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Abstract. In this paper, we introduce the concept of σ-fuzzy ideals in distributive p-algebras. It is proved that the set of all σ-fuzzy ideals forms a complete distributive lattice. Moreover, the class of all σ-fuzzy ideals of a distributive p-algebra is isomorphic to the class of fuzzy ideals of the lattice of all booster ideals. Finally, we prove that the image and pre-image of σ-fuzzy ideals are also σ-fuzzy ideals.

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1. Introduction

The theory of pseudo-complemented was introduced and extensively studied in semi-lattices and particularly in distributive lattices by O. Frink [3] and G. Birkhoff [2]. In recent time A. E. Badawy and M. S. Rao [1] introduced the concept of boosters and σ-ideals in a distributive p-algebra.

On the other hand, many papers on fuzzy algebras have been published since A. Rosenfeld [8] introduced the concept of fuzzy group in 1971. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. Uncertain data in many important applications in the areas such as economics, engineering, environment, medical sciences and business management could be caused by data randomness, information incompleteness, limitations of measuring instrument, delayed data updates etc. W. J. Liu [5] initiated the study of fuzzy subrings, and fuzzy ideals of a ring. B. Yuan and W. Wu [10] introduced the notion of fuzzy ideals and fuzzy congruences of distributive lattices. U. M. Swamy and D. V. Raju [9] studied properties of fuzzy ideals and congruences of lattices.

In this paper, we introduce the concept of σ-fuzzy ideal in a distributive p-algebra. It is proved that the set of all σ-fuzzy ideals forms a complete distributive lattice.

Moreover, the class of all σ-fuzzy ideals of a distributive p-algebra is isomorphic to the class of fuzzy ideals of the lattice of all booster ideals. Finally, we prove that the image and pre-image of σ-fuzzy ideals are also σ-fuzzy ideals.

2. Preliminaries

We refer to G. Birkhoff [2] for the elementary properties of lattices.

Definition 2.1 ([4])**.** An algebra $(L;\wedge,\vee,*,0,1)$ is a distributive p-algebra (or a pseudo-complemented lattice), if $(L;\wedge,\vee,0,1)$ is a distributive lattice and $*$ is the unary operation of pseudocomplementation.

Definition 2.2 ([4])**.** For any element a of a distributive lattice L with 0, the pseudo-complement a^* of a is an element satisfying the following property, for any $x \in L$,

$$a \wedge x = 0 \Leftrightarrow a^* \wedge x = x \Leftrightarrow x \leq a^*.$$

Remark 2.3. The pseudo-complement a^* of an element a is the greatest element disjoint from a, if such an element exists.

A distributive lattice L in which every element has a pseudo-complement is called a pseudo-complemented lattice.

Theorem 2.4. [4]

For any two elements a, b of a pseudo-complemented lattice, we have the following:

(1) $0^{**} = 0$ *and* $1^{**} = 1$,
(2) $a \wedge a^* = 0$,
(3) $a \leq b \Rightarrow b^* \leq a^*$
(4) $a \leq a^{**}$
(5) $a^{***} = a^*$
(6) $(a \vee b)^* = a^* \wedge b^*$
(7) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$
(8) $(a \vee b)^{**} = (a^* \wedge b^*)^* = (a^{**} \vee b^{**})^{**}$.

An element x of a p-algebra L is called closed, if $x^{**} = x$ and the set of all closed elements of L is denoted by $B(L)$.

Definition 2.5 ([1])**.** Let L be a distributive p-algebra. Then for every $a \in L$, define the booster of a as follows:

$$(a)^{\triangle} = \{x \in L : x \wedge a^* = 0\}.$$

$(a)^{\triangle}$ is an ideal of L containing a. It is obvious that that $(0)^{\triangle} = \{0\}$ and $(1)^{\triangle} = L$. For any $a \in L$, $(a)^{\triangle} = (a^{**})^{\triangle} = (a^{**}]$. The set of all boosters of a distributive p-algebra of L is denoted by $B_*(L)$.

$$B_*(L) = \{(x)^{\triangle} : x \in L\} = \{(x^{**})^{\triangle} : x \in L\}.$$

In [1], Badawy and Rao observed the following. In a distributive p-algebra L with least element 0 the set of all ideals of the form $(x)^{\triangle}$ can be made into a Boolean algebra $(B_*(L), \cap, \underline{\vee})$. For two boosters $(x)^{\triangle}$ and $(y)^{\triangle}$ their supremum in $B_*(L)$ is $(x)^{\triangle}\underline{\vee}(y)^{\triangle} = (x \vee y)^{\triangle}$ and also their infimum in $B_*(L)$ is $(x)^{\triangle} \cap (y)^{\triangle} = (x \wedge y)^{\triangle}$.

For an ideal I in L,

$$\sigma(I) = \{(x)^{\triangle} : x \in I\}$$

is an ideal in $B_*(L)$ and the set

$$\overleftarrow{\sigma}(J) = \{x \in L : (x)^{\triangle} \in J\}$$

is an ideal of L, when J is any ideal in $B_*(L)$. An ideal I of L is called a σ-ideal if $\overleftarrow{\sigma}\sigma(I) = I$.

Remember that, for any set A, a function $\mu : A \to ([0,1], \wedge, \vee)$ is called a fuzzy subset of A, where $[0,1]$ is a unit interval, $\alpha \wedge \beta = min(\alpha, \beta)$ and $\alpha \vee \beta = max(\alpha, \beta)$ for all $\alpha, \beta \in [0,1]$.

Definition 2.6 ([8])**.** Let μ and θ be fuzzy subsets of a set A. Define the fuzzy subsets $\mu \cup \theta$ and $\mu \cap \theta$ of A as follows: for each $x \in A$,

$$(\mu \cup \theta)(x) = \mu(x) \vee \theta(x) \text{ and } (\mu \cap \theta)(x) = \mu(x) \wedge \theta(x).$$

Then $\mu \cup \theta$ and $\mu \cap \theta$ are called the union and intersection of μ and θ, respectively.

For any collection, $\{\mu_i : i \in I\}$ of fuzzy subsets of X, where I is a nonempty index set, the least upper bound $\bigcup_{i \in I} \mu_i$ and the greatest lower bound $\bigcap_{i \in I} \mu_i$ of the μ_i's are given by for each $x \in X$,

$$(\bigcup_{i \in I} \mu_i)(x) = \bigvee_{i \in I} \mu_i(x) \text{ and } (\bigcap_{i \in I} \mu_i)(x) = \bigwedge_{i \in I} \mu_i(x),$$

respectively.

For each $t \in [0,1]$ the set

$$\mu_t = \{x \in A : \mu(x) \geq t\}$$

is called the level subset of μ at t [11].

Regarding fuzzy ideals and fuzzy filters of lattices, we refer [9].

