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Abstract. Rosenfeld introduced the notion of fuzzy subgroups of a group, and Kuroki, Davvaz, etc considered the theory of rough fuzzy groups and rough fuzzy subgroups. In the paper, we introduce the notion of L-subgroups of a group, define two rough operators on L-group by an L-normal subgroup, and investigate some of their properties.

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1. Introduction

In 1965, Zadeh proposed the pioneering work of fuzzy subsets of a set [20], and in 1971, Rosenfeld introduced the notion of fuzzy subgroups of a group [14] which led to the fuzzification of algebraic structures. In 1982, Pawlak initiated the rough set theory to study incomplete and insufficient information[13].

Dubois, Prade first investigated fuzzy rough set and rough fuzzy set in [7], which attracting many scholars attentions. From the view of the theory of groups, Budimirović, Davvaz, Kuroki, Mordeson, Tărnăuceanu,etc studied the notion of fuzzy subgroups of a group in [3, 6, 11, 12, 15]. In [2, 10, 19], Biswas, Kuroki, Yaqoob, etc considered the notions of rough groups and rough subgroups. Moreover, Davvaz, Kuroki, Yaqoob, etc investigated the theory of rough fuzzy groups and rough fuzzy subgroups in [6, 11, 18].

In [16], Wang and Chen investigated the theory of rough subgroups by means of a normal subgroup, and obtained some interesting results. As a generalization of [12, 16], in the paper, we define the notions of L-group, rough L-group, and investigate some of their properties.

The above contents are arranged into three parts, Section 3: **L**-group, and Section 4: Rough L-group. In Section 2, we give an overview of **L**-sets, group, rough sets, which surveys Preliminaries.

2. Preliminaries

The section is devoted to some main notions for each area, i.e., **L**-sets [1, 8], groups, rough sets [4, 5, 9, 13, 17].

2.1. L-sets.

The seminal paper on fuzzy sets is [20]. As a generalization, the notion of an **L**-set was introduced in [8]. An overview of the theory of **L**-sets and **L**-relations (i.e., fuzzy sets and relations in the framework of complete residuated lattices) can be found in [1]. In this paper, we assume that **L** is a complete Heyting algebra.

Definition 2.1. A complete Heyting algebra is a complete lattice **L**= $\langle L, \vee, \wedge, 0, 1\rangle$ which satisfies the following infinite distributive law:

$$a \wedge \bigvee B = \bigvee \{a \wedge b \mid b \in B\},$$

for all elements a and all subsets B.

Next, an **L**-set is defined in the following manner.

For a universe set X, an **L**-set in X is a mapping $A : X \to L$. $A(x)$ indicates the truth degree of "x belongs to A". We use the symbol L^X to denote the set of all **L**-sets in X. For instance: $1_X : X \to L$, $0_X : X \to L$ are defined as: for all $x \in X$, $1_X(x) = 1, 0_X(x) = 0$, respectively.

For $A, B \in L^X$, if $\forall x \in X$, we have $A(x) \leq B(x)$, then $A \subseteq B$. and $A = B$, if $A \subseteq B$, $B \subseteq A$.

Corresponding the operations $\vee, \wedge$ on **L**, two operations are defined on **L**-sets as follows.

Definition 2.2. Suppose $A, B \in L^X$, then $A \cap B$ and $A \cup B$ are defined as follows:

$$(A \cap B)(x) = A(x) \wedge B(x),\ (A \cup B)(x) = A(x) \vee B(x),$$

for every $x \in X$.

2.2. Rough Sets.

Pawlak proposed the rough set theory in [13]. Let (X, R) be an approximation space, and $R \subseteq X \times X$ be an equivalence relation, then for $A \subseteq X$, two subsets $\underline{R}(A)$ and $\overline{R}(A)$ of X are defined:

$$\underline{R}(A) = \{x \in X \mid [x]_R \subseteq A\},\ \overline{R}(A) = \{x \in X \mid [x]_R \cap A \neq \varnothing\},$$

where $[x]_R = \{y \in X \mid xRy\}$.

If $\underline{R}(A) = \overline{R}(A)$, A is called a definable set; if $\underline{R}(A) \neq \overline{R}(A)$, A is called an undefinable set, and $(\underline{R}(A), \overline{R}(A))$ is referred to as a pair of rough set. Therefore, $\underline{R}$ and $\overline{R}$ are called two rough operators.

