}) = \text{Ker}G_F(x^{**}) = \text{Ker}G_F(x)$ . Hence  $\text{Ker}G_F$  is  $(\circ, *)$ -fuzzy ideal of  $L$ .

Let  $\mu$  be any  $(\circ, *)$ -fuzzy ideal. If  $\text{Ker}G_F(x) = G_F(x, 0) = 1$ , then  $x^* = 0^*$ . We have  $x^{**} = 0^{**} = 0$ . This implies

$$\text{Ker}G_F(x) = \text{Ker}G_F(x^{**}) = 1 = \mu(0) = \mu(x^{**}) = \mu(x).$$

If  $\text{Ker}G_F(x) = 0$ , then  $\text{Ker}G_F(x) = 0 \leq \mu(x)$ . Thus for any  $x \in L$ ,  $\text{Ker}G_F(x) \leq \mu(x)$ . So  $\text{Ker}G_F \subseteq \mu$ . Hence  $\text{Ker}G_F$  is the smallest  $(\circ, *)$ -fuzzy ideal of  $L$ .  $\square$

**Corollary 5.8.**  *$(L, \circ, *)$  be a dpMS-algebra. The  $\text{Ker}G_F = \chi_{\{0\}}$  if and only if every fuzzy congruence kernel of  $L$  is a  $(\circ, *)$ -fuzzy ideal.*

*Proof.* (Necessity). Suppose  $\text{Ker}G_F = \chi_{\{0\}}$ . Then for every  $x \in L$ , we have  $x \wedge x^* = 0$ . Let  $\mu$  be the kernel of any fuzzy congruence  $\varphi$  of  $L$ , i.e.,  $\mu = \text{Ker}\varphi$ . Then  $\mu(x) = \text{Ker}\varphi(x) = \varphi(x, 0) \leq \varphi(x^{\circ}, 0) = \mu(x^{\circ})$ . Conversely,  $\mu(x^{\circ}) = \varphi(x^{\circ}, 0) \leq \varphi(x^*, 1) = \varphi(x^*, 1) \wedge \varphi(x, x) \leq \varphi(x \wedge x^*, x) = \varphi(0, x) = \mu(x)$ . Thus  $\mu(x) = \mu(x^{\circ})$ ,  $\forall x \in L$  and so  $\mu$  is a  $(\circ, *)$ -fuzzy ideal of  $L$ .

(Sufficiency). Suppose every fuzzy congruence kernel of  $L$  is a  $(\circ, *)$ -fuzzy ideal. Then by Theorem 5.7,  $\text{Ker}G_F(x) = 1 = \text{Ker}\chi_{\omega}(x)$ . Thus  $\text{Ker}G_F = \chi_{\{0\}}$ .  $\square$

**Corollary 5.9.** *Let  $(L, \circ, *)$  be a dpMS-algebra. If  $\varphi$  is a fuzzy congruence on  $L$  such that  $G_F \subseteq \varphi$ , then  $\text{Ker}\varphi$  is a  $(\circ, *)$ -fuzzy ideal of  $L$ .*

We denote the set of all  $(\circ, *)$ -fuzzy ideals of  $L$  by  $FI_{\circ*}(L)$  and the set of all fuzzy ideals of  $L$  by  $FI(L)$ .

**Theorem 5.10.** *If  $(L, ^\circ, *)$  a dpMS-algebra, then  $FI_{\circ*}(L)$  is a sublattice of the lattice  $FI(L)$  of fuzzy ideals of  $L$ .*

*Proof.* It is clear that  $KerG_F \in FI_{\circ*}(L)$ . Then  $FI_{\circ*}(L) \neq \emptyset$ . Let  $\mu, \nu \in FI_{\circ*}(L)$ . Then clearly,  $\mu \wedge \nu \in FI_{\circ*}(L)$ . We see that  $\mu \vee \nu \in FI_{\circ*}(L)$ . Thus

$$\begin{aligned} (\mu \vee \nu)(x) &= \sup\{\mu(i) \wedge \nu(j) : x = i \vee j\} \\ &\leq \sup\{\mu(i^{*\circ}) \wedge \nu(j^{*\circ}) : x^{*\circ} = i^{*\circ} \vee j^{*\circ}\} \\ &= (\mu \vee \nu)(x^{*\circ}). \end{aligned}$$

Conversely,

$$\begin{aligned} (\mu \vee \nu)(x^{*\circ}) &= \sup\{\mu(i) \wedge \nu(j) : x^{*\circ} = i \vee j\} \\ &\leq \sup\{\mu(i^{*\circ}) \wedge \nu(j^{*\circ}) : x^{*\circ\circ} = i^{*\circ} \vee j^{*\circ}\} \\ &= \sup\{\mu(i^{*\circ}) \wedge \nu(j^{*\circ}) : x^{**} = i^{*\circ} \vee j^{*\circ}\} \\ &= (\mu \vee \nu)(x^{**}). \end{aligned}$$

Clearly,  $KerG_F(x \wedge x^*) = 1$ ,  $KerG_F \subseteq \mu$  and  $KerG_F \subseteq \nu$ . So  $KerG_F \subseteq \mu \vee \nu$ . Since  $\mu \vee \nu$  is anti-tone,

$$\begin{aligned} (\mu \vee \nu)(x^{**}) &= (\mu \vee \nu)(x^{**}) \wedge (\mu \vee \nu)(x \wedge x^*) \\ &= (\mu \vee \nu)(x^{**} \vee (x \wedge x^*)) \\ &= (\mu \vee \nu)(x^{**} \vee x) \\ &\leq (\mu \vee \nu)(x). \end{aligned}$$

