Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 17, No. 2, (April 2019) pp. 191–204
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2019.17.2.191

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 17, No. 2, April 2019

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Received 13 December 2018; Revised 18 January 2019; Accepted 23 January 2019

Abstract. In this paper, we introduce the notions of L-fuzzy pre-proximities, L-fuzzy interior operators in complete residuated lattices. Moreover, we investigate the relations among the L-fuzzy pre-proximities, L-fuzzy interior operators and L-fuzzy topologies. We show that there is a Galois correspondence between the category of separated L-fuzzy interior spaces and that of separated L-fuzzy pre-proximity spaces.

2010 AMS Classification: 03E72, 04A72, 18B05

Keywords: Complete residuated lattice, L-fuzzy pre-proximity, L-fuzzy interior operators, Galois correspondence.

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1. Introduction

Ward et al. [30] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [3] investigated information systems and decision rules over complete residuated lattices. Höhle [13] introduced L-fuzzy topologies with algebraic structure L (cqm, quantales, MV-algebra). It has developed in many directions [5, 11, 29, 33].

Interior operators are very useful tools in several areas of mathematical structures with direct applications, both mathematical (e.g. topology, logic) and extra-mathematical (e.g. data mining, knowledge representation). In fuzzy set theory, several particular case as well as general theory of interior operators which operate with fuzzy sets (so called fuzzy interior operators) are studied ([2, 6]). Recently, Bělohlávek [4] outlined a general theory of fuzzy interior operators and fuzzy interior systems using the structure of the residuated lattice in place of the usual structure of truth value on $[0, 1]$. Ramadan [25, 27, 28, 26] studied the relationship between L-fuzzy interior systems and L- fuzzy topological spaces from a category viewpoint for a complete residuated lattice L.

Katsaras [16, 17, 18] introduced the concepts of fuzzy topogenous order and fuzzy topogenous structures in completely distributive lattice which are a unified approach to the three spaces: Chang's fuzzy topologies [7], Katsaras's fuzzy proximities [15, 19] and Hutton's fuzzy uniformities [14]. As an extension of Katsaras's definition, El-Dardery [9] introduced L-fuzzy topogenous order in view points of Sostak's fuzzy topology [31, 32] (see also [8, 10]) and Kim's L-fuzzy proximities [21] on strictly two-sided, commutative quantales. It has developed in many directions [8, 10, 12, 20, 22, 23, 24, 34].

In this paper, we introduce the notions of L-fuzzy pre-proximities, L-fuzzy interior operators in complete residuated lattices. Moreover, we investigate the relations among the L-fuzzy pre-proximities and L-fuzzy interior operators . We show that there is a Galois correspondence between the category of separated L-fuzzy interior spaces and that of separated L-fuzzy pre-proximity spaces.

The content of the paper is organized as follows. In section 2, we recall some fundamental concepts and related definitions of L-fuzzy interior and L-fuzzy topology in complete residuated lattices. In section 3, we investigates the relationships between L-fuzzy pre-proximities and L-fuzzy interior operators. In section 4, we investigates the relationships between L-fuzzy pre-proximities and L-fuzzy topologies. In section 5, there is a Galois correspondence between the category of L-fuzzy pre-proximity spaces and that of L-fuzzy interior spaces.

2. Preliminaries

Definition 2.1 ([3, 11, 30, 33])**.** An algebra $(L, \wedge, \vee, \odot, \rightarrow, \perp, \top)$ is called a complete residuated lattice, if it satisfies the following conditions:

(C1) $(L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$,

(C2) $(L, \odot, \top)$ is a commutative monoid,

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we assume that $(L, \leq, \odot, ^*)$ is a complete residuated lattice with an order reversing involution * which is defined by:

$$x \oplus y = (x^* \odot y^*)^*, \quad x^* = x \rightarrow \perp.$$

For $\alpha \in L, f \in L^X$, we denote $(\alpha \rightarrow f), (\alpha \odot f), \alpha_X \in L^X$ as $(\alpha \rightarrow f)(x) = \alpha \rightarrow f(x), (\alpha \odot f)(x) = \alpha \odot f(x), \alpha_X(x) = \alpha$,

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \perp, & \text{otherwise,} \end{cases} \quad \top_x^*(y) = \begin{cases} \perp, & \text{if } y = x, \\ \top, & \text{otherwise.} \end{cases}$$

Some basic properties of the binary operation $\odot$ and residuated operation $\rightarrow$ are collected in the following lemma, and they can be found in many works, for instance ([3, 11, 30, 33]).

Lemma 2.2. *For each $x, y, z, x_i, y_i, w \in L$, we have the following properties:*

(1) $\top \rightarrow x = x$, $\perp \odot x = \perp$,

(2) *if* $y \leq z$, *then* $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \rightarrow y \leq x \rightarrow z$ *and* $z \rightarrow x \leq y \rightarrow x$,

(3) $x \leq y$ *iff* $x \rightarrow y = \top$,

(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*, (\bigvee_i y_i)^* = \bigwedge_i y_i^*$,

(5) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$,
(6) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$,
(7) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$,
(8) $(\bigwedge_i x_i) \oplus y = \bigwedge_i (x_i \oplus y)$,
(9) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,
(10) $x \odot y = (x \to y^*)^*$, $x \oplus y = x^* \to y$ *and* $x \to y = y^* \to x^*$,
(11) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,
(12) $x \to y \leq (x \odot z) \to (y \odot z)$ *and* $(x \to y) \odot (y \to z) \leq x \to z$,
(13) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$,
(14) $x \odot (x \to y) \leq y$ *and* $y \leq x \to (x \odot y)$,
(15) $(x \vee y) \odot (z \vee w) \leq (x \vee z) \vee (y \odot w) \leq (x \oplus z) \vee (y \odot w)$,
(16) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$, $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$,
(17) $(x \odot y) \odot (z \oplus w) \leq (x \odot z) \oplus (y \odot w)$,
(18) $z \to x \leq (x \to y) \to (z \to y)$ *and* $y \to z \leq (x \to y) \to (x \to z)$.

Definition 2.3 ([4, 25])**.** A map $\mathcal{I} : L^X \to L^X$ is called an L-fuzzy interior operator on X, if $\mathcal{I}$ satisfies the following conditions:
(I1) $\mathcal{I}(\top_X) = \top_X$,
(I2) $\mathcal{I}(f) \leq f$, for all $f \in L^X$,
(I3) if $g \leq f$, then $\mathcal{I}(g) \leq \mathcal{I}(f)$, for all $f, g \in L^X$,
(I4) $\mathcal{I}(f \odot g) \geq \mathcal{I}(f) \odot \mathcal{I}(g)$.
The pair $(X, \mathcal{I})$ is called an L-Fuzzy interior space.