Furthermore, as generalizations, they also were defined by an arbitrary binary relation in [17], a mapping in [5, 9], and other methods. Dubois, Prade investigated fuzzy rough set and rough fuzzy set in [7].

2.3. Group.

We assume familiarity with the notion of a group as used in the intuitive set theory. Suppose G is a multiplicative group with an identity e, and A is a subgroup of G, if $\forall x, y \in A$, we have $xy \in A$.

N is a normal subgroup of G, if $\forall x \in G$, and $y \in N$, we have $xyx^{-1} \in N$.

3. L-Group

Suppose G is a group with an identity e and L is a complete Heyting algebra.

Definition 3.1. $A : G \to L$ is called an L-subgroup of G, if for every $x, y \in G$, we have $A(x) \wedge A(y) \leq A(xy)$ and $A(x) \leq A(x^{-1})$.

Example 3.2. Suppose $G = \{e, x, y, z\}$ with the operator as the following table:

$\cdot$	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	e	x
z	z	y	x	e

Then $A_1 = \{a/e, b/x, b/y, b/z\}$ is an L-subgroup of G, where $a, b \in L$, and $b \leq a$. In special, when $L = [0, 1]$, we choose $a = 0.6, b = 0.4$, $A_1 = \{0.6/e, 0.4/x, 0.4/y, 0.4/z\}$ (See [12]).

From [12], Propositions 3.3, 3.4 and 3.5 hold.

Proposition 3.3. *A is an L-subgroup of G if and only if $A(x^{-1}y) \geq A(x^{-1}) \wedge A(y)$, for all $x, y \in G$.*

Proposition 3.4. *Suppose A is an L-subgroup of G. Then for all $x \in G$,*
(1) $A(e) \geq A(x)$,
(2) $A(x) = A(x^{-1})$,
(3) $A(x^n) \geq A(x)$.

Proposition 3.5. *Suppose A, B are two L-subgroups of G. Then $A \cap B$ is also an L-subgroup of G.*

Definition 3.6. N is called a normal L-subgroup of G, if for every $x, y \in G$, $N(y) \leq N(xyx^{-1})$.

Clearly A_1 is a normal L-subgroup of G.
From [12], Propositions 3.7 and 3.8 hold.

Proposition 3.7. *Suppose N is an L-subgroup of G. Then the following conditions are equivalence:*
(1) *N is normal,*
(2) *$N(xy) = N(yx)$ for all $x, y \in G$,*
(3) *$N(xyx^{-1}) = N(y)$ for all $x, y \in G$.*

Proposition 3.8. *Suppose A, B are two normal L-subgroups of G. Then $A \cap B$ is also a normal L-subgroup of G.*

In the classical case, for two subsets A, B of G, $AB = \{z \mid z = xy, x \in A, y \in B\}$, as a generalization, we give the following definition.

Definition 3.9. For $A, B \in L^G$, we define AB as follows: for every $z \in G$,

$$(AB)(z) = \bigvee_{z=xy} A(x) \wedge B(y).$$

In special, $(\{a/x\}B)(w) = \bigvee_{w=st} \{a/x\}(s) \wedge B(t) = \bigvee_{w=xt} a \wedge B(t) = a \wedge B(x^{-1}w)$. $\{a/x\}\{b/y\} = \{c/z\}$, where $z = xy, c = a \wedge b$.

Example 3.10. From Example 3.2, clearly $A_2 = \{d/e, c/y\}$ is also an L-subgroup of G, where $c, d \in L$ and $c \leq d$.

In special, when $L = [0,1]$, let $c = 0.7, d = 0.5$, $A_2 = \{0.7/e, 0.5/y\}$, then $A_1A_2 = \{0.6/e, 0.4/x, 0.4/y, 0.4/z\}$ (See [12]).

4. Rough L-group

In the section, we introduce the notion of a rough L-group defined by a normal L-subgroup and investigate some of their properties.

First, we give the notion of a rough L-group.