Definition 2.7 ([9])**.** A proper fuzzy ideal μ of L is called prime fuzzy ideal of L, if for any two fuzzy ideals θ, η of L, $\theta \cap \eta \subseteq \mu \Rightarrow \theta \subseteq \mu$ or $\eta \subseteq \mu$.

Definition 2.8 ([7])**.** Let L be a lattice, $x \in L$ and $\alpha \in [0,1]$. Define a fuzzy subset α_x of L as:

$$\alpha_x(y) = \begin{cases} 1 \text{ , if } \ y \leq x \\ \alpha \text{ , if } \ y \nleq x \end{cases}$$

is a fuzzy ideal of L.

Remark 2.9 ([7])**.** α_x is called the α-level principal fuzzy ideal corresponding to x. Similarly, a fuzzy subset α^x of L defined

$$\alpha^x(y) = \begin{cases} 1 \text{ , if } \ x \leq y \\ \alpha \text{ , if } \ x \nleq y \end{cases}$$

is the α-level principal fuzzy filter corresponding to x.

Note that a fuzzy subset μ of L is nonempty, if there exists $x \in L$ such that $\mu(x) \neq 0$. The set of all fuzzy ideals and fuzzy filters of L are denoted by $FI(L)$ and $FF(L)$ respectively.

3. σ-fuzzy ideals

In this section, we study σ-fuzzy ideals of a distributive p-algebra and its properties. Throughout this section L denotes a distributive p-algebra.

Definition 3.1. For any fuzzy ideal μ of L, define a fuzzy subset $\sigma(\mu)$ of $B_*(L)$ as:

$$\sigma(\mu)((x)^{\triangle}) = sup\{\mu(y) : (y)^{\triangle} = (x)^{\triangle},\ y \in L\}.$$

Definition 3.2. For any fuzzy ideal θ of $B_*(L)$, define a fuzzy subset $\overleftarrow{\sigma}(\theta)$ of L as:

$$\overleftarrow{\sigma}(\theta)(x) = \theta((x)^{\triangle}).$$

Lemma 3.3. *For any fuzzy ideal μ of L, $\sigma(\mu)$ is a fuzzy ideal of $B_*(L)$.*

Proof. Let μ be a fuzzy ideal of L. Then clearly $\sigma(\mu)((0)^{\triangle}) = 1$. For any $(x)^{\triangle}, (y)^{\triangle} \in B_*(L)$,

$$\begin{aligned}
&\sigma(\mu)((x)^{\triangle}) \wedge \sigma(\mu)((y)^{\triangle})\\
&= sup\{\mu(a) : (a)^{\triangle} = (x)^{\triangle},\ a \in L\} \wedge sup\{\mu(b) : (b)^{\triangle} = (y)^{\triangle},\ b \in L\}\\
&= sup\{\mu(a) \wedge \mu(b) : (a)^{\triangle} = (x)^{\triangle},\ (b)^{\triangle} = (y)^{\triangle}\}\\
&\leq sup\{\mu(a \vee b) : (a \vee b)^{\triangle} = (x \vee y)^{\triangle}\}\\
&\leq sup\{\mu(c) : (c)^{\triangle} = (x \vee y)^{\triangle}\}\\
&= \sigma(\mu)((x)^{\triangle} \underline{\vee} (y)^{\triangle})
\end{aligned}$$

Again,

$$\begin{aligned}
\sigma(\mu)((x)^{\triangle}) &= sup\{\mu(a) : (a)^{\triangle} = (x)^{\triangle}\}\\
&\leq sup\{\mu(a \wedge y) : (a)^{\triangle} \wedge (y)^{\triangle} = (x)^{\triangle} \wedge (y)^{\triangle}\}\\
&\leq sup\{\mu(c) : (c)^{\triangle} = (x \wedge y)^{\triangle}\}\\
&= \sigma(\mu)((x)^{\triangle} \wedge (y)^{\triangle})
\end{aligned}$$

Similarly, $\sigma(\mu)((y)^{\triangle}) \leq \sigma(\mu)((x)^{\triangle} \wedge (y)^{\triangle})$. Thus

$$\sigma(\mu)((x)^{\triangle} \wedge (y)^{\triangle}) \geq \sigma(\mu)((x)^{\triangle}) \vee \sigma(\mu)((y)^{\triangle}).$$

So $\sigma(\mu)$ is a fuzzy ideal of $B_*(L)$. □

Lemma 3.4. *For any fuzzy ideal θ of $B_*(L)$, $\overleftarrow{\sigma}(\theta)$ is a fuzzy ideal of L.*

Proof. Let θ is a fuzzy ideal of $B_*(L)$. Since $(0)^{\triangle}$ is the least element of $B_*(L)$, $\overleftarrow{\sigma}(\theta)(0) = 1$. Then for any $x, y \in L$,

$$\overleftarrow{\sigma}(\theta)(x \vee y) = \theta((x)^{\triangle} \underline{\vee} (y)^{\triangle}) = \theta((x)^{\triangle}) \wedge \theta((y)^{\triangle}) = \overleftarrow{\sigma}(\theta)(x) \wedge \overleftarrow{\sigma}(\theta)(y).$$

Thus, $\overleftarrow{\sigma}(\theta)$ is a fuzzy ideal of L. □

Lemma 3.5. *If μ and θ are fuzzy ideals of L, then $\mu \subseteq \theta$ implies $\sigma(\mu) \subseteq \sigma(\theta)$.*

Lemma 3.6. *If μ, θ are fuzzy ideals of $B_*(L)$, then $\mu \subseteq \theta$ implies $\overleftarrow{\sigma}(\mu) \subseteq \overleftarrow{\sigma}(\theta)$.*

Lemma 3.7. *For any fuzzy ideal θ of $B_*(L)$, $\sigma\overleftarrow{\sigma}(\theta) = \theta$.*

Proof. Since θ is a fuzzy ideal of $B_*(L)$, $\overleftarrow{\sigma}(\theta)$ is a fuzzy ideal of L and $\sigma\overleftarrow{\sigma}(\theta)$ is a fuzzy ideal of $B_*(L)$. On the other hand,

$$\begin{aligned}\sigma\overleftarrow{\sigma}(\theta)((x)^{\triangle}) &= sup\{\overleftarrow{\sigma}(\theta)(a) : (a)^{\triangle} = (x)^{\triangle}\}\\ &= sup\{\theta((a)^{\triangle}) : (a)^{\triangle} = (x)^{\triangle}\}\\ &= \theta((x)^{\triangle}).\end{aligned}$$

Then the result holds. □

The proof of the following lemma is quite routine and will be omitted.

