Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 18, No. 1, (August 2019) pp. 15–30
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2019.18.1.15

@FMI

Inverse int-fuzzy soft bi-ideals over semigroups

Peerapong Suebsan

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 18, No. 1, August 2019

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 18, No. 1, (August 2019) pp. 15–30
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2019.18.1.15

@FMI

Inverse int-fuzzy soft bi-ideals over semigroups

Peerapong Suebsan

Received 12 February 2019; Revised 20 March 2019; Accepted 19 April 2019

Abstract. In this paper, we define inverse int-soft bi-ideals, inverse int-fuzzy soft bi-ideals over a semigroup and give some of their properties. Also, we introduce inverse int-prime fuzzy soft bi-ideals, inverse int-strongly prime fuzzy soft bi-ideals and inverse int-semiprime fuzzy soft bi-ideals over the semigroup. Moreover, we show that the images of inverse int-fuzzy soft bi-ideals over semigroups are the inverse int-fuzzy soft bi-ideals over semigroups under some conditions.

2010 AMS Classification: 20M15, 06D72

Keywords: Inverse int-soft bi-ideals, Inverse int-fuzzy soft bi-ideals, Inverse int-prime fuzzy soft bi-ideals, Inverse int-strongly prime fuzzy soft bi-ideals, Inverse int-semiprime fuzzy soft bi-ideals.

Corresponding Author: Peerapong Suebsan (`peerapong.su@up.ac.th`)

1. Introduction

Solving real world problems in many fields such as engineering, economics, computer science, environment, medical science involve data that contain uncertainties. Uncertainties may be dealt with using a wide range of existing theories such as fuzzy sets [25], rough sets [18], interval mathematics [7], intuitionistic fuzzy sets [2] and other mathematic tools. Accordingly, in 1999, Molodtsov [14] initiated a new mathematical tool which is soft set theory for modeling uncertainty. He pointed out directions of soft sets for the applications like soft analysis and game theory. Later, Maji et al. [11] defined soft subset and soft super set, equality of two soft sets. They presented soft binary operation, such as AND, OR, intersection, union and studied De morgan's Laws of soft set. In 2001, Maji et al. [13] extended the soft sets to fuzzy soft sets. They initiated the concept of fuzzy soft sets and defined fuzzy soft subsets, the intersection and union of fuzzy soft sets over common universe. In 2011, Neog and Sut [17] studied the intersection and union of fuzzy soft sets and presented some properties such as commutative property, associative property, idempotent property, absorption property, distributive property. After that the soft set theory and the fuzzy soft set theory have been developed by many researchers [1, 4, 5, 12].

The fuzzy soft sets are developed to fuzzy soft semigroups by Yang [24] (2011). He defined fuzzy soft [left, right] ideals over semigroups and fuzzy soft semigroups, and studied sufficient and necessary conditions for α-level set, intersection and union of fuzzy soft [left, right] ideals. Next, Siripitukdet and Suebsan [20] defined prime, semiprime and strongly prime fuzzy soft bi-ideals over semigroups and presented their properties. Since any fuzzy soft left [right] ideals over semigroups are fuzzy soft bi-ideals over semigroups but the converse of this property does not hold in general. Next, Suebsan and Siripitukdet [23] studied fuzzy soft bi-ideals over semigroups and their properties of them. Moreover, they proved that the image of fuzzy soft bi-ideals over semigroups are the fuzzy soft bi-ideals over semigroups under certain conditions. In 2014, Song et al. [21] introduced int-soft semigroups and int-soft left (right) ideals of semigroups. They proved that the soft intersection of int-soft left (right) ideals (int-soft semigroups) is also int-soft left (right) ideals (int-soft semigroups).

The soft sets and fuzzy soft sets are developed to inverse soft sets and inverse fuzzy soft sets over parameters by Çetkin et al. [6] (2016). They defined inverse soft set and inverse fuzzy soft sets and inverse fuzzy soft sets over parameters, and discussed the application in decision making problems of them. In 2019, Khalil and Hassan [9] studied an inverse fuzzy soft set and investigated their properties, characteristics, and operations. They constructed an algorithm using max-min and min-max decision of inverse fuzzy soft set for a fuzzy decision making problem. The concept of the inverse int-fuzzy soft set over semigroups are introduced by Suebsan. [22]. He defined inverse int-soft sets, int-fuzzy soft sets, inverse int-fuzzy soft sets, inverse int-soft subsemigroups, inverse int-fuzzy soft subsemigroups and inverse int-soft (left, right) ideals, inverse int-fuzzy soft (left, right) ideals over a semigroup and gave some of their properties. Moreover, we proved that the images of inverse int-fuzzy soft (left, right) ideals over semigroups are the inverse int-fuzzy soft (left, right) ideals over semigroups under some conditions.

In the structure of fuzzy semigroups, for any fuzzy left [right] ideal on semigroup is a fuzzy bi-ideal on semigroup but the converse of this property does not hold in general. Moreover, in the the structure of fuzzy soft semigroups for any fuzzy soft left [right] ideal on semigroup is a fuzzy soft bi-ideal on semigroup but the converse of this property does not hold in general. Motivated and inspired by the works above, we are interested in the inverse int-fuzzy soft bi-ideals over semigroups.

In this paper, we define inverse int-soft bi-ideals, inverse int-fuzzy soft bi-ideals over a semigroup and investigate some of their properties. Also, we introduce inverse int-prime fuzzy soft bi-ideals, inverse int-strongly prime fuzzy soft bi-ideals and inverse int-semiprime fuzzy soft bi-ideals over the semigroup. Moreover, we show that the images of inverse int-fuzzy soft bi-ideals over semigroups are the inverse int-fuzzy soft bi-ideals over semigroups under some conditions.

2. Preliminaries

In this section, we give some definitions and theorems which are use in this paper.

Let S be a semigroup. A semigroup S is said to be regular, if for every $x \in S$, there exists $a \in S$ such that $xax = x$. A non-empty subset T of a semigroup S is called a subsemigroup, if $T^2 \subseteq T$. A subsemigroup T of a semigroup S is called a left [right] ideal of S, if $ST \subseteq T[TS \subseteq T]$. A subsemigroup T on a semigroup S is called an ideal on S, if it is both a left and a right ideal on S. A subsemigroup B of S is called a bi-ideal of S, if $BSB \subseteq B$ [8].

The concept of fuzzy semigroups was introduced by Kuroki [10].

For any $a, b \in [0, 1]$, define $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Then $a \wedge b$ and $a \vee b$ are element in $[0, 1]$. A function f from a semigroup S to the unit interval $[0, 1]$ is called a fuzzy set on S. A fuzzy set f on S is called a fuzzy subsemigroup on S, if $f(xy) \geq f(x) \wedge f(y)$, for all $x, y \in S$.

A fuzzy set f on a semigroup S is called a fuzzy left [right] ideal on S, if $f(xy) \geq f(y)[f(xy) \geq f(x)]$, for all $x, y \in S$. A fuzzy set f on a semigroup S is called a fuzzy ideal on S, if it is both a fuzzy left and a fuzzy right ideal on S. We note that any fuzzy left [right] ideal on S is a fuzzy subsemigroup on S. A fuzzy subsemigroup f on a semigroup S is called a fuzzy bi-ideal on S, if $f(xyz) \geq f(x) \wedge f(z)$, for all $x, y, z \in S$. We note that any fuzzy left [right] ideal on S is a fuzzy bi-ideal on S. The converse of this property does not hold in general.

If f and g are fuzzy sets on a semigroup S, then $f \leq g$, $f \vee g$ and $f \wedge g$ (some authors [10, 15] use notations $f \subseteq g$, $f \cup g$ and $f \cap g$, respectively), respectively, are defined as follows:

(i) $f \leq g$, if $f(x) \leq g(x)$ for all $x \in S$,
(ii) $(f \vee g)(x) = f(x) \vee g(x)$ for all $x \in S$,
(iii) $(f \wedge g)(x) = f(x) \wedge g(x)$ for all $x \in S$,
(iv) $(f \circ g)(x) = \begin{cases} \bigvee_{yz=x}[f(y) \wedge g(z)] & \text{if } yz = x \\ 0 & \text{otherwise,} \end{cases}$

where $\bigvee\limits_{x=yz} [f(y) \wedge g(z)] = \sup\{f(y) \wedge g(z) | x = yz\}$.

