Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 17, No. 2, (April 2019) pp. 175–189
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2019.17.2.175

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 17, No. 2, April 2019

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Received 21 December 2018; Revised 19 January 2019; Accepted 23 January 2019

Abstract. In this paper, we introduce the notions of fuzzy soft interior spaces and fuzzy soft closure spaces. In particular, we prove the existence of initial fuzzy soft interior structures and initial fuzzy soft closure structures. From this fact, the category **FSC** is a topological category over **SET**.

2010 AMS Classification: 54A40, 54A10, 54C10

Keywords: Initial fuzzy soft interior spaces, Initial fuzzy soft closure spaces, Fuzzy soft topological spaces.

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1. Introduction and preliminaries

In 1999, Molodtsov [12] introduced the concept of soft set which is completely new approach for modelling uncertainties. Applications of soft set theory in other disciplines and real life problems are now catching momentum. Molodtsov [12] successfully applied the soft set theory into several directions, such as smoothness of functions, game theory, Riemann integration, theory of measurement, and so on. Maji et al. [11] gave first practical application of soft sets in decision making problems. They have also introduced the concept of fuzzy soft set, a more generalized concept, which is a combination of fuzzy set and soft set and also studied some of its properties. Ahmed and Kharal [4, 9] also made further contribution to the properties of fuzzy soft sets and fuzzy soft mappings. Soft set and fuzzy soft set theories have a rich potential for applications in several directions, a few of which have been shown by some authors [12, 13]. Moreover, Shabir and Naz [18] presented soft topological spaces and defined some concepts based on soft sets. Tanay and Kandemir [20] initially introduced the concept of fuzzy soft topological space using fuzzy soft sets, and studied the basic notions by following Chang's fuzzy topology [6]. Pazar Varol and Aygun [21] defined the fuzzy soft topology in sense of Lowen. A. Abdülkadir et al. [3] defined fuzzy soft topology in Šostak's sense [19]. G. Senel made a wide research on soft sets and its applications in [14, 15, 16, 17].

In this paper, we introduce the notions of fuzzy soft interior spaces and fuzzy soft closure spaces. We show the existence of initial fuzzy soft interior structures and initial fuzzy soft closure structures. From this fact, the category **FSC** is a topological category over **SET**. Furthermore, we can define the products of fuzzy soft interior spaces and fuzzy soft closure spaces. In particular, an initial structure of fuzzy soft topological spaces could be obtained by the initial structure of fuzzy soft closure spaces.

Throughout the paper, X refers to an initial universe, I^X is the set of all fuzzy sets on X (where $I = [0,1]$, $I_0 = (0,1]$ and $I_1 = [0,1)$, $I_{01} = (0,1)$). For $\alpha \in I$, $\overline{\alpha}(x) = \alpha$ for all $x \in X$, E is the set of all parameters. $f_E : E \to I^X$ is called a fuzzy soft set on the universe X. For $e \in E$ then f_e is a fuzzy subset of X. $\widetilde{(X,E)}$ denotes the collection of all fuzzy soft sets on X. A fuzzy soft set f_E is a fuzzy soft subset of a fuzzy soft set g_E, denoted by $f_E \sqsubseteq g_E$, if $f_e \leq g_e$ $\forall e \in E$. f_E is fuzzy soft equal to g_E if $f_E \sqsubseteq g_E$ and $g_E \sqsubseteq f_E$. The intersection $f_E \sqcap g_E$ of fuzzy soft sets f_E, g_E is defined as the soft set h_E such that $h_e = f_e \wedge g_e$ $\forall e \in E$, and the union $f_E \sqcup g_E$ is defined as the soft set h_E such that $h_e = f_e \vee g_e$ $\forall e \in E$. Let $x \in X$, denote a fuzzy soft point by x_E for which $x_e = x_1$ $\forall e \in E$ where x_1 is the fuzzy point with value 1 at x and 0 otherwise. A fuzzy soft point x_E is contained in a fuzzy soft set f_E if $1 = f_e(x)$ $\forall e \in E$, and thus x_E does not belong to f_E if $1 \neq f_e(x)$ for some $e \in E$. The fuzzy soft set f_E is called a null fuzzy soft set if $f_e = \overline{0}$ for any $e \in E$. The null fuzzy soft set will be denoted by $\tilde{\Phi}$. The soft set f_E is called an absolute fuzzy soft set if $f_e = \overline{1}$ for any $e \in E$. The absolute fuzzy soft set is denoted by $\tilde{E}$. The fuzzy soft complement f_E^c of a fuzzy soft set f_E is defined by: $(f_e)^c(x) = 1 - f_e(x)$ $\forall x \in X$ for any $e \in E$. It is evident that $(\tilde{E})^c = \tilde{\Phi}$ and $(\tilde{\Phi})^c = \tilde{E}$. A fuzzy soft set f_E on X is called α-absolute fuzzy soft set denoted by $\tilde{E}^\alpha$, if $f_e = \overline{\alpha}$ for each $e \in E$. Clearly $(\tilde{E}^\alpha)^c = \tilde{E}^{1-\alpha}$. All definitions and properties of fuzzy soft sets on X are found in [4, 20, 3, 7, 10, 5, 8, 1]. Connectedness in fuzzy soft topological spaces has been introduced by S.E. Abbas et al. in [2].