Definition 4.1. Suppose N is a L-normal subgroup of G and for every L-subset A of G, $A \neq 0_G$. We define $N^-(A)$ and $N_-(A)$ as follows, respectively: for every $x \in G$,

$$\begin{aligned} N^-(A)(x) &= \bigvee_{\{a/x\}\in M} \{a \mid \bigvee_{z\in G} (\{a/x\}N)(z) \wedge A(z) \neq 0\} \\ &= \bigvee_{\{a/x\}\in M} \{a \mid \bigvee_{z\in G} a \wedge N(x^{-1}z) \wedge A(z) \neq 0\}, \\ N_-(A)(x) &= \bigvee_{\{a/x\}\in M} \{a \mid \bigwedge_{z\in G} (\{a/x\}N)(z) \leq A(z)\} \\ &= \bigvee_{\{a/x\}\in M} \{a \mid \bigwedge_{z\in G} a \wedge N(x^{-1}z) \leq A(z)\}, \end{aligned}$$

where $M = \{\{a/x\} \mid x \in G, a \in L, a > 0\}$ of all fuzzy singletons .

Then $N^-(A), N_-(A)$ are called the upper approximation, the lower approximation of A with respect to the L-normal subgroup N, respectively.

If $L = 2$, then A is a classical subgroup, and $N^-(A) = \{x \mid xN \cap A \neq 0\}$ and $N_-(A) = \{x \mid xN \subseteq A\}$, which coincide with the definition in [16]. If $L = [0,1]$, the above definition coincides with [12].

Example 4.2. In Example 3.2, let $N = A_2$ be a normal L-subgroup of G. Then for A_1, we have

$$\begin{aligned} N^-(A_1)(e) &= \bigvee_{\{a/e\}\in M} \{a \mid \bigvee_{w\in G} (\{a/e\}N)(w) \wedge A_1(w) \neq 0\} = 1, \\ N^-(A_1)(x) &= \bigvee_{\{a/x\}\in M} \{a \mid \bigvee_{w\in G} (\{a/x\}N)(w) \wedge A_1(w) \neq 0\} = 1, \\ N^-(A_1)(y) &= \bigvee_{\{a/y\}\in M} \{a \mid \bigvee_{w\in G} (\{a/y\}N)(w) \wedge A_1(w) \neq 0\} = 1, \\ N^-(A_1)(z) &= \bigvee_{\{a/z\}\in M} \{a \mid \bigvee_{w\in G} (\{a/z\}N)(w) \wedge A_1(w) \neq 0\} = 1, \end{aligned}$$

that is, $N^-(A_1) = G$.

$$\begin{aligned} N_-(A_1)(e) &= \bigvee_{\{a/e\}\in M} \{a \mid \bigwedge_{w\in G} (\{a/e\}N)(w) \leq A_1(w)\} = 0.6, \\ N_-(A_1)(x) &= \bigvee_{\{a/x\}\in M} \{a \mid \bigwedge_{w\in G} (\{a/x\}N)(w) \leq A_1(w)\} = 0.4, \end{aligned}$$

$$N_-(A_1)(y) = \bigvee_{\{a/y\}\in M} \{a \mid \bigwedge_{w\in G} (\{a/y\}N)(w) \leq A_1(w)\} = 0.4,$$
$$N_-(A_1)(z) = \bigvee_{\{a/z\}\in M} \{a \mid \bigwedge_{w\in G} (\{a/z\}N)(w) \leq A_1(w)\} = 0.4,$$

that is, $N_-(A_1) = \{0.4/e, 0.4/x, 0.4/y, 0.4/z\}$.

Next, we discuss the following properties.

Proposition 4.3. *Suppose N is a normal L-subgroups of G and $A \in L^G$. Then we have:*
(1) $N_-(A) \subseteq A$,
(2) $N^-(A) \supseteq NA$.

Proof. (1) For every $w \in G$, we obtain $A(w) \wedge N(w^{-1}w) \leq A(w)$ but for $z \in G$ with $z \neq w$, $A(w) \wedge N(w^{-1}z) \leq A(z)$ may be not holds. Then

$$\begin{aligned} N_-(A)(w) &= \bigvee_{\{c/w\}\in M} \{c \mid \bigwedge_{z\in G} (\{c/w\}N)(z) \leq A(z)\} \\ &= \bigvee_{\{c/w\}\in M} \{c \mid \bigwedge_{z\in G} c \wedge N(w^{-1}z) \leq A(z)\} \\ &\leq \quad \bigvee \quad \{A(w) \mid A(w) \wedge N(w^{-1}w) \leq A(w)\} \\ &= A(w). \end{aligned}$$

