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Quasi-interior ideals and fuzzy quasi-interior ideals of Γ-semirings

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Abstract. In this paper,we introduce the notion of quasi-interior ideal and fuzzy quasi-interior ideal of Γ-semiring and we characterize the regular Γ-semiring in terms of fuzzy quasi-interior ideal of Γ-semiring.

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1. Introduction

In 1995, Murali Krishna Rao [12, 13, 14, 15, 17, 25] introduced the notion of a Γ-semiring as a generalization of Γ-ring, ring, ternary semiring and semiring. As a generalization of ring, the notion of a Γ-ring was introduced by Nobusawa [28] in 1964. Semigroup, as the basic algebraic structure was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. In 1981, Sen [30] introduced the notion of a Γ-semigroup as a generalization of semigroup.

Semiring was first introduced by Vandiver [32] in 1934 as a common generalization of ring and distributive lattice. Semiring is a universal algebra with two binary operations called addition and multiplication, where one of them distributive over the other. A natural example of semiring is the set of all natural numbers under usual addition and multiplication of numbers. Semiring as the basic algebraic structure was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata , coding theory and formal languages. Semiring theory has many applications in other branches of mathematics.

Ideals play an important role in advance studies and uses of algebraic structures. Generalization of ideals in algebraic structures is necessary for further study of algebraic structures. Many mathematicians proved important results and charecterizations of algebraic structures by using the concept and the properties of generalization of ideals in algebraic structures. The notion of a one sided ideal of any algebraic structure is a generalization of the notion of an ideal. The quasi ideals are generalization of left ideals and right ideals whereas the bi-ideals are generalization of quasi ideals. In 1952, the concept of bi-ideals was introduced by Good and Hughes [2] for semigroups. The notion of bi-ideals in rings and semgroupsrings were introduced by Lajos and Szasz [9, 10]. In 1956, Steinfeld [31] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki [4, 5, 6] introduced the concept of quasi ideal for a semiring. Henriksen [3] studied ideals in semirings. Quasi ideals in Γ-semirings studied by Jagtap and Pawar [7]. Rao and Rao et al. [17, 18, 19, 20, 21, 22, 26] introduced and studied bi- interior ideals, bi-quasi-ideals, bi-quasi-interior ideals in semigroups, semirings, Γ-semigroups and Γ-semirings.

The fuzzy set theory was developed by Zadeh [33] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. The fuzzification of algebraic structure was introduced by Rosenfeld [29] and he introduced the notion of fuzzy subgroups in 1971. Kuroki [8] studied fuzzy semigroups. Mandal [11] studied fuzzy ideals and fuzzy interior ideals in an ordered semiring. Rao and Rao et al. [16, 23, 24, 27] studied fuzzy k-ideals, $T-$fuzzy ideals of ordered Γ-semirings. In this paper, as a further generalization of ideals,we introduce the notion of quasi-interior ideal of Γ-semiring as a generalization of ideal, left ideal, right ideal, bi-ideal and interior ideal of Γ-semiring and the notion of fuzzy quasi-interior ideal of Γ-semiring. We characterize the regular $\Gamma-$semiring in terms of fuzzy quasi-interior ideal of Γ-semiring and studied some of their properties.

2. Preliminaries

In this section, we recall some of the fundamental concepts and definitions which are necessary for this paper.

Definition 2.1 ([1])**.** A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) is called a semiring, provided

(i) addition is a commutative operation,
(ii) multiplication distributes over addition both from the left and from the right,
(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$, for all $x \in S$.

Definition 2.2 ([12])**.** Let M and Γ be two non-empty sets. Then M is called a Γ-semigroup, if it satisfies

(i) $x\alpha y \in M$,
(ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Definition 2.3 ([12])**.** Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. A Γ-semigroup M is said to be Γ-semiring M, if it satisfies the following axioms, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

(i) $x\alpha(y+z) = x\alpha y + x\alpha z$,
(ii) $(x+y)\alpha z = x\alpha z + y\alpha z$,
(iii) $x(\alpha+\beta)y = x\alpha y + x\beta y$.

Every semiring M is a Γ-semiring with $\Gamma = M$ and a ternary operation as the usual semiring multiplication

Definition 2.4 ([12]). A Γ-semiring M is said to have zero element, if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0,$ for all $x \in M$.