Lemma 2.1 ([15])**.** *Let f and g be two fuzzy bi-ideal on S. Then $f \cap g$ is a fuzzy bi-ideal on S.*

Lemma 2.2 ([15])**.** *If f is any fuzzy set on a semigroup S and g is any fuzzy bi-ideal on S, then the products $f \circ g$ and $g \circ f$ are both fuzzy bi-ideals on S.*

The concept of prime fuzzy bi-ideals of semigroups are studied by Shabir et al. [19].

A fuzzy bi-ideal f on a semigroup S is called a prime fuzzy bi-ideal on S, if for any fuzzy bi-ideals g, h on S , $g \circ h \leq f$ implies $g \leq f$ or $h \leq f$. A fuzzy bi-ideal f on a semigroup S is called a strongly prime fuzzy bi-ideal, if for any fuzzy bi-ideals g and h on S, $g \circ h \wedge h \circ g \leq f$ implies either $g \leq f$ or $h \leq f$. A fuzzy bi-ideal g on S is said to be idempotent, if $g = g^2 = g \circ g$. A fuzzy bi-ideal f on a semigroup S is called a semiprime fuzzy bi-ideal, if for every fuzzy bi-ideal g on S, $g \circ g = g^2 \leq f$ implies that $g \leq f$.

The first concept of soft set over common universe are introduced by Molodtsov [14].

Definition 2.3 ([14])**.** Let U be a common universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U and $\varnothing \neq A \subseteq E$. A pair $(\widetilde{F}, A)$ is called a soft set over U, where $\widetilde{F}$ is a mapping $\widetilde{F} : A \to P(U)$.

Definition 2.4 ([11])**.** Let E be a set of parameters and let $\varnothing \neq A \subseteq E$. A pair (F, A) is called a fuzzy soft set over U, where F is a mapping given by $F : A \to \mathrm{Fuz}(U)$ and $\mathrm{Fuz}(U)$ is the set of all fuzzy sets on U.

The concept of the inverse fuzzy soft set over an initial universe set are introduced by Khalil et al. [9].

Definition 2.5 ([6])**.** Let U be an initial universe set and let E be a set of parameters. A pair $(\widehat{F}, U)$ is called an inverse soft set over E, where $\widehat{F}$ is a mapping given by $\widehat{F} : U \to P(E)$ and $P(E)$ is the power set of E.

Definition 2.6 ([9])**.** Let E be a set of parameters. A pair $(\mathcal{F}, U)$ is called an inverse fuzzy soft set over E, where $\mathcal{F}$ is a mapping given by $\mathcal{F} : U \to \mathrm{Fuz}(E)$ is a mapping and $\mathrm{Fuz}(E)$ is the set of all fuzzy parameter set on E.

Next, we give some definition of soft sets and fuzzy soft sets over semigroups.

Definition 2.7 ([24])**.** Let E be a set of parameters and let $\varnothing \neq A \subseteq E$. A pair (F, A) is called a fuzzy soft set over a semigroup S, where $F : A \to \mathrm{Fuz}(S)$ is a mapping and $\mathrm{Fuz}(S)$ is the set of all fuzzy sets on S.

Let (F, A) be a fuzzy soft set over a semigroup S. For $p \in A, F(p) \in \mathrm{Fuz}(S)$. Set $F_p := F(p)$. Then $F_p \in \mathrm{Fuz}(S)$.

Definition 2.8 ([24])**.** A fuzzy soft set (F, A) over a semigroup S is called a fuzzy soft subsemigroup, if F_p is a fuzzy subsemigroup on S for each $p \in A$.

Definition 2.9 ([24])**.** A fuzzy soft set (F, A) over a semigroup S is called a fuzzy soft left [right] ideal, if F_p is a fuzzy left [right] ideal on S for each $p \in A$.

Definition 2.10 ([24])**.** A fuzzy soft set (F, A) over a semigroup S is called a fuzzy soft ideal, if (F, A) is both a fuzzy soft left and a fuzzy soft right ideal on S.

Definition 2.11 ([16])**.** A fuzzy soft set (F, A) over a semigroup S is said to be a fuzzy soft bi-ideal over S, if F_p is a fuzzy bi-ideal on S for all $p \in A$.

Definition 2.12 ([3])**.** Let (F, A) be a fuzzy soft set over a semigroup S. For each $\alpha \in [0, 1]$, the set $(F, A)^\alpha$ is called an α-level set of (F, A), where

$$(F_p)^\alpha = \{x \in S \mid F_p(x) \geq \alpha\}, \text{ for all } p \in A.$$

Song et al. [21] provided new definitions and various results on int-soft set theory. In what follows, we take $E = S$, as a set of parameters, which is a semigroup unless otherwise specified.

Definition 2.13 ([21])**.** A soft set $(\widehat{f}, S)$ over U is called an int-soft semigroup over U, if it satisfies:

$$\widehat{f}(xy) \supseteq \widehat{f}(x) \cap \widehat{f}(y) \text{ for all } x, y \in S.$$

Definition 2.14 ([21])**.** A soft set $(\widehat{f}, S)$ over U is called an int-soft left [right] ideal over U, if it satisfies:

$$\widehat{f}(xy) \supseteq \widehat{f}(y)[\widehat{f}(xy) \supseteq \widehat{f}(x)] \text{ for all } x, y \in S.$$

If a soft set $(\widehat{f}, S)$ over U is both an int-soft left ideal and an int-soft right ideal over U, we say that $(\widehat{f}, S)$ is an int-soft two-sided ideal over U.

The concept of the inverse int-fuzzy soft set over semigroups are introduced by Suebsan. [22].

Definition 2.15 ([22])**.** Let U be an initial universe and let S be a semigroup. A pair $(\widehat{\mathbf{F}}, S)$ is called an int-fuzzy soft set over U, where $\widehat{\mathbf{F}}$ is a mapping given by $\widehat{\mathbf{F}} : S \to \mathrm{Fuz}(U)$ is a mapping and $\mathrm{Fuz}(U)$ is the set of all fuzzy sets on U.

Definition 2.16 ([22])**.** Let U be an initial universe and let S be a semigroup. A pair $(\widetilde{\mathcal{F}}, U)$ is called an inverse int-fuzzy soft set over S, where $\widetilde{\mathcal{F}}$ is a mapping given by $\widetilde{\mathcal{F}} : U \to \mathrm{Fuz}(S)$ is a mapping and $\mathrm{Fuz}(S)$ is the set of all fuzzy set on S.

Example 2.17. Let $U = \{u_1, u_2, u_3\}$ and $S = \{x, y, z\}$ be a semigroup having the following multiplication table:

$\cdot$	x	y	z
x	x	y	z
y	y	y	z
z	z	z	z

Table 1. Multiplication table of a semigroup S

We choose the inverse int-fuzzy soft set over S by Definition 2.16 as follows:

$$\begin{aligned}\widetilde{\mathcal{F}}_{u_1} &= \{x/0.1, y/0.4, z/0.8\},\\ \widetilde{\mathcal{F}}_{u_2} &= \{x/0.2, y/0.5, z/0.9\},\\ \widetilde{\mathcal{F}}_{u_3} &= \{x/0.3, y/0.6, z/0.9\}.\end{aligned}$$

Then we can view the inverse int-fuzzy soft set $(\widetilde{\mathcal{F}}, U)$ as a collection of approximations as follow: $(\widetilde{\mathcal{F}}, U) = \{\widetilde{\mathcal{F}}_{u_1} = \{x/0.1, y/0.4, z/0.8\}, \widetilde{\mathcal{F}}_{u_2} = \{x/0.2, y/0.5, z/0.9\}, \widetilde{\mathcal{F}}_{u_3} = \{x/0.3, y/0.6, z/0.9\}\}$.

Definition 2.18 ([22])**.** Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft sets over a semigroup S. $(\widetilde{\mathcal{F}}, U)$ is called an inverse int-fuzzy soft subset of $(\widetilde{\mathcal{G}}, V)$, denoted by $(\widetilde{\mathcal{F}}, U)\widetilde{\leq}(\widetilde{\mathcal{G}}, V)$, if

(i) $U \subseteq V$ and (ii) for all $u \in U$ there exists $v \in V$ such that $\widetilde{\mathcal{F}}_u \leq \widetilde{\mathcal{G}}_v$.