Thus we have $N_-(A) \subseteq A$.
(2) For every $w \in G$, if $(AN)(w) \neq 0$, then we have

$$\begin{aligned} N^-(A)(w) &= \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z\in G} c \wedge N(w^{-1}z) \wedge A(z) \neq 0\} \\ &= \bigvee_{\{c/w\}\in M} \{c \mid c \wedge [\bigvee_{z\in G} N(w^{-1}z) \wedge A(z)] \neq 0\} \\ &= \bigvee_{\{c/w\}\in M} \{c \mid c \wedge (AN)(zw^{-1}z) \neq 0\} \\ &\geq \quad \bigvee \quad \{(AN)(w) \mid (AN(w) \wedge (AN)(w) \neq 0\} \\ &\qquad\qquad \text{(Note: } c = (AN)(w), z = w) \\ &= (AN)(w). \end{aligned}$$

Thus $N^-(A) \supseteq NA$. □

Proposition 4.4. *Suppose $A, B \in L^G$ such that $A \subseteq B$ and N is a normal L-subgroup. Then*
(1) $N^-(A) \subseteq N^-(B)$,
(2) $N_-(A) \subseteq N_-(B)$.

Proof. By Definition 4.1, they can be easily proved. □

Proposition 4.5. *Suppose N is a normal L-subgroups of G and $A, B \in L^G$. Then we have:*
(1) $N^-(A \cup B) \supseteq N^-(A) \cup N^-(B)$,
(2) $N^-(A \cap B) \subseteq N^-(A) \cap N^-(B)$,
(3) $N_-(A \cup B) \supseteq N_-(A) \cup N_-(B)$,

(4) $N_-(A \cap B) \supseteq N_-(A) \cap N_-(B)$.

Proof. By Definition 4.1, they can be easily proved. □

Proposition 4.6. *Suppose N is a normal L-subgroups of G and A is an (normal) L-subgroup of G. Then $N^-(A)$ is an (normal) L-subgroups of G.*

Proof. Let $s, t \in G$. Then

$$\begin{aligned}
&N^-(A)(s) \wedge N^-(A)(t)\\
&= \bigvee_{\{a/s\}\in M} \{a \mid \bigvee_{x\in G} a \wedge N(s^{-1}x) \wedge A(x) \neq 0\}\\
&\quad \wedge \bigvee_{\{b/t\}\in M} \{b \mid \bigvee_{y\in G} b \wedge N(t^{-1}y) \wedge A(y) \neq 0\}\\
&= \bigvee_{\{a/s\}\in M} \bigvee_{\{b/t\}\in M} [\{a \mid \bigvee_{x\in G} a \wedge N(s^{-1}x) \wedge A(x) \neq 0\}\\
&\qquad \wedge \{b \mid \bigvee_{y\in G} b \wedge N(t^{-1}y) \wedge A(y) \neq 0\}]\\
&= \bigvee_{\{a/s\}\in M} \bigvee_{\{b/t\}\in M} \{a \wedge b \mid \bigvee_{x\in G} \bigvee_{y\in G} a \wedge b \wedge N(s^{-1}x) \wedge N(t^{-1}y) \wedge A(x) \wedge A(y) \neq 0\}\\
&= \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z=xy\in G} c \wedge N(s^{-1}x) \wedge N(t^{-1}y) \wedge A(x) \wedge A(y) \neq 0\}\\
&\leq \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z=xy\in G} c \wedge N(w^{-1}z) \wedge A(z) \neq 0\}\\
&= N^-(A)(w) \text{ (Note } w = st, z = xy).
\end{aligned}$$

Thus $N^-(A)$ is an L-subgroup of G.