An L-fuzzy interior space is called:
(T) topological, if $\mathcal{I}(\mathcal{I}(f)) = \mathcal{I}(f)$, $\forall f \in L^X$,
(S) stratified, if $\mathcal{I}(\alpha \odot f) \geq \alpha \odot \mathcal{I}(f)$, for all $f \in L^X$ and $\alpha \in L$,
(CST) co-stratified, if $\mathcal{I}(\alpha \to f) \geq \alpha \to \mathcal{I}(f)$, for all $f \in L^X$ and $\alpha \in L$,
(S) strong, if it is both stratified and co-stratified, i.e., $\mathcal{I}(\alpha \to f) = \alpha \to \mathcal{I}(f)$ for all $f \in L^X$ and $\alpha \in L$,
(SE) separated, if $\mathcal{I}(\top_x) = \top_x$, for all $x \in X$,
(GE) generalized, if $\mathcal{I}(f)(x) \geq \bigwedge_{x\in X} f(x)$,
(AL) Alexandrov, if $\mathcal{I}(\bigwedge_{i\in\Gamma} f_i) = \bigwedge_{i\in\Gamma} \mathcal{I}(f_i)$.

Let $(X, \mathcal{I}_X)$ and $(Y, \mathcal{I}_Y)$ be L-fuzzy interior spaces. Then $\varphi : (X, \mathcal{I}_X) \to (Y, \mathcal{I}_Y)$ is called an LF-interior map, if for each $g \in L^Y$,

$$\varphi^{\leftarrow}(\mathcal{I}_Y(g)) \leq \mathcal{I}_X(\varphi^{\leftarrow}(g)).$$

Remark 2.4. An L-fuzzy interior space $(X, \mathcal{I})$ is stratified if and only if $\mathcal{I}(\alpha \to f) \leq \alpha \to \mathcal{I}(f)$.

Definition 2.5 ([29])**.** A map $\mathcal{T} : L^X \to L$ is called an L-fuzzy topology on X, if it satisfies the following conditions:
(T1) $\mathcal{T}(\bot_X) = \mathcal{T}(\top_X) = \top$,
(T2) $\mathcal{T}(f \odot g) \geq \mathcal{T}(f) \odot \mathcal{T}(g)$, $\forall f, g \in L^X$,
(T3) $\mathcal{T}(\bigvee_i f_i) \geq \bigwedge_i \mathcal{T}(f_i)$, $\forall \{f_i\}_{i\in\Gamma} \subseteq L^X$.
The pair $(X, \mathcal{T})$ is called a L-fuzzy topological space.

A L-fuzzy topological space is said to be:
(ST) stratified, if $\mathcal{T}(\alpha \odot f) \geq \mathcal{T}(f)$,
(CST) co-stratified, if $\mathcal{T}(\alpha \to f) \geq \mathcal{T}(f)$,
(S) strong, if it is both stratified and co-stratified,
(AL) Alexandrov, if $\mathcal{T}(\bigwedge_i f_i) \geq \bigwedge_i \mathcal{T}(f_i),\ \forall\ \{f_i\}_{i\in\Gamma} \subseteq L^X$,
(SE) separated, if $\mathcal{T}(\top_x) = \top$, for all $x \in X$.

A mapping $\phi : X \longrightarrow Y$ between L-fuzzy topological spaces is called continuous from $(X, \mathcal{T}_X)$ to $(Y, \mathcal{T}_Y)$, if it holds that $\mathcal{T}_X(\varphi^{\leftarrow}(f)) \geq \mathcal{T}_Y(f)$, for all $f \in L^Y$.

3. The relationships between L-fuzzy pre-proximities and L-fuzzy interior spaces

Definition 3.1. A mapping $\delta : L^X \times L^X \to L$ is called a L-fuzzy pre-proximity on X, if it satisfies the following axioms:
(P1) $\delta(\top_X, \bot_X) = \delta(\bot_X, \top_X) = \bot$,
(P2) $\delta(f, g) \geq \bigvee_{x\in X}(f \odot g)(x)$,
(P3) if $f_1 \leq f_2, h_1 \leq h_2$, then $\delta(f_1, h_1) \leq \delta(f_2, h_2),\ \forall\ h \in L^X$,
(P4) for every $f_1, f_2, h_1, h_2 \in L^X$, we have

$$\delta(f_1 \odot f_2, h_1 \oplus h_2) \leq \delta(f_1, h_1) \oplus \delta(f_2, h_2),$$

$$\delta(f_1 \oplus f_2, h_1 \odot h_2) \leq \delta(f_1, h_1) \oplus \delta(f_2, h_2).$$

The pair (X, δ) is called a L-fuzzy pre-proximity space.

An L-fuzzy pre-proximity is called an L-fuzzy quasi-proximity on X, if
(Q) $\delta(f, g) \geq \bigwedge_h \{\delta(f, h) \oplus \delta(h^*, g)\}$,
An L-fuzzy quasi-proximity is called an L-fuzzy proximity on X, if
(P) $\delta^s = \delta$ where, $\delta^s(f, g) = \delta(g, f)$.

An L-fuzzy pre-proximity is called:
(St) stratified, if $\delta(\alpha \odot f, \alpha \to g) \leq \delta(f, g)$ and $\delta(\alpha \to f, \alpha \odot g) \leq \delta(f, g)$,
(SE) separated, if $\delta(\top_x, \top_x^*) = \delta(\top_x^*, \top_x) = \bot$, for each $x \in X$,
(AL) Alexandrov, if $\delta(\bigvee_{i\in\Gamma} f_i, g) \leq \bigvee_{i\in\Gamma} \delta(f_i, g)$, $\delta(f, \bigvee_{i\in\Gamma} g_i) \leq \bigvee_{i\in\Gamma} \delta(f, g_i)$,
(GL) generalized, if $\delta(f, g) \leq \bigvee_{x\in X} f(x) \odot \bigvee_{x\in X} g(x)$.

Let (X, δ_X) and (Y, δ_Y) be two L-fuzzy pre-proximity spaces. A mapping $\phi : (X, \delta_X) \to (Y, \delta_Y)$ is said to be L- proximity map, if

$$\delta_X(f, g) \leq \delta_Y(\phi^{\to}(f), \phi^{\to}(g)),\ \forall\ f, g \in L^X,$$

or equivalently, $\delta_X(\phi^{\leftarrow}(f), \phi^{\leftarrow}(g)) \leq \delta_Y(f, g)$.

From the following theorem, we obtain the L-fuzzy interior operator induced by an L-fuzzy pre-proximity.