Definition 1.1 ([1]). A map $\mathbf{I} : E \times \widetilde{(X,E)} \times I_0 \to \widetilde{(X,E)}$ is called a fuzzy soft interior operator on X, if $\mathbf{I}$ satisfies the following conditions for each $e \in E$, $f_E, g_E \in \widetilde{(X,E)}$, $r, s \in I_0$:

(I1) $\mathbf{I}(e, \tilde{E}, r) = \tilde{E}$,
(I2) $\mathbf{I}(e, f_E, r) \sqsubseteq f_E$,
(I3) if $f_E \sqsubseteq g_E$ and $r \leq s$, then $\mathbf{I}(e, f_E, r) \sqsubseteq \mathbf{I}(e, g_E, s)$,
(I4) $\mathbf{I}(e, f_E \sqcap g_E, r \wedge s) \sqsupseteq \mathbf{I}(e, f_E, r) \sqcap \mathbf{I}(e, g_E, s)$.

The pair $(X, \mathbf{I})$ is called a fuzzy soft interior space.

A fuzzy soft interior space $(X, \mathbf{I})$ is called topological provided that

$$\mathbf{I}(e, \mathbf{I}(e, f_E, r), r) = \mathbf{I}(e, f_E, r).$$

A fuzzy soft interior space $(X, \mathbf{I})$ is called weakly stratified, if for each $e \in E$,

$$\mathbf{I}(e, \tilde{E}^\alpha, r) \sqsupseteq \tilde{E}^\alpha \;\; \forall \tilde{E}^\alpha \in \widetilde{(X,E)},\; r \in I_0.$$

Let $(X, (\mathbf{I}_1)_E)$ and $(Y, (\mathbf{I}_2)_F)$ be fuzzy soft interior spaces. A function ϕ_ψ : $(X, (\mathbf{I}_1)_E) \to (Y, (\mathbf{I}_2)_F)$ is called fuzzy soft $\mathbf{I}$-map, if for all $f_E \in \widetilde{(Y,F)}$, $e \in E, r \in$

I_0,

$$\mathbf{I}_1(e, (\phi_\psi)^{-1}(f_E), r) \sqsupseteq (\phi_\psi)^{-1}(\mathbf{I}_2(\psi(e), f_E, r)).$$

Let $(\mathbf{I}_1)_E$ and $(\mathbf{I}_2)_E$ be fuzzy soft interior operators on X. We say that $(\mathbf{I}_1)_E$ is finer than $(\mathbf{I}_2)_E$ ($(\mathbf{I}_2)_E$ is coarser than $(\mathbf{I}_1)_E$), denoted by $(\mathbf{I}_2)_E \sqsubseteq (\mathbf{I}_1)_E$, if

$$\mathbf{I}_2(e, f_E, r) \sqsubseteq \mathbf{I}_1(e, f_E, r) \ \ \forall f_E \in \widetilde{(X, E)},\ e \in E, r \in I_0.$$