Definition 2.5 ([12]). Let M be a Γ-semiring. An element $a \in M$ is said to be regular element of M, if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

Definition 2.6 ([12]). Let M be a Γ-semiring. If every element of M is a regular, then M is said to be regular Γ-semiring.

Definition 2.7 ([12]). An element $a \in M$ is said to be idempotent of M, if $a = a\alpha a$, for some $\alpha \in \Gamma$.

Definition 2.8 ([12]). If every element of M is an idempotent of M, then M is said to be idempotent Γ-semiring M.

Definition 2.9 ([28]). Let M be a Γ-semiring. An element $1 \in M$ is said to be unity, if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.10 ([23]). A non-empty subset A of a Γ-semiring M is called:
(i) a Γ-subsemiring of M, if $(A,+)$ is a subsemigroup of $(M,+)$ and $A\Gamma A \subseteq A$,
(ii) a quasi ideal of M, if A is a Γ-subsemiring of M and $A\Gamma M \cap M\Gamma A \subseteq A$,
(iii) a bi-ideal of M, if A is a Γ-subsemiring of M and $A\Gamma M\Gamma A \subseteq A$,
(iv) an interior ideal of M, if A is a Γ-subsemiring of M and $M\Gamma A\Gamma M \subseteq A$,
(v) a left (right) ideal of M, if A is a Γ-subsemiring of M and $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$,
(vi) an ideal, if A is a Γ-subsemiring of $M, A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$,
(vii) a $k-$ideal, if A is a Γ-subsemiring of $M, A\Gamma M \subseteq A, M\Gamma A \subseteq A$ and $x \in M,\ x+y \in A, y \in A$ then $x \in A$,
(viii) a left(right) bi- quasi ideal of M, if A is a Γ-subsemiring of M and $M\Gamma A \cap A\Gamma M\Gamma A(A\Gamma M \cap A\Gamma M\Gamma A) \subseteq A$,
(ix) a bi-quasi ideal of M, if A is a left bi-quasi ideal and a right bi-quasi ideal of M

Definition 2.11 ([22]). A Γ-semiring M is a left (right) simple Γ-semiring, if M has no proper left (right) ideals of M

Definition 2.12 ([22]). A Γ-semiring M is said to be simple Γ-semring, if M has no proper ideals.

Definition 2.13 ([27]). Let M be a non-empty set. A mapping $f : M \to [0,1]$ is called a fuzzy subset of a Γ-semiring M. If f is not a constant function, then f is called a non-empty fuzzy subset

Definition 2.14 ([27]). Let f be a fuzzy subset of a non-empty set M. Then for $t \in [0,1]$, the set $f_t = \{x \in M \mid f(x) \geq t\}$ is called a level subset of M with respect to f.

Definition 2.15 ([27])**.** Let M be a Γ-semiring. A fuzzy subset μ of M is said to be fuzzy Γ-subsemiring of M, if it satisfies the following conditions:
(i) $\mu(x+y) \geq min\ \{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \geq min\ \{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.16. [27] A fuzzy subset μ of a Γ-semiring M is called a fuzzy left (right) ideal of M if for all $x, y \in M, \alpha \in \Gamma$, it satisfies the following conditions:
(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \geq \mu(y)\ (\mu(x))$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.17 ([27])**.** A fuzzy subset μ of a Γ-semiring M is called a fuzzy ideal of M, if it satisfies the following conditions:
(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \geq max\ \{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.18 ([27])**.** For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$, for all $x \in M$.

Definition 2.19 ([27])**.** Let f and g be fuzzy subsets of a Γ-semiring M. Then $f \circ g, f \cup g, f \cap g$, are defined by:

$$(f \circ g)(z) = \begin{cases} \sup\limits_{z=x\alpha y} \{min\{f(x), g(y)\}\}, \\ 0, \quad \text{otherwise.} \end{cases}$$

$$(f \cup g)(z) = max\{f(z), g(z)\} \ \ ; \ (f \cap g)(z) = min\{f(z), g(z)\}$$

$x, y \in M, \alpha \in \Gamma$, for all $z \in M$.

Definition 2.20 ([27])**.** Let A be a non-empty subset of M. The characteristic function of A is a fuzzy subset of M, defined by

$$\chi_{_A}(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$$

3. Quasi-interior ideals of Γ-semirings

In this section, we introduce the notion of quasi-interior ideal as a generalization of left(right)-ideal and interior ideal of Γ-semiring and study the properties of quasi-interior ideal of Γ-semiring. Throughout this paper$M$ is a Γ-semiring with unity element.