Furthermore, if A is a normal L-subgroup of G, then for $s, t \in G$, let $w = s^{-1}ts$. Then we have

$$\begin{aligned}
N^-(A)(s^{-1}ts) &= N^-(A)(w)\\
&= \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z\in G} c \wedge N(w^{-1}z) \wedge A(w) \neq 0\}\\
&= \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z\in G} c \wedge N((s^{-1}ts)^{-1}z) \wedge A(s^{-1}ts) \neq 0\}\\
&= \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z\in G} c \wedge N(st^{-1}s^{-1}z) \wedge A(t) \neq 0\}\\
&= \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z\in G} c \wedge N(st^{-1}zs^{-1}) \wedge A(t) \neq 0\}\\
&= \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z\in G} c \wedge N(t^{-1}z) \wedge A(t) \neq 0\}\\
&= \bigvee_{\{a/t\}\in M} \{a \mid \bigvee_{z\in G} a \wedge N(t^{-1}z) \wedge A(t) \neq 0\}\\
&= N^-(A)(t).
\end{aligned}$$

Thus $N^-(A)$ is a normal L-subgroup of G. □

In general, $N_-(A)$ is not an L-subgroup of G. But if $N_-(A)$ is an L-subgroup of G, and A is a normal L-subgroup of G, in the similar method, we can prove $N_-(A)$ is also a normal L-subgroup of G.

Proposition 4.7. *Suppose N, H are two normal L-subgroups of G and $A, B \in L^G$. Then we have:*
(1) $N^-(A)N^-(B) \subseteq N^-(AB)$,
(2) $N_-(A)N_-(B) \subseteq N_-(AB)$,
(3) $(N \cap H)^-(A) \supseteq N^-(A) \cap H^-(A)$,
(4) $(N \cap H)_-(A) \subseteq N_-(A) \cap H_-(A)$.

Proof. (1) For every $w \in G$,

$$\begin{aligned} N^-(AB)(w) &= \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z\in G} c \wedge N(w^{-1}z) \wedge (AB)(z) \neq 0\} \\ &= \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z\in G} c \wedge N(w^{-1}z) \wedge [\bigvee_{z=xy} A(x) \wedge B(y)] \neq 0\} \\ &= \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z=xy} c \wedge N(w^{-1}z) \wedge A(x) \wedge B(y) \neq 0\}, \end{aligned}$$

$$\begin{aligned} &(N^-(A)N^-(B))(w) \\ =& \bigvee_{w=st} N^-(A)(s) \wedge N^-(B)(t) \\ =& \bigvee_{w=st} [\bigvee_{\{a/s\}\in M} \{a \mid \bigvee_{x\in G} a \wedge N(s^{-1}x) \wedge A(x) \neq 0\} \\ &\wedge \bigvee_{\{b/t\}\in M} \{b \mid \bigvee_{y\in G} b \wedge N(t^{-1}y) \wedge B(y) \neq 0\}] \\ =& \bigvee_{w=st} \bigvee_{\{a/s\}\in M} \bigvee_{\{b/t\}\in M} [\{a \mid \bigvee_{x\in G} a \wedge N(s^{-1}x) \wedge A(x) \neq 0\} \\ &\wedge\{b \mid \bigvee_{y\in G} b \wedge N(t^{-1}y) \wedge B(y) \neq 0\}] \\ =& \bigvee_{w=st} \bigvee_{\{a/s\}\in M} \bigvee_{\{b/t\}\in M} \{a\wedge b \mid \bigvee_{x\in G} \bigvee_{y\in G} a\wedge N(s^{-1}x)\wedge A(x)\wedge b\wedge N(t^{-1}y)\wedge B(y) \neq 0\} \\ =& \bigvee_{w=st} \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z=xy} c \wedge N(s^{-1}x) \wedge N(t^{-1}y) \wedge A(x) \wedge B(y) \neq 0\} \\ &(\text{Note } \{c/w\} = \{a/s\}\{b/t\}) \\ \leq& \bigvee_{w=st} \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z=xy} c \wedge N(s^{-1}xt^{-1}y) \wedge A(x) \wedge B(y) \neq 0\} \\ =& \bigvee_{w=st} \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z=xy} c \wedge N(s^{-1}t^{-1}xy) \wedge A(x) \wedge B(y) \neq 0\} \\ =& \bigvee_{w=st} \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z=xy} c \wedge N(w^{-1}z) \wedge A(x) \wedge B(y) \neq 0\} \\ =& N^-(AB)(w). \end{aligned}$$