Definition 1.2 ([1]). A map $\mathbf{C} : E \times \widetilde{(X, E)} \times I_1 \to \widetilde{(X, E)}$ is called a fuzzy soft closure operator on X if $\mathbf{C}$ satisfies the following conditions for each $e \in E$, $f_E, g_E \in \widetilde{(X, E)}$, $r, s \in I_1$:

(C1) $\mathbf{C}(e, \tilde{\Phi}, r) = \tilde{\Phi}$,
(C2) $\mathbf{C}(e, f_E, r) \sqsupseteq f_E$,
(C3) if $f_E \sqsubseteq g_E$, then $\mathbf{C}(e, f_E, r) \sqsubseteq \mathbf{C}(e, g_E, r)$,
(C4) $\mathbf{C}(e, f_E \sqcup g_E, r) = \mathbf{C}(e, f_E, r) \sqcup \mathbf{C}(e, g_E, r)$,
(C5) $\mathbf{C}(e, f_E, r) \sqsubseteq \mathbf{C}(e, f_E, s)$, if $r \leq s$.

The pair $(X, \mathbf{C})$ is called a fuzzy soft closure space.

A fuzzy soft closure space $(X, \mathbf{C})$ is called topological provided that

$$\mathbf{C}(e, \mathbf{C}(e, f_E, r), r) = \mathbf{C}(e, f_E, r).$$

Let $(X, (\mathbf{C}_1)_E)$ and $(Y, (\mathbf{C}_2)_F)$ be fuzzy soft closure spaces. A function ϕ_ψ : $(X, (\mathbf{C}_1)_E) \to (Y, (\mathbf{C}_2)_F)$ is called fuzzy soft $\mathbf{C}$-map, if for all $f_E \in \widetilde{(X, E)}$, $e \in E, r \in I_1$,

$$\mathbf{C}_2(\psi(e), \phi_\psi(f_E), r) \sqsupseteq \phi_\psi(\mathbf{C}_1(e, f_E, r)).$$

Let $(\tau_1)_E$ and $(\tau_2)_E$ be fuzzy soft topologies on X. We say that $(\tau_1)_E$ is finer than $(\tau_2)_E$ ($(\tau_2)_E$ is coarser than $(\tau_1)_E$), denoted by $(\tau_2)_E \sqsubseteq (\tau_1)_E$, if

$$(\tau_2)_e(f_E) \leq (\tau_1)_e(f_E) \ \forall e \in E,\ \forall f_E \in \widetilde{(X, E)}.$$

Let $(\mathbf{C}_1)_E$ and $(\mathbf{C}_2)_E$ be fuzzy soft closure operators on X. We say that $(\mathbf{C}_1)_E$ is finer than $(\mathbf{C}_2)_E$ ($(\mathbf{C}_2)_E$ is coarser than $(\mathbf{C}_1)_E$), if

$$\mathbf{C}_2(e, f_E, r) \sqsupseteq \mathbf{C}_1(e, f_E, r) \ \ \forall f_E \in \widetilde{(X, E)},\ e \in E, r \in I_1.$$

Definition 1.3 ([3]). A mapping $\tau : E \to I^{\widetilde{(X,E)}}$ is called a fuzzy soft topology on X, if it satisfies the following conditions for each $e \in E$:

(O1) $\tau_e(\tilde{\Phi}) = \tau_e(\tilde{E}) = 1$,
(O2) $\tau_e(f_E \sqcap g_E) \geq \tau_e(f_E) \wedge \tau_e(g_E)$ for all $f_E, g_E \in \widetilde{(X, E)}$,
(O3) $\tau_e(\bigsqcup_{j \in J}(f_E)_j) \geq \bigwedge_{j \in J} \tau_e((f_E)_j)$ for all $(f_E)_j \in \widetilde{(X, E)}$, $j \in J$.