Definition 2.19 ([22])**.** Two inverse int-fuzzy soft sets $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ over a semigroup S is called an inverse int-fuzzy soft equal, if $(\widetilde{\mathcal{F}}, U)\widetilde{\leq}(\widetilde{\mathcal{G}}, V)$ and $(\widetilde{\mathcal{G}}, V)\widetilde{\leq}(\widetilde{\mathcal{F}}, U)$. We denote by $(\widetilde{\mathcal{F}}, U) = (\widetilde{\mathcal{G}}, V)$.

Definition 2.20 ([22])**.** Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft sets over a semigroup S with $U \cap V \neq \varnothing$. The intersection of them is denoted by:

$$(\widetilde{\mathcal{F}}, U)\widetilde{\bigwedge}(\widetilde{\mathcal{G}}, V) := (\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}}, U \cap V),$$

where $(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u \wedge \widetilde{\mathcal{G}}_u$, for each $u \in U \cap V$.
Then $(\widetilde{\mathcal{F}}, U)\widetilde{\bigwedge}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft set over S.

Definition 2.21 ([22])**.** Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft sets over a semigroup S. The union of them is denoted by:

$$(\widetilde{\mathcal{F}}, U)\tilde{\bigvee}(\widetilde{\mathcal{G}}, V) := (\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}}, U \cup V),$$

where $\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}}$ is defined as follows: for all $u \in U \cup V$,

$$(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u = \begin{cases} \widetilde{\mathcal{F}}_u & \text{if} \quad u \in U - V, \\ \widetilde{\mathcal{G}}_u & \text{if} \quad u \in V - U, \\ \widetilde{\mathcal{F}}_u \vee \widetilde{\mathcal{G}}_u & \text{if} \quad u \in U \cap V. \end{cases}$$

Then $(\widetilde{\mathcal{F}}, U)\tilde{\bigvee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft set over S.

Definition 2.22 ([22])**.** Let U be an initial universe set and let S be a semigroup. A collection of all inverse int-fuzzy soft sets over S is called an inverse int-fuzzy soft class and is denoted by $\widetilde{FS}(S, U)$.

That is, $\widetilde{FS}(S, U) = \{(\widetilde{\mathcal{F}}, U) | \widetilde{\mathcal{F}} : U \to \mathrm{Fuz}(S)\}$.

For any sets A and B, define $M(U, V) := \{f | f : U \to V\}$.
Let $\widetilde{FS}(S, U)$ and $\widetilde{FS}(T, V)$ be inverse int-fuzzy soft classes over a semigroup S and a semigroup T with the initial universes from U and V, respectively. That is,
$\widetilde{FS}(S, U) = \{(\widetilde{\mathcal{F}}, U) | \widetilde{\mathcal{F}} : U \to \mathrm{Fuz}(S)\}$,
$\widetilde{FS}(T, V) = \{(\widetilde{\mathcal{G}}, V) | \widetilde{\mathcal{G}} : V \to \mathrm{Fuz}(T)\}$.
Let $u : S \to T$ and $v : U \to V$ be two mappings.

We want to construct two functions

$$\Psi : \widetilde{FS}(S, U) \to \widetilde{FS}(T, V)$$

and

$$\Sigma : \widetilde{FS}(T, V) \to \widetilde{FS}(S, U).$$

(i) Let $(\widetilde{\mathcal{F}}, U) \in \widetilde{FS}(S, U)$ and let $\Omega : M(U, \mathrm{Fuz}(S)) \to M(v(U), \mathrm{Fuz}(T))$ be defined as follows: for each $y \in T$ and each $\beta \in v(U)$,

$$[\Omega(\widetilde{\mathcal{F}})]_\beta(y) = \begin{cases} \bigvee\limits_{x \in u^{-1}(y)} [\bigvee\limits_{\delta \in v^{-1}(\beta) \cap U} \widetilde{\mathcal{F}}_\delta(x)] & \text{if } u^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Psi(\widetilde{\mathcal{F}}, U) := (\Omega(\widetilde{\mathcal{F}}), v(U))$. Then $(\Omega(\widetilde{\mathcal{F}}), v(U))$ is an image of $(\widetilde{\mathcal{F}}, U)$ under the function Ψ.

(ii) Let $(\widetilde{\mathcal{G}}, V) \in \widetilde{FS}(V, T)$ and let $\Gamma : M(V, \mathrm{Fuz}(T)) \to M(v^{-1}(S), \mathrm{Fuz}(S))$ be defined as follows: for each $x \in S$ and each $\delta \in v^{-1}(V)$,

$$[\Gamma(\widetilde{\mathcal{G}})]_\delta(x) = \widetilde{\mathcal{G}}_{v(\delta)}(u(x)).$$

Let $\Sigma(\widetilde{\mathcal{G}}, V) := (\Gamma(\widetilde{\mathcal{G}}), v^{-1}(V))$.

Remark 2.23 ([22])**.** Let $\widetilde{\mathcal{FS}}(S, U)$ and $\widetilde{FS}(T, V)$ be two inverse int-fuzzy soft classes over semigroups S and T, respectively. Suppose that $u : S \to T$ and $v : U \to V$ are two mappings and $\Psi : \widetilde{FS}(S, U) \to \widetilde{FS}(T, V)$ and $\Sigma : \widetilde{FS}(T, V) \to \widetilde{FS}(S, U)$ are also two mappings. If u and v are bijections, then $\Sigma \circ \Psi = I$ and $\Psi \circ \Sigma = I$, where $\circ$ is a composite mapping and I is an identity mapping.(Hence $\Psi$and Σ are bijections.)

3. The inverse int-fuzzy soft bi-ideals over semigroups

In this section, we define inverse int-soft bi-ideals, inverse int-fuzzy soft bi-ideals over semigroups and investigate their properties and support examples. Moreover, we construct the image of inverse int-fuzzy soft bi-ideals over semigroups.

Definition 3.1. The product of two inverse int-fuzzy soft sets $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ over a semigroup S is denoted by:

$$(\widetilde{\mathcal{F}}, U)\widetilde{\odot}(\widetilde{\mathcal{G}}, V) := (\widetilde{\mathcal{F}} \circ \widetilde{\mathcal{G}}, U \cup V).$$

Then $(\widetilde{\mathcal{F}}, U)\widetilde{\odot}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft set over S, where for all $u \in U \cup V$,

$$(\widetilde{\mathcal{F}} \circ \widetilde{\mathcal{G}})_u = \begin{cases} \widetilde{\mathcal{F}}_u, & if \quad u \in U - V \\ \widetilde{\mathcal{G}}_u, & if \quad u \in V - U \\ \widetilde{\mathcal{F}}_u \circ \widetilde{\mathcal{G}}_u, & if \quad u \in U \cap V. \end{cases}$$

Definition 3.2. An inverse int-soft set $(\widehat{\mathcal{F}}, U)$ over a semigroup S is said to be an inverse int-soft bi-ideal over S, if $\widehat{\mathcal{F}}(u)$ is a bi-ideal of S, for all $u \in U$.

Definition 3.3. Let $(\widetilde{\mathcal{F}}, U)$ be an inverse int-fuzzy soft set over a semigroup S. $(\widetilde{\mathcal{F}}, U)$ is called an inverse int-fuzzy soft bi-ideal over S, if $\widetilde{\mathcal{F}}(u)$ is a fuzzy bi-ideal on S, for all $u \in U$.

Example 3.4. Let $U = \{u_1, u_2, u_3\}$ and $S = \{x, y, z\}$ be a semigroup having the same multiplication table as Example 2.17 (Table 1). Choose $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft set over S as:

$$\widetilde{\mathcal{F}}_{u_1} = \{x/0.1, y/0.4, z/0.8\},$$
$$\widetilde{\mathcal{F}}_{u_2} = \{x/0.2, y/0.5, z/0.9\},$$
$$\widetilde{\mathcal{F}}_{u_3} = \{x/0.3, y/0.6, z/0.9\}.$$

Then it is easy to verify that $\widetilde{\mathcal{F}}_{u_1}, \widetilde{\mathcal{F}}_{u_2}, \widetilde{\mathcal{F}}_{u_3}$ are both fuzzy bi-ideals on S. Thus $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft bi-ideal over S.