Definition 3.1. A non-empty subset B of a Γ-semiring M is said to be left quasi-interior ideal of M, if B is a Γ-subsemiring of M and $M\Gamma B\Gamma M\Gamma B \subseteq B$.

Definition 3.2. A non-empty subset B of a Γ-semiring M is said to be right quasi-interior ideal of M, if B is a Γ-subsemiring of M and $B\Gamma M\Gamma B\Gamma M \subseteq B$.

Definition 3.3. A non-empty subset B of a Γ-semiring M is said to be quasi-interior ideal of M, if B is a Γ-subsemiring of M and $B$is a left quasi-interior ideal and a right quasi-interior ideal of M.

Remark 3.4. A quasi-interior ideal of a Γ-semiring M need not be an interior ideal of a Γ-semiring M.

Example 3.5. Let Q be the set of all rational numbers and $M = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in Q \right\}$ and $\Gamma = M$ be additive abelian semigroups with respect to usual addition of matrices and a ternary operation is defined as the usual matrix multiplication. Then M is a Γ- semiring. If $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in Q \right\}$, then A is a right quasi-interior ideal but not a bi-ideal of the Γ-semiring M.

Example 3.6. Let N be a set of all even natural numbers and $\Gamma = \mathcal{N}$ be additive abelian semigroups. A ternary operation is defined as $(x, \alpha, y) \rightarrow x + \alpha + y$, where $+$ is the usual addition of integers. Then N is a Γ-semiring. Let $I = 4N$. Then I is not a quasi-interior ideal of N

In the following theorem, we mention some important properties and we omit the proofs since they are straight forward.

Theorem 3.7. *Let M be a Γ-semring. Then the following are hold.*

(1) *Every left ideal is a left quasi-interior ideal of M.*
(2) *Every right ideal is a right quasi-interior ideal of M.*
(3) *Every interior ideal is a quasi-interior ideal of M.*
(4) *every ideal is a quasi-interior ideal of M,*
(7) *If B is a quasi-interior ideal and T is an i.terior ideal of M, then $B \cap T$ is a quasi-interior ideal of ring M.*
(8) *Let M be a Γ-semiring and B be a Γ-subsemiring of M. If $M\Gamma M\Gamma M\Gamma B \subseteq B$ then B is a left quasi-interior ideal of M.*
(9) *Let M be a Γ-semiring and B be a Γ-subsemiring of M. If $M\Gamma M\Gamma M\Gamma B \subseteq B$ and $B\Gamma M\Gamma M\Gamma M \subseteq B$ then B is a quasi-interior ideal of M.*

Theorem 3.8. *If B be a left quasi-interior ideal of a Γ-semiring M, then B is a left bi-quasi ideal of M.*

Proof. Suppose B is a left quasi-interior ideal of the Γ-semiring M. Then $M\Gamma B\Gamma M\Gamma B \subseteq B$. Thus we have $B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B$. So

$$M\Gamma B \cap B\Gamma M\Gamma B \subseteq B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B.$$

Hence B is a left bi-quasi ideal of M. □

Corollary 3.9. *If B be a right quasi-interior ideal of a Γ-semiring M, then B is a right bi-quasi ideal of M.*

Corollary 3.10. *If B be a quasi-interior ideal of a Γ-semiring M, then B is a bi-quasi ideal of M.*

Theorem 3.11. *If B be a left quasi-interior ideal of a Γ-semiring M, then B is a bi–interior ideal of M.*

Proof. Suppose B is a left quasi-interior ideal of the Γ-semiring M. Then

$$M\Gamma B\Gamma M\Gamma B \subseteq B.$$

Thus we have

$$M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B.$$

So B is a bi–interior ideal of M. □

Corollary 3.12. *If B be a right quasi-interior ideal of a Γ-semiring M, then B is a bi-interior ideal of M.*

Corollary 3.13. *If B be a quasi-interior ideal of a Γ-semiring M, then B is a bi–interior ideal of M.*

Theorem 3.14. *Every left quasi-interior ideal of a Γ-semiring M is a bi-ideal of a Γ-semiring M.*

Proof. Let B be a left quasi-interior ideal of the Γ-semiring M. Then

$$B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B.$$

Thus $B\Gamma M\Gamma B \subseteq B$. So every left quasi-interior ideal of a Γ-semiring M is a bi-ideal of M. □