(2) For every $w \in G$,

$$\begin{aligned} N_-(AB)(w) &= \bigvee_{\{c/w\}\in M} \{c \mid \bigwedge_{z\in G} (\{c/w\}N)(z) \leq (AB)(z)\} \\ &= \bigvee_{\{c/w\}\in M} \{c \mid \bigwedge_{z\in G} (\{c/w\}N)(z) \leq \bigvee_{z=xy} A(x) \wedge B(y)\} \\ &= \bigvee_{z=xy} \bigvee_{w=st} [\bigvee_{\{a/s\}\in M} \{a \mid \bigwedge_{z\in G} (\{a/s\}N)(z) \leq A(x)\}] \\ &\quad \wedge[\bigvee_{\{b/t\}\in M} \{b \mid \bigwedge_{z\in G} (\{b/t\}N)(z) \leq B(y)\}] \\ &\quad (\text{Note } \{c/w\} = \{a/s\}\{b/t\}) \\ &= \bigvee_{z=xy} \bigvee_{w=st} [\bigvee_{\{a/s\}\in M} \{a \mid \bigwedge_{z\in G} a \wedge N(s^{-1}z) \leq A(x)\}] \end{aligned}$$

$$\begin{aligned}
&\wedge[\bigvee_{\{b/t\}\in M}\{b \mid \bigwedge_{z\in G} b\wedge N(t^{-1}z)\leq B(y)\}]\\
\geq &\bigvee_{w=st}[\bigvee_{\{a/s\}\in M}\{a \mid \bigwedge_{x\in G} a\wedge N(s^{-1}x)\leq A(x)\}]\\
&\wedge[\bigvee_{\{b/t\}\in M}\{b \mid \bigwedge_{y\in G} b\wedge N(t^{-1}y)\leq B(y)\}]\\
= &\bigvee_{w=st} N_-(A)(s)\wedge N_-(B)(t)\\
= &(N_-(A)N_-(B))(w).
\end{aligned}$$

Then $N_-(A)N_-(B)\subseteq N_-(AB)$.

(3) For every $w\in G$, we have

$$\begin{aligned}
(N\cap H)^-(A)(w) &= \bigvee_{\{c/w\}\in M}\{c \mid \bigvee_{z\in G}(\{c/w\}(N\cap H))(z)\wedge A(z)\neq 0\}\\
&= \bigvee_{\{c/w\}\in M}\{c \mid \bigvee_{z\in G} c\wedge (N\cap H)(w^{-1}z)\wedge A(z)\neq 0\}\\
&= \bigvee_{\{c/w\}\in M}\{c \mid \bigvee_{z\in G} c\wedge N(w^{-1}z)\wedge H(w^{-1}z)\wedge A(z)\neq 0\}\\
&\geq [\bigvee_{\{c/w\}\in M}\{c \mid \bigvee_{z\in G} c\wedge N(w^{-1}z)\wedge A(z)\neq 0\}]\\
&\quad\wedge[\bigvee_{\{c/w\}\in M}\{c \mid \bigvee_{z\in G} c\wedge H(w^{-1}z)\wedge A(z)\neq 0\}]\\
&= N^-(A)(w)\wedge H^-(A)(w)\\
&= (N^-(A)\cap H^-(A))(w).
\end{aligned}$$

(4) For every $w\in G$, we have

$$\begin{aligned}
(N\cap H)_-(A)(w) &= \bigvee_{\{c/w\}\in M}\{c \mid \bigwedge_{z\in G}(\{c/w\}(N\cap H))(z)\leq A(z)\}\\
&= \bigvee_{\{c/w\}\in M}\{c \mid \bigwedge_{z\in G} c\wedge (N\cap H)(w^{-1}z)\leq A(z)\}\\
&= \bigvee_{\{c/w\}\in M}\{c \mid \bigwedge_{z\in G} c\wedge N(w^{-1}z)\wedge H(w^{-1}z)\leq A(z)\}\\
&\leq \bigvee_{\{c/w\}\in M}\{c \mid \bigwedge_{z\in G} c\wedge N(w^{-1}z)\leq A(z)\}\\
&\quad\wedge \bigvee_{\{c/w\}\in M}\{c \mid \bigwedge_{z\in G} c\wedge H(w^{-1}z)\leq A(z)\}\\
&= N_-(A)(w)\wedge H_-(A)(w)\\
&= (N_-(A)\cap H_-(A))(w).
\end{aligned}$$

□

Proposition 4.8. *Suppose N, H are two normal L-subgroups of G. Then for every L-subgroup A of G, we have $N^-(A)H^-(A)\subseteq (NH)^-(A)$.*

