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The category of hesitant H-fuzzy sets

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Abstract. We redefine the hesitant fuzzy empty set, the hesitant fuzzy whole set, the intersection and the union of two hesitant fuzzy sets, and prove that the family $HS(X)$ of all hesitant fuzzy sets in a set X is a Boolean algebra. Next, we introduce the category $\mathbf{HSet}(H)$ consisting of hesitant H-fuzzy spaces and preserving mappings between them and study the category $\mathbf{HSet}(H)$ in the sense of a topological universe and prove that it is Cartesian closed over **Set**, where **Set** denotes the category consisting of ordinary sets and ordinary mappings between them.

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1. Introduction

In 1965, Zadeh [34] introduced a fuzzy set as the generalization of an ordinary set. After that time, many researchers [4, 7, 8, 12, 22, 26, 27] investigated fuzzy sets in the sense of category theory, for instance, $\mathbf{Set}(H)$, $\mathbf{Set_f}(H)$, $\mathbf{Set_g}(H)$, $\mathbf{Fuz}(H)$. In particular, Carrega [4], Eytan [7], Goguen [8], Pittes [26], Ponasse [27] studied $\mathbf{Set}(H)$ in topos view-point.

In 1984, Nel [23] introduced the concept of a topological universe which implies quasitopos [1]. Its notion has already been put to effective use several areas of mathematics in [19, 20, 24].

Hur [12] investigated $\mathbf{Set}(H)$ in topological universe view-point. Lim et al [22] introduced the new category $\mathbf{VSet}(H)$ and investigated it in the sense of topological universe. Moreover, by using the concept of an intutionistic fuzzy set introduced by Atanasossov [2], Hur et al. [13] introduced the category $\mathbf{ISet}(H)$ and studied it in a view point of topological universe. Recently, Hur et al [14, 15] studied the categories $\mathbf{NSet}(H)$ of neutrosophic sets which were introduced by Smarandache

[29] and **NCSet**(H) of neutrosophic crisp set defined by Salama and Smarandache [28] in the sense of topological view-point, respectively. Moreover, Kim et al. [18] investigated the category **NCSet** of intuitionistic sets introduced by Coker [5] in the sense of topological universe. Also Lee et al. [21] studied the category **BPSet** of bipolar fuzzy sets defined by Zhang [35] in the same sense.

In 2010, Torra [31] introduced the notion of a hesitant fuzzy set (Refer to [25, 30]). After then, Jun et al. [9] studied hesitant fuzzy bi-ideals in semigroups. Xia and Xu [32] applied hesitant fuzzy set for decision making. Furthermore, Deepark and John [6] investigated hesitant fuzzy rough sets through hesitant fuzzy relations.

In this paper, we redefine the hesitant fuzzy empty set, the hesitant fuzzy whole set, the intersection and the union of two hesitant fuzzy sets, and prove that the family $HS(X)$ of all hesitant fuzzy sets in a set X is a Boolean algebra. Next, we introduce the category **HSet**(H) consisting of hesitant H-fuzzy spaces and preserving mappings between them and study the category **HSet**(H) in the sense of a topological universe and prove that it is Cartesian closed over **Set** (See Theorem 4.15), where **Set** denotes the category consisting of ordinary sets and ordinary mappings between them.

2. Preliminaries

In this section, we list some basic definitions and well-known results with respect to category theory from [10, 17, 23, 31] which are needed in the next sections. Let us recall that a concrete category is a category of sets which are endowed with an unspecified structure.

Definition 2.1 ([17])**.** Let **A** be a concrete category and $((Y_j, \xi_j))_J$ a family of objects in **A** indexed by a class J. For any set X, let $(f_j : X \to Y_j)_J$ be a source of mappings indexed by J. Then an **A**-structure ξ on X is said to be initial with respect to (in short, w.r.t.) $(X, (f_j), ((Y_j, \xi_j))_J)$, if it satisfies the following conditions:

(i) for each $j \in J$, $f_j : (X, \xi) \to (Y_j, \xi_j)$ is an **A**-morphism,

(ii) if (Z, ρ) is an **A**-object and $g : Z \to X$ is a mapping such that for each $j \in J$, the mapping $f_j \circ g : (Z, \rho) \to (Y_j, \xi_j)$ is an **A**-morphism, then $g : (Z, \rho) \to (X, \xi)$ is an **A**-morphism.

In this case, $(f_j : (X, \xi) \to (Y_j, \xi_j))_J$ is called an initial source in **A**.

Dual notion: cotopological category.

Definition 2.2 ([17])**.** Let **A** be a concrete category.

(i) The **A**-fibre of a set X is the class of all **A**-structures on X.

(ii) **A** is said to be properly fibred over **Set**, if it satisfies the following:

(a) (Fibre-smallness) for each set X, the **A**-fibre of X is a set,

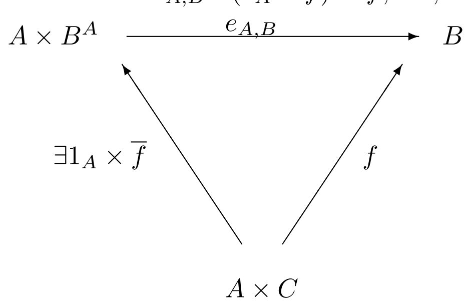
(b) (Terminal separator property) for each singleton set X, the **A**-fibre of X has precisely one element,

(c) if ξ and η are **A**-structures on a set X such that $id : (X, \xi) \to (X, \eta)$ and $id : (X, \eta) \to (X, \xi)$ are **A**-morphisms, then $\xi = \eta$.

Definition 2.3 ([10])**.** A category **A** is said to be Cartesian closed if it satisfies the following conditions:

(i) for each **A**-object A and B, there exists a product $A \times B$ in **A**,

(ii) exponential objects exist in A, i.e., for each **A**-object A, the functor $A \times - : A \to A$ has a right adjoint, i.e., for any **A**-object B, there exist an **A**-object B^A and a **A**-morphism $e_{A,B} : A \times B^A \to B$ (called the evaluation) such that for any **A**-object C and any **A**-morphism $f : A \times C \to B$, there exists a unique **A**-morphism $\bar{f} : C \to B^A$ such that $e_{A,B} \circ (1_A \times \bar{f}) = f$, i.e., the diagram commutes:



Definition 2.4 ([23])**.** A category **A** is called a topological universe over **Set** if it satisfies the following conditions:

(i) **A** is well-structured, i.e. (a) **A** is concrete category; (b) fibre-smallness condition; (c) **A** has the terminal separator property,

(ii) **A** is cotopological over **Set**,

(iii) final episinks in A are preserved by pullbacks, i.e., for any episink $(g_j : X_j \to Y)_J$ and any **A**-morphism $f : W \to Y$, the family $(e_j : U_j \to W)_J$, obtained by taking the pullback f and g_j, for each $j \in J$, is again a final episink.

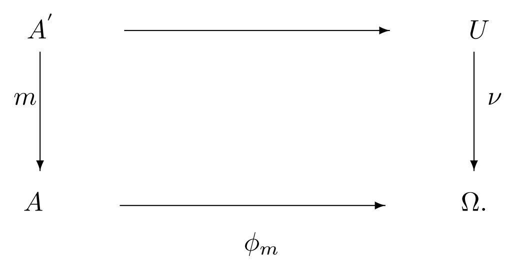
Definition 2.5 ([27])**.** A category **A** is called a topos, if the following conditions hold:

(i) There is a terminal object U in **A**,

(ii) **A** has equalizers,

(iii) **A** is Cartesian closed,

(iv) There is a subobject classifier in **A**, i.e., there is an Ω and a morphism ν from U to Ω such that for each morphism m from A' to A, there exists a unique morphism ϕ_m from A to Ω such that the diagram is a pullback:



Definition 2.6 ([31])**.** Let X be a reference set and let $P[0,1]$ denote the power set of $[0,1]$. Then a mapping $h : X \to P[0,1]$ is called a hesitant fuzzy set in X.

The empty hesitant fuzzy set (denoted by $h_{\mathbf{0}}$) and the full hesitant fuzzy set (denoted by $h_{\mathbf{1}}$) are defined by: for each $x \in X$,

$$h_{\mathbf{0}}(x) = \{0\} \text{ and } h_{\mathbf{1}}(x) = \{1\}.$$

We will denote the set of all hesitant fuzzy sets in X as $HFS(X)$.