Hence  $\mu \vee \nu \in FI_{\circ*}(L)$ . Therefore  $FI_{\circ*}(L)$  is a sublattice of  $FI(L)$ .  $\square$

**Theorem 5.11.** *If  $(L, ^\circ, *)$  is a dpMS-algebra then for each  $\mu \in FI_{\circ*}(L)$ , there exists a smallest fuzzy congruence on  $L$  with kernel  $\mu$  given by the followings:*

$$\delta(\mu)(x, y) = \sup\{\mu(i) : (x \vee i^{\circ\circ}) \wedge i^\circ = (y \vee i^{\circ\circ}) \wedge i^\circ\}.$$

*Proof.* Reflexive and symmetric are clear. Next, we prove that  $\delta(\mu)$  is transitive. Suppose that  $(x \vee i^{\circ\circ}) \wedge i^\circ = (y \vee i^{\circ\circ}) \wedge i^\circ$ ,  $(y \vee j^{\circ\circ}) \wedge j^\circ = (z \vee j^{\circ\circ}) \wedge j^\circ$ , for any  $x, y, i, j \in L$ . Put  $r = i \vee j$ . Then we have

$$\begin{aligned} (x \vee r^{\circ\circ}) \wedge r^\circ &= (x \vee (i^{\circ\circ} \vee j^{\circ\circ})) \wedge (i^\circ \wedge j^\circ) \\ &= ((x \vee i^{\circ\circ}) \vee j^{\circ\circ}) \wedge (i^\circ \wedge j^\circ) \\ &= ((x \vee i^{\circ\circ}) \wedge (i^\circ \wedge j^\circ)) \vee (j^{\circ\circ} \wedge (i^\circ \wedge j^\circ)) \\ &= ((y \vee i^{\circ\circ}) \wedge (i^\circ \wedge j^\circ)) \vee (j^{\circ\circ} \wedge (i^\circ \wedge j^\circ)) \\ &= ((y \vee j^{\circ\circ}) \wedge (i^\circ \wedge j^\circ)) \vee (i^{\circ\circ} \wedge (i^\circ \wedge j^\circ)) \\ &= ((z \vee j^{\circ\circ}) \wedge (i^\circ \wedge j^\circ)) \vee (i^{\circ\circ} \wedge (i^\circ \wedge j^\circ)) \\ &= (z \vee (i^{\circ\circ} \vee j^{\circ\circ})) \wedge (i^\circ \wedge j^\circ) \\ &= (z \vee r^{\circ\circ}) \wedge r^\circ. \end{aligned}$$

Thus  $\delta(\mu)(x, y) \wedge \delta(\mu)(y, z)$   
 $= \sup\{\mu(i) : (x \vee i^{\circ\circ}) \wedge i^\circ = (y \vee i^{\circ\circ}) \wedge i^\circ\} \wedge \sup\{\mu(j) : (y \vee j^{\circ\circ}) \wedge j^\circ = (z \vee j^{\circ\circ}) \wedge j^\circ\}$   
 $= \sup\{\mu(i) \wedge \mu(j) : (x \vee i^{\circ\circ}) \wedge i^\circ = (y \vee i^{\circ\circ}) \wedge i^\circ, (y \vee j^{\circ\circ}) \wedge j^\circ = (z \vee j^{\circ\circ}) \wedge j^\circ\}$   
 $\leq \sup\{\mu(r) : (x \vee r^{\circ\circ}) \wedge r^\circ = (z \vee r^{\circ\circ}) \wedge r^\circ\}$   
 $= \delta(\mu)(x, z).$

So  $\delta(\mu)(x, y) \wedge \delta(\mu)(y, z) \leq \delta(\mu)(x, z)$ . Hence  $\delta(\mu)$  is transitive.

For any  $z \in L$ ,

$$\begin{aligned} &\delta(\mu)(x, y) \\ &= \delta(\mu)(x, y) \wedge \delta(\mu)(z, z) \end{aligned}$$

$= \sup\{\mu(i) : (x \vee i^{\circ\circ}) \wedge i^{\circ} = (y \vee i^{\circ\circ}) \wedge i^{\circ}\} \wedge \sup\{\mu(j) : (z \vee j^{\circ\circ}) \wedge j^{\circ} = (z \vee j^{\circ\circ}) \wedge j^{\circ}\}$   
 $= \sup\{\mu(i) \wedge \mu(j) : (x \vee i^{\circ\circ}) \wedge i^{\circ} = (y \vee i^{\circ\circ}) \wedge i^{\circ}, (z \vee j^{\circ\circ}) \wedge j^{\circ} = (z \vee j^{\circ\circ}) \wedge j^{\circ}\}.$   
 Put  $r = i \vee j$ . Then we have  $r^{\circ} = i^{\circ} \wedge j^{\circ}$ ,  $r^{\circ\circ} = i^{\circ\circ} \vee j^{\circ\circ}$ . Thus

$$\begin{aligned}
 (x \vee z) \vee r^{\circ\circ} \wedge r^{\circ} &= ((x \vee z) \vee (i^{\circ\circ} \vee j^{\circ\circ})) \wedge (i^{\circ} \wedge j^{\circ}) \\
 &= ((x \vee i^{\circ\circ}) \vee (z \vee j^{\circ\circ})) \wedge (i^{\circ} \wedge j^{\circ}) \\
 &= ((x \vee i^{\circ\circ}) \wedge (i^{\circ} \wedge j^{\circ})) \vee ((z \vee j^{\circ\circ}) \wedge (i^{\circ} \wedge j^{\circ})) \\
 &= ((y \vee i^{\circ\circ}) \wedge (i^{\circ} \wedge j^{\circ})) \vee ((z \vee j^{\circ\circ}) \wedge (j^{\circ} \wedge j^{\circ})) \\
 &= ((y \vee z) \vee (i^{\circ\circ} \vee j^{\circ\circ})) \wedge (i^{\circ} \wedge j^{\circ}) \\
 &= (y \vee z) \vee r^{\circ\circ} \wedge r^{\circ}.
 \end{aligned}$$