Corollary 3.15. *Every right quasi-interior ideal of a Γ-semiring M is a bi-ideal of M.*

Corollary 3.16. *Every quasi-interior ideal of a Γ-semiring M is a bi-ideal of M.*

Theorem 3.17. *Let M be a regular Γ-semiring. Then $M\Gamma B\Gamma M\Gamma B = B$ for all left quasi-interior ideals B of M.*

Proof. Suppose M is a regular Γ-semiring, B is a left quasi-interior ideal of M and $x \in B$. Then $M\Gamma B\Gamma M\Gamma B \subseteq B$ and there exist $y \in M,\ \alpha, \beta \in \Gamma$ such that

$$x = x\alpha y\beta x\alpha y\beta x \in M\Gamma B\Gamma M\Gamma B.$$

Thus $x \in M\Gamma B\Gamma M\Gamma B$. So $M\Gamma B\Gamma M\Gamma B = B$, for all left quasi-interior ideals B of the regular Γ-semiring M. □

Corollary 3.18. *Let M be a regula Γ-semiring. Then $B\Gamma M\Gamma B\Gamma M = B$, for all right quasi-interior ideals B of the regular Γ-semiring M.*

Corollary 3.19. *Let M be a regular Γ-semiring. Then $B\Gamma M\Gamma B\Gamma M = B$ and $M\Gamma B\Gamma M\Gamma B = B$, for all quasi-interior ideals B of the regular Γ-semiring M.*

Theorem 3.20. *M is a regular Γ-semiring if and only if $A\Gamma B = A \cap B$ for any right ideal A and left ideal B of a Γ-semiring M.*

Theorem 3.21. *Let M be a right simple Γ-semiring. If $M\Gamma B\Gamma M\Gamma B = B$, for all left quasi-interior ideals B of M. then M is a regular Γ-semiring,*

Proof. Suppose that $M\Gamma B\Gamma M\Gamma B = B$, for all left quasi-interior ideals B of M. Let $B = R \cap L$, where R is an ideal and L is a left ideal of M. Then B is a quasi-interior ideal of M.

Thus $M\Gamma(R\cap L)\Gamma M\Gamma(R\cap L) = R\cap L$

$$\begin{aligned}
R\cap L &= M\Gamma(R\cap L)\Gamma M\Gamma(R\cap L)\\
&\subseteq R\Gamma M\Gamma L\\
&\subseteq R\Gamma L\\
R\cap L &= M\Gamma((R\cap L)\Gamma M\Gamma(R\cap L)\\
&\subseteq M\Gamma R\Gamma L\Gamma M\Gamma R\Gamma L\\
&\subseteq R\Gamma L\\
&\subseteq R\cap L \text{ (since } R\Gamma L\subseteq L \text{ and } R\Gamma L\subseteq R).
\end{aligned}$$

So $R\cap L = R\Gamma L$. Hence by Theorem 3.20, M is a regular Γ-semiring. □

4. Fuzzy quasi interior ideals of Γ-Semirings

In this section, we introduce the notion of fuzzy right(left) quasi-interior ideal as a generalization of fuzzy bi-ideal of a Γ-semiring and study the properties of fuzzy right(left) bi-quasi ideals.

Definition 4.1. A fuzzy subset μ of a Γ-semiring M is called a fuzzy left (right) quasi-interior ideal, if

(i) $\mu(x+y)\geq \min\{\mu(x),\mu(y)\}$, for all $x,y\in M$,
(ii) $\chi_M\circ\mu\circ\chi_M\circ\mu\subseteq\mu\big(\mu\circ\chi_M\circ\mu\circ\chi_M\subseteq\mu\big)$.

A fuzzy subset μ of a Γ-semiring M is called a fuzzy quasi-interior ideal if it is both a left quasi-interior ideal and a right quasi-interior ideal of M.

Example 4.2. Let Q be the set of all rational numbers and $M = \left\{\begin{pmatrix} a & 0\\ b & 0\end{pmatrix} \mid a,b\in Q\right\}$ and $\Gamma = M$ be additive abelian semigroups with respect to usual addition of matrices and a ternary operation is defined as the usual matrix multiplication. Then M is a Γ-semiring. If $A = \left\{\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix} \mid a\in Q\right\}$, then A is a right quasi-interior ideal but not a bi-ideal of the Γ-semiring M. Define $\mu : M\to[0,1]$ such that

$$\mu(x) = \begin{cases} 1 & \text{if } x\in A,\\ 0, & \text{otherwise.}\end{cases}$$

Then μ is a fuzzy right quasi-interior ideal of the Γ-semiring M.