For each $h \in HFS(X)$, the lower bound and upper bound of h, denoted by h^- and h^+, are defined by: for each $x \in X$,

$$h^-(x) = inf\ h(x) = \bigwedge h(x) \text{ and } h^+(x) = sup\ h(x) = \bigvee h(x).$$

Then clearly, h^- and h^+ are fuzzy sets in X introduced by Zadeh [34].

Definition 2.7 ([31])**.** Let $h \in HFS(X)$. Then the complement of h, denoted by h^c, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$h^c(x) = \bigcup_{\gamma \in h(x)} \{1 - \gamma\}.$$

Result 2.8 ([31], Proposition 3.8)**.** *The complement is involutive, i.e.,*

$$(h^c)^c = h.$$

Definition 2.9 ([31])**.** Let $h_1,\ h_2 \in HFS(X)$. Then

(i) the union of h_1 and h_2, denoted by $h_1 \cup h_2$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$\begin{aligned}(h_1 \cup h_2)(x) &= \{\gamma \in (h_1(x) \cup h_2(x) : \gamma \geq max(h_1^-(x), h_2^-(x))\} \\ &= \textstyle\bigcup_{\gamma_1 \in h_1(x),\ \gamma_2 \in h_2(x)} (\gamma_1 \vee \gamma_2),\end{aligned}$$

(ii) the intersection of h_1 and h_2, denoted by $h_1 \cap h_2$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$\begin{aligned}(h_1 \cap h_2)(x) &= \{\gamma \in (h_1(x) \cup h_2(x) : \gamma \leq min(h_1^+(x), h_2^+(x))\} \\ &= \textstyle\bigcup_{\gamma_1 \in h_1(x),\ \gamma_2 \in h_2(x)} (\gamma_1 \wedge \gamma_2).\end{aligned}$$

Example 2.10. Let $X = \{a, b, c\}$ and let $h_1,\ h_2 \in HFS(X)$ given by:

$$h_1(a) = [0.3, 0.7],\ h_1(b) = \{0.1, 0.5\},\ h_1(c) = \{0.1, 0.8\}$$

and

$$h_2(a) = \{0.3, 0.5\},\ h_2(b) = [0.1, 0.6],\ h_2(c) = \{0.4, 0.8\}.$$

Then

$$(h_1 \cup h_2)(a) = \bigcup_{\gamma_1 \in h_1(a),\ \gamma_2 \in h_2(a)} (\gamma_1 \vee \gamma_2) = [0.3, 0.7]$$

and

$$(h_1 \cap h_2)(a) = \bigcup_{\gamma_1 \in h_1(a),\ \gamma_2 \in h_2(a)} (\gamma_1 \wedge \gamma_2) = [0.3, 0.5].$$

Similarly, we can calculate the following:

$$(h_1 \cup h_2)(b) = [0.1, 0.6],\ (h_1 \cup h_2)(c) = \{0.4, 0.8\}$$

and

$$(h_1 \cap h_2)(b) = [0.1, 0.5],\ (h_1 \cap h_2)(c) = \{0.1, 0.4, 0.8\}.$$

Remark 2.11. From Definition 2.9, we can easily see that for any $h_1,\ h_2 \in HFS(X)$, if $h_1(x) = \phi$ or $h_2(x) = \phi$ for some $x \in X$, then $h_1 \cup h_2$ and $h_1 \cap h_2$ are not defined.

Result 2.12 ([32], Theorem 1)**.** *Let $h_1,\ h_2 \in HFS(X)$. Then*

(1) $(h_1 \cap h_2)^c = h_1^c \cup h_2^c$,

(2) $(h_1 \cup h_2)^c = h_1^c \cap h_2^c$.

Definition 2.13 ([6])**.** Let $h_1,\ h_2 \in HFS(X)$. Then
we say that h_1 is a subset of h_2, denoted by $h_1 \subseteq h_2$, if $h_1(x) \subseteq h_2(x)$, for each $x \in X$,
(ii) we say that h_1 is equal to h_2, denoted by $h_1 = h_2$, if $h_1 \subseteq h_2$ and $h_2 \subseteq h_1$.

Definition 2.14 ([6])**.** Let $h_1,\ h_2 \in HFS(X)$. Then we say that h_1 is a proper subset of h_2, denoted by $h_1 \subset h_2$, if

$$h_1(x) \subseteq h_2(x), \text{ for each } x \in X$$

and

$$h_1(x) \neq h_2(x), \text{ for some } x \in X.$$

Result 2.15 ([6], Lemma 2.15)**.** *Let $h,\ h_1,\ h_2,\ h_3 \in HFS(X)$. Then*
(1) $(h \cup h_{\mathbf{1}}) = h_{\mathbf{1}},\ (h \cap h_{\mathbf{1}}) = h$,
(2) $(h \cup h_{\mathbf{0}}) = h,\ (h \cap h_{\mathbf{0}}) = h_{\mathbf{0}}$,
(3) $h \cup h^c \neq h_{\mathbf{1}},\ h \cap h^c \neq h_{\mathbf{0}}$, *in general,*
(4) (Commutative laws): $h_1 \cup h_2 = h_2 \cup h_1,\ h_1 \cap h_2 = h_2 \cap h_1$,
(5) (Associative laws): $h_1 \cup (h_2 \cup h_3) = (h_1 \cup h_2) \cup h_3,\ h_1 \cap (h_2 \cap h_3) = (h_1 \cap h_2) \cap h_3$,
(5) (Distributive law): $h_1 \cap (h_2 \cup h_3) = (h_1 \cap h_2) \cup (h_1 \cap h_3)$,

Remark 2.16. For any $h_1,\ h_2 \in HFS(X)$, in general, the following do not hold:

$$h_1 \cap h_2 \subseteq h_1,\ h_1 \cap h_2 \subseteq h_2 \text{ and } h_1 \subseteq h_1 \cup h_2,\ h_2 \subseteq h_1 \cup h_2.$$

Consider Example 2.10,

$$(h_1 \cap h_2)(a) = [0.3, 0.5] \not\subseteq \{0.3, 0.5\} = h_2(a)$$

and

$$(h_1 \cap h_2)(b) = [0.1, 0.5] \not\subseteq \{0.1, 0.5\} = h_1(b).$$

Also

$$h_1(c) = \{0.1, 0.8\} \not\subseteq \{0.4, 0.8\} = (h_1 \cup h_2)(c).$$

Definition 2.17 ([31])**.** For a hesitant fuzzy set h in X, we define the intuitionistic fuzzy set as the envelope of h, denoted by $A_{env}(h)$, is represented by the pair (μ, ν) defined as follows: for each $x \in X$,

$$\mu(x) = h^-(x) \text{ and } \nu(x) = 1 - h^+(x).$$

The following is the relationship between hesitant fuzzy sets and fuzzy multisets introduced by Yager [33].

Remark 2.18 ([31], Lemma 14)**.** All hesitant fuzzy sets can be represented as fuzzy multisets.

3. Further properties of hesitant fuzzy sets

In this section, we find some properties of hesitant fuzzy sets, and redefine the intersection and the union of hesitant fuzzy sets.

Proposition 3.1. *Let $h_1,\ h_2,\ h_3 \in HFS(X)$. Then*
(1) $h_1 \cap (h_2 \cup h_3) = (h_1 \cap h_2) \cup (h_1 \cap h_3)$,
(2) $h_1 \cup (h_2 \cap h_3) = (h_1 \cup h_2) \cap (h_1 \cup h_3)$.

Proof. (1) The proof is well-known in [6]. But we will prove it differently.