Theorem 3.2. *Let (X, δ) be a L-fuzzy pre-proximity space. Define two mapping $\mathcal{I}_\delta : L^X \to L^X$ as follows:*

$$\mathcal{I}_\delta(f)(x) = \bigvee_{g\in L^X} \{\delta^*(g, g^*) \odot g(x) \mid g \leq f\}.$$

Then
(1) $(X, \mathcal{I}_\delta)$ *is an* L*-fuzzy interior space,*
(2) *if* δ *is stratified, then so is* $\mathcal{I}_\delta$*,*
(3) *if* δ *is separated, then* $\mathcal{I}_\delta$ *is separated.*

Proof. (1) (I1) Since $\delta(\top_X, \perp_X) = \perp$,

$$\begin{aligned}\mathcal{I}_\delta(\top_X)(x) &= \bigvee_{g \in L^X} \{\ \delta^*(g, g^*) \odot g(x) \mid g \leq f\} \\ &\geq (\delta^*(\top_X, \perp_X) \odot \top_X(x)) = \top_X(x).\end{aligned}$$

(I2)

$$\begin{aligned}\mathcal{I}_\delta(f)(x) &= \bigvee_{g \in L^X} \{\delta^*(g, g^*) \odot g(x) \mid g \leq f\} \\ &\leq \bigvee_{g \in L^X} \{\bigwedge_{x \in X} (g(x) \to g(x)) \odot g(x) \mid g \leq f\} \leq f(x).\end{aligned}$$

(I3) If $f \leq g$,

$$\begin{aligned}\mathcal{I}_\delta(f)(x) &= \bigvee_{h \in L^X} \{\ \delta^*(h, h^*) \odot h(x)\ \mid h \leq f\ \} \\ &\leq \bigvee_{h \in L^X} \{\ \delta^*(h, h^*) \odot h(x)\ \mid h \leq g\ \} = \mathcal{I}_\delta(g)(x).\end{aligned}$$

(I4) From Lemma 2.2, we obtain

$$\begin{aligned}\mathcal{I}_\delta(f) \odot \mathcal{I}_\delta(h) &= (\bigvee_{g \leq f} (\delta^*(g, g^*) \odot g)) \odot (\bigvee_{k \leq h} (\delta^*(k, k^*) \odot k)) \\ &= \bigvee_{g \leq f, k \leq h} (\delta^*(g, g^*) \odot \delta^*(k, k^*) \odot g \odot k) \\ &\leq \bigvee_{g \odot k \leq f \odot h} (\delta^*(g \odot k, g^* \oplus k^*) \odot (g \odot k) \leq \mathcal{I}_\delta(f \odot h).\end{aligned}$$

(2) If δ is a stratified, then $\mathcal{I}_\delta$ is stratified by

$$\begin{aligned}\alpha \odot \mathcal{I}_\delta(f) &= \alpha \odot \bigvee_{g \leq f} (\delta^*(g, g^*) \odot g) \\ &= \bigvee_{g \leq f} (\delta^*(g, g^*) \odot (\alpha \odot g)) \\ &\leq \bigvee_{(\alpha \odot g) \leq (\alpha \odot f)} (\delta^*(\alpha \odot g, \alpha \to g^*) \odot (\alpha \odot g)) = \mathcal{I}_\delta(\alpha \odot f).\end{aligned}$$

(3) By (I2) and $\mathcal{I}_\delta(\top_x) = \bigvee_{g \leq \top_x} (\delta^*(g, g^*) \odot g) \geq \delta^*(\top_x, \top_x^*) \odot \top_x = \top_x$, we have $\mathcal{I}_\delta(\top_x) = \top_x$. □

Example 3.3. (1) Define $\delta_1 : L^X \times L^X \to L$ as $\delta_1(f, g) = \bigvee_{x, y \in X} (f(x) \odot g(y))$. Then (P1), (P2) and (P3) are easily proved.
(P4) For all $f_1, f_2, h_1, h_2 \in L^X$,

$$\begin{aligned}\delta_1(f_1 \odot f_2, h_1 \oplus h_2) &= \bigvee_{x, y \in X} ((f_1 \odot f_2)(x) \odot (h_1 \oplus h_2)(y)) \\ &= \bigvee_{x, y \in X} ((f_1 \odot f_2)(x) \odot (h_1 \oplus h_2)(y)) \\ &\leq \bigvee_{x, y \in X} ((f_1(x) \odot h_1(y)) \oplus (f_2(x) \odot h_2(y))) \\ &\leq \delta_1(f_1, h_1) \oplus \delta_1(f_2, h_2).\end{aligned}$$

Thus δ_1 is a L-fuzzy pre-proximity on X.
Since $\delta_1(\top_x, \top_x^*) = \top$, δ_1 is not separated. By Theorem 3.2, we have

$$\mathcal{I}_{\delta_1}(f) = \bigvee_{g \leq f} ((\bigwedge_{x, y \in X} (g(x) \to g(y))) \odot g) \leq f.$$

(2) Define $\delta_2 : L^X \times L^X \to L$ as $\delta_2(f, g) = \bigvee_{x \in X} (f(x) \odot g(x))$. Then (P1), (P2) and (P3) are easily proved.

(P4) For all $f_1, f_2, h_1, h_2 \in L^X$,

$$\begin{array}{ll} \delta_2(f_1 \odot f_2, h_1 \oplus h_2) & = \bigvee_{x\in X}((f_1 \odot f_2)(x) \odot (h_1 \oplus h_2)(x)) \\ & = \bigvee_{x\in X}((f_1 \odot f_2)(x) \odot (h_1 \oplus h_2)(x)) \\ & \leq \bigvee_{x\in X}((f_1(x) \odot h_1(x)) \oplus (f_2(x) \odot h_2(x))) \\ & \leq \delta_2(f_1, h_1) \oplus \delta_2(f_2, h_2). \end{array}$$

(Q)

$$\begin{array}{l} \bigwedge_{h\in L^X}(\delta_2(f,h) \oplus \delta_2(h^*, g)) \\ = \bigwedge_{h\in L^X}(\bigvee_{x\in X}(f(x) \odot h(x)) \oplus \bigvee_{x\in X}(h^*(x) \odot g(x))) \\ (\text{Put } h = g,) \\ \leq \bigvee_{x\in X}(f(x) \odot g(x)) \oplus \bigvee_{x\in X}(g^*(x) \odot g(x)) \\ = \bigvee_{x\in X}(f(x) \odot g(x)) \oplus \perp = \delta_2(f,g). \end{array}$$

Thus δ_2 is an L-fuzzy proximity on X. Since $\delta_2(\top_x, \top_x^*) = \perp$, δ_2 is separated. By Theorem 3.2, we have

$$\mathcal{I}_{\delta_2}(f) = \bigvee_{g\leq f}((\bigwedge_{x\in X}(g(x) \to g(x))) \odot g = f.$$

(3) Let $R \in L^{X\times X}$ be a reflexive fuzzy relation on X such that

$$R(x,y) \leq R(x,y) \odot R(x,y).$$

Define $\delta_3(f,g) = \bigvee_{x,y\in X}(R(x,y) \odot f(x) \odot g(y))$. Then (P1), (P2) and (P3)

(P4) For all $f_1, f_2, h_1, h_2 \in L^X$,

$$\begin{array}{l} \delta_3(f_1, h_1) \oplus \delta_3(f_2, h_2) \\ = \Big(\bigvee_{x,y\in X}(R(x,y) \odot f_1(x) \odot h_1(y)\Big) \\ \oplus\Big(\bigvee_{x,y\in X}(R(x,y) \odot f_2(x) \odot h_2(y)\Big) \\ = \bigvee_{x,y\in X}\Big(R(x,y) \odot f_1(x) \odot h_1(y)\Big) \oplus \Big(R(x,y) \odot f_2(x) \odot h_2(y)\Big) \\ \geq \bigvee_{x,y\in X}(R(x,y) \odot R(x,y) \odot f_1(x) \odot f_2(x)) \odot (h_1(y) \oplus h_2(y)) \\ \geq \bigvee_{x,y\in X}(R(x,y) \odot (f_1 \odot f_2)(x) \odot (h_1 \oplus h_2(y))) = \delta_3(f_1 \odot f_2, h_1 \oplus h_2). \end{array}$$

It is easy to see that δ_3 is Alexandrov L-fuzzy pre-proximity on X. Since

$$\begin{array}{ll} \delta_3(\alpha \odot f, \alpha \to g) & = \bigvee_{x,y\in X}((\alpha \odot f)(x) \odot (\alpha \to g)(y)) \\ & \leq \bigvee_{x,y\in X}(f(x) \odot g(y)) = \delta_3(f,g), \end{array}$$

δ_3 stratified.