Lemma 3.8. *For any fuzzy ideals μ, θ of L, we have the following:*

(1) $\mu \subseteq \overleftarrow{\sigma}\sigma(\mu)$,
(2) $\mu \subseteq \theta \Rightarrow \overleftarrow{\sigma}\sigma(\mu) \subseteq \overleftarrow{\sigma}\sigma(\theta)$,
(3) $\overleftarrow{\sigma}\sigma(\overleftarrow{\sigma}\sigma(\mu)) = \overleftarrow{\sigma}\sigma(\mu)$.

Theorem 3.9. *The set $FI(B_*(L))$ of all fuzzy ideals of $B_*(L)$ forms a complete distributive lattice, where the infimum and supremum of any family $\{\mu_\gamma : \gamma \in I\}$ of fuzzy ideals is given by:*

$$\bigwedge \mu_\gamma = \bigcap \mu_\gamma \text{ and } \bigvee \mu_\gamma = \langle \bigcup \mu_\gamma \rangle.$$

Theorem 3.10. *The mapping σ is a homomorphism of $FI(L)$ into $FI(B_*(L))$.*

Proof. Let μ, θ be two fuzzy ideals of L. It is enough to prove that

$$\sigma(\mu \cap \theta) = \sigma(\mu) \cap \sigma(\theta) \text{ and } \sigma(\mu \vee \theta) = \sigma(\mu)\underline{\vee}\sigma(\theta).$$

Since σ is an isotone, we get $\sigma(\mu \cap \theta) \subseteq \sigma(\mu) \cap \sigma(\theta)$. Then for any $(x)^{\triangle} \in B_*(L)$,

$$\begin{aligned}\sigma(\mu)((x)^{\triangle}) \wedge \sigma(\theta)((x)^{\triangle}) &= sup\{\mu(a) : (a)^{\triangle} = (x)^{\triangle}\} \wedge sup\{\theta(b) : (b)^{\triangle} = (x)^{\triangle}\}\\ &\leq sup\{\mu(a \wedge b) : (a \wedge b)^{\triangle} = (x)^{\triangle}\}\\ &\quad \wedge sup\{\theta(a \wedge b) : (a \wedge b)^{\triangle} = (x)^{\triangle}\}\\ &= sup\{\mu(a \wedge b) \wedge \theta(a \wedge b) : (a \wedge b)^{\triangle} = (x)^{\triangle}\}\\ &= sup\{(\mu \cap \theta)(a \wedge b) : (a \wedge b)^{\triangle} = (x)^{\triangle}\}\\ &\leq sup\{(\mu \cap \theta)(c) : (c)^{\triangle} = (x)^{\triangle}\}\\ &= \sigma(\mu \cap \theta)((x)^{\triangle})\end{aligned}$$

Thus $\sigma(\mu) \cap \sigma(\theta) \subseteq \sigma(\mu \cap \theta)$. So, $\sigma(\mu) \cap \sigma(\theta) = \sigma(\mu \cap \theta)$. Again, clearly $\sigma(\mu)\underline{\vee}\sigma(\theta) \subseteq \sigma(\mu \vee \theta)$. Now for any $(x)^{\triangle} \in B_*(L)$,

$$\begin{aligned}\sigma(\mu \vee \theta)((x)^{\triangle}) &= sup\{(\mu \vee \theta)(a) : (a)^{\triangle} = (x)^{\triangle}\}\\ &= sup\{sup\{\mu(y) \wedge \theta(z) : a = y \vee z\} : (y \vee z)^{\triangle} = (x)^{\triangle}\}\\ &\leq sup\{sup\{\mu(b_1) \wedge \theta(b_2) : (b_1)^{\triangle} = (y)^{\triangle},\ (b_2)^{\triangle} = (z)^{\triangle}\} : (y \vee z)^{\triangle} = (x)^{\triangle}\}\\ &= sup\{sup\{\mu(b_1) : (b_1)^{\triangle} = (y)^{\triangle}\}\\ &\quad \wedge sup\{\theta(b_2) : (b_2)^{\triangle} = (z)^{\triangle}\},\ (y \vee z)^{\triangle} = (x)^{\triangle}\}\\ &= sup\{\sigma(\mu)(y)^{\triangle} \wedge \sigma(\theta)(z)^{\triangle} : (y \vee z)^{\triangle} = (x)^{\triangle}\}\\ &= sup\{\sigma(\mu)(y)^{\triangle} \wedge \sigma(\theta)(z)^{\triangle} : (y)^{\triangle}\underline{\vee}(z)^{\triangle} = (x)^{\triangle}\}\\ &= (\sigma(\mu)\underline{\vee}\sigma(\theta))((x)^{\triangle})\end{aligned}$$

Then $\sigma(\mu \vee \theta) \subseteq \sigma(\mu)\underline{\vee}\sigma(\theta)$. Thus $\sigma(\mu \vee \theta) = \sigma(\mu)\underline{\vee}\sigma(\theta)$. So, σ is a homomorphism. □

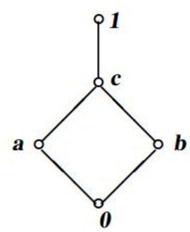
Corollary 3.11. *For any two fuzzy ideals μ and θ, we have*

$$\overleftarrow{\sigma}\sigma(\mu\cap\theta)=\overleftarrow{\sigma}\sigma(\mu)\cap\overleftarrow{\sigma}\sigma(\theta).$$

Proof. For any $x\in L$, $\overleftarrow{\sigma}\sigma(\mu\cap\theta)(x)=\sigma(\mu\cap\theta)((x)^{\triangle})$. Since $\sigma(\mu\cap\theta)=\sigma(\mu)\cap\sigma(\theta)$, we have $\overleftarrow{\sigma}\sigma(\mu\cap\theta)(x)=\overleftarrow{\sigma}\sigma(\mu)(x)\wedge\overleftarrow{\sigma}\sigma(\theta)(x)$. Then $\overleftarrow{\sigma}\sigma(\mu\cap\theta)=\overleftarrow{\sigma}\sigma(\mu)\cap\overleftarrow{\sigma}\sigma(\theta)$. □

Definition 3.12. A fuzzy ideal of μ of L is called σ-fuzzy ideal, if $\mu=\overleftarrow{\sigma}\sigma(\mu)$.

Example 3.13. Consider the distributive lattice $L=\{0,a,b,c,1\}$ whose Hasse diagram is given below.



Define a fuzzy subset μ of L as follows: $\mu(0)=1,\ \mu(a)=0.5$ and $\mu(b)=\mu(c)=\mu(1)=0.4$. Then it can be easily verified that μ is a σ-fuzzy ideal of L.