Let $x \in X$ and let $\gamma \in [h_1 \cap (h_2 \cup h_3)](x)$. Then there are $\gamma_1 \in h_1(x)$, $\gamma_2 \in (h_2 \cup h_3)(x)$ such that $\gamma = \gamma_1 \wedge \gamma_2$. Since $\gamma_2 \in (h_2 \cup h_3)(x)$, there are $\gamma_{2_1} \in h_2(x)$, $\gamma_{2_2} \in h_3(x)$ such that $\gamma_2 = \gamma_{2_1} \vee \gamma_{2_2}$. Thus

$$\gamma = \gamma_1 \wedge \gamma_2 = \gamma_1 \wedge (\gamma_{2_1} \vee \gamma_{2_2}) = (\gamma_1 \wedge \gamma_{2_1}) \vee (\gamma_1 \wedge \gamma_{2_2}).$$

Since $\gamma_1 \in h_1(x),\ \gamma_{2_1} \in h_2(x),\ \gamma_{2_2} \in h_3(x)$,

$$\gamma_1 \wedge \gamma_{2_1} \in (h_1 \cap h_2)(x) \text{ and } \gamma_1 \wedge \gamma_{2_2} \in (h_1 \cap h_3)(x).$$

So $\gamma \in [(h_1 \cap h_2) \cup (h_1 \cap h_3)](x)$. Hence $[h_1 \cap (h_2 \cup h_3)](x) \subseteq [(h_1 \cap h_2) \cup (h_1 \cap h_3)](x)$.

Similarly, we can show that $[(h_1 \cap h_2) \cup (h_1 \cap h_3)](x) \subseteq [h_1 \cap (h_2 \cup h_3)](x)$. Therefore the result holds.

(2) Let $x \in X$ and let $\gamma \in [h_1 \cup (h_2 \cap h_3)](x)$. Then there are $\gamma_1 \in h_1(x)$, $\gamma_2 \in (h_2 \cap h_3)(x)$ such that $\gamma = \gamma_1 \vee \gamma_2$. Since $\gamma_2 \in (h_2 \cap h_3)(x)$, there are $\gamma_{2_1} \in h_2(x)$, $\gamma_{2_2} \in h_3(x)$ such that $\gamma_2 = \gamma_{2_1} \wedge \gamma_{2_2}$. Thus

$$\gamma = \gamma_1 \vee \gamma_2 = \gamma_1 \vee (\gamma_{2_1} \wedge \gamma_{2_2}) = (\gamma_1 \vee \gamma_{2_1}) \wedge (\gamma_1 \vee \gamma_{2_2}).$$

Since $\gamma_1 \in h_1(x),\ \gamma_{2_1} \in h_2(x),\ \gamma_{2_2} \in h_3(x)$,

$$\gamma_1 \vee \gamma_{2_1} \in (h_1 \cup h_2)(x) \text{ and } \gamma_1 \vee \gamma_{2_2} \in (h_1 \cup h_3)(x).$$

So $\gamma \in [(h_1 \cup h_2) \cap (h_1 \cup h_3)](x)$. Hence $[h_1 \cup (h_2 \cap h_3)](x) \subseteq [(h_1 \cup h_2) \cap (h_1 \cup h_3)](x)$.

Similarly, we can show that $[(h_1 \cup h_2) \cap (h_1 \cup h_3)](x) \subseteq [h_1 \cup (h_2 \cap h_3)](x)$. Therefore the result holds. □

Definition 3.2. Let $(h_j)_{j \in J} \subset HFS(X)$. Then

(i) the union of $(h_j)_{j \in J}$, denoted by $\bigcup_{j \in J} h_j$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$(\bigcup_{j \in J} h_j)(x) = \bigcup_{\gamma_j \in h_j(x)} \bigvee_{j \in J} \gamma_j,$$

(ii) the intersection of $(h_j)_{j \in J}$, denoted by $\bigcap_{j \in J} h_j$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$(\bigcap_{j \in J} h_j)(x) = \bigcup_{\gamma_j \in h_j(x)} \bigwedge_{j \in J} \gamma_j.$$

Example 3.3. Let $X = \{a, b, c\}$ and let $h_1,\ h_2,\ h_3,\ h_4 \in HFS(X)$ given by:

$h_1(a) = \{0.1\} \cup [0.2, 0.6],\ h_1(b) = \{0.3, 0.7\},\ h_1(c) = [0, 1]$,
$h_2(a) = \{0.3, 0.7\},\ h_2(b) = [0.4, 0.6],\ h_2(c) = \{0.2, 0.8\}$,
$h_3(a) = \{0.8\},\ h_3(b) = [0, 1],\ h_3(c) = \{0.6\}$,
$h_4(a) = [0, 1],\ h_4(b) = \{0.2, 0, 8\},\ h_4(c) = \{0.1, 0.6\}$.

Let $\gamma_j \in h_j(a)$, $(j = 1,\ 2,\ 3,\ 4)$. Then

$\gamma_a = \bigvee_{j=1}^{4} \gamma_j = \gamma_1 \vee \gamma_2 \vee \gamma_3 \vee \gamma_4$

and

$\gamma_a \in h_1(a) \cup h_2(a) \cup h_3(a) \cup h_4(a) = [0, 1]$.

Since $\gamma_3 \in h_3(a)$, $\gamma_3 = 0.8$. Thus $\gamma_a \in [0.8, 0.1]$. So $(\bigcup_{j=1}^{4} h_j)(a) = [0.8, 1]$.

Similarly, we can see that $(\bigcup_{j=1}^{4} h_j)(b) = [0.3, 1]$, $(\bigcup_{j=1}^{4} h_j)(c) = [0.1, 1]$. Also we can confirm that $(\bigcap_{j=1}^{4} h_j)(a) = (\bigcap_{j=1}^{4} h_j)(b) = (\bigcap_{j=1}^{4} h_j)(c) = [0, 0.6]$.

Remark 3.4. For any $h_1,\ h_2 \in HFS(X)$, in general, the following does not hold:

$$h_1 \cap h_2 \subseteq h_1 \cup h_2.$$

Let $h_1,\ h_2 \in HFS(X)$ in Example 3.3. Then clearly,

$$(h_1 \cap h_2)(a) = \{0.1\} \cup [0.2, 0.6] \not\subseteq [0.3, 0.6] \cup \{0.7\} = (h_1 \cup h_2)(a).$$

Proposition 3.5 (Generalized Distributive Laws). *Let $h \in HFS(X)$ and let $(h_j)_{j\in J} \subset HFS(X)$. Then*
(1) $h \cap (\bigcup_{j\in J} h_j) = \bigcup_{j\in J}(h \cap h_j)$,
(2) $h \cup (\bigcap_{j\in J} h_j) = \bigcap_{j\in J}(h \cup h_j)$.

Proof. (1) Let $x \in X$ and let $\gamma \in [h \cap (\bigcup_{j\in J} h_j)](x)$. Then there are $\gamma_1 \in h(x)$ and $\gamma_2 \in (\bigcup_{j\in J} h_j)(x)$ such that $\gamma = \gamma_1 \wedge \gamma_2$. Since $\gamma_2 \in (\bigcup_{j\in J} h_j)(x)$, for each $j \in J$, there is $\gamma_j \in h_j(x)$ such that $\gamma_2 = \bigvee_{j\in J} \gamma_j$. Thus

$$\gamma = \gamma_1 \wedge \gamma_2 = \gamma_1 \wedge (\bigvee_{j\in J} \gamma_j) = \bigvee_{j\in J} (\gamma_1 \wedge \gamma_j).$$

Furthermore, $\gamma_1 \wedge \gamma_j \in (h \cap h_j)(x)$, for each $j \in J$. So $\gamma \in [\bigcup_{j\in J}(h \cap h_j)](x)$, i.e.,

$$[h \cap (\bigcup_{j\in J} h_j)](x) \subseteq [\bigcup_{j\in J}(h \cap h_j)](x).$$

Similarly, we can prove that $[\bigcup_{j\in J}(h \cap h_j)](x) \subseteq [h \cap (\bigcup_{j\in J} h_j)](x)$. Hence the result holds.

(2) The proof is similar to (1). □

Proposition 3.6 (Generalized DeMorgan's Laws). *Let $(h_j)_{j\in J} \subset HFS(X)$. Then*
(1) $(\bigcup_{j\in J} h_j)^c = \bigcap_{j\in J} h_j^c$,
(2) $(\bigcap_{j\in J} h_j)^c = \bigcup_{j\in J} h_j^c$.