In the following theorem, we show that an inverse int-fuzzy soft set over a semigroup is an inverse int-fuzzy soft bi-ideal over a semigroup if and only if an α-level set of inverse int-fuzzy soft set over the semigroup is an inverse int-soft left bi-ideal over the semigroup for all $\alpha \in [0, 1]$.

Theorem 3.5. *Let $(\widetilde{\mathcal{F}}, U)$ be an inverse int-fuzzy soft set over a semigroup S. Then $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft bi-ideal over S if and only if $(\widetilde{\mathcal{F}}, U)^\alpha$ is an inverse int-soft bi-ideal over S, for all $\alpha \in [0, 1]$.*

Proof. Suppose $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft bi-ideal over S. Let $\alpha \in [0, 1]$, $u \in U, y, z \in (\widetilde{\mathcal{F}}_u)^\alpha$ and $x \in S$. Then $\widetilde{\mathcal{F}}_u(y) \geq \alpha$ and $\widetilde{\mathcal{F}}_u(z) \geq \alpha$. By the hypothesis, $\widetilde{\mathcal{F}}_u$ is a fuzzy bi-ideal on S. Thus we have

$$\widetilde{\mathcal{F}}_u(y \cdot z) \geq \widetilde{\mathcal{F}}_u(y) \wedge \widetilde{\mathcal{F}}_u(z) \geq \alpha.$$

So $y \cdot z \in (\widetilde{\mathcal{F}}_u)^\alpha$. Hence $(\widetilde{\mathcal{F}}_u)^\alpha$ is a subsemigroup of S.

Also, we have

$$\widetilde{\mathcal{F}}_u(y \cdot x \cdot z) \geq \widetilde{\mathcal{F}}_u(y) \wedge \widetilde{\mathcal{F}}_u(z) \geq \alpha.$$

Then $y \cdot x \cdot z \in (\widetilde{\mathcal{F}}_u)^\alpha$. Thus $(\widetilde{\mathcal{F}}_u)^\alpha$ is a bi-ideal of S. So $(\widetilde{\mathcal{F}}, U)^\alpha$ is an inverse int-soft bi-ideal over S.

Conversely, suppose $(\widetilde{\mathcal{F}}, U)^\alpha$ is an inverse int-soft bi-ideal over S, for each $\alpha \in [0,1]$. Let $u \in U$ and $y, z \in S$. Choose $\alpha := \widetilde{\mathcal{F}}_u(y) \wedge \widetilde{\mathcal{F}}_u(z)$. Then $y, z \in (\widetilde{\mathcal{F}}_u)^\alpha$. Since $(\widetilde{\mathcal{F}}_u)^\alpha$ is a subsemigroup of S, we have $y \cdot z \in (\widetilde{\mathcal{F}}_u)^\alpha$. This implies that

$$\widetilde{\mathcal{F}}_u(y \cdot z) \geq \alpha = \widetilde{\mathcal{F}}_u(y) \wedge \widetilde{\mathcal{F}}_u(z).$$

Thus $\widetilde{\mathcal{F}}_u$ is a fuzzy subsemigroup on S. By the assumption, $y \cdot x \cdot z \in (\widetilde{\mathcal{F}}_u)^\alpha$ for all $x \in S$. Thus

$$\widetilde{\mathcal{F}}_u(y \cdot x \cdot z) \geq \alpha = \widetilde{\mathcal{F}}_u(y) \wedge \widetilde{\mathcal{F}}_u(z).$$

So $\widetilde{\mathcal{F}}_u$ is a fuzzy bi-ideal on S. Hence $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft bi-ideal over S. □

The following theorem shows that $(\widetilde{\mathcal{F}}, U)\tilde{\bigwedge}(\widetilde{\mathcal{G}}, V)$ with $U \cap V \neq \varnothing$ is an inverse int-fuzzy soft bi-ideal over a semigroup.

Theorem 3.6. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy bi-ideals over a semigroup S. If $U \cap V \neq \varnothing$, then $(\widetilde{\mathcal{F}}, U)\tilde{\bigwedge}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft bi-ideal over S.*

Proof. Let $u \in U \cap V \neq \varnothing$. Then $(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u \wedge \widetilde{\mathcal{G}}_u$. Let $x, y, z \in S$. Since $\widetilde{\mathcal{F}}_u$ and $\widetilde{\mathcal{G}}_u$ are both fuzzy subsemigroups on S, we have

$$\begin{aligned}(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u(x \cdot y) &= \widetilde{\mathcal{F}}_u(x \cdot y) \wedge \widetilde{\mathcal{G}}_u(x \cdot y)\\ &\geq \{\widetilde{\mathcal{F}}_u(x) \wedge \widetilde{\mathcal{F}}_u(y)\} \wedge \{\widetilde{\mathcal{G}}_p(x) \wedge \widetilde{\mathcal{G}}_u(y)\}\\ &= \{(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u(x) \wedge (\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{F}})_u(y)\}.\end{aligned}$$

Thus $(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_p$ is a fuzzy subsemigroup on S.

Also, since $\widetilde{\mathcal{F}}_u$ and $\widetilde{\mathcal{G}}_u$ are both fuzzy bi-ideals on S, we have

$$\begin{aligned}(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u(x \cdot y \cdot z) &= \widetilde{\mathcal{F}}_u(x \cdot y \cdot z) \wedge \widetilde{\mathcal{G}}_p(x \cdot y \cdot z)\\ &\geq \{\widetilde{\mathcal{F}}_u(x) \wedge \widetilde{\mathcal{F}}_u(z)\} \wedge \{\widetilde{\mathcal{G}}_u(x) \wedge \widetilde{\mathcal{G}}_u(z)\}\\ &= \{(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u(x) \wedge (\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u(z)\}.\end{aligned}$$

So $(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u$ is a fuzzy bi-ideal on S. Hence $(\widetilde{\mathcal{F}}, U)\tilde{\bigwedge}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft bi-ideal over S. □

The next theorem shows that $(\widetilde{\mathcal{F}}, U)\tilde{\bigvee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft bi-ideals over a semigroup under some conditions.

Theorem 3.7. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy bi-ideals over a semigroup S. If $U \subseteq V$ or $V \subseteq U$, then $(\widetilde{\mathcal{F}}, U)\tilde{\bigvee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft bi-ideal over S.*

Proof. It suffices to show the theorem for the case $U \subseteq V$. Let $u \in U \cup V$. Since $U \subseteq V$, we have $U \cup V = V$.

If $u \in V - U$, then $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{G}}_u$.

If $u \in U \cap V = U$, then $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u$.
Thus $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u$ is a fuzzy bi-ideal on S. So $(\widetilde{\mathcal{F}}, U)\tilde{\bigvee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft bi-ideal over S. □

Theorem 3.8. *Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft bi-ideals over a semigroup S. If $U \cap V = \varnothing$, then $(\widetilde{\mathcal{F}}, U)\tilde{\bigvee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft bi-ideal over S.*

Proof. Suppose $U \cap V = \varnothing$. Then $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u$ or $\widetilde{\mathcal{G}}_u$. Since $\widetilde{\mathcal{F}}_u$ and $\widetilde{\mathcal{G}}_u$ are both fuzzy bi-ideals on S, $(\widetilde{\mathcal{F}} \vee \widetilde{\mathcal{G}})_u$ is a fuzzy bi-ideal on S. Then $(\widetilde{\mathcal{F}}, U)\tilde{\bigvee}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft bi-ideal over S. □

The next theorem shows that the product of two inverse int-fuzzy soft bi-ideals over a semigroup is an inverse int-fuzzy soft bi-ideal over the semigroup.