Theorem 3.14. *For a nonempty fuzzy subset μ of L , μ is a σ-fuzzy ideal if and only if each level subset of μ is a σ-ideal of L.*

Proof. Let μ be a σ-fuzzy ideal of L. Then $\mu_t=(\overleftarrow{\sigma}\sigma(\mu))_t$. To prove each level subset of μ is a σ-ideal of L, it is enough to show $\overleftarrow{\sigma}\sigma(\mu_t)=(\overleftarrow{\sigma}\sigma(\mu))_t$. Since $\mu_t\subseteq\overleftarrow{\sigma}\sigma(\mu_t)$, $(\overleftarrow{\sigma}\sigma(\mu))_t\subseteq\overleftarrow{\sigma}\sigma(\mu_t)$. Let $x\in\overleftarrow{\sigma}\sigma(\mu_t)$. Then $(x)^{\triangle}\in\sigma(\mu_t)$ and there is $y\in\mu_t$ such that $(x)^{\triangle}=(y)^{\triangle}$. Which implies $sup\{\mu(a):(x)^{\triangle}=(y)^{\triangle}\}\geq t$. This shows that $x\in(\overleftarrow{\sigma}\sigma(\mu))_t$. Thus $\mu_t=\overleftarrow{\sigma}\sigma(\mu_t)$. So each level subset of μ is a σ-ideal of L.

Conversely, assume that each level subset of μ is a σ-ideal. Then clearly, $\mu\subseteq\overleftarrow{\sigma}\sigma(\mu)$. Let $t=\overleftarrow{\sigma}\sigma(\mu)(x)=sup\{\mu(y):(y)^{\triangle}=(x)^{\triangle}\}$. Then for each $\epsilon>0$, there is $a\in L$, $(a)^{\triangle}=(x)^{\triangle}$ such that $\mu(a)>t-\epsilon$. This implies that $x\in\beta\alpha(\mu_{t-\epsilon})=\mu_{t-\epsilon}$. Thus $x\in\bigcap_{\epsilon>0}\mu_{t-\epsilon}=\mu_t$. So $\overleftarrow{\sigma}\sigma(\mu)\subseteq\mu$. Hence, μ is a σ-fuzzy ideal of L. □

Lemma 3.15. *For a nonempty subset I of L, I is a σ-fuzzy ideal if and only if χ_I is a σ-fuzzy ideal of L.*

Proof. Take a σ-ideal I of L. Then $\overleftarrow{\sigma}\sigma(I)=\{x\in L:(x)^{\triangle}\in\sigma(I)\}=I$. Let $x\in L$. If $x\in I$, then $\overleftarrow{\sigma}\sigma(\chi_I)(x)=1=\chi_I(x)$. Let $x\notin I$. Assume that $\overleftarrow{\sigma}\sigma(\chi_I)(x)=1$. Then there is $y\in I$ such that $(y)^{\triangle}=(x)^{\triangle}\in\sigma(I)$. Since I is a σ-ideal, $x\in I$. Which is contradiction. Thus, $\overleftarrow{\sigma}\sigma(\chi_I)(x)=0$. So, χ_I is a σ-fuzzy ideal.

Conversely, assume that χ_I is a σ-fuzzy ideal of L. Then clearly, $I\subseteq\overleftarrow{\sigma}\sigma(I)$. Let $x\in\overleftarrow{\sigma}\sigma(I)$. Since χ_I is a σ-fuzzy ideal, $\overleftarrow{\sigma}\sigma(\chi_I)(x)=1=\chi_I(x)$. Thus $x\in I$. So I is a σ-ideal of L. □

Theorem 3.16. *For a fuzzy ideal μ of L. μ is a σ-fuzzy ideal if and only if $\mu(x)=\mu(y)$ for each $x,y\in L$ such that $(x)^{\triangle}=(y)^{\triangle}$.*

Let us denote the set of all σ-fuzzy ideals of L by $FI_{\sigma}(L)$.

Theorem 3.17. *The class $FI_\sigma(L)$ of all σ-fuzzy ideals of L forms a complete distributive lattice with respect to set inclusion.*

Proof. Clearly $(FI_\sigma(L), \subseteq)$ is a partially ordered set. For $\mu, \theta \in FI_\sigma(L)$, define

$$\mu \wedge \theta = \mu \cap \theta \text{ and } \mu \underline{\vee} \theta = \overleftarrow{\sigma}\sigma(\mu \vee \theta).$$

Then clearly $\mu \wedge \theta,\ \mu\underline{\vee}\theta \in FI_\sigma(L)$. We need to show $\mu\underline{\vee}\theta$ is the least upper bound of $\{\mu,\ \theta\}$. Since $\theta,\ \mu \subseteq \mu \vee \theta \subseteq \mu\underline{\vee}\theta$, $\mu\underline{\vee}\theta$ is an upper bound of $\{\mu,\ \theta\}$. Let η be any upper bound for $\{\mu,\ \theta\}$ in $FI_\sigma(L)$. Thus $\mu \vee \theta \subseteq \eta$. Which implies that $\overleftarrow{\sigma}\sigma(\mu \vee \theta) \subseteq \overleftarrow{\sigma}\sigma(\eta) = \eta$. So, $\overleftarrow{\sigma}\sigma(\mu \vee \theta)$ is the supremum of $\{\mu,\ \theta\}$ in $FI_\sigma(L)$. Hence $(FI_\sigma(L), \wedge, \underline{\vee})$ is a lattice.

Now we prove the distributivity. Let $\mu,\ \theta \in FI_\sigma(L)$. Then

$$\begin{aligned}
\mu\underline{\vee}(\theta \cap \eta) &= \overleftarrow{\sigma}\sigma((\mu \vee \theta) \cap (\mu \vee \eta)) \\
&= \overleftarrow{\sigma}\sigma(\mu \vee \theta) \cap \overleftarrow{\sigma}\sigma(\mu \vee \eta) \\
&= (\mu\underline{\vee}\theta) \cap (\mu\underline{\vee}\eta).
\end{aligned}$$

Thus, $FI_\sigma(L)$ is a distributive lattice.

Next we prove the completeness. Since $\{0\}$ and L are σ-ideals, $\chi_{\{0\}}$ and χ_L are least and greatest elements of $FI_\sigma(L)$ respectively. Let $\{\mu_i : i \in I\} \subseteq FI_\sigma(L)$. Then $\bigcap_{i\in I}\mu_i$ is a fuzzy ideal of L and $\bigcap_{i\in I}\mu_i \subseteq \overleftarrow{\sigma}\sigma(\bigcap_{i\in I}\mu_i)$.

$$\begin{aligned}
\bigcap_{i\in I}\mu_i \subseteq \mu_i,\ \forall i \in I \quad &\Rightarrow \quad \overleftarrow{\sigma}\sigma(\bigcap_{i\in I}\mu_i) \subseteq \mu_i,\ \forall i \in I \\
&\Rightarrow \quad \overleftarrow{\sigma}\sigma(\bigcap_{i\in I}\mu_i) \subseteq \bigcap_{i\in I}\mu_i.
\end{aligned}$$

Thus $\overleftarrow{\sigma}\sigma(\bigcap_{i\in I}\mu_i) = \bigcap_{i\in I}\mu_i$. So $(FI_\sigma(L), \wedge, \underline{\vee})$ is a complete distributive lattice. □

Theorem 3.18. *The class $FI_\sigma(L)$ of all σ-fuzzy ideals of L are isomorphic to the class $FI(B_*(L))$ of all fuzzy ideals of $B_*(L)$.*

Proof. Define $f : FI_\sigma(L) \longrightarrow FI(B_*(L)),\ f(\mu) = \sigma(\mu),\ \forall \mu \in FI_\sigma(L)$. Let $\mu, \theta \in FI_\sigma(L)$ and $f(\mu) = f(\theta)$. Then $\sigma(\mu) = \sigma(\theta)$. Thus $\overleftarrow{\sigma}\sigma(\mu) = \overleftarrow{\sigma}\sigma(\theta)$. So $\mu = \theta$. Hence f is one to one.