Proof. For every $w\in G$, we have

$$(NH)^-(A)(w) = \bigvee_{\{c/w\}\in M}\{c \mid \bigvee_{z\in G}(\{c/w\}(NH))(z)\wedge A(z)\neq 0\}$$

$$= \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z\in G} c\wedge (NH)(w^{-1}z)\wedge A(z)\neq 0\},$$

$$\begin{aligned}
&(N^{-}(A)H^{-}(A))(w)\\
&= \bigvee_{w=st} N^{-}(A)(s)\wedge H^{-}(A)(t)\\
&= \bigvee_{w=st}[\bigvee_{\{a/s\}\in M}\{a \mid \bigvee_{x\in G}(\{a/s\}N)(x)\wedge A(x)\neq 0\}]\\
&\quad \wedge[\bigvee_{\{b/t\}\in M}\{b \mid \bigvee_{y\in G}(\{b/t\}H)(y)\wedge A(y)\neq 0\}]\\
&= \bigvee_{w=st}[\bigvee_{\{a/s\}\in M}\{a \mid \bigvee_{x\in G} a\wedge N(s^{-1}x)\wedge A(x)\neq 0\}]\\
&\quad \wedge[\bigvee_{\{b/t\}\in M}\{b \mid \bigvee_{y\in G} b\wedge H(t^{-1}y)\wedge A(y)\neq 0\}]\\
&= \bigvee_{w=st}\bigvee_{\{a/s\}\in M}\bigvee_{\{b/t\}\in M}\{a\wedge b \mid \bigvee_{x\in G}\bigvee_{y\in G} a\wedge b\wedge N(s^{-1}x)\wedge H(t^{-1}y)\wedge A(x)\wedge A(y)\neq 0\}\\
&= \bigvee_{\{c/w\}\in M}\{c \mid \bigvee_{x\in G}\bigvee_{y\in G} c\wedge N(s^{-1}x)\wedge H(t^{-1}y)\wedge A(x)\wedge A(y)\neq 0\}\\
&= \bigvee_{\{c/w\}\in M}\{c \mid \bigvee_{z=xy\in G} c\wedge N(s^{-1}x)\wedge H(t^{-1}y)\wedge A(x)\wedge A(y)\neq 0\}\\
&= \bigvee_{\{c/w\}\in M}\{c \mid \bigvee_{z=xy\in G} c\wedge (NH)(w^{-1}z)\wedge A(x)\wedge A(y)\neq 0\}\quad (w=st)\\
&\leq \bigvee_{\{c/w\}\in M}\{c \mid \bigvee_{z=xy\in G} c\wedge (NH)(w^{-1}z)\wedge A(z)\neq 0\}\\
&= (NH)^{-}(A)(w).
\end{aligned}$$

□

Proposition 4.9. *Suppose N, H are two normal L-subgroups of G. Then for every L-subgroup A of G, we have $(NH)^{-}(A)\supseteq (N^{-}(A))H\cap(H^{-}(A))N$.*