Example 4.3. Let N be a set of all even natural numbers and $\Gamma = \mathcal{N}$ be additive abelian semigroups. A ternary operation is defined as $(x,\alpha,y)\to x+\alpha+y$, where $+$ is the usual addition of integers. Then N is a Γ-semiring. Let $I = 4N$. Then I is a tri-ideal of N but not quasi-ideal, interior ideal, bi-interior ideal and bi-quasi ideal of a Γ-semiring N. Define $\mu : N\to[0,1]$ such that

$$\mu(x) = \begin{cases} 1 & \text{if } x\in I,\\ 0, & \text{otherwise.}\end{cases}$$

Then μ is a fuzzy right quasi-interior ideal of the Γ-semiring.M.

Theorem 4.4. *Every fuzzy right ideal of a Γ-semiring M is a tri-ideal of M.*

Proof. Let μ be a fuzzy right ideal of the Γ-semiring M and $x \in M$. Then

$$\begin{aligned}(\mu \circ \chi_M)(x) &= \sup_{x=a\alpha b} \min\{\mu(a), \chi_M(b)\} a, b \in M, \alpha \in \Gamma \\ &= \sup_{x=a\alpha b} \mu(a) \\ &\leq \sup_{x=a\alpha b} \mu(a\alpha b) \\ &= \mu(x).\end{aligned}$$

Thus $(\mu \circ \chi_M)(x) \leq \mu(x)$. Now

$$\begin{aligned}(\mu \circ \chi_M \circ \mu \circ \chi_M)(x) &= \sup_{x=u\alpha v\beta s} \min\{(\mu \circ \chi_M)(u\alpha v), (\mu \circ \chi_M)(s)\} \\ &\leq \sup_{x=u\alpha v\beta s} \min\{\mu(u\alpha v), \mu(s)\} \\ &= \mu(x).\end{aligned}$$

So μ is a fuzzy right quasi-interior ideal of the Γ-semiring M. □

Corollary 4.5. *Every fuzzy left ideal of a* Γ*-semiring* M *is a fuzzy left quasi-interior ideal of* M.

Corollary 4.6. *Every fuzzy ideal of a* Γ*-semiring* M *is a fuzzy quasi-interior ideal of* M.

Theorem 4.7. *Let* M *be a* Γ*-semiring and* μ *be a non-empty fuzzy subset of* M. *A fuzzy subset* μ *is a fuzzy left quasi-interior ideal of a* Γ*-semiring* M *if and only if the level subset* μ_t *of* μ *is a left quasi-interior ideal of a* Γ*- semiring* M *for every* $t \in [0, 1]$, *where* $\mu_t \neq \phi$.

Proof. Suppose μ is a fuzzy left quasi-interior ideal of the Γ-semiring M, $\mu_t \neq \phi, t \in [0, 1]$ and $a, b \in \mu_t$. Then

$$\begin{aligned}&\mu(a) \geq t, \mu(b) \geq t \\ \Rightarrow &\mu(a + b) \geq \min\{\mu(a), \mu(b)\} \geq t \\ \Rightarrow &a + b \in \mu_t.\end{aligned}$$

Let $x \in M\Gamma\mu_t\Gamma M\Gamma\mu_t$. Then $x = b\alpha a\beta d\gamma c$, where $b, d \in M, a, c \in \mu_t, \alpha, \beta$ and $\gamma \in \Gamma$. Then

$$\begin{aligned}&(\chi_M \circ \mu \circ \chi_M \circ \mu)(x) \geq t \\ \Rightarrow &\mu(x) \geq \chi_M \circ \mu \circ \chi_M \circ \mu(x) \geq t.\end{aligned}$$

Thus $x \in \mu_t$. So μ_t is a left quasi-interior ideal of the Γ-semiring M.

Conversely, suppose that μ_t is a left quasi-interior ideal of the Γ-semiring M, for all $t \in Im(\mu)$. Let $x, y \in M, \alpha \in \Gamma, \mu(x) = t_1, \mu(y) = t_2$ and $t_1 \geq t_2$. Then $x, y \in \mu_{t_2}$. Thus $x + y \in \mu_{t_2}$ and $x\alpha y \in \mu_{t_2}$. So $\mu(x + y) \geq t_2 = \min\{t_1, t_2\} = \min\{\mu(x), \mu(y)\}$. Hence $\mu(x + y) \geq t_2 = \min\{\mu(x), \mu(y)\}$.