Theorem 3.9. *If $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ are two inverse int-fuzzy soft bi-ideals over a semigroup S, then $(\widetilde{\mathcal{F}}, U)\widetilde{\odot}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft bi-ideal over S.*

Proof. Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be two inverse int-fuzzy soft bi-ideals over S and $u \in U \cup V$. Then

$$(\widetilde{\mathcal{F}} \circ \widetilde{\mathcal{G}})_u = \begin{cases} \widetilde{\mathcal{F}}_u & \text{if} \quad p \in U - V \\ \widetilde{\mathcal{G}}_u & \text{if} \quad p \in V - U \\ \widetilde{\mathcal{F}}_u \circ \widetilde{\mathcal{G}}_u & \text{if} \quad u \in U \cap V. \end{cases}$$

Then by the hypothesis and by Lemma 2.2, $(\widetilde{\mathcal{F}} \circ \widetilde{\mathcal{G}})_u$ is a fuzzy bi-ideal on S. Thus $(\widetilde{\mathcal{F}}, U)\widetilde{\odot}(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft bi-ideal over S. □

The next theorem shows that a semigroup is regular if and only if $(\widetilde{\mathcal{F}}, U)\tilde{\bigwedge}(\widetilde{\mathcal{G}}, V)\tilde{\leq}(\widetilde{\mathcal{F}}, U)\widetilde{\odot}(\widetilde{\mathcal{G}}, V)$ for every inverse int-fuzzy soft bi-ideal $(\widetilde{\mathcal{F}}, U)$ over S and for every inverse int-fuzzy soft left [right] ideal $(\widetilde{\mathcal{G}}, V)$ over S with $U \cap V \neq \varnothing$.

Theorem 3.10. *LetS be a semigroup. S is regular if and only if $(\widetilde{\mathcal{F}}, U)\tilde{\bigwedge}(\widetilde{\mathcal{G}}, V)\tilde{\leq}(\widetilde{\mathcal{F}}, U)\widetilde{\odot}(\widetilde{\mathcal{G}}, V)$ for every inverse int-fuzzy soft bi-ideal $(\widetilde{\mathcal{F}}, U)$ over S and for every inverse int-fuzzy soft left [right] ideal $(\widetilde{\mathcal{G}}, V)$ over S with $U \cap V \neq \varnothing$.*

Proof. We prove only inverse int-fuzzy soft left ideal [inverse int-fuzzy soft right ideal is similar].

Suppose S is regular. Let $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$ be any inverse int-fuzzy soft bi-ideal and any inverse int-fuzzy soft left ideal over S, respectively. Let $u \in U \cap V$. Then $U \cap V \subseteq U \cup V$. By Theorem 3.1.11 (3) [15],

$$(\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u \wedge \widetilde{\mathcal{G}}_u \leq \widetilde{\mathcal{F}}_u \circ \widetilde{\mathcal{G}}_u = (\widetilde{\mathcal{F}} \circ \widetilde{\mathcal{G}})_u.$$

Thus $(\widetilde{\mathcal{F}}, U)\tilde{\bigwedge}(\widetilde{\mathcal{G}}, V)\tilde{\leq}(\widetilde{\mathcal{F}}, U)\widetilde{\odot}(\widetilde{\mathcal{G}}, V)$.

On the other hand, let f and g be fuzzy bi-ideal and fuzzy left ideal on S, respectively. Let U be a non-empty set. Define $\widetilde{\mathcal{F}}_u = f$ and $\widetilde{\mathcal{G}}_u = g$, for all $u \in U$. Then (F, A) and (G, B) are fuzzy soft bi-ideal and fuzzy soft left ideal over S, respectively. By the hypothesis,

$$(\widetilde{\mathcal{F}}, U)\tilde{\bigwedge}(\widetilde{\mathcal{G}}, V)\tilde{\leq}(\widetilde{\mathcal{F}}, U)\widetilde{\odot}(\widetilde{\mathcal{G}}, V).$$

Fix $u \in U$. Thus $\widetilde{\mathcal{F}}_u \wedge \widetilde{\mathcal{G}}_u = (\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{G}})_u \leq (\widetilde{\mathcal{F}} \circ \widetilde{\mathcal{G}})_u = \widetilde{\mathcal{F}}_u \circ \widetilde{\mathcal{G}}_u$.
So $f \wedge g \leq f \circ g$. Hence by Theorem 3.1.11 (3) [15], S is regular. □

Definition 3.11. Let $(\widetilde{\mathcal{F}}, U)$ be an inverse int-fuzzy soft set over a semigroup S. $(\widetilde{\mathcal{F}}, U)$ is called an inverse int-prime fuzzy soft bi-ideal over S, if $\widetilde{\mathcal{F}}(u)$ is a prime fuzzy bi-ideal on S, for all $u \in U$.

Theorem 3.12. *Let $(\widetilde{\mathcal{F}}, U)$, $(\widetilde{\mathcal{G}}, V)$ and $(\widetilde{\mathcal{H}}, W)$ be inverse int-fuzzy soft bi-ideals over a semigroup S. Suppose that $(\widetilde{\mathcal{F}}, U)$ is an inverse int-prime fuzzy soft bi-ideal with $(\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{H}}, V \cup W) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$. Then the following statement holds:*
(1) *if $V \cap W = \varnothing$, then $(\widetilde{\mathcal{G}}, V) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{H}}, W) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$,*
(2) *if $|V \cap W| = 1$, then $(\widetilde{\mathcal{G}}, V) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$ or $(\widetilde{\mathcal{H}}, W) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$.*

Proof. Suppose $(\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{H}}, V \cup W) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$. Then $V \cup W \subseteq U$ and $(\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{H}})_p \leq \widetilde{\mathcal{F}}_p$, for all $p \in V \cup W$. Since $V \cup W \subseteq U$, we have $V \subseteq U$ and $W \subseteq U$.

(1) Suppose $V \cap W = \varnothing$. Let $p \in V$. Then $\widetilde{\mathcal{G}}_p = (\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{H}})_p \leq \widetilde{\mathcal{F}}_p$. Thus $(\widetilde{\mathcal{G}}, V) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$.

Similarly, let $q \in W$. Then $\widetilde{\mathcal{H}}_q = (\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{H}})_q \leq \widetilde{\mathcal{F}}_q$. Thus $(\widetilde{\mathcal{H}}, W) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$.

(2) Suppose $|V \cap W| = 1$. Assume that $(\widetilde{\mathcal{G}}, V) \widetilde{\nleq} (\widetilde{\mathcal{F}}, U)$. Then there exists $q \in V$ such that $\widetilde{\mathcal{G}}_q \nleq \widetilde{\mathcal{F}}_q$. If $q \notin W$, then $\widetilde{\mathcal{G}}_q = (\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{H}})_q \leq \widetilde{\mathcal{F}}_q$. Thus $q \in V \cap W$. Let $r \in W$. We must show that $\widetilde{\mathcal{H}}_r \leq \widetilde{\mathcal{F}}_r$.

Case 1: Suppose $r \in W - V$. Then $\widetilde{\mathcal{H}}_r = (\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{G}})_r \leq \widetilde{\mathcal{F}}_r$.

Case 2: Suppose $r \in V \cap W$. Then $r = q$. Thus $\widetilde{\mathcal{G}}_r \circ \widetilde{\mathcal{H}}_r = (\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{H}})_r \leq \widetilde{\mathcal{F}}_r$. Since $(\widetilde{\mathcal{F}}, U)$ is a prime fuzzy soft bi-ideal over a semigroup S, we have $\widetilde{\mathcal{G}}_r \leq \widetilde{\mathcal{F}}_r$ or $\widetilde{\mathcal{H}}_r \leq \widetilde{\mathcal{F}}_r$. But $\widetilde{\mathcal{G}}_r \nleq \widetilde{\mathcal{F}}_r$. So $\widetilde{\mathcal{H}}_r \leq \widetilde{\mathcal{F}}_r$. Hence $(\widetilde{\mathcal{H}}, W) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$. □

Definition 3.13. Let $(\widetilde{\mathcal{F}}, U)$ be an inverse int-fuzzy soft set over a semigroup S. $(\widetilde{\mathcal{F}}, U)$ is called an inverse int-strongly prime fuzzy soft bi-ideal over S if $\widetilde{\mathcal{F}}(u)$ is a strongly prime fuzzy bi-ideal on S, for all $u \in U$.