This implies  $\delta(\mu)(x, y) = \delta(\mu)(x, y) \wedge \delta(\mu)(z, z) \leq \delta(\mu)(x \vee z, y \vee z)$ . Similarly,  $\delta(\mu)(x, y) \leq \delta(\mu)(x \wedge z, y \wedge z)$ . So  $\delta(\mu)$  is lattice fuzzy congruence of  $L$ .

Next, we see that  $\delta(\mu)(x, y) \leq \delta(\mu)(x^{\circ}, y^{\circ}) \wedge \delta(\mu)(x^*, y^*)$ .

$$\begin{aligned}
 (x \vee i^{\circ\circ}) \wedge i^{\circ} &= (y \vee i^{\circ\circ}) \wedge i^{\circ} & (*) \\
 \Rightarrow (x^{\circ} \wedge i^{\circ}) \vee i^{\circ\circ} &= (y^{\circ} \wedge i^{\circ}) \vee i^{\circ\circ} \\
 \Rightarrow (x^{\circ} \vee i^{\circ\circ}) \wedge (i^{\circ} \vee i^{\circ\circ}) &= (y^{\circ} \vee i^{\circ\circ}) \wedge (i^{\circ} \vee i^{\circ\circ}) \\
 \Rightarrow (x^{\circ} \vee i^{\circ\circ}) \wedge (i^{\circ} \vee i^{\circ\circ}) \wedge i^{\circ} &= (y^{\circ} \vee i^{\circ\circ}) \wedge (i^{\circ} \vee i^{\circ\circ}) \wedge i^{\circ} \\
 \Rightarrow (x^{\circ} \vee i^{\circ\circ}) \wedge i^{\circ} &= (y^{\circ} \vee i^{\circ\circ}) \wedge i^{\circ}.
 \end{aligned}$$

Also from (\*), we have

$$\begin{aligned}
 (x^* \wedge i^{\circ\circ*}) \vee i^{\circ*} &= (y^* \wedge i^{\circ\circ*}) \vee i^{\circ*} \\
 \Rightarrow (x^* \vee i^{\circ*}) \wedge (i^{\circ\circ*} \vee i^{\circ*}) &= (y^* \vee i^{\circ*}) \wedge (i^{\circ\circ*} \vee i^{\circ*}) \\
 \Rightarrow (x^* \vee i^{\circ\circ*}) \wedge (i^{\circ\circ*} \vee i^{\circ\circ*}) &= (y^* \vee i^{\circ\circ*}) \wedge (i^{\circ\circ*} \vee i^{\circ*}).
 \end{aligned}$$

Put  $j = i^{\circ*}$ . Then

$$\begin{aligned}
 (x^* \vee j^{\circ\circ}) \wedge (j^{\circ} \vee j^{\circ\circ}) &= (y^* \vee j^{\circ\circ}) \wedge (j^{\circ} \vee j^{\circ\circ}) \\
 \Rightarrow (x^* \vee j^{\circ\circ}) \wedge (j^{\circ} \vee j^{\circ\circ}) \wedge j^{\circ} &= (y^* \vee j^{\circ\circ}) \wedge (j^{\circ} \vee j^{\circ\circ}) \wedge j^{\circ} \\
 \Rightarrow (x^* \vee j^{\circ\circ}) \wedge j^{\circ} &= (y^* \vee j^{\circ\circ}) \wedge j^{\circ}.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 \delta(\mu)(x, y) &= \sup\{\mu(i) : (x \vee i^{\circ\circ}) \wedge i^{\circ} = (y \vee i^{\circ\circ}) \wedge i^{\circ}\} \\
 &\leq \sup\{\mu(i) : (x^{\circ} \vee i^{\circ\circ}) \wedge i^{\circ} = (y^{\circ} \vee i^{\circ\circ}) \wedge i^{\circ}\} \\
 &= \delta(\mu)(x^{\circ}, y^{\circ}).
 \end{aligned}$$

Also,

$$\begin{aligned}
 \delta(\mu)(x, y) &= \sup\{\mu(i) : (x \vee i^{\circ\circ}) \wedge i^{\circ} = (y \vee i^{\circ\circ}) \wedge i^{\circ}\} \\
 &\leq \sup\{\mu(j) : (x^* \vee j^{\circ\circ}) \wedge j^{\circ} = (y^* \vee j^{\circ\circ}) \wedge j^{\circ}\} \\
 &= \delta(\mu)(x^*, y^*).
 \end{aligned}$$

Thus  $\delta(\mu)$  is fuzzy congruence of  $L$ .