On the other hand, we have $M\Gamma\mu_l\Gamma M\Gamma\mu_l \subseteq \mu_t$, for all $l \in Im(\mu)$.

Suppose $t = \min\{Im(\mu)\}$. Then $M\Gamma\mu_t\Gamma M\Gamma\mu_t \subseteq \mu_t$. Thus $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$. So μ is a fuzzy left quasi-interior ideal of the Γ-semiring M. □

Corollary 4.8. *Let M be a Γ-semiring and μ be a non-empty fuzzy subset of M. A fuzzy subset μ is a fuzzy right quasi-interior ideal of a $\Gamma-$ semiring if and only if the level subset μ_t of μ is a right quasi-interior ideal of a Γ-semiring M for every $t \in [0,1]$, where $\mu_t \neq \phi$.*

Theorem 4.9. *Let I be a non-empty subset of a Γ-semiring M and χ_I be the characteristic function of I. Then I is a right quasi-interior ideal of a Γ-semiring M if and only if χ_I is a fuzzy right quasi-interior ideal of a Γ-semiring M.*

Proof. Let I be a non-empty subset of the Γ-semiring M and χ_I be the characteristic function of I. Suppose I is a right quasi-interior ideal of the Γ-semiring M. Obviously χ_I is a fuzzy Γ-subsemiring of M. We have $I\Gamma M\Gamma I\Gamma M \subseteq I$. Then

$$\chi_I \circ \chi_M \circ \chi_I \circ \chi_M = \chi_{I\Gamma M\Gamma I\Gamma M} \subseteq \chi_I.$$

Thus χ_I is a fuzzy right quasi-interior ideal of the Γ-semiring M.

Conversely, suppose that χ_I is a fuzzy right quasi-interior ideal of M. Then I is a Γ-subsemiring of M. Thus we have

$$\chi_I \circ \chi_M \circ \chi_I \circ \chi_M \subseteq \chi_I.$$

So $\chi_{I\Gamma M\Gamma I\Gamma M} \subseteq \chi_I$. Hence $I\Gamma M\Gamma I\Gamma M \subseteq I$. Therefore I is a right quasi-interior ideal of the Γ-semiring M. □

Corollary 4.10. *Let I be a non-empty subset of a Γ-semiring M and χ_I be the characteristic function of I. Then I is a left quasi-interior ideal of a Γ-semiring M if and only if χ_I is a fuzzy left quasi-interior ideal of a Γ-semiring M.*

Theorem 4.11. *If μ and λ are fuzzy left quasi-interior ideals of a Γ-semiring M, then $\mu \cap \lambda$ is a fuzzy left quasi-interior ideal of a Γ-semiring M.*

Proof. Let μ and λ be left quasi-interior ideals of the Γ-semiring $M, x, y \in M$ and $\alpha \in \Gamma, \beta \in \Gamma$. Then

$$\begin{aligned}(\mu \cap \lambda)(x+y) &= \min\{\mu(x+y), \lambda(x+y)\}\\ &\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\}\\ &= \min\{\min\{\mu(x), \lambda(x)\}, \min\{\mu(y), \lambda(y)\}\}\\ &= \min\{(\mu \cap \lambda(x)), (\mu \cap \lambda)(y)\}\end{aligned}$$

and

$$\begin{aligned}(\chi_M \circ \mu \cap \lambda)(x) &= \sup_{x=a\alpha b} \min\{\chi_M(a), (\mu \cap \lambda)(b)\}\\ &= \sup_{x=a\alpha b} \min\{\chi_M(a), \min\{\mu(b), \lambda(b)\}\}\\ &= \sup_{x=a\alpha b} \min\{\min\{\chi_M(a), \mu(b)\}, \min\{\chi_M(a), \lambda(b)\}\}\\ &= \min\{\sup_{x=a\alpha b} \min\{\chi_M(a), \mu(b)\}, \sup_{x=a\alpha b} \min\{\chi_M(a), \lambda(b)\}\}\\ &= \min\{(\chi_M \circ \mu)(x).(\chi_M \circ \lambda)(x)\}\\ &= (\chi_M \circ \mu \cap \chi_M \circ \lambda)(x).\end{aligned}$$