Let $\eta \in FI(B_*(L))$. Then $\overleftarrow{\sigma}(\eta)$ is a fuzzy ideal of L. We show that $\overleftarrow{\sigma}(\eta)$ is a σ-fuzzy ideal of L. Let $x \in L$. Then $\overleftarrow{\sigma}\sigma(\overleftarrow{\sigma}(\eta))(x) = \sigma\overleftarrow{\sigma}(\eta)((x)^\triangle)$. Then by Lemma 3.7,

$$\sigma\overleftarrow{\sigma}(\eta)((x)^\triangle) = \eta((x)^\triangle) = \overleftarrow{\sigma}(\eta)(x).$$

Thus $\overleftarrow{\sigma}(\eta) = \overleftarrow{\sigma}\sigma(\overleftarrow{\sigma}(\eta))$. So for each $\eta \in FI(B_*(L)),\ f(\overleftarrow{\sigma}(\eta)) = \eta$. Hence, f is onto.

Now for any $\mu, \theta \in FI_\sigma(L)$, $f(\mu\underline{\vee}\theta) = f(\overleftarrow{\sigma}\sigma(\mu\vee\theta)) = \sigma(\overleftarrow{\sigma}\sigma(\mu\vee\theta)) = \sigma(\mu\vee\theta) = \sigma(\mu) \vee \sigma(\theta) = f(\mu) \vee f(\theta)$. Similarly $f(\mu \cap \theta) = f(\mu) \cap f(\theta)$. Therefore, f is an isomorphism of $FI_\sigma(L)$ onto the lattice of fuzzy ideals of $B_*(L)$ □

Lemma 3.19. *Let μ be a fuzzy ideal of L. Then the fuzzy subset μ^* of L defined by:*

$$\mu^*(x) = sup\{\mu(a) : a^* \leq x,\ a \in L\}$$

is a fuzzy filter of L.

Proof. Let μ be a fuzzy ideal of L. Since $0^* = 1$, we get $\mu^*(1) = 1$. For any $x, y \in L$,

$$\begin{aligned}\mu^*(x) \wedge \mu^*(y) &= sup\{\mu(a) : a^* \leq x,\ a \in L\} \wedge sup\{\mu(b) : b^* \leq y,\ b \in L\}\\ &= sup\{\mu(a) \wedge \mu(b) : a^* \leq x,\ b^* \leq y\}\end{aligned}$$

Since $a^* \leq x$ and $b^* \leq y$, we get $(a \vee b)^* \leq x \wedge y$. Using this fact, we have

$$\begin{aligned}\mu^*(x) \wedge \mu^*(y) &\leq sup\{\mu(a \vee b) : (a \vee b)^* \leq x \wedge y\}\\ &\leq sup\{\mu(c) : c^* \leq x \wedge y\}\\ &= \mu^*(x \wedge y)\end{aligned}$$

Again, $\mu^*(x) = sup\{\mu(a) : a^* \leq x,\ a \in L\} \leq sup\{\mu(a) : a^* \leq x \vee y\} = \mu^*(x \vee y)$. Similarly, $\mu^*(y) \leq \mu^*(x \vee y)$. Then $\mu^*(x \vee y) \geq \mu^*(x) \vee \mu^*(y)$. Thus μ^* is a fuzzy filter of L. □

Lemma 3.20. *Let μ be a fuzzy ideal of L. Then the fuzzy subset μ_{**} of L defined by:*

$$\mu_{**}(x) = sup\{\mu(a) : x \leq a^{**},\ a \in L\}$$

is a fuzzy ideal of L.

Proof. Let μ be a fuzzy ideal of L. Since $0 \leq 0^{**}$, we get $\mu_{**}(0) = 1$. For any $x, y \in L$,

$$\begin{aligned}\mu_{**}(x) \wedge \mu_{**}(y) &= sup\{\mu(a) : x \leq a^{**},\ a \in L\} \wedge sup\{\mu(b) : y \leq b^{**},\ b \in L\}\\ &= sup\{\mu(a) \wedge \mu(b) : x \leq a^{**},\ y \leq b^{**}\}\\ &\leq sup\{\mu(a \vee b) : x \vee y \leq (a \vee b)^{**}\}\\ &\leq sup\{\mu(c) : x \vee y \leq c^{**}\}\\ &= \mu_{**}(x \vee y)\end{aligned}$$

Again, $\mu_{**}(x) = sup\{\mu(a) : x \leq a^{**},\ a \in L\} \leq sup\{\mu(a) : x \wedge y \leq a^{**}\} = \mu_{**}(x \wedge y)$. Similarly, $\mu_{**}(y) \leq \mu_{**}(x \wedge y)$. Then $\mu_{**}(x \wedge y) \geq \mu_{**}(x) \vee \mu_{**}(y)$. Thus μ_{**} is a fuzzy ideal of L. □

Theorem 3.21. *Let μ be a fuzzy ideal of L. Then*

(1) *$\mu = \mu_{**}$ if and only if $\mu(a) = \mu(a^{**})$ for each $a \in L$,*
(2) *μ is σ-fuzzy ideal if and only if $\mu = \mu_{**}$.*

Proof. Let μ be a fuzzy ideal of L. Then

(1) Let $\mu = \mu_{**}$. Since $x \leq x^{**}$, for each $x \in L$ and μ is a fuzzy ideal, we get $\mu(x) \geq \mu(x^{**})$. For any $x \in L$, $\mu(x^{**}) = sup\{\mu(a) : x^{**} \leq a^{**},\ a \in L\} \geq \mu(x)$. Then $\mu(x) = \mu(x^{**})$, for each $x \in L$.