Since $\delta_2(\top_x, \top_x^*) = \perp$, δ_3 is separated. If R is a transitive, δ_3 is L-fuzzy quasi-proximity on X from

(Q)

$$\begin{array}{l} \bigwedge_{h\in L^X}(\delta_3(f,h) \oplus \delta_3(h^*, g)) \\ = \bigwedge_{h\in L^X}(\bigvee_{x,y\in X}(R(x,y) \odot (f(x) \odot h(y)) \oplus \bigvee_{y,z\in X} R(y,z) \odot (h^*(y) \odot g(z))) \\ (\text{Put } h(y) = \Big(\bigvee_{z\in X}(R(x,y) \odot f(x))\Big)^*) \\ \leq \bigvee_{y\in X}(h^*(y) \odot h(y)) \oplus \bigvee_{y,z\in X}(R(y,z) \odot \bigvee_{z\in X}(R(x,y) \odot f(x) \odot g(z)))) \\ = \perp \oplus \bigvee_{x,z\in X}(\bigvee_{y\in X}(R(y,z) \odot (R(x,y))(f(x) \odot g(z)))) \\ = \bigvee_{x,z\in X}(R(x,z) \odot f(x) \odot g(z) = \delta_3(f,g). \end{array}$$

For $R(x,y) = \top_{X\times X}$, we have

$$\delta_3(f,g) = \bigvee_{x,y\in X} (\top_{X\times X}(x,y) \odot f(x) \odot g(y) = \bigvee_{x,y\in X} (f(x) \odot g(y)) = \delta_1.$$

For $R(x,y) = \triangle_{X\times X}$,

$$\triangle_{X\times X}(x,y) = \begin{cases} \top, & \text{if } y = x, \\ \bot, & \text{otherwise,} \end{cases}$$

$$\delta_3(f,g) = \bigvee_{x,y\in X} \triangle_{X\times X}(x,y) \odot f(x) \odot g(y) = \bigvee_{x\in X} (f(x) \odot g(x)) = \delta_2.$$

Thus $\delta_2 \le \delta_3 \le \delta_1$. By Theorem 3.2, we have

$$\mathcal{I}_{\delta_3}(f) \le \bigvee_{g\le f} ((\bigwedge_{x\in X} (g(x) \to g(x))) \odot g) = f.$$

From the following theorem, we obtain the L-fuzzy pre-proximity induced by an L-fuzzy interior operator.

Theorem 3.4. *Let $(X,\mathcal{I})$ be an L-fuzzy interior space. Define a map $\delta_{\mathcal{I}} : L^X \times L^X \to L$ by:*

$$\delta_{\mathcal{I}}(f,g) = \bigvee_{x\in X} (f(x) \odot \mathcal{I}^*(g^*)(x)) \quad \forall\ f,g \in L^X.$$

Then we have the following properties.
(1) *$\delta_{\mathcal{I}}$ is an L-fuzzy pre-proximity,*
(2) *if $\mathcal{I}$ is a stratified then, then so is $\delta_{\mathcal{I}}$,*
(3) *$\delta_{\mathcal{I}}(f,g) \le \bigvee_{h\in L^X} (\delta_{\mathcal{I}}(f,h) \odot \delta_{\mathcal{I}}(h^*,g))$, the equality holds if $\mathcal{I}$ is topological,*
(4) *if $\mathcal{I}$ is topological, then $\delta_{\mathcal{I}}$ is a L-fuzzy quasi-proximity on X,*
(5) *$\mathcal{I} \ge \mathcal{I}_{\delta_{\mathcal{I}}}$, the equality holds if $\mathcal{I}$ is topological,*
(6) *if $\mathcal{I}$ is separated, then $\delta_{\mathcal{I}}$ is separated,*
(7) *$\delta_{\mathcal{I}_\delta} \le \delta$,*
(8) *if $\mathcal{I}$ is generalized (resp. Alexandrov), then $\delta_{\mathcal{I}}$ is generalized (resp. Alexandrov).*

Proof. (1) (P1) Since $\mathcal{I}(\bot_X) = \bot_X$ and $\mathcal{I}(\top_X) = \top_X$, we have

$$\begin{array}{l} \delta_{\mathcal{I}}(\top_X, \bot_X) = \bigvee_{x\in X} (\top_X(x) \odot \mathcal{I}^*(\bot_X^*)(x)) = \bot, \\ \delta_{\mathcal{I}}(\bot_X, \top_X) = \bigvee_{x\in X} (\bot_X(x) \odot \mathcal{I}^*(\top_X^*)(x)) = \bot. \end{array}$$

(P2) Since $\mathcal{I}(g) \le g$, we have

$$\delta_{\mathcal{I}}(f,g) = \bigvee_{x\in X} (f(x) \odot \mathcal{I}^*(g^*)(x)) \ge \bigvee_{x\in X} (f(x) \odot g(x)).$$

(P3) If $g \le g_1$, $f \le f_1$ and by (I3), then $\mathcal{I}^*(g^*) \le \mathcal{I}^*(g_1^*)$. Thus

$$\begin{array}{rl} \delta_{\mathcal{I}}(f,g) & = \bigvee_{x\in X} (f(x) \odot \mathcal{I}^*(g^*)(x)) \\ & \le \bigvee_{x\in X} (f_1(x) \odot \mathcal{I}^*(g_1^*)(x)) = \delta_{\mathcal{I}}(f_1, g_1). \end{array}$$

(P4) For $f_1, f_2, g_1, g_2 \in L^X$,

$$\begin{aligned} \delta_{\mathcal{I}}(f_1, g_1) \oplus \delta_{\mathcal{I}}(f_2, g_2) &= \bigvee_{x \in X}(f_1(x) \odot \mathcal{I}^*(g_1^*)(x)) \\ &\oplus \bigvee_{x \in X}(f_2(x) \odot \mathcal{I}^*(g_2^*)(x)) \\ &\geq \bigvee_{x \in X}(f_1(x) \odot f_2(x) \odot (\mathcal{I}^*(g_1^*)(x) \oplus \mathcal{I}^*(g_2^*)(x)) \\ &\quad \text{(by Lemma 2.2 (11))} \\ &\geq \bigvee_{x \in X}(f_1(x) \odot f_2(x) \odot \mathcal{I}^*((g_1 \oplus g_2)^*)(x)) \\ &= \delta_{\mathcal{I}}(f_1 \odot f_2, g_1 \oplus g_2). \end{aligned}$$

So $\delta_{\mathcal{I}}$ is a L-fuzzy pre-proximity.