Next, we show that  $\text{Ker}\delta(\mu) = \mu$ . It is obvious that  $\mu \subseteq \text{Ker}\delta(\mu)$ . Conversely,

$$\begin{aligned}
 \text{Ker}\delta(\mu)(x) &= \delta(\mu)(x, 0) \\
 &= \sup\{\mu(i) : (x \vee i^{\circ\circ}) \wedge i^{\circ} = i^{\circ\circ} \wedge i^{\circ}\} \\
 &\leq \sup\{\mu(i) : (x^{**} \vee i^{**}) \wedge i^{\circ**} = i^{**} \wedge i^{\circ**}\}.
 \end{aligned}$$

Now  $(x^{**} \vee i^{**}) \wedge i^{\circ**} = i^{**} \wedge i^{\circ**}$ . Then  $(x^{**} \wedge i^{\circ**}) \vee (i^{**} \wedge i^{\circ**}) = i^{**} \wedge i^{\circ**}$ . We observe that  $x^{**} \wedge i^{\circ**} \leq i^{**} \wedge i^{\circ**} \leq i^{**}$ . Since  $\mu \in FI_{\circ*}(L)$ ,

$$\mu(i) = \mu(i^{**}) = \mu(i^{\circ*}) \leq \mu(x^{**} \wedge i^{\circ**}).$$

Thus

$$\begin{aligned}
 Ker\delta(\mu)(x) &\leq \sup\{\mu(i) : (x^{**} \vee i^{**}) \wedge i^{\circ**} = i^{**} \wedge i^{\circ**}\} \\
 &= \sup\{\mu(i^{\circ*}) \wedge \mu(x^{**} \wedge i^{\circ**}) : (x^{**} \vee i^{**}) \wedge i^{\circ**} = i^{**} \wedge i^{\circ**}\} \\
 &= \sup\{\mu(i^{\circ*} \vee (x^{**} \wedge i^{\circ**})) : (x^{**} \vee i^{**}) \wedge i^{\circ**} = i^{**} \wedge i^{\circ**}\} \\
 &= \sup\{\mu(i^{\circ*} \vee x^{**}) : (x^{**} \vee i^{**}) \wedge i^{\circ**} = i^{**} \wedge i^{\circ**}\} \\
 &\leq \mu(x^{**}) = \mu(x).
 \end{aligned}$$

So  $\mu = Ker\delta(\mu)$ .

Let  $\phi$  be any fuzzy congruence on  $L$  with  $Ker\phi = \mu$ . We prove that  $\delta(\mu) \subseteq \phi$ .  
Now

$$\begin{aligned}
 \delta(\mu)(x, y) &= \sup\{\mu(i) : (x \vee i^{\circ\circ}) \wedge i^{\circ} = (y \vee i^{\circ\circ}) \wedge i^{\circ}\} \\
 &= \sup\{\phi(i, 0) : (x \vee i^{\circ\circ}) \wedge i^{\circ} = (y \vee i^{\circ\circ}) \wedge i^{\circ}\}.
 \end{aligned}$$

Since  $\phi$  is fuzzy congruence on  $L$ ,  $\phi(i, 0) \leq \phi(i^{\circ}, 1) \leq \phi(i^{\circ\circ}, 0)$ . Then

$$\phi(i, 0) \leq \phi(i^{\circ}, 1) \leq \phi(x, (x \vee i^{\circ\circ}) \wedge i^{\circ}) \wedge \phi((y \vee i^{\circ\circ}) \wedge i^{\circ}, y) \leq \phi(x, y).$$

Thus  $\delta(\mu) \subseteq \phi$ . So  $\delta(\mu)$  is as our required.  $\square$

**Corollary 5.12.** Let  $(L, ^{\circ}, *)$  be a dpMS-algebra. If  $\mu \in FI_{\circ*}(L)$ , then  $\bar{\Theta}[\mu] = \delta(\mu)$ .

*Proof.* By Theorem 5.11, it is enough to show that  $Ker\bar{\Theta}[\mu] = \mu$ . It is clear that  $\mu \subseteq Ker\bar{\Theta}[\mu]$ . Conversely, Theorem 4.7,

$$\begin{aligned}
 Ker\bar{\Theta}[\mu](x) &= \bar{\Theta}[\mu](0, x) \\
 &= \sup_n (\sup_{z_1, z_2, \dots, z_{2n}} (\bar{\Theta}_{lat}[\mu](0, z_1) \wedge \bar{\Theta}_{lat}[\mu^{\geq}](z_1, z_2) \\
 &\quad \wedge \bar{\Theta}_{lat}[\mu](z_2, z_3) \wedge \dots \wedge \bar{\Theta}_{lat}[\mu](z_{2n}, x))) \\
 &\leq \bar{\Theta}_{lat}[\mu](0, z_1) \\
 &= \sup\{\mu(i) : i = i \vee z_1\} \\
 &= \sup\{\mu(i \vee z_1) : i = i \vee z_1\} \\
 &\leq \mu(z_1).
 \end{aligned}$$

Or

$$\begin{aligned}
 Ker\bar{\Theta}[\mu](x) &= \bar{\Theta}[\mu](0, x) \\
 &\leq \bar{\Theta}_{lat}[\mu^{\geq}](0, z_1) \\
 &= \sup\{\mu(j) : 0 = z_1 \wedge j^{\circ}, j \in L\} \\
 &\leq \sup\{\mu(j) : 0 = z_1^{\circ*} \wedge j^*, j \in L\} \\
 &= \sup\{\mu(j^{**}) : 0 = z_1^{\circ*} \wedge j^*, j \in L\} \\
 &\leq \mu(z_1^{\circ*}) \\
 &= \mu(z_1),
 \end{aligned}$$

since  $z_1^{\circ*} \leq j^{**}$ .