Proof. (1) Let $x \in X$ and let $1 - \gamma \in (\bigcup_{j\in J} h_j)^c(x)$. Then $\gamma \in (\bigcup_{j\in J} h_j)(x)$. Thus $\gamma = \bigvee_{j\in J} \gamma_j$ and $\gamma_j \in h_j(x)$, for each $j \in J$. So $1 - \gamma_j \in h_j^c(x)$, for each $j \in J$. Moreover, $1 - \gamma = 1 - \bigvee_{j\in J} \gamma_j = \bigwedge_{j\in J}(1 - \gamma_j)$. Hence $1 - \gamma \in (\bigcap_{j\in J} h_j^c)(x)$, i.e.,

$$(\bigcup_{j\in J} h_j)^c(x) \subseteq (\bigcap_{j\in J} h_j^c)(x).$$

Similarly, we can show that $(\bigcap_{j\in J} h_j^c)(x) \subseteq (\bigcup_{j\in J} h_j)^c(x)$. Therefore the result holds.

(2) The proof is similar to (1). □

Definition 3.7. Let X and Y be a nonempty sets, let $h_X \in HFS(X)$ and $h_Y \in HFS(Y))$ and let $f : X \to Y$ be a mapping. Then

(i) the image of h_X under f, denoted by $f(h_X)$, is a hesitant fuzzy set in Y defined as follows: for each $y \in Y$,

$$f(h_X)(y) = \begin{cases} \bigcup_{x\in f^{-1}(y)} h_X(x) & \text{if } f^{-1}(y) \neq \phi \\ \phi & \text{otherwise,} \end{cases}$$

(ii) the preimage of h_Y under f, denoted by $f^{-1}(h_Y)$, is a hesitant fuzzy set in Y defined as follows: for each $x \in X$,

$$f^{-1}(h_Y)(x) = h_Y \circ f(x).$$

Proposition 3.8. *Let $f : X \to Y$ be a mapping, and let $h_X,\ h_{X1},\ h_{X2} \in HFS(X)$, $(h_{X_j})_{j\in J} \subset HFS(X)$, $h_Y,\ h_{Y1},\ h_{Y2} \in HFS(Y)$ and $(h_{Y_j})_{j\in J} \subset HFS(Y)$. Then*

(1) *if $h_{X1} \subseteq h_{X2}$, then $f(h_{X1}) \subseteq f(h_{X2})$,*

(2) *$f(h_{X1} \cup h_{X2}) = f(h_{X1}) \cup f(h_{X2})$, $f(\bigcup_{j\in J} h_{X_j}) = \bigcup_{j\in J} f(h_{X_j})$,*

(3) *$f(h_{X1} \cap h_{X2}) \subseteq f(h_{X1}) \cap f(h_{X2})$, $f(\bigcap_{j\in J} h_{X_j}) \subseteq \bigcap_{j\in J} f(h_{X_j})$,*

(3)$'$ *if f is injective, then $f(h_{X1} \cap h_{X2}) = f(h_{X1}) \cap f(h_{X2})$, $f(\bigcap_{j\in J} h_{X_j}) = \bigcap_{j\in J} f(h_{X_j})$,*

(4) *$f(A) = h_{\mathbf{0}}$ if and only if $A = h_{\mathbf{0}}$,*

(5) *if $h_{Y1} \subseteq h_{Y2}$, then $f^{-1}(h_{Y1}) \subseteq f^{-1}(h_{Y2})$,*

(6) *$f^{-1}(h_{Y1} \cup h_{Y2}) = f^{-1}(h_{Y1}) \cup f^{-1}(h_{Y2})$, $f^{-1}(\bigcup_{j\in J} h_{Y_j}) = \bigcup_{j\in J} f^{-1}(h_{Y_j})$,*

(7) *$f^{-1}(h_{Y1} \cap h_{Y2}) \subseteq f^{-1}(h_{Y1}) \cap f^{-1}(h_{Y2})$, $f^{-1}(\bigcap_{j\in J} h_{Y_j}) \subseteq \bigcap_{j\in J} f^{-1}(h_{Y_j})$,*

(8) *$f^{-1}(h_Y) = h_{\mathbf{0}}$ if and only if $h_Y \cap f(h_{\mathbf{1}}) = h_{\mathbf{1}}$,*

(9) *$h_X \subset f^{-1} \circ f(h_X)$; in particular, $h_X = f^{-1} \circ f(h_X)$, if f is injective,*

(10) *$f \circ f^{-1}(h_Y) \subset h_Y$; in particular, $f \circ f^{-1}(h_Y) = h_Y$, if f is sujective.*

Proof. (1) Suppose that $h_{X1} \subseteq h_{X2}$ and let $y \in Y$ such that $f^{-1}(y) \neq \phi$. Then

$$[f(h_{X1})](y) = \bigcup_{x\in f^{-1}(y)} h_{X1}(x) \subseteq \bigcup_{x\in f^{-1}(y)} h_{X2}(x) = [f(h_{X2})](y).$$

Thus $f(h_{X1}) \subseteq f(h_{X2})$.

(2) Let $y \in Y$ such that $f^{-1}(y) \neq \phi$ and let $\gamma \in [f(h_{X1} \cup h_{X2})](y)$. Since

$$[f(h_{X1} \cup h_{X2})](y) = \bigcup_{x\in f^{-1}(y)} (h_{X1} \cup h_{X2})(x),$$

for each $x \in f^{-1}(y)$, there is $\gamma_x \in (h_{X1} \cup h_{X2})(x)$ such that $\gamma = \bigvee_{x\in f^{-1}(y)} \gamma_x$. Since $\gamma_x \in (h_{X1} \cup h_{X2})(x)$, there are $\gamma_{x_1} \in h_{X1}(x)$ and $\gamma_{x_2} \in h_{X2}(x)$ such that $\gamma_x = \gamma_{x_1} \vee \gamma_{x_2}$. Then

$$\gamma = \bigvee_{x\in f^{-1}(y)} \gamma_x = \bigvee_{x\in f^{-1}(y)} (\gamma_{x_1} \vee \gamma_{x_2}) = (\bigvee_{x\in f^{-1}(y)} \gamma_{x_1}) \vee (\bigvee_{x\in f^{-1}(y)} \gamma_{x_2}).$$

Furthermore,

$$\bigvee_{x\in f^{-1}(y)} \gamma_{x_1} \in \bigcup_{x\in f^{-1}(y)} h_{X1}(x) = f(h_{X1})(y)$$

and

$$\bigvee_{x\in f^{-1}(y)} \gamma_{x_2} \in \bigcup_{x\in f^{-1}(y)} h_{X2}(x) = f(h_{X2})(y).$$

Thus $\gamma \in f(h_{X1})(y) \cup f(h_{X2})(y)$. So $[f(h_{X1} \cup h_{X2})](y) \subseteq f(h_{X1})(y) \cup f(h_{X2})(y)$.

Similarly, we can prove that $f(h_{X1})(y) \cup f(h_{X2})(y) \subseteq [f(h_{X1} \cup h_{X2})](y)$. Hence the result holds.

The proof of the second part also can be proved, similarly.

The proofs of the remainder are omitted. □

For any ordinary subsets $A,\ B$ of a set X, the following hold always:

$$A \cap B \subseteq A,\ A \cap B \subseteq B \text{ and } A \subseteq A \cup B,\ B \subseteq A \cup B.$$

But from Remark 2.16, we can see that for any hesitant fuzzy sets $h_1,\ h_2$ in X, the following do not hold, in general:

$$h_1 \cap h_2 \subseteq h_1,\ h_1 \cap h_2 \subseteq h_2 \text{ and } h_1 \subseteq h_1 \cap h_2,\ h_2 \subseteq h_1 \cap h_2. \tag{4.1}$$

Then we will define newly the intersection and the union of hesitant fuzzy sets so that (4.1) holds.