(2)

$$\begin{aligned} \delta_{\mathcal{I}}(\alpha \odot f, \alpha \to g) &= \bigvee_{x \in X}(\alpha \odot f(x) \odot \mathcal{I}^*(\alpha \odot g^*)(x)) \\ &\leq \bigvee_{x \in X}(\alpha \odot f(x) \odot (\alpha \to \mathcal{I}^*(g^*)(x)) \\ &\leq \bigvee_{x \in X}(f(x) \odot \mathcal{I}^*(g^*)(x)) = \delta_{\mathcal{I}}(f, g). \end{aligned}$$

(3) For $f, g, h \in L^X$,

$$\begin{aligned} &\delta^*_{\mathcal{I}}(f, h) \odot \delta^*_{\mathcal{I}}(h^*, g) \\ &= \Big(\bigvee_{x \in X}(f(x) \odot \mathcal{I}^*(h^*)(x))\Big)^* \odot \Big(\bigvee_{x \in X}(h^*(x) \odot \mathcal{I}^*(g^*)(x))\Big)^* \\ &= \bigwedge_{x \in X}(f(x) \to \mathcal{I}(h^*)(x)) \odot \bigwedge_{x \in X}(h^*(x) \to \mathcal{I}(g^*)(x)) \qquad (\text{Since } \mathcal{I}(h^*) \leq h^*) \\ &\leq \bigwedge_{x \in X}(f(x) \to h^*(x)) \odot \bigwedge_{x \in X}(h^*(x) \to \mathcal{I}(g^*)(x)) \\ &\leq \bigwedge_{x \in X}(f(x) \to \mathcal{I}(g^*)(x)) = \delta^*_{\mathcal{I}}(f, g). \end{aligned}$$

Then $\delta_{\mathcal{I}}(f, g) \leq \bigwedge_{h \in L^X}(\delta_{\mathcal{I}}(f, h) \oplus \delta_{\mathcal{I}}(h^*, g))$.

If $\mathcal{I}$ is topological, then

$$\begin{aligned} &\bigvee_{h \in L^X}(\delta^*_{\mathcal{I}}(f, h) \odot \delta^*_{\mathcal{I}}(h^*, g)) \\ &= \bigvee_{h \in L^X}(\bigwedge_{x \in X}(f(x) \to \mathcal{I}(h^*)(x)) \odot \bigwedge_{x \in X}(h^*(x) \to \mathcal{I}(g^*)(x))) \quad (\text{Put } h^* = \mathcal{I}(g^*)) \\ &\geq \bigwedge_{x \in X}(f(x) \to \mathcal{I}(\mathcal{I}(g^*)(x))) \odot \bigwedge_{x \in X}(\mathcal{I}(g^*)(x) \to \mathcal{I}(g^*)(x)) \\ &= \bigwedge_{x \in X}(f(x) \to \mathcal{I}(g^*)(x)) = \delta^*_{\mathcal{I}}(f, g). \end{aligned}$$

(4) By (2), it is trivial.

(5)

$$\begin{aligned} \mathcal{I}_{\delta_{\mathcal{I}}}(f)(x) &= \bigvee_{g \leq f}(\delta^*_{\mathcal{I}}(g, g^*) \odot g(x))) \\ &= \bigvee_{g \leq f} \bigwedge_{x \in X}(g(x) \to \mathcal{I}(g)(x)) \odot g(x) \\ &\leq \bigvee_{\mathcal{I}(g) \leq \mathcal{I}(f)} \mathcal{I}(g)(x) = \mathcal{I}(f)(x). \end{aligned}$$

If $\mathcal{I}$ is topological, then

$$\begin{aligned} \mathcal{I}_{\delta_{\mathcal{I}}}(f)(x) &= \bigvee_{g \leq f}(\delta^*_{\mathcal{I}}(g, g^*) \odot g) \\ &= \bigvee_{g \leq f}(\bigwedge_{x \in X}(g(x) \to \mathcal{I}(g)) \odot g(x)) \\ &\geq \bigwedge_{x \in X}(\mathcal{I}(f)(x) \to \mathcal{I}(\mathcal{I}(f)(x)) \odot \mathcal{I}(f)(x)) \quad (\text{Put } g = \mathcal{I}(g)) \\ &= \mathcal{I}(f)(x). \end{aligned}$$

(6) Let $\mathcal{I}$ be separated, then $\delta_{\mathcal{I}}(\top_z, \top_z^*) = \bigvee_{x \in X}(\top_z(x) \odot \mathcal{I}^*(\top_z)(x)) = \bot$.

(7)

$$\begin{aligned} \delta_{\mathcal{I}_\delta}(f, g) &= \bigvee_{x \in X}(f(x) \odot \mathcal{I}^*_\delta(g^*)(x)) \\ &= \bigvee_{x \in X}(f(x) \odot \Big(\bigvee_{h \leq g^*}(\delta^*(h, h^*) \odot h(x))\Big)^*) \\ &\leq \bigvee_{x \in X}(f(x) \odot \Big(\bigvee_{h \leq g^*}(\bigwedge_{x \in X}(h(x) \to h(x)) \odot h(x))\Big)^*) \\ &\leq \bigvee_{x \in X}(f(x) \odot g(x)) \leq \delta(f, g). \end{aligned}$$

□

Corollary 3.5. *Let $(X,\mathcal{I})$ be an L-fuzzy interior space. Define a map $\delta^s_{\mathcal{I}} : L^X \times L^X \to L$ by:*

$$\delta^s_{\mathcal{I}}(f,g) = \bigvee_{x\in X} (g(x) \odot \mathcal{I}^*(f^*)(x)) \quad \forall\ f,g \in L^X.$$

Then we have the following properties.

(1) *$\delta^s_{\mathcal{I}}$ is an L-fuzzy pre-proximity. If $\mathcal{I}$ is stratified, then $\delta^s_{\mathcal{I}}$ is stratified.*
(2) *$\delta^s_{\mathcal{I}}(f,g) \le \bigvee_{h\in L^X}(\delta^s_{\mathcal{I}}(f,h) \odot \delta^s_{\mathcal{I}}(h^*,g))$, the equality holds if $\mathcal{I}$ is topological.*
(3) *If $\mathcal{I}$ is topological, then $\delta^s_{\mathcal{I}}$ is a L-fuzzy quasi-proximity on X.*
(4) *$\mathcal{I} \ge \mathcal{I}_{\delta^s_{\mathcal{I}}}$, the equality holds, if $\mathcal{I}$ is topological.*
(5) *If $\mathcal{I}$ is separated, then $\delta^s_{\mathcal{I}}$ is separated.*
(6) *$\delta^s_{\mathcal{I}_\delta} \le \delta^s$.*
(7) *If $\mathcal{I}$ is generalized (resp. Alexandrov), then $\delta^s_{\mathcal{I}}$ is generalized (resp. Alexandrov).*

Example 3.6. (1) Define $\mathcal{I}_1 : L^X \to L^X$ as $\mathcal{I}_1(f) = \bigwedge_{x\in X} f(x)$. Then (I1), (I2), (I3) and (I4) are easily proved. Thus $\mathcal{I}_1$ is a topological L-fuzzy interior operator on X. Since $\mathcal{I}_1(\top_x) = \perp_X$ and $\delta_{\mathcal{I}_1}(\top_x, \top^*_x) = \top$, $\mathcal{I}_1$ and $\delta_{\mathcal{I}_1}$ are not separated. So by Theorem 3.5, we have

$$\begin{aligned}
\delta_{\mathcal{I}_1}(f,g) &= \textstyle\bigvee_{x\in X}(f(x) \odot \mathcal{I}^*_1(g^*)(y)) = \bigvee_{x,y\in X}(f(x) \odot g(y)),\\
\delta^s_{\mathcal{I}_1}(f,g) &= \textstyle\bigvee_{x\in X}(g(x) \odot \mathcal{I}^*_1(f^*)(y)) = \bigvee_{x,y\in X}(f(y) \odot g(x)).
\end{aligned}$$

Since $\mathcal{I}_1$ is topological, $\mathcal{I}_{\delta_{\mathcal{I}_1}} = \mathcal{I}_1$.