Suppose that for  $r \leq k$ ,  $\mu(i) \leq \mu(z_r)$ , for  $z_r \vee i = z_{r+1} \vee i$ . Then

$$\begin{aligned}
 &\bar{\Theta}[\mu](0, x) \\
 &= \sup_{z_1, z_2, \dots, z_{2n}} (\bar{\Theta}_{lat}[\mu](0, z_1) \wedge \bar{\Theta}_{lat}[\mu^{\geq}](z_1, z_2) \wedge \bar{\Theta}_{lat}[\mu](z_2, z_3) \wedge \dots \wedge \bar{\Theta}_{lat}[\mu](z_{2n}, x)) \\
 &\leq \bar{\Theta}_{lat}[\mu](z_k, z_{k+1}) \\
 &= \sup\{\mu(i) : i \vee z_k = i \vee z_{k+1}\} \\
 &= \sup\{\mu(i) \wedge \mu(z_k) : i \vee z_k = i \vee z_{k+1}\} \\
 &= \sup\{\mu(i \vee z_k) : i \vee z_k = i \vee z_{k+1}\} \\
 &= \sup\{\mu(i \vee z_{k+1}) : i \vee z_k = i \vee z_{k+1}\} \\
 &\leq \mu(z_{k+1}).
 \end{aligned}$$

On the other hand, since  $j^\circ \wedge z_k = j^\circ \wedge z_{k+1}$  and  $j^{**} \vee j^* = 1$ , we have

$$j^{**} \wedge z_k^{\circ*} = j^{**} \wedge z_{k+1}^{\circ*} \geq z_{k+1}^{\circ*}.$$

Thus

$$\begin{aligned} \bar{\Theta}[\mu](0, x) &= \bar{\Theta}_{lat}[\mu_\circ^>](z_k, z_{k+1}) \\ &= \sup\{\mu(j) : j^\circ \wedge z_k = j^\circ \wedge z_{k+1}, j \in L\} \\ &\leq \mu(z_k^{\circ*}) \\ &= \mu(z_k) \end{aligned}$$

So it follows by induction that  $Ker\bar{\Theta}[\mu](x) \leq \mu(x)$ . This implies  $\mu = Ker\bar{\Theta}[\mu]$ . Hence the required equality holds.  $\square$

**Theorem 5.13.** *If  $(L, \circ, *)$  be a dpMS-algebra, then for each  $\mu \in FI_{\circ*}(L)$ , the greatest fuzzy congruence on  $L$  with kernel  $\mu$  is the relation  $\sigma(\mu)$  given by:*

$$\sigma(\mu)(x, y) = \mu((x^{**} \wedge y^*) \vee (x^* \wedge y^{**})).$$

*Proof.* We prove that  $\sigma(\mu)$  is a fuzzy congruence on  $L$ . It is obvious that  $\sigma(\mu)$  is reflexive and symmetric. We show that  $\sigma(\mu)$  is transitive.

$$\begin{aligned} \mu(x^{**} \wedge z^*) &= \mu((x^{**} \wedge z^*) \wedge (y^* \vee y^{**})) \\ &= \mu((x^{**} \wedge z^* \wedge y^*) \vee ((x^{**} \wedge z^* \wedge y^{**}))) \\ &= \mu((x^{**} \wedge z^* \wedge y^*) \wedge \mu((x^{**} \wedge z^* \wedge y^{**}))) \\ &\geq \mu(x^{**} \wedge y^*) \wedge \mu(z^* \wedge y^{**}) \\ &\geq \mu(x^{**} \wedge y^*) \wedge \mu(x^* \wedge y^{**}) \wedge \mu(y^* \wedge z^{**}) \wedge \mu(z^* \wedge y^{**}) \\ &= \mu((x^{**} \wedge y^*) \vee (x^* \wedge y^{**})) \wedge \mu((y^* \wedge z^{**}) \vee (z^* \wedge y^{**})) \\ &= \sigma(\mu)(x, y) \wedge \sigma(\mu)(y, z). \end{aligned}$$

Similarly,  $\mu(x^* \wedge z^{**}) \geq \sigma(\mu)(x, y) \wedge \sigma(\mu)(y, z)$ . This implies

$$\mu(x^{**} \wedge z^*) \wedge \mu(x^* \wedge z^{**}) \geq \sigma(\mu)(x, y) \wedge \sigma(\mu)(y, z).$$

Then  $\sigma(\mu)(x, z) \geq \sigma(\mu)(x, y) \wedge \sigma(\mu)(y, z)$ . Also

$$\begin{aligned} \sigma(\mu)(x \wedge a, y \wedge a) &= \mu(((x \wedge a)^{**} \wedge (y \wedge a)^*) \vee ((x \wedge a)^* \wedge (y \wedge a)^{**})) \\ &= \mu((x^{**} \wedge y^* \wedge a^{**}) \vee (x^* \wedge y^{**} \wedge a^{**})) \\ &= \mu(((x^{**} \wedge y^*) \vee (x^* \wedge y^{**})) \wedge a^*) \\ &\geq \mu((x^{**} \wedge y^*) \wedge (x^* \wedge y^{**})) \\ &= \sigma(\mu)(x, y). \end{aligned}$$

Similarly,  $\sigma(\mu)(x \vee a, y \vee a) \geq \sigma(\mu)(x, y)$ . Thus  $\sigma(\mu)$  is a lattice of fuzzy congruence.