Definition 3.9. Let $h_1,\ h_2 \in HFS(X)$ and let $(h_j)_{j \in J} \subset HFS(X)$. Then

(i) the intersection of h_1 and h_2, denoted by $h_1 \widetilde{\cap} h_2$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$(h_1 \widetilde{\cap} h_2)(x) = h_1(x) \cap h_2(x),$$

(ii) the intersection of $(h_j)_{j \in J}$, denoted by $\widetilde{\bigcap}_{j \in J} h_j$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$(\widetilde{\bigcap}_{j \in J} h_j)(x) = \bigcap_{j \in J} h_j(x),$$

(iii) the union of h_1 and h_2, denoted by $h_1 \widetilde{\cup} h_2$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$(h_1 \widetilde{\cup} h_2)(x) = h_1(x) \cup h_2(x),$$

(iv) the union of $(h_j)_{j \in J}$, denoted by $\widetilde{\bigcup}_{j \in J} h_j$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$(\widetilde{\bigcup}_{j \in J} h_j)(x) = \bigcup_{j \in J} h_j(x).$$

Example 3.10. (1) Let $h_1,\ h_2$ be hesitant fuzzy sets in X given in Example 2.10. Then

$$(h_1 \widetilde{\cap} h_2)(a) = h_1(a) \cap h_2(a) = \{0.3, 0.7\}$$

and

$$(h_1 \widetilde{\cup} h_2)(a) = h_1(a) \cup h_2(a) = [0.3, 0.7].$$

Similarly, we can calculate the following:

$(h_1 \widetilde{\cap} h_2)(b) = \{0.1, 0.5\},\ (h_1 \widetilde{\cap} h_2)(c) = \{0.8\}$

and

$(h_1 \widetilde{\cup} h_2)(b) = [0.1, 0.6],\ (h_1 \widetilde{\cup} h_2)(c) = \{0.1, 0.4, 0.8\}$.

(2) Consider $h_1,\ h_2\ h_3\ h_4$ be hesitant fuzzy sets in X given in Example 3.3. Then

$$(\widetilde{\bigcap}_{j=1}^{4} h_j)(a) = \bigcap_{j=1}^{4} h_j(a) = \phi$$

and

$$(\widetilde{\bigcup}_{j=1}^{4} h_j)(a) = \bigcup_{j=1}^{4} h_j(a) = [0, 1].$$

Similarly, we can calculate the following:

$(\widetilde{\bigcap}_{j=1}^{4} h_j)(b) = (\widetilde{\bigcap}_{j=1}^{4} h_j)(b) = \phi$

and

$(\widetilde{\bigcup}_{j=1}^{4} h_j)(b) = (\widetilde{\bigcup}_{j=1}^{4} h_j)(c) = [0, 1].$

Remark 3.11 (Compare to Result 2.15 (1) and (2))**.** For each $h \in HFS(X)$, the following do not hold, in general:

$$h \widetilde{\cup} h_{\mathbf{1}} = h_{\mathbf{1}},\ h \widetilde{\cap} h_{\mathbf{1}} = h$$

and

$$h \widetilde{\cup} h_{\mathbf{0}} = h_{\mathbf{0}},\ h \widetilde{\cap} h_{\mathbf{0}} = h_{\mathbf{0}}.$$

Then we will redefine the hesitant fuzzy empty set and the hesitant fuzzy whole set as follows.

Definition 3.12. Let X be a nonempty set. Then the hesitant fuzzy empty [resp. whole] set, denoted by $h^{\mathbf{0}}$ [resp. $h^{\mathbf{1}}$], is a hesitant fuzzy set in X defined as: for each $x \in X$,

$$h^{\mathbf{0}}(x) = \phi \text{ [resp. } h^{\mathbf{1}}(x) = [0, 1]].$$

In this case, we will denote the set of all hesitant fuzzy sets in X as $HS(X)$.

Definition 3.13. Let X be a nonempty set and let $h \in HS(X)$. Then the complement of h, denoted by h^c, is a hesitant fuzzy set in X defined as: for each $x \in X$,

$$h^c(x) = h(x)^c = [0, 1] \setminus h(x).$$

The following are the immediate results of Definitions 2.13, 3.9, 3.12 and 3.13.

Proposition 3.14. *Let X be a nonempty set, let $h,\ h_1,\ h_2,\ h_3 \in HS(X)$ and let $(h_j)_{j\in J} \subset HS(X)$. Then*

(1) (Idempotent laws): $h \widetilde{\cup} h = h$, $h \widetilde{\cap} h = h$,
(2) (Commutative laws): $h_1 \widetilde{\cup} h_2 = h_2 \widetilde{\cup} h_1$, $h_1 \widetilde{\cap} h_2 = h_2 \widetilde{\cap} h_1$,
(3) (Associative laws): $h_1 \widetilde{\cup} (h_2 \widetilde{\cup} h_3) = (h_1 \widetilde{\cup} h_2) \widetilde{\cup} h_3$, $h_1 \widetilde{\cap} (h_2 \widetilde{\cap} h_3) = (h_1 \widetilde{\cap} h_2) \widetilde{\cap} h_3$,
(4) (Distributive laws): $h_1 \widetilde{\cup} (h_2 \widetilde{\cap} h_3) = (h_1 \widetilde{\cup} h_2) \widetilde{\cap} (h_1 \widetilde{\cup} h_3)$,
$h_1 \widetilde{\cap} (h_2 \widetilde{\cup} h_3) = (h_1 \widetilde{\cap} h_2) \widetilde{\cup} (h_1 \widetilde{\cap} h_3)$,
(4)$'$ (Generalized Distributive laws): $h \widetilde{\cup} (\widetilde{\bigcap}_{j\in J} h_j) = \widetilde{\bigcap}_{j\in J} (h \widetilde{\cup} h_j)$,
$h \widetilde{\cap} (\widetilde{\bigcup}_{j\in J} h_j) = \widetilde{\bigcup}_{j\in J} (h \widetilde{\cap} h_j)$,
(5) (Absorption laws): $h_1 \widetilde{\cup} (h_1 \widetilde{\cap} h_2) = h_1$, $h_1 \widetilde{\cap} (h_1 \widetilde{\cup} h_2) = h_1$.
(6) (DeMorgan's laws): $(h_1 \widetilde{\cup} h_2)^c = h_1^c \widetilde{\cap} h_2^c$, $(h_1 \widetilde{\cap} h_2)^c = h_1^c \widetilde{\cup} h_2^c$,
(6)$'$ (Generalized DeMorgan's laws): $(\widetilde{\bigcup}_{j\in J} h_j)^c = \widetilde{\bigcap}_{j\in J} h_j^c$, $(\widetilde{\bigcap}_{j\in J} h_j)^c = \widetilde{\bigcup}_{j\in J} h_j^c$,
(7) $(h^c)^c = h$,
(8) $h_1 \widetilde{\cap} h_2 \subseteq h_1$ *and* $h_2 \widetilde{\cap} h_1 \subseteq h_2$,
(9) $h_1 \subseteq h_2 \widetilde{\cup} h_1$ *and* $h_1 \subseteq h_2 \widetilde{\cup} h_2$,
(10) *if* $h_1 \subseteq h_2$ *and* $h_2 \subseteq h_3$, *then* $h_1 \subseteq h_3$,
(11) *if* $h_1 \subseteq h_2$, *then* $h_1 \widetilde{\cap} h \subseteq h_2 \widetilde{\cap} h$ *and* $h_1 \widetilde{\cup} h \subseteq h_2 \widetilde{\cup} h$,
(12) $h^{\mathbf{0}} \subseteq h \subseteq h^{\mathbf{1}}$,
(13) $h \widetilde{\cap} h^{\mathbf{0}} = h^{\mathbf{0}}$, $h \widetilde{\cup} h^{\mathbf{0}} = h$, $h \widetilde{\cap} h^{\mathbf{1}} = h$, $h \widetilde{\cup} h^{\mathbf{1}} = h^{\mathbf{1}}$.

From the above proposition, we can easily see that $(HS(X), \widetilde{\cap}, \widetilde{\cup}, ^c)$ is a Boolean algebra with the least element $h^{\mathbf{0}}$ and the largest element $h^{\mathbf{1}}$.

Now we redefine the image of a hesitant fuzzy set under a mapping.

Definition 3.15. Let X and Y be a nonempty sets, let $h_X \in HS(X)$ and $h_Y \in HS(Y)$ and let $f : X \to Y$ be a mapping. Then the image of h_X under f, denoted by $f(h_X)$, is a hesitant fuzzy set in Y defined as follows: for each $y \in Y$,

$$f(h_X)(y) = \begin{cases} \widetilde{\bigcup}_{x\in f^{-1}(y)} h_X(x) & \text{if } f^{-1}(y) \neq \phi \\ \phi & \text{otherwise.} \end{cases}$$

From the above Definition, we have the same result of Proposition 3.8.