Let $\widetilde{(X, \tau)}$ and $\widetilde{(Y, \tau^*)}$ be fuzzy soft topological spaces. A fuzzy soft mapping ϕ_ψ from $\widetilde{(X, E)}$ into $\widetilde{(Y, F)}$ is called a fuzzy soft continuous mapping, if

$$\tau_e(\phi_\psi^{-1}(g_E)) \geq \tau^*_{\psi(e)}(g_E) \ \ \forall g_E \in \widetilde{(Y, F)},\ e \in E.$$

Theorem 1.4 ([1]). *Let (X, τ_E) be a fuzzy soft topological space. For each $f_E \in I^{\widetilde{(X,E)}}$, $e \in E, r \in I_0$, we define an operator $\mathsf{C}_\tau : E \times \widetilde{(X,E)} \times I_0 \to \widetilde{(X,E)}$ as follows:*

$$\mathsf{C}_\tau(e, f_E, r) = \sqcap\{g_E \in \widetilde{(X,E)} : f_E \sqsubseteq g_E, \tau_e(g_E^c) \geq r\}.$$

Then, $(X, (\mathsf{C}_\tau)_E)$ is a topological fuzzy soft closure space, and if $r = \vee\{s \in I : \mathsf{C}_\tau(e, f_E, s) = f_E\}$, then $\mathsf{C}_\tau(e, f_E, r) = f_E$.

Theorem 1.5. *Let (X, C_E) be a fuzzy soft closure space. Define the function $\tau_{\mathsf{C}} : E \to I^{\widetilde{(X,E)}}$ on X by: For each $e \in E$,*

$$(\tau_{\mathsf{C}})_e(f_E) = \bigvee\{r \in I : \mathsf{C}(e, f_E^c, r) = f_E^c\}.$$

Then

(1) *$(\tau_{\mathsf{C}})_e$ is a fuzzy soft topology on X,*
(2) *We have $\mathsf{C} = \mathsf{C}_{\tau_{\mathsf{C}}}$ iff a fuzzy soft closure space (X, C_E) satisfies the following conditions:*
 (a) *it is topological,*
 (b) *if $r = \vee\{s \in I : \mathsf{C}(e, f_E, s) = f_E\}$, then $\mathsf{C}(e, f_E, r) = f_E$.*

Proof. Direct. □

Example 1.6. Let $X = \{x, y, z\}$, $E = \{e_1, e_2\}$, $f_E = \{(e, x_1 \vee y_1) : e \in E\}$ and $g_E = \{(e, z_1) : e \in E\}$.

Define $\mathsf{C} : E \times \widetilde{(X,E)} \times I_0 \to \widetilde{(X,E)}$ for all $e \in E$ as follows:

$$\mathsf{C}(e, h_E, r) = \begin{cases} \tilde{\Phi} & \text{if } h_E = \tilde{\Phi}, r \in I_0 \\ f_E & \text{if } h_E = x_t^e \in f_E, t \in I_0, 0 < r \leq \frac{1}{2} \\ g_E & \text{if } h_E = z_s^e \in g_E, s \in I_0, 0 < r \leq \frac{1}{2} \\ \tilde{E} & \text{otherwise.} \end{cases}$$

Then (X, C_E) is a fuzzy soft closure space. Since $\mathsf{C}(e, x_t^e, \frac{1}{3}) = f_E$ and $\mathsf{C}(e, f_E, \frac{1}{3}) = \tilde{E}$, $\mathsf{C}(e, \mathsf{C}(e, x_t^e, \frac{1}{3}), \frac{1}{3}) \neq \mathsf{C}(e, x_t^e, \frac{1}{3})$. Thus (X, C_E) is not topological fuzzy soft closure space.