Theorem 3.14. *Let $(\widetilde{\mathcal{F}}, U)$, $(\widetilde{\mathcal{G}}, V)$ and $(\widetilde{\mathcal{H}}, W)$ be inverse int-fuzzy soft bi-ideals over a semigroup S. Suppose that $(\widetilde{\mathcal{F}}, U)$ is an inverse int-strongly prime fuzzy soft bi-ideal with $|V \cap W| = 1$ and $(\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{H}}, V \cup W) \widetilde{\bigwedge} (\widetilde{\mathcal{H}} \circ \widetilde{\mathcal{G}}, W \cup V) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$. Then $(\widetilde{\mathcal{G}}, V) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$ or $(\widetilde{\mathcal{H}}, W) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$.*

Proof. By the hypothesis, we have $(V \cup W) \subseteq U$ and $(\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{H}})_p \wedge (\widetilde{\mathcal{H}} \circ \widetilde{\mathcal{G}})_p \leq \widetilde{\mathcal{F}}_p$, for all $p \in V \cup W$. Since $V \cup W \subseteq U$, we have $V \subseteq U$ and $W \subseteq U$.

Suppose $(\widetilde{\mathcal{G}}, V) \widetilde{\nleq} (\widetilde{\mathcal{F}}, U)$. Then there exists $q \in V$ such that $\widetilde{\mathcal{G}}_q \nleq \widetilde{\mathcal{F}}_q$. If $q \notin W$, then $\widetilde{\mathcal{G}}_q = (\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{H}})_q \wedge (\widetilde{\mathcal{H}} \circ \widetilde{\mathcal{G}})_q \leq \widetilde{\mathcal{F}}_q$. Thus $q \in V \cap W$. Let $r \in W$. We must show that $\widetilde{\mathcal{H}}_r \leq \widetilde{\mathcal{F}}_r$.

Case 1: Suppose $r \in W - V$. Then $\widetilde{\mathcal{H}}_r = (\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{H}})_r \wedge (\widetilde{\mathcal{H}} \circ \widetilde{\mathcal{G}})_r \leq \widetilde{\mathcal{F}}_r$.

Case 2 Suppose $r \in V \cap W$. Then $r = q$. Thus $\widetilde{\mathcal{G}}_r \circ \widetilde{\mathcal{H}}_r = (\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{H}})_r \wedge (\widetilde{\mathcal{H}} \circ \widetilde{\mathcal{G}})_r \leq \widetilde{\mathcal{F}}_r$. Since $(\widetilde{\mathcal{F}}, U)$ is an inverse int-strongly prime fuzzy soft bi-ideal over a semigroup S, we have $\widetilde{\mathcal{G}}_r \leq \widetilde{\mathcal{F}}_r$ or $\widetilde{\mathcal{H}}_r \leq \widetilde{\mathcal{F}}_r$. But $\widetilde{\mathcal{G}}_r \nleq \widetilde{\mathcal{F}}_r$. So $\widetilde{\mathcal{H}}_r \leq \widetilde{\mathcal{F}}_r$.
Hence $(\widetilde{\mathcal{H}}, W) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$. □

Definition 3.15. An inverse int-fuzzy soft subsemigroup $(\widetilde{\mathcal{G}}, V)$ over a semigroup S is said to be an inverse int-fuzzy soft idempotent, if $\widetilde{\mathcal{G}}_p$ is idempotent, for all $p \in V$.

Note that $(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft idempotent if and only if

$$(\widetilde{\mathcal{G}}, V) = (\widetilde{\mathcal{G}}, V)^2 = (\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{G}}, V).$$

Definition 3.16. An inverse int-fuzzy soft set $(\widetilde{\mathcal{F}}, U)$ over a semigroup S is said to be an inverse int-semiprime fuzzy soft bi-ideal over S, if $\widetilde{\mathcal{F}}_p$ is a semiprime fuzzy bi-ideal on S for all $p \in U$.

Theorem 3.17. *Let $(\widetilde{\mathcal{F}}, U)$ be an inverse int-fuzzy soft bi-ideal over a semigroup S. Then $(\widetilde{\mathcal{F}}, U)$ is an inverse int-semiprime fuzzy soft bi-ideal over S if and only if for every inverse int-fuzzy soft bi-ideal $(\widetilde{\mathcal{G}}, V)$ over S, $(\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{G}}, V) = (\widetilde{\mathcal{G}}, V)^2 \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$ implies that $(\widetilde{\mathcal{G}}, V) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$.*

Proof. Let $(\widetilde{\mathcal{G}}, V)$ be an inverse int-fuzzy soft bi-ideal over S. Let $p \in V$. Suppose $(\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{G}}, V) = (\widetilde{\mathcal{G}}, V)^2 \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$. Then we have $V \subseteq U$ and $(\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{G}})_p \leq \widetilde{\mathcal{F}}_p$. Thus $\widetilde{\mathcal{G}}_p = (\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{G}})_p \leq \widetilde{\mathcal{F}}_p$. So $(\widetilde{\mathcal{G}}, V) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$.

Conversely, let $p \in U$ and $(\widetilde{\mathcal{G}}_p)^2 \leq \widetilde{\mathcal{F}}_p$. By the hypothesis, $(\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{G}}, U) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$. Thus $(\widetilde{\mathcal{G}}, U) \widetilde{\leq} (\widetilde{\mathcal{F}}, U)$. So $\widetilde{\mathcal{G}}_p \leq \widetilde{\mathcal{F}}_p$. Hence $(\widetilde{\mathcal{F}}, U)$ is an inverse int-semiprime fuzzy soft bi-ideal over S. □

The following theorem shows that the image of an inverse int-fuzzy soft bi-ideal over a semigroup is an inverse int-fuzzy soft bi-ideal over the semigroup under some conditions.

Theorem 3.18. *Let $\widetilde{FS}(S, U)$ and $\widetilde{FS}(T, V)$ be two inverse int-fuzzy soft classes over semigroups S and T, respectively. Suppose that $\Psi : \widetilde{FS}(S, U) \to \widetilde{FS}(T, V)$ is a mapping, $v : U \to V$ is an injection and u is an isomorphism from S to T. If $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft bi-ideal over S, then $\Psi(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft bi-ideal over T.*

Proof. Suppose $(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft bi-ideal over S. By Theorem 3.5, $(\widetilde{\mathcal{F}}, U)^\alpha$ is an inverse int-soft bi-ideal over S, for all $\alpha \in [0, 1]$. We shall show that $\Psi(\widetilde{\mathcal{F}}, U)^\alpha = (\Omega(\widetilde{\mathcal{F}}), v(U))^\alpha$ is an inverse int-soft bi-ideal over T for all $\alpha \in [0, 1]$.

Let $p \in v(U)$. There exists $q \in U$ such that $p = v(q)$. Let $z_1, z_2 \in ([\Omega(\widetilde{\mathcal{F}})]_p)^\alpha$. Then

$$\alpha \leq [\Omega(\widetilde{\mathcal{F}})]_p(z_1) = \bigvee_{s \in u^{-1}(z_1)} [\bigvee_{\beta \in v^{-1}(p) \cap U} \widetilde{\mathcal{F}}_\beta(s)] = \bigvee_{s \in u^{-1}(z_1)} \widetilde{\mathcal{F}}_q(s),$$

and

$$\alpha \leq [\Omega(\widetilde{\mathcal{F}})]_p(z_2) = \bigvee_{s \in u^{-1}(z_2)} [\bigvee_{\beta \in v^{-1}(p) \cap U} \widetilde{\mathcal{F}}_\beta(s)] = \bigvee_{s \in u^{-1}(z_2)} \widetilde{\mathcal{F}}_q(s).$$

This implies that there exist $s_1, s_2 \in S$ such that

$$u(s_1) = z_1, u(s_2) = z_2 \text{ and } \widetilde{\mathcal{F}}_q(s_1) \geq \alpha, \widetilde{\mathcal{F}}_q(s_2) \geq \alpha.$$

Thus

$$\begin{aligned}[\Omega(\widetilde{\mathcal{F}})]_p(z_1 \cdot z_2) &= \bigvee_{s\in u^{-1}(z_1\cdot z_2)} [\bigvee_{\beta\in v^{-1}(p)\cap U} \widetilde{\mathcal{F}}_\beta(s)] \\ &= \bigvee_{s\in u^{-1}(z_1\cdot z_2)} \widetilde{\mathcal{F}}_q(s) \\ &\geq \widetilde{\mathcal{F}}_q(s_1 \cdot s_2) \\ &\geq \widetilde{\mathcal{F}}_q(s_1) \wedge \widetilde{\mathcal{F}}_q(s_2) \geq \alpha.\end{aligned}$$

So $z_1 \cdot z_2 \in ([\Omega(\widetilde{\mathcal{F}})]_p)^\alpha$.