Proof. For every $w\in G$, we have

$$\begin{aligned}
&((N^{-}(A))H\cap(H^{-}(A))N)(w)\\
&= ((N^{-}(A))H)(w)\wedge((H^{-}(A))N)(w)\\
&= [\bigvee_{w=st}(N^{-}(A)(s)\wedge H(t)]\wedge[\bigvee_{w=st} H^{-}(A)(t)\wedge N(s)]\\
&= [\bigvee_{w=st}\bigvee_{\{a/s\}\in M}\{a \mid \bigvee_{x\in G} a\wedge N(s^{-1}x)\wedge A(x)\neq 0\}\wedge H(t)]\\
&\quad \wedge[\bigvee_{w=st}\bigvee_{\{b/t\}\in M}\{b \mid \bigvee_{y\in G} b\wedge H(t^{-1}y)\wedge A(y)\neq 0\}\wedge N(s)]\\
&= [\bigvee_{w=st}\bigvee_{\{a/s\}\in M}\{a\wedge H(t) \mid \bigvee_{x\in G} a\wedge N(s^{-1}x)\wedge A(x)\neq 0\}]\\
&\quad \wedge[\bigvee_{w=st}\bigvee_{\{b/t\}\in M}\{b\wedge N(s) \mid \bigvee_{y\in G} b\wedge H(t^{-1}y)\wedge A(y)\neq 0\}]\\
&= \bigvee_{w=st}\bigvee_{\{a/s\}\in M}\bigvee_{\{b/t\}\in M}[\{a\wedge H(t) \mid \bigvee_{x\in G} a\wedge N(s^{-1}x)\wedge A(x)\neq 0\}]\\
&\quad \wedge[\{b\wedge N(s) \mid \bigvee_{y\in G} b\wedge H(t^{-1}y)\wedge A(y)\neq 0\}]\\
&= \bigvee_{w=st}\bigvee_{\{a/s\}\in M}\bigvee_{\{b/t\}\in M}\{a\wedge H(t)\wedge b\wedge N(s) \mid\\
&\qquad \bigvee_{x\in G}\bigvee_{y\in G} a\wedge N(s^{-1}x)\wedge A(x)\wedge b\wedge H(t^{-1}y)\wedge A(y)\neq 0\}\\
&= \bigvee_{w=st}\bigvee_{\{c/w\}\in M}\{c\wedge H(t)\wedge N(s) \mid \bigvee_{z=xy} c\wedge N(s^{-1}x)\wedge A(x)\wedge H(t^{-1}y)\wedge A(y)\neq 0\}\\
&\leq \bigvee_{w=st}\bigvee_{\{c/w\}\in M}\{c \mid \bigvee_{z=xy} c\wedge N(s^{-1}x)\wedge H(t^{-1}y)\wedge A(z)\neq 0\}\quad (A(x)\wedge A(y)\leq A(z))
\end{aligned}$$

$$= \bigvee_{\{c/w\}\in M} \{c \mid \bigvee_{z\in G} c \wedge (NH)(w^{-1}z) \wedge A(z) \neq 0\}$$
$$= (NH)^{-}(A)(w). \qquad \square$$

Proposition 4.10. *Suppose N, H are two normal L-subgroups of G. Then for every L-subgroup A of G, we have $N_{-}(A)H_{-}(A) \subseteq (NH)_{-}(A)$.*

Proof. For every $w \in G$,

$$\begin{aligned}
&(N_{-}(A)H_{-}(A))(w)\\
&= \bigvee_{w=st} N_{-}(A)(s) \wedge H_{-}(A)(t)\\
&= \bigvee_{w=st} [\bigvee_{\{a/s\}\in M} \{a \mid \bigvee_{x\in G} a \wedge N(s^{-1}x) \leq A(x)\}]\\
&\quad \wedge [\bigvee_{\{b/t\}\in M} \{b \mid \bigvee_{y\in G} b \wedge H(t^{-1}y) \leq A(y)\}]\\
&= \bigvee_{w=st} \bigvee_{\{a/s\}\in M} \bigvee_{\{b/t\}\in M} [\{a \mid \bigvee_{x\in G} a \wedge N(s^{-1}x) \leq A(x)\}]\\
&\qquad \wedge [\{b \mid \bigvee_{y\in G} b \wedge H(t^{-1}y) \leq A(y)\}]\\
&= \bigvee_{w=st} \bigvee_{\{a/s\}\in M} \bigvee_{\{b/t\}\in M} [\{a \wedge b \mid \bigvee_{x\in G} \bigvee_{y\in G} a \wedge b \wedge N(s^{-1}x) \wedge H(t^{-1}y) \leq A(x) \wedge A(y)\}]\\
&= \bigvee_{w=st} \bigvee_{\{c/w\}\in M} [\{c \mid \bigvee_{z=xy\in G} c \wedge N(s^{-1}x) \wedge H(t^{-1}y) \leq A(x) \wedge A(y)\}]\\
&= \bigvee_{w=st} \bigvee_{\{c/w\}\in M} [\{c \mid \bigvee_{z=xy\in G} c \wedge (NH)(w^{-1}z) \leq A(x) \wedge A(y)\}]\\
&\leq \bigvee_{\{c/w\}\in M} [\{c \mid \bigvee_{z=xy\in G} c \wedge (NH)(w^{-1}z) \leq A(z)\}]\\
&= (NH)_{-}(A)(w). \qquad \square
\end{aligned}$$

5. Conclusion

In the paper, we investigated two problems. One is generalized the notion of a group in fuzzy setting; The other is defined two rough operators on an L-group, and discussed some of their properties.

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