Conversely, assume that $\mu(a) = \mu(a^{**})$, for each $a \in L$. Clearly $\mu \subseteq \mu_{**}$. Let $x \in L$. Then $\mu_{**}(x) = sup\{\mu(a) : x \leq a^{**},\ a \in L\}$. For each $a \in L$ which satisfies $x \leq a^{**}$, we have $\mu(x) \geq \mu(a^{**})$. Since $\mu(a) = \mu(a^{**})$, we get $\mu(x) \geq \mu(a^{**})$, for each $a \in L$ such that $x \leq a^{**}$. This implies $\mu(x)$ is an upper bound of $\{\mu(a) : x \leq a^{**},\ a \in L\}$. Which implies $\mu(x) \geq \mu_{**}(x)$. This shows that $\mu = \mu_{**}$.

(2) Let $\mu = \mu_{**}$. Then clearly $\mu \subseteq \overleftarrow{\sigma}\sigma(\mu)$. For any $x \in L$,

$$\overleftarrow{\sigma}\sigma(\mu)(x) = sup\{\mu(a) : (a)^{\triangle} = (x)^{\triangle},\ a \in L\} \leq sup\{\mu(a) : x \leq a^{**}\} = \mu(x).$$

Thus μ is a σ-fuzzy ideal.

Conversely, suppose that $\mu = \overleftarrow{\sigma}\sigma(\mu)$. Now we prove $\mu(a) = \mu(a^{**})$, for each $a \in L$. For any $x \in L$,

$$\mu(x^{**}) = sup\{\mu(a) : (x^{**})^{\triangle} = (a)^{\triangle}\} = sup\{\mu(a) : (x)^{\triangle} = (a)^{\triangle}\} = \mu(x).$$

Then by (1), we get $\mu = \mu_{**}$. □

Lemma 3.22. *The α-level principal fuzzy ideal α_a is a σ-fuzzy ideal if and only if a is a closed element of L.*

Proof. Let "a" be a closed element. For any $x \in L$,

$$\overleftarrow{\sigma}\sigma(\alpha_a)(x) = sup\{\alpha_a(b) : (b)^{\triangle} = (x)^{\triangle}\}.$$

If $x \leq a$, then $\alpha_a(x) = \overleftarrow{\sigma}\sigma(\alpha_a)(x) = 1$. If $x \nleq a$, then $\alpha_a(x) = \alpha$. Assume that $\overleftarrow{\sigma}\sigma(\alpha_a)(x) = 1$. Then there exists $b \in L$ such that $b \leq a$ and $(b)^{\triangle} = (x)^{\triangle}$. This implies that $b^{**} \leq a$ and $b^{**} = x^{**}$. This shows that $x \leq a$. Which is a contradiction. Thus $\overleftarrow{\sigma}\sigma(\alpha_a)(x) = \alpha$. Hence α_a is a σ-fuzzy ideal.

Conversely, suppose that α_a is a σ-fuzzy ideal. Then by theorem (3.21(1)), $\alpha_a = (\alpha_a)_{**}$. This implies that $\alpha_a(a^{**}) = 1$. This shows that $a^{**} \leq a$. Thus $a^{**} = a$. So a is a closed element. □

Theorem 3.23. *Let μ be a σ-fuzzy ideal of L and λ be a fuzzy filter of L such that $\mu \cap \lambda \leq \alpha$, $\alpha \in [0, 1)$. Then there exists a prime σ-fuzzy ideal θ of L such that $\mu \subseteq \theta$ and $\theta \cap \lambda \leq \alpha$.*

Proof. Put $\mathcal{P} = \{\eta \in FI_{\sigma}(L) : \mu \subseteq \eta \texttt{ and } \eta \cap \lambda \leq \alpha\}$. Since $\mu \in \mathcal{P}$, $\mathcal{P}$ is nonempty and it forms a poset together with the inclusion ordering of fuzzy sets. Let $\mathcal{A} = \{\mu_i\}_{i \in I}$ be any chain in $\mathcal{P}$. We need to prove $\bigcup_{i\in I} \mu_i \in \mathcal{A}$. Then clearly $(\bigcup_{i\in I} \mu_i)(0) = 1$. For any $x, y \in L$,

$$\begin{aligned}(\bigcup_{i\in I} \mu_i)(x) \wedge (\bigcup_{i\in I} \mu_i)(y) &= sup\{\mu_i(x) : i \in I\} \wedge sup\{\mu_j(y) : j \in I\} \\ &= sup\{\mu_i(x) \wedge \mu_j(y) : i, j \in I\} \\ &\leq sup\{(\mu_i \cup \mu_j)(x) \wedge (\mu_i \cup \mu_j)(y) : i, j \in I\}.\end{aligned}$$

Since $\mathcal{A}$ is a chain, either $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$. Without loss of generality, we can assume that $\mu_i \subseteq \mu_j$. Thus $\mu_i \cup \mu_j = \mu_j$. This shows that

$$\begin{aligned}(\bigcup_{i\in I} \mu_i)(x) \wedge (\bigcup_{i\in I} \mu_i)(y) &\leq sup\{\mu_j(x) \wedge \mu_j(y) : j \in I\} \\ &= sup\{\mu_j(x \vee y) : j \in I\} \\ &= (\bigcup_{i\in I} \mu_i)(x \vee y).\end{aligned}$$

Again $(\bigcup_{i\in I} \mu_i)(x) = sup\{\mu_i(x) : i \in I\} \leq sup\{\mu_i(x \wedge y) : i \in I\} = (\bigcup_{i\in I} \mu_i)(x \wedge y)$. Similarly, $(\bigcup_{i\in I} \mu_i)(y) \leq (\bigcup_{i\in I} \mu_i)(x \wedge y)$. Thus $\bigcup_{i\in I} \mu_i$ is a fuzzy ideal of L. It remains to show $\bigcup_{i\in I}\mu_i$ is a σ-fuzzy ideal.

$$\begin{aligned}\overleftarrow{\sigma}\sigma(\bigcup_{i\in I}\mu_i)(x) &= sup\{(\bigcup_{i\in I}\mu_i)(a) : (a)^{\triangle} = (x)^{\triangle}\}\\ &= sup\{sup\{\mu_i(a) : i\in I\} : (a)^{\triangle} = (x)^{\triangle}\}\\ &= sup\{sup\{\mu_i(a) : (a)^{\triangle} = (x)^{\triangle}\} : i\in I\}\\ &= sup\{\mu_i(x) : i\in I\}\\ &= (\bigcup_{i\in I}\mu_i)(x).\end{aligned}$$

Then $\bigcup_{i\in I}\mu_i$ is a σ-fuzzy ideal. Since $\mu_i\cap\eta\leq\alpha$, for each $i\in I$,

$$\begin{aligned}((\bigcup_{i\in I}\mu_i)\cap\eta)(x) &= (\bigcup_{i\in I}\mu_i)(x)\wedge\eta(x)\\ &= sup\{\mu_i(x)\wedge\eta(x) : i\in I\}\\ &= sup\{(\mu_i\cap\eta)(x) : i\in I\}\leq\alpha.\end{aligned}$$

Thus $(\bigcup_{i\in I}\mu_i)\cap\eta\leq\alpha$. So $\bigcup_{i\in I}\mu_i\in\mathcal{A}$. By applying Zorn's lemma, we get a maximal element, let say $\theta\in\mathcal{P}$, that is, θ is a σ-fuzzy ideal of L such that $\mu\subseteq\theta$ and $\theta\cap\eta\leq\alpha$.