Let $y \in T$. Since u is surjective, there exists $x \in S$ such that $u(x) = y$. Since $\widetilde{\mathcal{F}}_q$ is an inverse int-fuzzy bi-ideal on S, we have

$$\begin{aligned}[\Omega(\widetilde{\mathcal{F}})]_p(z_1 \cdot y \cdot z_2) &= \bigvee_{s\in u^{-1}(z_1\cdot y\cdot z_1)} [\bigvee_{\delta\in v^{-1}(p)\cap U} \widetilde{\mathcal{F}}_\delta(s)] \\ &= \bigvee_{s\in u^{-1}(z_1\cdot y\cdot z_1)} \widetilde{\mathcal{F}}_q(s) \\ &= \widetilde{\mathcal{F}}_q(s_1 \cdot x \cdot s_1) \\ &\geq \widetilde{\mathcal{F}}_q(s_1) \wedge \widetilde{\mathcal{F}}_q(s_1) \geq \alpha,\end{aligned}$$

which implies that $z_1 \cdot y \cdot z_2 \in ([\Omega(\widetilde{\mathcal{F}})]_p)^\alpha$. Hence $([\Omega(\widetilde{\mathcal{F}})]_p)^\alpha$ is a bi-ideal of T. Therefore $\Psi(\widetilde{\mathcal{F}}, U)^\alpha = (\Omega(\widetilde{\mathcal{F}}), v(U))^\alpha$ is an inverse int-soft bi-ideal over T for all $\alpha \in [0, 1]$. By Theorem 3.5, $\Psi(\widetilde{\mathcal{F}}, U)$ is an inverse int-fuzzy soft bi-ideal over T. □

The following example shows that if u is not an isomorphism from S to T, then the Theorem 3.18 is not true.

Example 3.19. Let $S = \{c_1, c_2, c_3, c_4\}$ and $T = \{d_1, d_2, d_3\}$ be two semigroups having the multiplication tables, respectively,

$\cdot$	c_1	c_2	c_3	c_4
c_1	c_1	c_2	c_3	c_4
c_2	c_2	c_2	c_3	c_4
c_3	c_3	c_3	c_3	c_4
c_4	c_4	c_4	c_4	c_4

Table 2. Multiplication table of a semigroup S

$\bullet$	d_1	d_2	d_3
d_1	d_1	d_2	d_3
d_2	d_2	d_2	d_3
d_3	d_3	d_3	d_3

TABLE 3. Multiplication table of a semigroup T

Let $U = \{u_1, u_2, u_3\}$, $V = \{v_1, v_2, v_3\}$, and let $\widetilde{FS}(S,U)$ and $\widetilde{FS}(T,V)$ be two inverse int-fuzzy soft classes over semigroups S and T, respectively.
Let $\Psi : \widetilde{FS}(S,U) \to \widetilde{FS}(T,V),\ u : S \to T$ and $v : U \to V$ be mappings defined by:

$$u(c_1) = d_1,\ u(c_2) = d_2,\ u(c_3) = d_3,\ u(c_4) = d_1,\ v(u_1) = v_1,\ v(u_2) = v_2,\ v(u_3) = v_3.$$

Then it is easy to check that u is not an isomorphism.

Let $(\widetilde{\mathcal{F}}, U)$ be an inverse int-fuzzy soft bi-ideal over S

$$\begin{aligned}
\widetilde{\mathcal{F}}_{u_1} &= \{c_1/0.2, c_2/0.4, c_3/0.6, c_4/0.8\},\\
\widetilde{\mathcal{F}}_{u_2} &= \{c_1/0.3, c_2/0.5, c_3/0.7, c_4/0.9\},\\
\widetilde{\mathcal{F}}_{u_3} &= \{c_1/0.4, c_2/0.6, c_3/0.8, c_4/0.9\}.
\end{aligned}$$

Then $[\Omega(F)]_{v_1} = \{d_1/0.8, d_2/0.4, d_3/0.6\}$, which implies that $\Psi(\widetilde{\mathcal{F}}, U) = (\Omega(\widetilde{\mathcal{F}}), v(U))$. Since $[\Omega(\widetilde{\mathcal{F}})]_{v_1}(d_1 \bullet d_2 \bullet d_1) = [\Omega(\widetilde{\mathcal{F}})]_{v_1}(d_2) = 0.4 < 0.8 = [\Omega(\widetilde{\mathcal{F}})]_{v_1}(d_1) \wedge [\Omega(\widetilde{\mathcal{F}})]_{v_1}(d_1)$, it follows that $[\Omega(F)]_{v_1}$ is not a fuzzy bi-ideal on T. Thus $\Psi(F, A)$ is not an inverse int-fuzzy soft bi-ideal over T.

The following theorem shows that $\Sigma(G, B)$ of an inverse int-fuzzy soft bi-ideal over a semigroup is an inverse int-fuzzy soft bi-ideal over the semigroup under some conditions.

Theorem 3.20. *Let $\widetilde{FS}(S,U)$ and $\widetilde{FS}(T,V)$ be two an inverse int-fuzzy soft classes over semigroups S and T, respectively. Suppose that $\Sigma : \widetilde{FS}(T,V) \to \widetilde{FS}(S,U)$ is a mapping, $v : U \to V$ is a mapping and u is a homomorphism from S to T. If $(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft bi-ideal over T then $\Sigma(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft bi-ideal over S.*

Proof. Suppose $(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft bi-ideal over T. Let $\delta \in v^{-1}(V)$. Then we have $v(\delta) \in V$. This implies that $\widetilde{\mathcal{G}}_{v(\delta)}$ is a fuzzy bi-ideal on T. Let $x, y, z \in S$. Then

$$\begin{aligned}
[\Gamma(\widetilde{\mathcal{G}})]_\delta(x \cdot y) &= \widetilde{\mathcal{G}}_{v(\delta)}(u(x \cdot y))\\
&= \widetilde{\mathcal{G}}_{v(\delta)}(u(x) \cdot u(y))\\
&\geq \widetilde{\mathcal{G}}_{v(\delta)}(u(x)) \wedge \widetilde{\mathcal{G}}_{v(\delta)}(u(y))\\
&= [\Gamma(\widetilde{\mathcal{G}})]_\delta(x) \wedge [\Gamma(\widetilde{\mathcal{G}})]_\delta(y).
\end{aligned}$$

This implies that $[\Gamma(\widetilde{\mathcal{G}})]_\delta$ is a fuzzy subsemigroup on S.

Also, we have

$$\begin{aligned}
[\Gamma(\widetilde{\mathcal{G}})]_\delta(x \cdot y \cdot z) &= \widetilde{\mathcal{G}}_{v(\delta)}(u(x \cdot y \cdot z))\\
&= \widetilde{\mathcal{G}}_{v(\delta)}(u(x) \cdot u(y) \cdot u(z))\\
&\geq \widetilde{\mathcal{G}}_{v(\delta)}(u(x)) \wedge \widetilde{\mathcal{G}}_{v(\delta)}(u(z))\\
&= [\Gamma(\widetilde{\mathcal{G}})]_\delta(x) \wedge [\Gamma(\widetilde{\mathcal{G}})]_\delta(z).
\end{aligned}$$

Thus $[\Gamma(\widetilde{\mathcal{G}})]_\delta$ is a fuzzy bi-ideal on S. So $\Sigma(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft bi-ideal over S. □

The following example satisfies the conditions of Theorem 3.20.