Proposition 3.16. *Let $f : X \to Y$ be a mapping, and let $h_X,\ h_{X1},\ h_{X2} \in HS(X)$, $(h_{X_j})_{j\in J} \subset HS(X)$, $h_Y,\ h_{Y1},\ h_{Y2} \in HS(Y)$ and $(h_{Y_j})_{j\in J} \subset HS(Y)$. Then*

(1) *if $h_{X1} \subseteq h_{X2}$, then $f(h_{X1}) \subseteq f(h_{X2})$,*
(2) $f(h_{X1}\widetilde{\cup}h_{X2}) = f(h_{X1})\widetilde{\cup}f(h_{X2})$, $f(\widetilde{\bigcup}_{j\in J}h_{X_j}) = \widetilde{\bigcup}_{j\in J}f(h_{X_j})$,
(3) $f(h_{X1}\widetilde{\cap}h_{X2}) \subseteq f(h_{X1})\widetilde{\cap}f(h_{X2})$, $f(\widetilde{\bigcap}_{j\in J}h_{X_j}) \subseteq \widetilde{\bigcap}_{j\in J}f(h_{X_j})$,
(3)′ *if f is injective, then* $f(h_{X1}\widetilde{\cap}h_{X2}) = f(h_{X1})\widetilde{\cap}f(h_{X2})$, $f(\widetilde{\bigcap}_{j\in J}h_{X_j}) = \widetilde{\bigcap}_{j\in J}f(h_{X_j})$,
(4) $f(A) = h^{\mathbf{0}}$ *if and only if* $A = h^{\mathbf{0}}$,
(5) *if $h_{Y1} \subseteq h_{Y2}$, then $f^{-1}(h_{Y1}) \subseteq f^{-1}(h_{Y2})$,*
(6) $f^{-1}(h_{Y1}\widetilde{\cup}h_{Y2}) = f^{-1}(h_{Y1})\widetilde{\cup}f^{-1}(h_{Y2})$, $f^{-1}(\widetilde{\bigcup}_{j\in J}h_{Y_j}) = \widetilde{\bigcup}_{j\in J}f^{-1}(h_{Y_j})$,
(7) $f^{-1}(h_{Y1}\widetilde{\cap}h_{Y2}) \subseteq f^{-1}(h_{Y1})\widetilde{\cap}f^{-1}(h_{Y2})$, $f^{-1}(\widetilde{\bigcap}_{j\in J}h_{Y_j}) \subseteq \widetilde{\bigcap}_{j\in J}f^{-1}(h_{Y_j})$,
(8) $f^{-1}(h_Y) = h^{\mathbf{1}}$ *if and only if* $h_Y\widetilde{\cap}f(h^{\mathbf{1}}) = h^{\mathbf{1}}$,
(9) $h_X \subset f^{-1} \circ f(h_X)$*; in particular,* $h_X = f^{-1} \circ f(h_X)$*, if f is injective,*
(10) $f \circ f^{-1}(h_Y) \subset h_Y$*; in particular,* $f \circ f^{-1}(h_Y) = h_Y$*, if f is sujective.*

4. Properties of the category **HSet**(H)

Definition 4.1 ([3, 16])**.** A lattice H is called a complete Heyting algebra, if it satisfies the following conditions:

(i) it is a complete lattice,
(ii) for any $a, b \in H$, the set $\{x \in H : x \wedge a \leq b\}$ has a greatest element denoted by $a \to b$ (called the relative pseudo-complement of a in b), i.e., $x \wedge a \leq b$ if and only if $x \leq (a \to b)$.

In particular, if H is a complete Heyting algebra with the least element 0, then for each $a \in H$, $N(a) = a \to 0$ is called negation or the pseudo-complement of a. Moreover, if H is a complete Heyting algebra with the least element 0 and largest element 1, then for each $a \in H$, $a \wedge N(a) = 0$ but $a \vee N(a) = 1$ does not hold, in general.

Result 4.2 ([3], Ex. 6 in p. 46)**.** *Let H be a complete Heyting algebra and $a, b \in H$.*

(1) *If $a \leq b$, then $N(b) \leq N(a)$.*

In fact, $N : H \to H$ is an order reversing operation in $(H, \leq)$.

(2) $a \leq NN(a)$.
(3) $N(a) = NNN(a)$.
(4) $N(a \vee b) = N(a) \wedge N(b)$ *and* $N(a \wedge b) = N(a) \vee N(b)$.

Definition 4.3. Let X be a nonempty set. Then a mapping $h : X \to P(H)$ is called an hesitant H-fuzzy set, where $P(H)$ denote the power set of H. The pair (X, h) is called a hesitant H-fuzzy space.

Definition 4.4. Let (X, h_X) and (Y, h_Y) be two hesitant H-fuzzy spaces. Then a mapping $f : (X, h_X) \to (Y, h_Y)$ is a preserving mapping, if for each $x \in X$,

$$h_X(x) \subseteq h_Y \circ f(x), \text{ i.e., } h_X \subseteq h_Y \circ f.$$

Proposition 4.5. *Let (X, h_X), (Y, h_Y) and (Z, h_Z) be three hesitant H-fuzzy spaces.*
(1) *The identity mapping $1_X : (X, h_X) \to (X, h_X)$ is a preserving mapping.*
(2) *If $f : (X, h_X) \to (Y, h_Y)$ and $g : (Y, h_Y) \to (Z, h_Z)$ are preserving mappings, then $g \circ f : (X, h_X) \to (Z, h_Z)$ is a preserving mapping.*

Proof. (1) The proof is clear.
(2) Suppose $f : (X, h_X) \to (Y, h_Y)$ and $g : (Y, h_Y) \to (Z, h_Z)$ are preserving mappings and let $x \in X$. Then

$$\begin{aligned} h_Z \circ (g \circ f)(x) &= h_Z \circ (g \circ f(x)) \\ &\supseteq h_Y \circ (f(x)) \text{ [Since } g \text{ is a preserving mapping]} \\ &= h_Y \circ f(x) \\ &\supseteq h_X(x). \text{ [Since } f \text{ is a preserving mapping]} \end{aligned}$$

Thus $g \circ f$ is a preserving mapping. □

We will denote the collection consisting of all hesitant H-fuzzy spaces and all preserving mappings between any two hesitant H-fuzzy spaces as **HSet**(H). Then from Proposition 4.5, we can easily see that **HSet**(H) forms a concrete category. In the sequel, a preserving mapping between any two hesitant H-fuzzy spaces will be called a **HSet**(H)-mapping.

Lemma 4.6. *The category* **HSet**(H) *is topological over* **Set**.

Proof. Let X be a set and let $(X_j, h_j)_{j \in J}$ be any family of hesitant H-fuzzy spaces indexed by a class J. Suppose $(f_j : X \to X_j)_J$ be a source of mappings. We define a mapping $h_X : X \to P(H)$ as follows: for each $x \in X$,

$$h_X(x) = [\widetilde{\bigcap}_{j \in J} f_j^{-1}(h_j)](x) = \bigcap_{j \in J} h_j \circ f_j(x).$$

Then clearly, $f_j : (X, h_X) \to (X_j, h_j)$ is a **HSet**(H)-mapping, for each $j \in J$.
For any object (Y, h_Y), let $g : Y \to X$ be any mapping for which $f_j \circ f : (Y, h_Y) \to (X_j, h_j)$ is a **HSet**(H)-mapping, for each $j \in J$ and let $y \in Y$. Then for each $j \in J$,

$$h_Y(y) \subseteq h_j \circ (f_j \circ f)(y) = (h_j \circ f_j) \circ g(y).$$

Thus by the definition of h_X,

$$h_Y(y) \subseteq (\bigcap_{j \in J} h_j \circ f_j) \circ g(y) = h_X \circ g(y).$$

So $g : (Y, h_Y) \to (X, h_X)$ is a **HSet**(H)-mapping. Hence $(f_j : (X, h_X) \to (X_j, h_j))_J$ is an initial source in **HSet**(H). □

Example 4.7. (1) (**Inverse image of a hesitant H-fuzzy set structure**) Let X be a set, let (Y, h_Y) be a hesitant H-fuzzy space and let $f : X \to Y$ be a mapping. Then there exists a unique initial hesitant H-fuzzy set structure h_X in X for which $f : (X, h_X) \to (Y, h_Y)$ is a **HSet**(H)-mapping. In fact,

$$h_X = f^{-1}(h_Y) = h_Y \circ f.$$

In this case, h_X is called the inverse image under f of the hesitant H-fuzzy set structure h_Y in Y.