Now we proceed to show θ is a prime fuzzy ideal. Assume that θ is not prime fuzzy ideal. Let $\gamma_1\cap\gamma_2\subseteq\theta$ such that $\gamma_1\nsubseteq\theta$ and $\gamma_2\nsubseteq\theta$, $\gamma_1,\ \gamma_2\in FI(L)$. If we put $\theta_1=\overleftarrow{\sigma}\sigma(\gamma_1\vee\theta)$ and $\theta_2=\overleftarrow{\sigma}\sigma(\gamma_2\vee\theta)$, then both θ_1 and θ_2 are σ-fuzzy ideals of L properly containing θ. Since θ is maximal in $\mathcal{P}$, we get $\theta_1\notin\mathcal{P}$ and $\theta_2\notin\mathcal{P}$. This shows that $\theta_1\cap\eta\nleq\alpha$ and $\theta_2\cap\eta\nleq\alpha$. This implies there exist $x,y\in L$ such that $(\theta_1\cap\eta)(x)>\alpha$ and $(\theta_2\cap\eta)(y)>\alpha$. Which implies $((\theta_1\cap\theta_2)\cap\eta)(x\wedge y)>\alpha\Rightarrow(\overleftarrow{\sigma}\sigma(\theta\vee(\gamma_1\wedge\gamma_2))(x\wedge y)\wedge\eta(x\wedge y)>\alpha$. This shows that $(\theta\cap\eta)(x\wedge y)>\alpha$. Which is a contradiction. Hence θ is prime σ-fuzzy ideal of L. □

Corollary 3.24. *Let μ be a σ-fuzzy ideal of L, $a\in L$ and $\alpha\in[0,1)$. If $\mu(a)\leq\alpha$, then there exists a prime σ-fuzzy ideal θ of L such that $\mu\subseteq\theta$ and $\theta(a)\leq\alpha$.*

Corollary 3.25. *Every proper σ-fuzzy ideal of L is the intersection of all prime σ-fuzzy ideals containing it.*

Proof. Let μ be a proper σ-fuzzy ideal of L. Consider the following.

$$\mu_0=\bigcap\{\eta : \eta \textit{ is a prime } \sigma\textit{-fuzzy ideal and } \mu\subseteq\eta\}.$$

Clearly $\mu\subseteq\mu_0$. Suppose $\mu_0\nsubseteq\mu$. Then there is $a\in L$ such that $\mu_0(a)>\mu(a)$. Let $\mu(a)=\alpha$. Consider the set $\mathcal{P}=\{\eta\in FI_\sigma(L) : \mu\subseteq\eta \texttt{ and } \eta(a)\leq\alpha\}$. By the above corollary, we can find a prime σ-fuzzy ideal θ of L such that $\mu\subseteq\theta$ and $\theta(a)\leq\alpha$. This implies $\mu_0\subseteq\theta$. This shows that $\mu_0(a)\leq\alpha$. Which is a contradiction. Thus $\mu_0\subseteq\mu$. So $\mu_0=\mu$. □

4. σ-fuzzy ideals and homomorphism

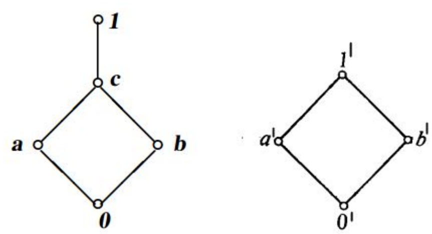
In this section, some properties of the homomorphic images and the inverse images of σ-fuzzy ideals are studied. Throughout this section L and M denotes distributive p-algebras with least elements 0 and $0^{'}$ respectively.

In [6], Rao obsered that, let L and $L^{'}$ be two pseudo-complemented distributive lattices with pseudo-complementation $*$ and $f : L \longrightarrow M$ an onto homomorphism and $Kerf = \{0\}$. Then $f(x^*) = (f(x))^*$, for all $x \in L$.

Theorem 4.1. *Let $f : L \longrightarrow M$ be an onto homomorphism and $Kerf = \{0\}$. Then the image of a σ-fuzzy ideal of L is a σ-fuzzy ideal of M.*

Proof. Let μ be a σ-fuzzy ideal of L. Then $f(\mu)$ is a fuzzy ideal of M. Now we prove $f(\mu)(y) = f(\mu)(y^{**})$, for each $y \in M$. Since $f(\mu)$ is a fuzzy ideal and $y \leq y^{**}$, we get $f(\mu)(y) \geq f(\mu)(y^{**})$, for each $y \in M$. Again, $f(\mu)(y^{**}) = sup\{\mu(a) : a \in f^{-1}(y^{**}),\ a \in L\} \geq \mu(b^{**})$, for each $b \in f^{-1}(y)$. Since μ is a σ-fuzzy ideal, we have $\mu(x) = \mu(x^{**})$, for each $x \in L$. Based on this fact we have, $f(\mu)(y^{**}) \geq \mu(b)$, for each $b \in f^{-1}(y)$. This implies $f(\mu)(y^{**})$ is an upper bound of $\{\mu(b) : b \in f^{-1}(y)\}$. This shows that $f(\mu)(y^{**}) \geq f(\mu)(y)$, for each $y \in M$. Thus $f(\mu)(y^{**}) = f(\mu)(y)$, for each $y \in M$ and by Theorem 3.21, $f(\mu) = f(\mu)_{**}$. So $f(\mu)$ is a σ-fuzzy ideal of M. □

Example 4.2. Consider the distributive p-algebras $L = \{0, a, b, c, 1\}$ and $M = \{0^{'}, a^{'}, b^{'}, 1^{'}\}$ given by the following diagrams.



Define $f : L \longrightarrow M$, such that $f(0) = 0^{'}, f(a) = a^{'}, f(b) = b^{'}$ and $f(c) = 1^{'} = f(1)$. Then f is an onto homomorphism and $Ker\ f = \{0\}$. If we define a fuzzy subset μ of L as $\mu(0) = 1,\ \mu(a) = 0.5$ and $\mu(b) = \mu(c) = \mu(1) = 0.4$, then by Example 3.13, μ is a σ-fuzzy ideal of L. Since μ is a fuzzy ideal of L, $f(\mu)$ is a fuzzy ideal of M. Now $\overleftarrow{\sigma}\sigma(f(\mu))(0^{'}) = sup\{f(\mu)(x) : (x)^{\triangle} = (0^{'})^{\triangle},\ x \in M\} = f(\mu)(0^{'})$, $\overleftarrow{\sigma}\sigma(f(\mu))(a^{'}) = f(\mu)(a^{'})$, $\overleftarrow{\sigma}\sigma(f(\mu))(b^{'}) = f(\mu)(b^{'})$ and $\overleftarrow{\sigma}\sigma(f(\mu))(1^{'}) = f(\mu)(1^{'})$. This shows that $\overleftarrow{\sigma}\sigma(f(\mu)) = f(\mu)$. Thus $f(\mu)$ is a σ-fuzzy ideal of M.