On the other hand,

$$\begin{aligned} \sigma(\mu)(x^*, y^*) &= \mu((x^{***} \wedge y^{**}) \vee (x^{**} \wedge y^{***})) \\ &= \mu((x^* \wedge y^{**}) \vee (x^{**} \wedge y^*)) \\ &= \sigma(\mu)(x, y) \end{aligned}$$

and

$$\begin{aligned} \sigma(\mu)(x, y) &= \mu((x^{**} \wedge y^*) \vee (x^* \wedge y^{**})) \\ &= \mu(((x^{**} \wedge y^*) \vee (x^* \wedge y^{**}))^{\circ*}) \\ &= \mu((x^{\circ**} \wedge y^{\circ**}) \vee (x^{\circ**} \wedge y^{\circ**})) \\ &= \mu((x^{\circ*} \wedge y^{\circ**}) \vee (x^{\circ**} \wedge y^{\circ*})) \\ &= \sigma(\mu)(x^\circ, y^\circ). \end{aligned}$$

So  $\sigma(\mu)$  is a fuzzy congruence of a dpMS-algebra  $L$ .

Also  $Ker\sigma(\mu)(x) = \sigma(\mu)(x, 0) = \mu((x^{**} \wedge 0^*) \vee (x^* \wedge 0^{**})) = \mu(x^{**}) = \mu(x)$ . Then  $Ker\sigma(\mu) = \mu$ . Thus  $\sigma(\mu)$  is a fuzzy congruence of  $L$  with kernel  $\mu$ .

Finally, let  $\varphi$  be a congruence on  $L$  with  $\text{Ker}\varphi = \mu$ . Then

$$\varphi(x, y) \leq \varphi(x^*, y^*) \leq \varphi(x^{**}, y^{**}).$$

Thus

$$\begin{aligned} \varphi(x, y) &\leq \varphi(x^*, y^*) \\ &= \varphi(x^*, y^*) \wedge \varphi(x^{**}, y^{**}) \wedge (x^{**}, x^{**}) \wedge \varphi(x^*, x^*) \\ &\leq \varphi(x^* \wedge x^{**}, y^* \wedge y^{**}) \wedge \varphi(x^{**} \wedge x^*, y^{**} \wedge x^*) \\ &\leq \varphi(((x^* \wedge x^{**}) \vee (x^{**} \wedge x^*)), ((y^* \wedge y^{**}) \wedge (y^{**} \wedge x^*))) \\ &= \varphi(0, ((y^* \wedge x^{**}) \wedge (y^{**} \wedge x^*))) \\ &= \text{Ker}\varphi((y^* \wedge x^{**}) \wedge (y^{**} \wedge x^*)) \\ &= \mu((y^* \wedge x^{**}) \wedge (y^{**} \wedge x^*)) \\ &= \sigma(\mu)(x, y). \end{aligned}$$

So  $\varphi \subseteq \sigma(\mu)$ . Hence  $\sigma(\mu)$  is the greatest fuzzy congruence of  $L$  with kernel  $\mu$ .  $\square$

**Theorem 5.14.** *If  $(L, \circ, *)$  be a dpMS-algebra then for each  $\mu, \nu \in FI_{\circ*}(L)$ , the following statements hold:*

- (1)  $\mu \subseteq \nu \Leftrightarrow \sigma(\mu) \subseteq \sigma(\nu)$ ,
- (2)  $\sigma(\text{Ker}G_F) = G_F$ ,
- (3)  $G_F \subseteq \sigma(\mu)$ .

As constituted in Theorems 5.11 and 5.13, for every  $\mu \in FI_{\circ*}(L)$ , there exists the smallest fuzzy congruence  $\delta(\mu)$  and the biggest fuzzy congruence  $\sigma(\mu)$  on  $L$  with kernel  $\mu$ . The relationship between these two congruences can be characterized as follows.

**Theorem 5.15.** *Let  $(L, \circ, *)$  be a dpMS-algebra. If  $\mu \in FI_{\circ*}(L)$ , then  $\sigma(\mu) = G_F \vee \delta(\mu)$ .*

*Proof.* By Theorems 5.11 and 5.13, we have clearly seen that  $G_F \vee \delta(\mu) \leq \sigma(\mu)$ . To show the reverse inequality, put  $t = (x^{**} \wedge y^*) \vee (x^* \wedge y^{**})$ . Then  $t^{\circ\circ} = t^{**} = t$  and  $x^{**} \vee t = y^{**} \vee t = x^{**} \vee y^{**}$ . This implies  $(x^{**} \vee t) \wedge t^{\circ} = (y^{**} \vee t) \wedge t^{\circ}$  and also  $(x^{**} \vee t^{\circ\circ}) \wedge t^{\circ} = (y^{**} \vee t^{\circ\circ}) \wedge t^{\circ}$  as  $t^{\circ\circ} = t$ . Now

$$\begin{aligned} \sigma(\mu)(x, y) &= \mu((x^{**} \wedge y^*) \vee (x^* \wedge y^{**})) \\ &= \mu(t) \\ &\leq \sup\{\mu(t) : (x^{**} \vee t^{\circ\circ}) \wedge t^{\circ} = (y^{**} \vee t^{\circ\circ}) \wedge t^{\circ}\} \\ &= \delta(\mu)(x^{**}, y^{**}). \end{aligned}$$

Clearly  $G_F(z, z^{**}) = 1$ , for all  $z \in L$ . On the other hand,

$$\begin{aligned} \delta(\mu)(x^{**}, y^{**}) &= \delta(\mu)(x^{**}, y^{**}) \wedge G_F(y^{**}, y) \wedge G_F(x^{**}, x) \\ &\leq (\delta(\mu) \circ G_F)(x, y) \leq (\delta(\mu) \vee G_F)(x, y). \end{aligned}$$

Thus the result is holds.  $\square$

**Theorem 5.16.** *Let  $(L, \circ, *)$  is a dpMS-algebra. A  $(\circ, *)$ -fuzzy ideal of  $L$  is precisely the kernel of the fuzzy congruence  $\varphi$  on  $L$  if and only if  $(L/\varphi, *)$  is boolean.*