In particular, if $X \subset Y$ and $f : X \to Y$ is the inclusion mapping, then the inverse image h_X of h_Y under f is called a hesitant H-fuzzy subset of (Y, h_Y). In fact,

$$h_X(x) = h_Y(x), \text{ for each } x \in X.$$

(2) (**Hesitant H-fuzzy product structure**) Let $((X_j, h_j))_{j\in J}$ be any family of hesitant H-fuzzy spaces and let $X = \Pi_{j\in J} X_j$. For each $j \in J$, let $pr_j : X \to X_j$ be the ordinary projection. Then there exists a unique hesitant H-fuzzy set h_X in X for which $pr_j : (X, h_X) \to (X_j, h_j)$ is a **HSet**(H)-mapping, for each $j \in J$. In this case, h_X is called the hesitant H-fuzzy product of $(h_j)_{j\in J}$ and (X, h_X) is called the hesitant H-fuzzy product space of $((X_j, h_j))_{j\in J}$, and denoted as the following, respectively:

$$h_X = \Pi_{j\in J} h_j \text{ and } (X, h_X) = (\Pi_{j\in J} X_j, \Pi_{j\in J} h_j).$$

In fact, $h_X(x) = \widetilde{\bigcap}_{j\in J} h_j \circ pr_j(x)$, for each $x \in X$.

In particular, if $J = \{1, 2\}$, then for each $(x, y) \in X_1 \times X_2$,

$$(h_1 \times h_2)(x, y) = h_1(x) \cap h_2(y).$$

The following is obvious from Lemma 5.4 and Theorem 1.6 in [17] or Proposition in Section 1 in [11].

Corollary 4.8. *The category* **HSet**(H) *is complete and cocomplete over* **Set***.*

It is well-known that a concrete category is topological if and only if it is cotopological (See Theorem 1.5 in [17]). But we prove directly that **HSet**(H) is cotopological.

Lemma 4.9. *The category* **HSet**(H) *is cotopological over* **Set***.*

Proof. Let X be any set and let $((X_j, h_j))_{j\in J}$ be any family of hesitant H-fuzzy spaces indexed by a class J. Suppose $(f_j : X_j \to X)_{j\in J}$ is a sink of mappings. We define a mapping $h_X : X \to P(H)$ as follows: for each $x \in X$,

$$h_X(x) = \bigcup_{j\in J} [\widetilde{\bigcup}_{x_j \in f^{-1}(x)} h_j](x_j) = \bigcup_{j\in J} \bigcup_{x_j \in f^{-1}(x)} h_j(x_j).$$

Then clearly, $f_j : (X_j, A_j) \to (X, A)$ is a **HSet**(H)-mapping, for each $j \in J$.

For any hesitant H-fuzzy space (Y, h_Y), let $g : X \to Y$ be any mapping such that $g \circ f_j : (X_j, h_j) \to (Y, h_Y)$ is a **HSet**(H)-mapping, for each $j \in J$ and let $x \in X$. Then for each each $j \in J$ and each $x_j \in f^{-1}(x)$,

$$h_j(x_j) \subseteq (h_Y \circ (g \circ f_j))(x_j) = h_Y \circ (g(f_j)(x_j)) = h_Y \circ g(x).$$

Thus by the definition of h_X, $h_X(x) \subseteq h_Y \circ g(x)$. So $g : (X, h_X) \to (Y, h_Y)$ is a **HSet**(H)-mapping. Hence **HSet**(H) is cotopological over **Set**. □

Example 4.10. (**Hesitant H-fuzzy quotient structure**) Let (X, h_X) be a hesitant H-fuzzy space, let $\sim$ be an equivalence relation on X and let $\pi : X \to X/\sim$ be the canonical mapping. We define a mapping $h^{-}_{X/\sim} : X/\sim \to P(H)$ as follows: for each $[x] \in X/\sim$,

$$h_{X/\sim}([x]) = [\widetilde{\bigcup}_{x^{'}\in\pi^{-1}([x])} h_X](x^{'}) = \bigcup_{x^{'}\in\pi^{-1}([x])} h_X(x^{'}).$$

Then $h_{X/\sim} \in HS(X/\sim)$. Furthermore, $\pi : (X, h_X) \to (X/\sim, h_{X/\sim})$ is a **HSet**-mapping. Thus $h_{X/\sim}$ is the final hesitant H-fuzzy set in $X/\sim$.

In this case, $h_{X/\sim}$ is called the hesitant H-fuzzy quotient set in X by $\sim$.

Definition 4.11 ([11])**.** Let **A** be a concrete category and let $f, g : A \to B$ be two **A**-morphisms. Then a pair (E, e) is called an equalizer in **A** of f and g, if the following conditions hold:

(i) $e : E \to A$ is an **A**-morphism,

(ii) $f \circ e = g \circ e$,

(iii) for any **A**-morphism $e^{'} : E^{'} \to A$ such that $f \circ e^{'} = g \circ e^{'}$, there exists a unique **A**-morphism $\bar{e} : E^{'} \to E$ such that $e^{'} = e \circ \bar{e}$.

In this case, we say that **A** has equalizers.

Dual notion: Coequalizer.

Proposition 4.12. **HSet**(H) *has equalizers.*

Proof. Let $f, g : (X, h_X) \to (Y, h_Y)$ be two **HSet**(H)-mappings. Let $E = \{a \in X : f(a) = g(a)\}$ and define a mapping $h_E : E \to P(H)$ as follows: for each $a \in E$,

$$h_E(a) = h_X(a).$$

Then clearly, $h_E \subseteq h_X$ and h_E is a hesitant H-fuzzy set in E. Consider the inclusion mapping $i : E \to X$. Then clearly, $i : (E, h_E) \to (X, h_X)$ is a **HSet**(H)-mapping and $f \circ i = g \circ i$.

Let $k : (E^{'}, h_{E^{'}}) \to (X, h_E)$ be a **HSet**(H)-mapping such that $f \circ k = g \circ k$. We define a mapping $\bar{k} : E^{'} \to E$ as follows: for each $e^{'} \in E^{'}$,

$$\bar{k}(e^{'}) = i^{-1} \circ k(e^{'}).$$

Then clearly, $k = i \circ \bar{k}$.

Let $e^{'} \in E^{'}$. Since $k : (E^{'}, h_{E^{'}}) \to (X, h_E)$ is a **HSet**(H)-mapping,

$$\begin{aligned} h_E \circ \bar{k}(e^{'}) &= h_E \circ \bar{k}(e^{'}) \\ &= h_E \circ (i^{-1} \circ k(e^{'})) \\ &= h_E \circ k(e^{'}) \\ &\subseteq h_{E^{'}}(e^{'}). \end{aligned}$$

Thus $\bar{k} : (E^{'}, h_{E^{'}}) \to (E, h_E)$ is a **HSet**(H)-mapping. The uniqueness of $\bar{k}$ can be easily proved. So **HSet**(H) has equalizers. □

Lemma 4.13. *Final episinks in* **HSet**(H) *are preserved by pullbacks.*

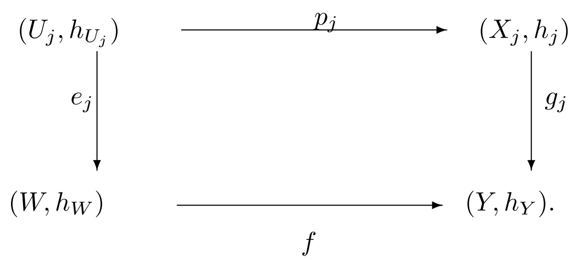
Proof. Let $(g_j : (X_j, h_j) \to (Y, h_Y))_{j\in J}$ be any final episink in **HSet**(H) and let $f : (W, h_W) \to (Y, h_Y)$ be any **HSet**(H)-mapping. For each $j \in J$, let

$$U_j = \{(w, x_j) \in W \times X_j : f(w) = g_j(x_j)\},$$

and let us consider a mapping $h_{U_j} : U_j \to P(H)$ as follows: for each $(w, x_j) \in U_j$,

$$h_{U_j}(w, x_j) = (h_W \times h_j)(w, x_j) = h_W(w) \cap h_j(x_j), \text{ i.e., } h_{U_j} = (h_W \times h_j)\mid_{U_j \times U_j}.$$