(3) Define $\mathcal{I}_2 : L^X \to L^X$ as $\mathcal{I}_2(f) = f$. Then (I1), (I2), (I3) and (I4) are easily proved. Thus $\mathcal{I}_2$ is a topological L-fuzzy interior operator on X. Since $\mathcal{I}_2(\top_x) = \top_X$ and $\delta_{\mathcal{I}_2}(\top_x, \top^*_x) = \perp$, $\mathcal{I}_2$ and $\delta_{\mathcal{I}_2}$ are separated. So by Theorem 3.5, we have

$$\begin{aligned}
\delta_{\mathcal{I}_1}(f,g) &= \textstyle\bigvee_{x\in X}(f(x) \odot \mathcal{I}^*_1(g^*)(x)) = \bigvee_{x\in X}(f(x) \odot g(x)),\\
\delta^s_{\mathcal{I}_1}(f,g) &= \textstyle\bigvee_{x\in X}(g(x) \odot \mathcal{I}^*_1(f^*)(x)) = \bigvee_{x\in X}(g(x) \odot f(x)).
\end{aligned}$$

Since $\mathcal{I}_2$ is topological, $\mathcal{I}_{\delta_{\mathcal{I}_2}} = \mathcal{I}_2$.

4. The relationships between L-fuzzy pre-proximities and L-fuzzy topologies

Theorem 4.1. *Let δ be Alexandrov L-fuzzy pre-proximity on X. Define a mapping $\mathcal{T}_\delta : L^X \to L$ by: $\mathcal{T}_\delta(f) = \delta^*(f, f^*)$. Then*

(1) *$\mathcal{T}_\delta$ is an L-fuzzy topology on X,*
(2) *if δ is stratified, then so is $\mathcal{T}_\delta$,*
(3) *if δ is separated, then so is $\mathcal{T}_\delta$.*

Proof. (1) (T1) $\mathcal{T}_\delta(\perp_X) = \delta^*(\perp_X, \top_X) = \top$, $\mathcal{T}_\delta(\top_X) = \delta^*(\top_X, \perp_X) = \top$.

(T2) $\mathcal{T}_\delta(f \odot g) = \delta^*(f \odot g, f^* \oplus g^*) \ge \delta^*(f, f^*) \odot \delta^*(g, g^*) = \mathcal{T}_\delta(f) \odot \mathcal{T}_\delta(g)$.

(T3) $\mathcal{T}_\delta(\bigvee_{i\in\Gamma} f_i) = \delta^*(\bigvee_{i\in\Gamma} f_i, \bigwedge_{i\in\Gamma} f^*_i) \ge \bigwedge_{i\in\Gamma} \delta^*(f_i, f^*_i) = \bigwedge_{i\in\Gamma} \mathcal{T}_\delta(f_i)$.

(2) $\mathcal{T}_\delta(\alpha \odot f) = \delta^*(\alpha \odot f, \alpha \to f^*) \ge \delta^*(f, f^*) = \mathcal{T}_\delta(f)$,

$$\mathcal{T}_\delta(\alpha \to f) = \delta^*(\alpha \to f, \alpha \odot f^*) \ge \delta^*(f, f^*) = \mathcal{T}_\delta(f).$$

(3) It is easy. $\square$

Theorem 4.2. *Let $(X, \mathcal{I})$ be an L-fuzzy interior space. Define a mapping $\mathcal{T}_{\mathcal{I}_\delta} : L^X \to L$ by: $\mathcal{T}_{\mathcal{I}_\delta}(f) = \bigwedge_{x \in X}(f(x) \to \mathcal{I}_\delta(f)(x))$. Then*

(1) $\mathcal{T}_{\mathcal{I}_\delta}$ is an L-fuzzy topology on X,

(2) if $\mathcal{I}$ is Alexandrov (resp. strong, separated), then $\mathcal{T}_{\delta_\mathcal{I}}$ is Alexandrov (resp. strong, separated).

Proof. (1) (T1) $\mathcal{T}_{\mathcal{I}_\delta}(\top_X) = \bigwedge_{x \in X}(\top_X(x) \to \mathcal{I}_\delta(\top_X)) = \top$,

$$\mathcal{T}_{\mathcal{I}_\delta}(\bot_X) = \bigwedge_{x \in X}(\bot_X(x) \to \mathcal{I}_\delta(\bot_X)) = \top.$$

(T2)

$$\begin{aligned}
\mathcal{T}_{\mathcal{I}_\delta}(f \odot g) &= \bigwedge_{x \in X}(f \odot g)(x) \to \mathcal{I}_\delta(f \odot g)(x)) \\
&\geq \bigwedge_{x \in X}(f(x) \odot g(x) \to (\mathcal{I}_\delta(f)(x) \odot \mathcal{I}_\delta(g)(x))) \\
&\quad \text{(by Lemma 2.2 (11))} \\
&\geq \bigwedge_{x \in X}(f(x) \to \mathcal{I}_\delta(f)(x)) \odot \bigwedge_{x \in X}(g(x) \to \mathcal{I}_\delta(g)(x)) \\
&= \mathcal{T}_{\mathcal{I}_\delta}(f) \odot \mathcal{T}_{\mathcal{I}_\delta}(g).
\end{aligned}$$

(T3)

$$\begin{aligned}
\mathcal{T}_{\mathcal{I}_\delta}(\bigvee_{i \in \Gamma} f_i) &= \bigwedge_{x \in X}(\bigvee_{i \in \Gamma} f_i(x) \to \mathcal{I}_\delta(\bigvee_{i \in \Gamma} f_i)(x)) \\
&\geq \bigwedge_{x \in X}(\bigvee_{i \in \Gamma} f_i(x) \to \bigvee_{i \in \Gamma} \mathcal{I}_\delta(f_i)(x)) \\
&\quad \text{(by Lemma 2.2 (16))} \\
&\geq \bigwedge_{i \in \Gamma} \bigwedge_{x \in X}(f_i(x) \to \mathcal{I}_\delta(f_i)(x))) = \bigwedge_{i \in \Gamma} \mathcal{T}_{\mathcal{I}_\delta}(f_i).
\end{aligned}$$