*Proof.* Suppose  $\mu$  is a  $(\circ, *)$ -fuzzy ideal. Then by Theorem 5.14,  $G_F \leq \sigma(\mu)$ . Since for each  $x \in L$ ,  $1 = \text{Ker}G_F(x \wedge x^*) \leq \text{Ker}\sigma(\mu)(x \wedge x^*)$  and  $1 = \text{Coker}G_F(x \vee x^*) \leq \text{Coker}\sigma(\mu)(x \vee x^*)$ , we have

$$1 = \text{Ker}\varphi(x \wedge x^*) = \varphi(x \wedge x^*, 0)$$

and

$$1 = \text{Coker}\sigma(\mu)(x \vee x^*) = \sigma(\mu)(x \vee x^*, 1).$$

Thus  $\sigma(\mu)_x \wedge (\sigma(\mu)_x)^* = \sigma(\mu)_0$  and  $\sigma(\mu)_x \vee (\sigma(\mu)_x)^* = \sigma(\mu)_1$ . So  $(L/\sigma(\mu), *)$  is boolean.

Conversely, suppose that  $\varphi$  is a congruence on  $L$  such that  $(L/\varphi, *)$  is boolean. We prove that  $\text{Ker}\varphi$  is a  $(\circ, *)$ -fuzzy ideal of  $L$ . Let  $x \in L$ . Then

$$\text{Ker}\varphi(x) = \varphi(x, 0) \leq \varphi(x^{\circ*}, 0) = \text{Ker}\varphi(x^{\circ*}).$$

Also  $\text{Ker}\varphi(x^{\circ*}) = \varphi(x^{\circ*}, 0) \leq \varphi(x^*, 1)$ . Since  $(L/\sigma(\mu), *)$  is boolean,  $\varphi_x \wedge (\varphi_x)^* = \varphi_0$ . Thus  $\varphi(x \wedge x^*, 0) = 1$ . On the other hand,

$$\begin{aligned} \text{Ker}\varphi(x^{\circ*}) &\leq \varphi(x^*, 1) \wedge \varphi(x, x) \leq \varphi(x \wedge x^*, x) \\ &= \varphi(x \wedge x^*, x) \wedge 1 \\ &\leq \varphi(x, 0) = \text{Ker}\varphi(x). \end{aligned}$$

So  $\text{Ker}\varphi$  is  $(\circ, *)$ -fuzzy ideal.  $\square$

**Example 5.17.** As indicated in Example 5.6,  $\text{Ker}G_F = \mu \in FI_{\circ*}(L)$ . We can be easily verified that  $(G_F)_0 = (G_F)_a = (G_F)_b = (G_F)_c = (G_F)_d = (G_F)_f$ ,  $L/G_F = \{(G_F)_0, (G_F)_e, (G_F)_g, (G_F)_h, (G_F)_1\}$  and  $(L/G_F, *)$  is boolean.

**Theorem 5.18.** If  $(L, \circ, *)$  is a dpMS-algebra, then  $FI_{\circ*}(L) \simeq [G_F, \chi_l]$ .

*Proof.* Let  $\mu, \nu \in FI_{\circ*}(L)$ . Then  $\sigma(\mu \wedge \nu)(x, y) = (\mu \wedge \nu)((x^{**} \wedge y^*) \vee (x^* \wedge y^{**})) = \mu((x^{**} \wedge y^*) \vee (x^* \wedge y^{**})) \wedge \nu((x^{**} \wedge y^*) \vee (x^* \wedge y^{**})) = \sigma(\mu)(x, y) \wedge \sigma(\nu)(x, y)$ . Thus  $\mu \longrightarrow \sigma(\mu)$  is  $\wedge$ -monomorphism.

By Theorem 5.14, we see that the mapping  $\mu \longrightarrow \sigma(\mu)$  is an isotone from  $FI_{\circ*}(L) \longrightarrow [G_F, \chi_l]$ , so that for  $\mu, \nu \in FI_{\circ*}(L)$ , we have

$$(5.1) \quad \sigma(\mu) \vee \sigma(\nu) \leq \sigma(\mu \vee \nu).$$

The converse inequality of (5.1):

$$\begin{aligned} \sigma(\mu \vee \nu)(x, y) &= (\mu \vee \nu)((x^{**} \wedge y^*) \vee (x^* \wedge y^{**})) \\ &= \sup\{\mu(i) \wedge \nu(j) : i \vee j = (x^{**} \wedge y^*) \vee (x^* \wedge y^{**})\}, \text{ i.e.,} \end{aligned}$$

$$(5.2) \quad \sigma(\mu \vee \nu)(x, y) = \sup\{\mu(i) \wedge \nu(j) : i \vee j = (x^{**} \wedge y^*) \vee (x^* \wedge y^{**})\}.$$

Put  $t = (x^{**} \wedge y^*) \vee (x^* \wedge y^{**}) = i \vee j$ . Then  $t = t^{**} = i^{**} \vee j^{**}$  and  $t \wedge i^* = j^{**} \wedge i^*$  and also observe that

$$\begin{aligned} &[(x \vee i)^* \wedge (y \vee i)^{**}] \vee [(x \vee i)^{**} \wedge (y \vee i)^*] \\ &= [x^* \wedge i^* \wedge (y^{**} \vee i^{**})] \vee [(x^{**} \vee i^{**}) \wedge y^* \wedge i^*] \\ &= (x^* \wedge y^{**} \wedge i^*) \vee (x^{**} \wedge y^* \wedge i^*) = [(x^* \wedge y^{**}) \vee (x^{**} \wedge y^*)] \wedge i^* \\ &= t \wedge i^*. \end{aligned} \tag{5.3}$$

$$\begin{aligned} &[z^* \wedge (z \vee i)^{**}] \vee [z^{**} \wedge (z \vee i)^*] \\ &= [(z^* \wedge z^{**}) \vee (z^* \wedge i^{**})] \vee [z^{**} \wedge z^{**} \wedge z^* \wedge i^*] \\ &= z^* \wedge i^{**}, \forall z \in L (z = x \text{ or } z = y). \end{aligned} \tag{5.4}$$