From Theorem 1.5, we obtain $\tau_{\mathsf{C}} : E \to I^{\widetilde{(X,E)}}$ for all $e \in E$ as follows:

$$(\tau_{\mathsf{C}})_e(h_E) = \begin{cases} 1 & \text{if } h_E = \tilde{\Phi} \text{ or } h_E = \tilde{E} \\ \frac{1}{2} & \text{if } h_E = f_E \\ 0 & \text{otherwise} \end{cases}$$

and so

$$\mathsf{C}_{\tau_{\mathsf{C}}}(e, h_E, r) = \begin{cases} \tilde{\Phi} & \text{if } h_E = \tilde{\Phi}, r \in I_0 \\ g_E & \text{if } h_E = z_s^e \in g_E, s \in I_0, 0 < r \leq \frac{1}{2} \\ \tilde{E} & \text{otherwise.} \end{cases}$$

Hence $\mathsf{C} \neq \mathsf{C}_{\tau_{\mathsf{C}}}$.

Theorem 1.7. *Let τ_E be a fuzzy soft topology and C_τ a fuzzy soft closure operator on X. Then τ_{C_τ} is a fuzzy soft topology on X such that $\tau = \tau_{\mathsf{C}_\tau}$.*

Proof. Clear. □

Definition 1.8 ([3]). A mapping $\beta : E \to I^{\widetilde{(X,E)}}$ is called a fuzzy soft base on X, if it satisfies the following conditions: for each $e \in E$,

(B1) $\beta_e(\tilde{\Phi}) = \beta_e(\tilde{E}) = 1$,

(B2) $\beta_e(f_E \sqcap g_E) \;\geq\; \beta_e(f_E) \wedge \beta_e(g_E)$, for all $f_E, g_E \in \widetilde{(X,E)}$.

Theorem 1.9 ([3]). *Let β be a fuzzy soft base on X. Define a map $\tau_\beta : E \to I^{\widetilde{(X,E)}}$ as follows:*

$$(\tau_\beta)_e(f_E) \;=\; \bigvee\{\bigwedge_{j\in J} \beta_e((f_E)_j) \;:\; f_E = \bigsqcup_{j\in J}(f_E)_j\},\ \forall e \in E.$$

Then τ_β is the coarsest fuzzy soft topology on X for which $(\tau_\beta)_e(f_E) \geq \beta_e(f_E)$, for all $e \in E$, $f_E \in \widetilde{(X,E)}$.

Theorem 1.10 ([3]). *Let $\{(X_j, (\tau_j)_{E_j})\}_{j\in J}$ be a family of fuzzy soft topological spaces, X be a set, E be a parameter set and for each $j \in J$, $\phi_j : X \to X_j$ and $\psi_j : E \to E_j$ be maps. Define $\beta : E \to I^{\widetilde{(X,E)}}$ on X by:*

$$\beta_e(f_E) \;=\; \bigvee\{\bigwedge_{j=1}^{n}(\tau_{k_j})_{\psi_{k_j}(e)}((f_E)_{k_j}) \;:\; f_E = \bigsqcup_{j=1}^{n}(\phi_\psi)^{-1}_{k_j}((f_E)_{k_j})\},$$

where $\bigvee$ is taken over all finite subsets $K = \{k_1, k_2, \cdots, k_n\} \subseteq J$. Then

(1) β is a fuzzy soft base on X,

(2) the fuzzy soft topology τ_β generated by β is the coarsest fuzzy soft topology on X for which all $(\phi_\psi)_j$, $j \in J$ are fuzzy soft continuous maps,

(2) a map $\phi_\psi : (Y, \delta_F) \to (X, (\tau_\beta)_E)$ is fuzzy soft continuous iff for each $j \in J$, $(\phi_\psi)_j \circ \phi_\psi \;:\; (Y, \delta_F) \to (X_j, (\tau_j)_{E_j})$ is a fuzzy soft continuous map.