(2) By Lemma 2.2 (textcolorred16), we have

$$\begin{aligned}
\mathcal{T}_{\mathcal{I}_\delta}(\bigwedge_{i \in \Gamma} f_i) &= \bigwedge_{x \in X}(\bigwedge_{i \in \Gamma} f_i(x) \to \mathcal{I}_\delta(\bigwedge_{i \in \Gamma} f_i)(x)) \\
&= \bigwedge_{x \in X}(\bigwedge_{i \in \Gamma} f_i(x) \to \bigwedge_{i \in \Gamma} \mathcal{I}_\delta(f_i)(x))) \\
&\geq \bigwedge_{i \in \Gamma}(\bigwedge_{x \in X} f_i(x) \to \mathcal{I}_\delta(f_i)(x))) = \bigwedge_{i \in \Gamma} \mathcal{T}_{\mathcal{I}_\delta}(f_i).
\end{aligned}$$

Hence, $\mathcal{T}_{\mathcal{I}_\delta}$ is Alexandrov L-fuzzy topology on X. By Lemma 2.2 (14) and (18), we have

$$\begin{aligned}
\mathcal{T}_{\mathcal{I}_\delta}(\alpha \odot f) &= \bigwedge_{x \in X}((\alpha \odot f(x)) \to (\mathcal{I}_\delta(\alpha \odot f)(x))) \\
&\geq \bigwedge_{x \in X}((\alpha \odot f(x)) \to (\alpha \odot \mathcal{I}_\delta(f)(x))) \\
&\geq \bigwedge_{x \in X}(f(x) \to \mathcal{I}_\delta(f)(x)) = \mathcal{T}_{\mathcal{I}_\delta}(f).
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_{\mathcal{I}_\delta}(\alpha \to f) &= \bigwedge_{x \in X}((\alpha \to f(x)) \to \mathcal{I}_\delta(\alpha \to f)(x)) \\
&= \bigwedge_{x \in X}((\alpha \to f(x)) \to (\alpha \to \mathcal{I}_\delta(f)(x))) \\
&\geq \bigwedge_{x \in X}(f(x) \to \mathcal{I}_\delta(f)(x)) = \mathcal{T}_{\mathcal{I}_\delta}(f).
\end{aligned}$$

Other cases are easily proved. $\square$

Theorem 4.3. *Let (X, δ) be an L-fuzzy pre-proximity space. Define a mapping $\mathcal{T}_\delta^{(1)} : L^X \to L$ by: $\mathcal{T}_\delta^{(1)}(f) = \bigwedge_{x \in X}(f(x) \to \delta^*(\top_x, f^*))$. Then*

(1) $\mathcal{T}_\delta^{(1)}$ is a L-fuzzy topology on X,

(2) if δ is Alexandrov and $\delta(\alpha \odot f, g) \geq \alpha \odot \delta(f, g)$, then so is $\mathcal{T}^{(1)}$ and $\mathcal{T}_\delta^{(1)} \geq \mathcal{T}_\delta$,

(3) if δ is separated, then $\mathcal{T}_\delta^{(1)}$ is separated.

Proof. (1) (T1)

$$\begin{array}{ll}\mathcal{T}_\delta^{(1)}(\bot_X) & = \top, \\ \mathcal{T}_\delta^{(1)}(\top_X) & = \bigwedge_{x\in X}(\top_X(x) \to \delta^*(\top_x, \top_X^*)) = (\bigvee_{x\in X}\delta(\top_x, \top_X^*))^* = \top.\end{array}$$

(T2)

$$\begin{array}{ll}\mathcal{T}_\delta^{(1)}(f\odot g) & = \bigwedge_{x\in X}((f\odot g)(x) \to \delta^*(\top_x, f^*\oplus g^*)) \\ & \geq \bigwedge_{x\in X}((f\odot g)(x) \to \delta^*(\top_x, f^*)\odot\delta^*(\top_x, g^*)) \\ & \geq \bigwedge_{x\in X}((f(x) \to \delta^*(\top_x, f^*))\odot(g(x)\to\delta^*(\top_x, g^*))) \\ & \geq \mathcal{T}_\delta^{(1)}(f)\odot\mathcal{T}_\delta^{(1)}(g).\end{array}$$

(T3)

$$\begin{array}{ll}\mathcal{T}_\delta^{(1)}(\bigvee_{i\in\Gamma} f_i) & = \bigwedge_{x\in X}(\bigvee_{i\in\Gamma} f_i(x) \to \delta^*(\top_x, \bigwedge_{i\in\Gamma} f_i^*)) \\ & \geq \bigwedge_{x\in X}(\bigvee_{i\in\Gamma} f_i(x) \to \bigvee_{i\in\Gamma}\delta^*(\top_x, f_i^*)) \\ & \geq \bigwedge_{i\in\Gamma}\bigwedge_{x\in X}(f_i(x)\to\delta^*(\top_x, f_i^*)) = \bigwedge_{i\in\Gamma}\mathcal{T}_\delta^{(1)}(f_i).\end{array}$$

(2) If δ is Alexandrov, then

$$\begin{array}{ll}\mathcal{T}_\delta^{(1)}(\bigwedge_{i\in\Gamma} f_i) & = \bigwedge_{x\in X}(\bigwedge_{i\in\Gamma} f_i(x) \to \delta^*(\top_x, \bigvee_{i\in\Gamma} f_i^*)) \\ & = \bigwedge_{x\in X}(\bigwedge_{i\in\Gamma} f_i(x) \to \bigwedge_{i\in\Gamma}\delta^*(\top_x, f_i^*)) \\ & \geq \bigwedge_{i\in\Gamma}\bigwedge_{x\in X}(f_i(x)\to\delta^*(\top_x, f_i^*)) = \bigwedge_{i\in\Gamma}\mathcal{T}_\delta^{(1)}(f_i).\end{array}$$

Thus $\mathcal{T}_\delta^{(1)}$ is Alexandrov L-fuzzy topology on X.

If $\delta(\alpha\odot f, g)\geq \alpha\odot\delta(f,g)$, we have

$$\begin{array}{ll}\mathcal{T}_\delta^{(1)}(f) & = \bigwedge_{x\in X}(f(x)\to\delta^*(\top_x, f^*)) \\ & = (\bigvee_{x\in X}(f(x)\odot\delta(\top_x, f^*)))^* \\ & \geq (\delta(\bigvee_{x\in X}(f(x)\odot\top_x), f^*))^* \\ & = \delta^*(f, f^*) = \mathcal{T}_\delta(f).\end{array}$$

(3) It is easily proved. □

5. Galois correspondences

Theorem 5.1. *Let (X,δ_X) and (Y,δ_Y) be L-fuzzy pre-proximity spaces and $\phi : (X,\delta_X)\to(Y,\delta_Y)$ is a LF-proximity map. Then*
(1) $\varphi : (X,\mathcal{I}_{\delta_X})\to(Y,\mathcal{I}_{\delta_Y})$ *is a LF-interior map,*
(2) $\varphi : (X,\mathcal{T}_{\delta_X})\to(Y,\mathcal{T}_{\delta_Y})$ *is a LF-continuous map,*
(3) $\varphi : (X,\mathcal{T}^{(1)}_{\delta_X})\to(Y,\mathcal{T}^{(1)}_{\delta_Y})$ *is a LF-continuous map.*