For each $j \in J$, let $e_j : U_j \to W$ and $p_j : U_j \to X_j$ be the usual projections. Then clearly, $e_j : (U_j, h_{U_j}) \to (W, h_W)$ and $p_j : (U_j, h_{U_j}) \to (X_j, h_j)$ are $\mathbf{HSet}(H)$-mappings and $g_j \circ p_j = f \circ e_j$, for each $j \in J$. Thus we have the following pullback square in $\mathbf{HSet}(H)$:



We will prove that $(e_j : (U_j, h_{U_j}) \to (W, h_W))_{j\in J}$ is a final episink in $\mathbf{HSet}(H)$. Let $w \in W$. Since $(g_j)_{j\in J}$ is an episink in $\mathbf{HSet}(H)$, there is $j \in J$ such that $g_j(x_j) = f(w)$, for some $x_j \in X_j$. Thus $(w, x_j) \in U_j$ and $e_j(w, x_j) = w$. So $(e_j)_{j\in J}$ is an episink in $\mathbf{HSet}(H)$.

Finally, let us show that $(e_j)_J$ is final in $\mathbf{HSet}(H)$. Let h^*_W be the final structure in W w.r.t. $(e_j)_{j\in J}$ and let $w \in W$. Then

$$\begin{aligned} h_W(w) &= h_W(w) \cap h_W(w) \\ &\subseteq h_W(w) \cap h_Y \circ f(w) \\ &\quad \text{[Since } f : (W, h_W) \to (Y, h_Y) \text{ is a } \mathbf{HSet}(H)\text{-mapping]} \\ &= h_W(w) \cap [\textstyle\bigcup_{j\in J} \bigcup_{x_j \in g_j^{-1}(f(w))} h_j(x_j)] \\ &\quad \text{[Since } (g_j : (X_j, h_j) \to (Y, h_Y))_{j\in J} \text{ is a final episink in } \mathbf{HSet}(H)] \\ &= \textstyle\bigcup_{j\in J} \bigcup_{x_j \in g_j^{-1}(f(w))} [h_W(w) \cap h_j(x_j)] \\ &= \textstyle\bigcup_{j\in J} \bigcup_{(w,x_j) \in e_j^{-1}(w)} [h_W(w) \cap h_j(x_j)] \\ &= \textstyle\bigcup_{j\in J} \bigcup_{(w,x_j) \in e_j^{-1}(w)} [h_{U_j}(w, x_j)] \\ &= h^*_W(w). \end{aligned}$$

Thus $h_W(w) \subseteq (h^*_W)(w)$. So $h_W \subseteq h^*_W$. Since $(e_j : (U_j, h_{U_j}) \to (W, h_W))_{j\in J}$ is final, $1_W : (W, h^*_W) \to (W, h_W)$ is a $\mathbf{HSet}(H)$-mapping and thus $h^*_W \subseteq h_W$. Hence $h^*_W = h_W$. Therefore $(e_j)_{j\in J}$ is final. This completes the proof. □

For any singleton set $\{a\}$, since the hesitant fuzzy set $h_{\{a\}}$ in $\{a\}$ is not unique, the category $\mathbf{HSet}(H)$ is not properly fibred over **Set**. Then From Definitions 2.2 and 2.4, Lemmas 4.7 and 4.11, we have the following result.

Theorem 4.14. *The category* $\mathbf{HSet}(H)$ *satisfies all the conditions of a topological universe over* **Set** *except the terminal separator property.*

Theorem 4.15. *The category* $\mathbf{HSet}(H)$ *is Cartesian closed over* **Set**.

Proof. From Lemma 4.4, it is clear that $\mathbf{HSet}(H)$ has products. Then it is sufficient to prove that $\mathbf{HSet}(H)$ has exponential objects.

For any hesitant H-fuzzy spaces $\mathbf{X} = (X, h_X)$ and $\mathbf{Y} = (X, h_Y)$, let Y^X be the set of all ordinary mappings from X to Y. For each $f \in Y^X$, let

$$D(f) = \{x \in X : h_X(x) \supset h_Y \circ f(x)\}.$$

We define a mapping $h_{Y^X} : Y^X \to P(H)$ as follows: for each $f \in Y^X$,

$$h_{Y^X}(f) = \begin{cases} \bigcap_{x \in D(f)} h_Y \circ f(x) & \text{if } D(f) \neq \phi \\ \text{H} & \text{if } D(f) = \phi. \end{cases}$$

Then clearly, h_{Y^X} is a hesitant H-fuzzy set in Y^X.

Let $\mathbf{Y^X} = (Y^X, h_{Y^X})$ and let us define a mapping $e_{X,Y} : X \times Y^X \to Y$ as follows: for each $(x, f) \in X \times Y^X$,

$$e_{X,Y}(x, f) = f(x).$$

Let $(x, f) \in X \times Y^X$.

Case (i): Suppose $D(f) = \phi$. Then

$$\begin{aligned} h_X \times h_{Y^X}(x, f) &= h_X(x) \cap h_{Y^X}(f) \\ &= h_X(x) \text{ [Since } D(f) = \phi,\ h_{Y^X}(f) = H.] \\ &\subseteq h_Y \circ f(x) \text{ [Since } D(f) = \phi] \\ &= h_Y \circ e_{X,Y}(x, f). \end{aligned}$$

Case (ii) Suppose $D(f) \neq \phi$. Then

$$\begin{aligned} h_X \times h_{Y^X}(x, f) &= h_X(x) \cap h_{Y^X}(f) \\ &= h_X(x) \cap [\textstyle\bigcap_{a \in (f)} h_Y \circ f(a)] \text{ [Since } D(f) \neq \phi] \\ &\subseteq h_Y \circ f(x) \\ &= h_Y \circ e_{X,Y}(x, f). \end{aligned}$$

Thus in all cases, $e_{X,Y} : \mathbf{X} \times \mathbf{Y^X} \to \mathbf{Y}$ is a $\mathbf{HSet}(H)$-mapping, where $\mathbf{X} \times \mathbf{Y^X} = (X \times Y^X, h_X \times h_{Y^X})$.

For any hesitant H-fuzzy space $\mathbf{Z} = (Z, h_Z)$, let $k : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ be a $\mathbf{HSet}(H)$-mapping. We define a mapping $\bar{k} : Z \to Y^X$ as follows: for each $z \in Z$ and each $x \in X$,

$$[\bar{k}(z)](x) = k(x, z).$$

Then by the similar arguments of proof of Theorem $\bar{k}$ is a $\mathbf{HSet}(H)$-mapping. Moreover, we can see that $\bar{k}$ is a unique $\mathbf{Het}(H)$-mapping such that $e_{X,Y} \circ (1_X \times \bar{k}) = k$. This completes the proof. □

Remark 4.16. The category $\mathbf{HSet}(H)$ is not a topos, since it has no subobject classifier.

Example 4.17. Let $I = \{0, 1\}$ be two points chain, respectively and let $X = \{a\}$. Let h_1 and h_2 be the hesitant H-fuzzy sets in X defined by:

$$h_1(a) = \{0\} \text{ and } h_2(a) = \{1\}.$$

Let $1_X : (X, h_1) \to (X, h_2)$ be the identity mapping. Then clearly, 1_X is both monomorphism and epimorphism in $\mathbf{HSet}(H)$. But 1_X is not an isomorphism in $\mathbf{HSet}(H)$. Thus $\mathbf{HSet}(H)$ has no subobject classifier.

5. Conclusions

We constructed the category **HSet**(H) consisting of hesitant fuzzy spaces and preserving mappings between them and studied it in a view of a topological universe. In particular, we obtained an exponential objects in **HSet**(H) (See Theorems 4.15) and we confirmed that **HSet**(H) is not a topos (See Remark 4.16 and Example 4.17).

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