Thus $\chi_M \circ \mu \cap \chi_M \circ \lambda = \chi_M \circ \mu \cap \chi_M \circ \lambda$. On the other hand,

$$\begin{aligned}&(\chi_M \circ \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda)(x)\\ &= \sup_{x=a\alpha b\beta c} \min\{(\chi_M \circ \mu \cap \chi_M \circ \lambda)(a), (\chi_M \circ \mu \cap \lambda)(b\beta c)\}\\ &= \sup_{x=a\alpha b\beta c} \min\{\chi_M \circ \mu \cap \lambda(a), \chi_M \circ \mu \cap \chi_M \circ \lambda(b\beta c)\}\end{aligned}$$

$= \sup\limits_{x=a\alpha b\beta c} \min\{\chi_M \circ \mu(a), \chi_M \circ \lambda(a)\}\}, \min\{(\chi_M \circ \mu)(b\beta c), (\chi_M \circ \lambda)(b\beta c)\}\}$
$= \sup\limits_{x=a\alpha\beta bc} \min\{\min\{(\chi_M \circ \mu)(a), \chi_M \circ \mu(b\beta c)\}, \min\{(\chi_M \circ \lambda)(a), (\chi_M \circ \lambda)(b\beta c)\}\}$
$\min\{ \sup\limits_{x=a\alpha b\beta c} \min\{\chi_M \circ \mu(a), \chi_M \circ \mu(b\beta c)\}, \sup\limits_{x=a\alpha b\beta c} \min\{\chi_M \circ \lambda(a), \chi_M \circ \lambda(b\beta c)\}\}$
$= \min\{\chi_M \circ \mu \circ \chi_M \circ \mu(x), \chi_M \circ \lambda \circ \chi_M \circ \lambda(x)\}$
$= \chi_M \circ \mu \circ \chi_M \circ \mu \cap \chi_M \circ \lambda \circ \chi_M \circ \lambda(x).$
So $\chi_M \circ \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = \chi_M \circ \mu \circ \chi_M \circ \mu \cap \chi_M \circ \lambda \circ \chi_M \circ \lambda$. Hence

$$\chi_M \circ \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = \chi_M \circ \mu \circ \chi_M \circ \mu \cap \chi_M \circ \lambda \circ \chi_M \circ \lambda \subseteq \mu \cap \lambda.$$

Therefore $\mu \cap \lambda$ is a left fuzzy quasi-interior ideal of M. This completes the proof. □

Corollary 4.12. *If μ and λ are fuzzy right quasi-interior ideals of a Γ-semiring M, then $\mu \cap \lambda$ is a fuzzy right quasi-interior ideal of a Γ-semiring M.*

Corollary 4.13. *Let μ and λ be fuzzy quasi-interior ideals of a Γ-semiring M. Then $\mu \cap \lambda$ is a fuzzy quasi-interior ideal of a Γ-semiring M.*

Theorem 4.14. *Let M be a regular Γ-semiring. Then μ is a fuzzy left quasi-interior ideal of M if and only if μ is a fuzzy quasi ideal of M.*

Proof. Let μ be a fuzzy left quasi-interior ideal of the Γ-semiring M and $x \in M$. Then
$\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$. Suppose $(\chi_M \circ \mu)(x) > \mu(x)$ and $(\mu \circ \chi_M)(x) > \mu(x)$. Since M is a regular, there exist $y \in M, \alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x$. Then

$$\begin{aligned}(\mu \circ \chi_M)(x) &= \sup_{x=x\alpha y\beta x} \min\{\mu(x), \chi_M(y\beta x)\}\\ &= \sup_{x=x\alpha y\beta x} \min\{\mu(x), 1\}\\ &= \sup_{x=x\alpha y\beta x} \mu(x)\\ &> \mu(x)\end{aligned}$$

and

$$\begin{aligned}(\mu \circ \chi_M \circ \mu \circ \chi_M)(x) &= \sup_{x=x\alpha y\beta x} \min\{(\mu \circ \chi_M)(x), (\mu \circ \chi_M)(y\beta x)\})\\ &> \sup_{x=x\alpha y\beta x} \min\{\mu(x), \mu(y\beta x)\}\\ &= \mu(x).\end{aligned}$$

This is a contradiction. Thus μ is a fuzzy quasi ideal of M. The converse is true. □