Proof. For each $f\in L^Y$,
(1)

$$\begin{array}{ll}\varphi^{\leftarrow}(\mathcal{I}_{\delta_Y}(f))(x) & = \varphi^{\leftarrow}(\bigvee_{g\leq f}\delta_Y^*(g, g^*)\odot g(x) \\ & = \bigvee_{g\leq f}\delta_Y^*(g,g)\odot g(\varphi(x))) \\ & \leq \bigvee_{g\leq f}\delta_X^*(\varphi^{\leftarrow}(g), \varphi^{\leftarrow}(g^*))\odot g(\phi(x)) \\ & \leq \bigvee_{\varphi^{\leftarrow}(g)\leq\phi^{\leftarrow}(f)}(\delta_X^*(\varphi^{\leftarrow}(g), \varphi^{\leftarrow}(g^*)\odot\phi^{\leftarrow}(g)(x)) = \mathcal{I}_{\delta_X}(\varphi^{\leftarrow}(f))(x).\end{array}$$

(2) $\mathcal{T}_{\delta_X}(\varphi^{\leftarrow}(f)) = \delta_X^*(\varphi^{\leftarrow}(f), \varphi^{\leftarrow}(f^*))\geq\delta_Y^*(f,f^*) = \mathcal{T}_{\delta_Y}(f).$

(3)

$$\begin{aligned}\mathcal{T}^{(1)}_{\delta_X}(\varphi^{\leftarrow}(f)) &= \bigwedge_{x\in X}(\varphi^{\leftarrow}(f)(x) \to \delta^*_X(\varphi^{\leftarrow}(\top_{\varphi(x)}), \varphi^{\leftarrow}(f^*)))\\ &= \bigwedge_{x\in X}(f(\varphi(x)) \to \delta^*_X(\varphi^{\leftarrow}(\top_{\varphi(x)}), \varphi^{\leftarrow}(f^*)))\\ &\geq \bigwedge_{x\in X}(f(\varphi(x)) \to \delta^*_Y((\top_{\varphi(x)}), f^*))\\ &\geq \bigwedge_{y\in Y}(f(y) \to \delta^*_Y(\top_y, f^*)) = \mathcal{T}^{(1)}_{\delta_Y}(f).\end{aligned}$$

□

Theorem 5.2. *Let $(X, \mathcal{I}_X)$ and $(Y, \mathcal{I}_Y)$ be L-fuzzy interior spaces and $\varphi : (X, \mathcal{I}_X) \to (Y, \mathcal{I}_Y)$ be an LF-interior map. Then $\varphi : (X, \delta_{\mathcal{I}_X}) \to (Y, \delta_{\mathcal{I}_Y})$ is a LF-proximity map.*

Proof. Since $\mathcal{I}_Y(g^*)(\varphi(x)) \leq \mathcal{I}_X(\varphi^{\leftarrow}(g^*))(x)$, we have

$$\begin{aligned}\delta_{\mathcal{I}_X}(\varphi^{\leftarrow}(f), \varphi^{\leftarrow}(g)) &= \bigvee_{x\in X}\Big(\varphi^{\leftarrow}(f)(x) \odot \mathcal{I}^*_X(\varphi^{\leftarrow}(g)^*)(x)\Big)\\ &\leq \bigvee_{x\in X}\Big(f(\varphi(x)) \odot \mathcal{I}^*_Y(g^*)(\varphi(x))\Big)\\ &\leq \bigvee_{y\in Y}\Big(f(y) \odot \mathcal{I}^*_Y(g^*)(y)\Big) = \delta_{\mathcal{I}_Y}(f, g).\end{aligned}$$

□

Definition 5.3 ([1])**.** Suppose that $F : \mathcal{D} \to \mathcal{C}, G : \mathcal{C} \to \mathcal{D}$ are concrete functors. The pair (F, G) is called a Galois correspondence between $\mathcal{C}$ and $\mathcal{D}$, if for each $Y \in \mathcal{C}, id_Y : F \circ G(Y) \to Y$ is a $\mathcal{C}$-morphism, and for each $X \in \mathcal{D}, id_X : X \to G \circ F(X)$ is a $\mathcal{D}$-morphism.

If (F, G) is a Galois correspondence, then it is easy to check that F is a left adjoint of G, or equivalently that G is a right adjoint of F.

The category of separated L-fuzzy pre-proximity spaces with LF-proximity mappings as morphisms is denoted by **SPROX**.

The category of separated L-fuzzy interior spaces with LF-interior mappings as morphisms is denoted by **SFI**.

From Theorems 3.2 and 5.1, we obtain a concrete functor $\Upsilon : \mathbf{SPROX} \to \mathbf{SFI}$ defined as

$$\Upsilon(X, \delta) = (X, \mathcal{I}_\delta), \Upsilon(\varphi) = \varphi.$$

From Theorems 3.4 and 5.2, we obtain a concrete functor $\Omega : \mathbf{SFI} \to \mathbf{SPROX}$ defined as

$$\Omega(X, \mathcal{I}) = (X, \delta_{\mathcal{I}}), \Omega(\varphi) = \varphi.$$

Theorem 5.4. *$\Omega : \boldsymbol{SFI} \to \boldsymbol{SPROX}$ is a left adjoint of $\Upsilon : \boldsymbol{SPROX} \to \boldsymbol{SFI}$, i.e., (Υ, Ω) is a Galois correspondence.*

Proof. By Theorem 3.4 (5), if $\mathcal{I}_X$ is an separated L-fuzzy interior operator on a set X, then $\Upsilon(\Omega(I_X)) = \mathcal{I}_{\delta_{\mathcal{I}_X}} \leq \mathcal{I}_X$. Thus the identity map $id_X : (X, \mathcal{I}_X) \to (X, \mathcal{I}_{\mathcal{T}_{\mathcal{I}_X}}) = (X, \Upsilon(\Omega(I_X)))$ is an LF-interior map. Moreover, if δ_Y is a separated L-fuzzy pre-proximity on a set Y, by Theorem 3.4 (7), $\Omega(\Upsilon(\delta_Y)) = \delta_{\mathcal{I}_{\delta_Y}} \leq \delta_Y$. So the identity map $id_Y : (Y, \delta_{\mathcal{I}_{\delta_Y}}) \to (Y, \delta_Y)$ is LF-proximity map. Hence (Υ, Ω) is a Galois correspondence. □

6. Conclusions

In this paper, L-fuzzy pre-proximities and L-fuzzy interior operators in complete residuated lattice are investigated. From a given L-fuzzy pre-proximity δ, we can obtain an L-fuzzy interior operator $\mathcal{I}_\delta$ (see Theorem 3.2). Conversely, for given L-fuzzy interior space $\mathcal{I}$, we obtain L-fuzzy pre-proximity $\delta_\mathcal{I}$ (see Theorem 3.4) and L-fuzzy topologies $\mathcal{T}_\delta$ and $\mathcal{T}_{\mathcal{I}_\delta}$ (Theorems 4.1, 4.2 and 4.3).

It is also shown that there is a Galois correspondence between the category of (separated) L-fuzzy interior spaces and that of (separated) L-fuzzy pre-proximity spaces (theorem 5.4).

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