From (5.2) and (5.3),

$$\begin{aligned} \sigma(\mu \vee \nu)(x, y) &= (\mu \vee \nu)((x^{**} \wedge y^*) \vee (x^* \wedge y^{**})) \\ &= \sup\{\mu(i) \wedge \nu(j) : i \vee j = (x^{**} \wedge y^*) \vee (x^* \wedge y^{**})\} \\ &\leq \nu(j^{**} \wedge i^*) = \nu(t \wedge i^*) \\ &= \nu([(x \vee i)^* \wedge (y \vee i)^{**}] \vee [(x \vee i)^{**} \wedge (y \vee i)^*]) \end{aligned}$$

$$= \sigma(\nu)(x \vee i, y \vee i). \quad (5.5)$$

From (5.2) and (5.4),

$$\begin{aligned} \sigma(\mu \vee \nu)(x, y) &= (\mu \vee \nu)((x^{**} \wedge y^*) \vee (x^* \wedge y^{**})) \\ &= \sup\{\mu(i) \wedge \nu(j) : i \vee j = (x^{**} \wedge y^*) \vee (x^* \wedge y^{**})\} \\ &\leq \mu(x^* \wedge i^{**}) \\ &= \mu([x^* \wedge (x \vee i)^{**}] \vee [x^{**} \wedge (x \vee i)^*]) \\ &= \sigma(\mu)(x, x \vee i). \end{aligned} \quad (5.6)$$

From (5.5) and (5.6),

$$\sigma(\mu \vee \nu)(x, y) \leq \sigma(\nu)(x \vee i, y \vee i) \wedge \sigma(\mu)(x, x \vee i) \leq (\sigma(\nu) \vee \sigma(\mu))(x, y \vee i).$$

Similarly,  $\sigma(\mu \vee \nu)(x, y) \leq (\sigma(\nu) \vee \sigma(\mu))(x \vee i, y)$ . Thus we obtain

$$\sigma(\mu \vee \nu)(x, y) \leq (\sigma(\nu) \vee \sigma(\mu))(x, y).$$

So the reverse inequality of (5.2) holds. This implies  $\mu \longrightarrow \sigma(\mu)$  is a  $\vee$ -morphism. In addition to this it is an injective by Theorem 5.14 (1).

Next we see that it is a surjective. Let  $\varphi \in [G_F, \chi_\iota]$ . Then by Corollary 5.9,  $\text{Ker}\varphi$  is a  $(\circ, *)$ -fuzzy ideal of  $L$ . Now  $\varphi(x, y) \leq \varphi(x^*, y^*) \leq \varphi(x^* \wedge x^{**}, y^* \wedge x^{**}) = \varphi(0, y^* \wedge x^{**})$  and  $\varphi(x, y) \leq \varphi(0, x^* \wedge y^{**})$ . Then

$$\begin{aligned} \varphi(x, y) &\leq \varphi(0, ((x^* \wedge y^{**}) \wedge (y^* \wedge x^{**}))) \\ &= \text{Ker}(\varphi)((x^* \wedge y^{**}) \wedge (y^* \wedge x^{**})) \\ &= \sigma(\text{Ker}\varphi)(x, y). \end{aligned}$$

Conversely,

$$\begin{aligned} \sigma(\text{Ker}\varphi)(x, y) &= \text{Ker}(\varphi)((x^* \wedge y^{**}) \vee (y^* \wedge x^{**})) \\ &\leq \text{Ker}(\varphi)(y^* \wedge x^{**}) \\ &= \varphi(0, y^* \wedge x^{**}) \\ &\leq \varphi(1, y^{**} \vee x^*). \end{aligned}$$

Since  $G_F \leq \varphi$ , we have  $\varphi(x, x^{**}) = 1$ . Thus

$$\begin{aligned} \sigma(\text{Ker}\varphi)(x, y) &\leq \varphi(1, y^{**} \wedge x^*) \\ &\leq \varphi(x, (y^{**} \wedge x^*) \wedge x^{**}) \\ &\leq \varphi(x, x^{**} \wedge y^{**}). \end{aligned}$$

Similarly,  $\sigma(\text{Ker}\varphi)(x, y) \leq \varphi(y, x^{**} \wedge y^{**})$ . It follow that  $\sigma(\text{Ker}\varphi)(x, y) \leq \varphi(x, y)$ . So we have got  $\sigma(\text{Ker}\varphi) = \varphi$ . Hence  $\mu \longrightarrow \sigma(\mu)$  is a surjective. Therefore it is a lattice isomorphism.  $\square$

## 6. CONCLUSION AND FUTURE WORK

In this paper, we have studied the fuzzy congruences, kernel fuzzy ideals and  $(\circ, *)$ -fuzzy ideas of demi-Pseudocomplemented MS-algebras and their Properties. In [14], Filters congruences of Pseudocomplemented MS-algebras have been studied. One of the most promising ideas could be the investigation of fuzzy setting applied to filters congruences of Pseudocomplemented MS-algebras.

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