Example 3.21. Let $S = \{c_1, c_2, c_3\}$, $T = \{d_1, d_2, d_3\}$ be two semigroups having the multiplication tables, respectively,

$\cdot$	c_1	c_2	c_3
c_1	c_1	c_2	c_3
c_2	c_2	c_2	c_3
c_3	c_3	c_3	c_3

TABLE 4. Multiplication table of a semigroup S

$\bullet$	d_1	d_2	d_3
d_1	d_1	d_2	d_3
d_2	d_2	d_2	d_3
d_3	d_3	d_3	d_3

TABLE 5. Multiplication table of a semigroup T

Let $U = \{u_1, u_2, u_3\}$, $V = \{v_1, v_2, v_3, v_4\}$ and let $\widetilde{FS}(S, U)$ and $\widetilde{FS}(T, U)$ be two inverse int-fuzzy soft classes over semigroups S and T, respectively. Suppose that $\Sigma : \widetilde{FS}(T, V) \to \widetilde{FS}(S, U)$, $u : S \to T$ and $v : U \to V$ are mappings defined by:

$$u(c_1) = d_2,\ u(c_2) = d_2,\ u(c_3) = d_3,\ v(u_1) = v_1,\ v(u_2) = v_1,\ v(u_3) = v_3.$$

Then it is easy to check that u is a homomorphism from S to T.

Let $(\widetilde{\mathcal{G}}, V)$ be an inverse int-fuzzy soft bi-ideal over T such that

$$\begin{aligned}\widetilde{\mathcal{G}}_{v_1} &= \{d_1/0.4, d_2/0.5, d_3/0.6\},\\ \widetilde{\mathcal{G}}_{v_2} &= \{d_1/0.6, d_2/0.7, d_3/0.8\},\\ \widetilde{\mathcal{G}}_{v_3} &= \{d_1/0.4, d_2/0.5, d_3/0.6\}.\end{aligned}$$

Then $[\Gamma(G)]_{u_1} = \{c_1/0.5, c_2/0.5, c_3/0.6\}, [\Gamma(G)]_{u_2} = \{c_1/0.7, c_2/0.7, c_3/0.8\}$, $[\Gamma(G)]_{u_3} = \{c_1/0.5, c_2/0.5, c_3/0.6\}$, which implies that $\Sigma(\widetilde{\mathcal{G}}, V) = (\Gamma(\widetilde{\mathcal{G}}), v^{-1}(V))$. Thus it is easy to verify that $[\Gamma(\widetilde{\mathcal{G}})]_{u_1}, [\Gamma(\widetilde{\mathcal{G}})]_{u_2}$ and $[\Gamma(\widetilde{\mathcal{G}})]_{u_3}$ are both fuzzy bi-ideals on S. So $\Sigma(\widetilde{\mathcal{G}}, V)$ is an inverse int-fuzzy soft bi-ideal over S.

The following corollary shows that $(\Sigma \circ \Psi)(\widetilde{\mathcal{F}}, U) = (\widetilde{\mathcal{F}}, U)$ and $(\Psi \circ \Sigma)(\widetilde{\mathcal{G}}, V) = (\widetilde{\mathcal{G}}, V)$ for every inverse inverse int-fuzzy soft bi-ideals $(\widetilde{\mathcal{F}}, U)$ and $(\widetilde{\mathcal{G}}, V)$, where $\circ$ is a composite mapping, under certain conditions.

Corollary 3.22. *Let $\widetilde{FS}(S,U)$ and $\widetilde{FS}(T,V)$ be two inverse int-fuzzy soft classes over semigroups S and T, respectively. Suppose that $u : S \to T$ and $v : U \to V$ are two mappings and $\Psi : \widetilde{FS}(S,U) \to FS(T,V)$ and $\Sigma : \widetilde{FS}(T,V) \to \widetilde{FS}(S,U)$ are also two mappings. If u and v are bijections, then $(\Sigma \circ \Psi)(\widetilde{\mathcal{F}},U) = (\widetilde{\mathcal{F}},U)$ and $(\Psi \circ \Sigma)(\widetilde{\mathcal{G}},V) = (\widetilde{\mathcal{G}},V)$ for every inverse int-fuzzy soft bi-ideals $(\widetilde{\mathcal{F}},U)$ and $(\widetilde{\mathcal{G}},V)$, where $\circ$ is a composite mapping.*

Proof. It follows from Remark 2.23. $\square$

4. Conclusions

We defined inverse int-soft bi-ideals, inverse int-fuzzy soft bi-ideals over a semigroup and give some of their properties. Also, we introduced inverse int-prime fuzzy soft bi-ideals, inverse int-strongly prime fuzzy soft bi-ideals and inverse int-semiprime fuzzy soft bi-ideals over the semigroup. Moreover, we proved that the images of inverse int-fuzzy soft bi-ideals over semigroups are the inverse int-fuzzy soft bi-ideals over semigroups under some conditions.

Acknowledgements. The author is highly grateful to the reviewers for their valuable suggestions helpful in improving this paper. This research work of the author was supported by University of Phayao.

References

[1] H. Aktaş and N. Çağman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2726–2735.
[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and Systems 20 (1986) 87–96.
[3] A. Aygünoğlu and H. Aygün, Introduction to fuzzy soft groups, Comput. Math. Appl. 58 (2009) 1279–1286.
[4] N. Çağman, F. Çitak and S. Enginoğlu, FP-soft set theory and its applications, Ann. Fuzzy Math. Inform. 2 (2) (2011) 219–226.
[5] N. Çağman, S. Enginoğlu and F. Çitak, Fuzzy soft set theory and its applications, Iran. J. Fuzzy Syst. 8 (3) (2011) 137–147.
[6] V. Çetkin, A. Aygünoğlu and H. Aygün, A new approach in handling soft decision making problems, J. Nonlinear Sci. Appl. 9 (2016) 231–239.
[7] M. B. Gorzalzany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, Fuzzy sets and Systems 21 (1987) 1–17.
[8] J. M. Howie, An Introduction to Semigroup Theory. Acad. Press, London 1976.
[9] A. M. Khalil and N. Hassan, Inverse fuzzy soft set and its application in decision making, Int. J. Inform. Decis. Sci. 11 (1) (2019) 73–92.
[10] N. Kuroki, On fuzzy ideals and fuzzy bi-ideals in semigroups, Fuzzy sets and Systems 5 (1981) 203–215.
[11] P. K. Maji, R. Biswas and A. R. Roy, Fuzzy soft set, J. Fuzzy Math. 9 (3) (2001) 589–602.
[12] P. K. Maji, R. Biswas and A. R. Roy, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002) 1077–1083.
[13] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[14] D. Molodtsov, Soft set theory-First results, Comput. Math. Appl. 37 (1999) 19–31.
[15] J. N. Mordeson, D. S. Malik and N. Kuroki, Fuzzy Semigroups. Springer-Verlag, Berlin Heidelberg 2010.
[16] M. Naz, M. Shabir and M. I. Ali, On fuzzy soft semigroups, World Appl. Sci. J. (Special Issue of Applied Math). 22 (2003) 62–83.
[17] T.J. Neog and D. K. Sut, On union and intersection of fuzzy soft set, Int. J. Comput. Tech. Appl. 2 (5) (2011) 1160–1176.
[18] Z. Pawlak, Rough sets, Int. J. Inf. Comput. Sci. 11 (1982) 341–356.

[19] M. Shabir, Y. B. Jun and M. Bano, On prime fuzzy bi-ideals of semigroups, Iran. J. Fuzzy Syst. 7 (3) (2010) 115–128.
[20] M. Siripitukdet and P. Suebsan, On fuzzy soft bi-ideals over semigroups, Songkl. J. Sci. Tech. 37 (5) (2015) 237–249.
[21] S. Z. Song, H. S. Kim and Y. B. Jun, Ideal theory in semigroups based on inter sectional soft sets, The Sci. World J. Article ID 136424 (2014) 1–7.
[22] P. Suebsan, Inverse int-fuzzy soft (left, right) ideals over semigroups, Submitted.
[23] P. Suebsan and M. Siripitukdet, The mapping on classes of fuzzy soft bi-ideals over semigroups, Azerb. J. Math. 8 (2) (2018) 43–59.
[24] C. F. Yang, Fuzzy soft semigroups and fuzzy soft ideals, Comput. Math. Appl. 44 (2011) 255–261.
[25] L. A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965) 338–353.

Peerapong Suebsan (peerapong.su@up.ac.th)
Department of Mathematics, School of Science, University of Phayao, postal code 56000, Phayao, Thailand