Theorem 4.3. *Let $f : L \longrightarrow M$ be a homomorphism. Then the pre-image of a σ-fuzzy ideal of M is a σ-fuzzy ideal of L.*

Proof. Let θ be a σ-fuzzy ideal of M. Then $\theta(y) = \theta(y^{**})$, for all $y \in M$ and $f^{-1}(\theta)$ is a fuzzy ideal of L. Now we prove $f^{-1}(\theta)(a) = f^{-1}(\theta)(a^{**})$, for all $a \in L$. For any any $a \in L$, $f^{-1}(\theta)(a) = \theta(f(a)) = \theta(f(a)^{**}) = f^{-1}(\theta)(a^{**})$. Thus $f^{-1}(\theta)(a) = f^{-1}(\theta)(a^{**})$, for all $a \in L$. So $f^{-1}(\theta)$ is a σ-fuzzy ideal of L. □

Theorem 4.4. *Let $f : L \longrightarrow M$ be an onto homomorphism and $Kerf = \{0\}$. Then there is a homomorphism between $FI_{\sigma}(L)$ and $FI_{\sigma}(M)$.*

Proof. Define $g : FI_\sigma(L) \longrightarrow FI_\sigma(M)$ by $g(\mu) = f(\mu)$, for each $\mu \in FI_\sigma(L)$. Clearly, $\chi_{\{0\}},\ \chi_L \in FI_\sigma(L)$. Since Ker$f = \{0\}$ and f is onto, we get $f(\chi_{\{0\}}) = \chi_{\{0'\}}$ and $f(\chi_L) = \chi_M$. This implies $g(\chi_{\{0\}}) = \chi_{\{0'\}}$ and $g(\chi_L) = \chi_M$. Let $\mu,\ \theta \in FI_\sigma(L)$. Then $\mu \cap \theta$ and $\mu \underline{\vee} \theta$ are σ-fuzzy ideals. Thus $f(\mu \cap \theta)$ and $f(\mu \underline{\vee} \theta)$ are σ-fuzzy ideals of M. Since $\mu \cap \theta \subseteq \mu$ and $\mu \cap \theta \subseteq \theta$, we have $f(\mu \cap \theta) \subseteq f(\theta) \cap f(\mu)$. For any $y \in M$, $(f(\mu) \cap f(\theta))(y) = sup\{\mu(a) : a \in f^{-1}(y),\ a \in L\} \wedge\ sup\{\theta(b) : b \in f^{-1}(y),\ b \in L\}$. Since f is a homomorphism and $f(a) = y,\ f(b) = y$, we get $f(a \wedge b) = y$. Using this fact, we have

$$\begin{aligned}
(f(\mu) \cap f(\theta))(y) &\leq sup\{\mu(a \wedge b) : a \wedge b \in f^{-1}(y)\} \\
&\quad \wedge sup\{\theta(a \wedge b) : a \wedge b \in f^{-1}(y)\} \\
&= sup\{\mu(a \wedge b) \wedge \theta(a \wedge b) : a \wedge b \in f^{-1}(y)\} \\
&= sup\{(\mu \cap \theta)(a \wedge b) : a \wedge b \in f^{-1}(y)\} \\
&\leq sup\{(\mu \cap \theta)(c) : c \in f^{-1}(y)\} \\
&= f(\mu \cap \theta)(y).
\end{aligned}$$

So $f(\mu) \cap f(\theta) = f(\mu \cap \theta)$. Again, clearly $f(\mu) \underline{\vee} f(\theta) \subseteq f(\mu \underline{\vee} \theta)$. For any $y \in M$,

$$\begin{aligned}
f(\mu \underline{\vee} \theta)(y) &= sup\{(\mu \underline{\vee} \theta)(x) : x \in f^{-1}(y),\ x \in L\} \\
&= sup\{\overleftarrow{\sigma}\sigma(\mu \vee \theta)(x) : x \in f^{-1}(y),\ x \in L\} \\
&= sup\{sup\{(\mu \vee \theta)(a) : a^* = x^*\} : f(x) = y\} \\
&= sup\{sup\{sup\{\mu(a_1) \wedge \theta(a_2) : a_1 \vee a_2 = a\} : a^* = x^*\} : f(x) = y\}
\end{aligned}$$

and

$$\begin{aligned}
(f(\mu) \underline{\vee} f(\theta))(y) &= Sup\{(f(\mu) \vee f(\theta))(b) : b^* = y^*\} \\
&= Sup\{f(\mu \vee \theta)(b) : b^* = y^*\} \\
&= Sup\{Sup\{(\mu \vee \theta)(c) : c \in f^{-1}(b)\} : b^* = y^*\} \\
&= Sup\{Sup\{Sup\{\mu(c_1) \wedge \theta(c_2) : c_1 \vee c_2 = c\} : f(c) = b\} : b^* = y^*\}.
\end{aligned}$$

Now, put $A = \{(a_1, a_2) \in L \times L : (a_1 \vee a_2)^* = x^*,\ f(x) = y\}$ and $B = \{(c_1, c_2) \in L \times L : f(c_1 \vee c_2) = b,\ b^* = y^*\}$. If $(a_1, a_2) \in A$, then $(a_1 \vee a_2)^* = x^*$ and $f(x) = y$. This implies $(f(a_1 \vee a_2))^* = y^*$. Put $b = f(a_1 \vee a_2)$. Then $b^* = y^*$. Thus $(a_1, a_2) \in B$ and $A \subseteq B$. So $f(\mu \underline{\vee} \theta) \subseteq f(\mu) \underline{\vee} f(\theta)$. Hence $f(\mu \underline{\vee} \theta) = f(\mu) \underline{\vee} f(\theta)$. □

5. Conclusions

In this work, we studied the concept of σ-fuzzy ideals in distributive p-algebras. We prove that the set of all σ-fuzzy ideals forms a complete distributive lattice. We also show that every proper σ-fuzzy ideal is the intersection of all prime σ-fuzzy ideals containing it. Finally, we prove that the image and pre-image of σ-fuzzy ideals are also σ-fuzzy ideals. Our future work will focus on σ-fuzzy ideals in a 0-1 distributive lattice.

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