Corollary 4.15. *Let M be a regular Γ-semiring. Then μ is a fuzzy right quasi-interior ideal of M if and only if μ is a fuzzy quasi ideal of M.*

Theorem 4.16. *Let M be a regular Γ-semiring. If μ is a fuzzy left quasi-interior ideal of a $\Gamma-$semiring M, then $\mu = \chi_M \circ \mu \circ \chi_M \circ \mu$,* *Proof.* Let μ be a fuzzy left quasi-interior ideal of the regular Γ-semiring M and $x, y \in M, \alpha, \beta \in \Gamma$. Then $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$. Thus

$$\begin{aligned}(\chi_M \circ \mu \circ \chi_M \circ \mu)(x) &= \sup_{x = x\alpha y\beta x} \{\min\{(\chi_M \circ \mu)(x), (\chi_M \circ \mu)(y\beta x)\}\} \\ &\geq \sup_{x = x\alpha y\beta x} \{\min\{\mu(x), \mu(x)\}\} \\ &= \mu(x).\end{aligned}$$

So $\mu \subseteq \chi_M \circ \mu \circ \chi_M \circ \mu$. Hence $\mu = \chi_M \circ \mu \circ \chi_M \circ \mu$, for any fuzzy quasi-interior ideal μ of the $\Gamma-$semiring M. □

Theorem 4.17. *Let M be a right simple Γ-semiring.If $\mu = \chi_M \circ \mu \circ \chi_M \circ \mu$, for any fuzzy left quasi-interior ideal μ of a $\Gamma-$semiring M, then M is a regular.*

Proof. Suppose that $\mu = \chi_M \circ \mu \circ \chi_M \circ \mu$, for any fuzzy quasi-interior ideal μ of the $\Gamma-$semiring M. Let B be a left quasi-interior ideal of the Γ-semiring M. Then by Theorem 4.10, χ_B be a fuzzy left quasi-interior ideal of the Γ-semiring M. Thus

$$\chi_B = \chi_M \circ \chi_B \circ \chi_M \circ \chi_B = \chi_{M\Gamma B\Gamma M\Gamma B}.$$

and

$$B = M\Gamma B\Gamma M\Gamma B.$$

So by Theorem 3.21, M is a regular Γ-semiring. □

Theorem 4.18. *Let M be a regular Γ-semiring. If fuzzy left quasi-interior ideal γ and fuzzy ideal μ of Γ-semiring M, then $\mu \cap \gamma \subseteq \gamma \circ \mu \circ \gamma \cap \mu \circ \gamma \circ \mu$.*

Proof. Let M be a regular Γ-semiring and $x \in M$. Then there exist $y \in M, \alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x$. Thus

$$\begin{aligned}&(\mu \circ \gamma \circ \mu \circ \gamma)(x) \\ &= \sup_{x = x\alpha y\beta x} \{\min\{(\mu \circ \gamma)(x\alpha y), (\mu \circ \gamma)(x)\}\} \\ &= \min\{\sup_{x\alpha y = x\alpha y\beta x\alpha y} \{\min\{\mu(x), \gamma(y\beta x\alpha y)\}, \sup_{x\alpha y = x\alpha y\beta x\alpha y} \{\min\{\mu(x), \gamma(y\beta x\alpha y)\}\}\} \\ &\geq \min\{\min\{\mu(x), \gamma(x)\}, min\{\mu(x), \gamma(x)\}\} \\ &= \min\{\mu(x), \gamma(x)\} = (\mu \cap \gamma)(x).\end{aligned}$$

So $\mu \cap \gamma \subseteq \mu \circ \gamma \circ \mu \circ \gamma$. □

Corollary 4.19. *Let M be a regular Γ-semiring. If fuzzy right quasi-interior ideal γ and fuzzy ideal μ of a Γ-semiring M, then $\mu \cap \gamma \subseteq \gamma \circ \mu \circ \gamma \cap \mu \circ \gamma \circ \mu$.*

5. Conclusion

In this paper, we introduced the notion of right (left) quasi-interior ideal of a Γ-semiring and fuzzy right (left) quasi-interior ideal of a Γ-semiring and characterized the regular Γ-semiring in terms of fuzzy right(left)quasi-interior ideals of a Γ-semiring and studied some of their properties. In continuation of this paper, we propose to introduce the notion of quasi-interior ideal of an ordered semiring and study the characterization of ordered semiring in terms of fuzzy quasi-interior ideal of ordered semirings.

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