2. Initial fuzzy soft interior spaces

Theorem 2.1. *Let $\{(X_j, (\boldsymbol{I}_j)_{E_j})\}_{j\in J}$ be a family of fuzzy soft interior spaces, X be a set, E, E_j are parameter sets and for each $j \in J$, $\phi_j : X \to X_j$ and $\psi_j : E \to E_j$ be maps. Define a map $\mathsf{I} : E \times \widetilde{(X,E)} \times I_0 \to \widetilde{(X,E)}$ on X by:*

$$\mathsf{I}(e, f_E, r) = \bigsqcup_{\sqcap_{k=1}^{n}(\phi_\psi)^{-1}_{j_k}((f_E)_{j_k}) \sqsubseteq f_E} \{\sqcap_{k=1}^{n}(\phi_\psi)^{-1}_{j_k}(\boldsymbol{I}_{j_k}(\psi_{j_k}(e), (f_E)_{j_k}, r))\},$$

for all finite subsets $K = \{j_1, j_2, \cdots, j_n\} \subseteq J$. Then I is the coarsest fuzzy soft interior operator on X for which each $j \in J$, $(\phi_\psi)_j$ is a fuzzy soft I-map.

Proof. (I1) Since $\mathsf{I}(e, \tilde{E}, r) \sqsupseteq (\phi_\psi)^{-1}_j(\mathbf{I}_j(\psi_j(e), \tilde{E}_j, r)) = \tilde{E}$, we have $\mathsf{I}(e, \tilde{E}, r) = \tilde{E}$.

(I2) For all finite subsets $K = \{j_1, j_2, \cdots, j_n\} \subseteq J$, we have

$$\begin{aligned}
\mathsf{I}(e, f_E, r) &= \bigsqcup_{\sqcap_{k=1}^{n}(\phi_\psi)^{-1}_{j_k}((f_E)_{j_k}) \sqsubseteq f_E} \{\sqcap_{k=1}^{n}(\phi_\psi)^{-1}_{j_k}(\mathbf{I}_{j_k}(\psi_{j_k}(e), (f_E)_{j_k}, r))\} \\
&\sqsubseteq \bigsqcup_{\sqcap_{k=1}^{n}(\phi_\psi)^{-1}_{j_k}((f_E)_{j_k}) \sqsubseteq f_E} \{\sqcap_{k=1}^{n}(\phi_\psi)^{-1}_{j_k}((f_E)_{j_k})\} \\
&\sqsubseteq f_E.
\end{aligned}$$

(I3) It is easily proved from the definition of I.

(I4) For all finite subsets $K=\{k_1,k_2,\cdots,k_p\}$, $L=\{l_1,l_2,\cdots,l_q\}$ of J such that $\sqcap_{t=1}^{p}(\phi_\psi)_{k_t}^{-1}((f_E)_{k_t})\sqsubseteq f_E$ and $\sqcap_{t=1}^{q}(\phi_\psi)_{l_t}^{-1}((g_E)_{l_t})\sqsubseteq g_E$, we have

$$(\sqcap_{t=1}^{p}(\phi_\psi)_{k_t}^{-1}((f_E)_{k_t}))\sqcap(\sqcap_{t=1}^{q}(\phi_\psi)_{l_t}^{-1}((g_E)_{l_t}))\sqsubseteq f_E\sqcap g_E.$$

Furthermore, we have for each $t\in K\cap L$,

$$(\phi_\psi)_t^{-1}((f_E)_t)\sqcap(\phi_\psi)_t^{-1}((g_E)_t)=(\phi_\psi)_t^{-1}((f_E)_t\sqcap(g_E)_t),$$

$$\begin{aligned}(\phi_\psi)_t^{-1}(\mathbf{I}_t(\psi_t(e),(f_E)_t,r)) \quad&\sqcap\quad (\phi_\psi)_t^{-1}(\mathbf{I}_t(\psi_t(e),(g_E)_t,s))\\ &\sqsubseteq\quad (\phi_\psi)_t^{-1}(\mathbf{I}_t(\psi_t(e),(f_E)_t\sqcap(g_E)_t,r\wedge s)),\ e\in E.\end{aligned}$$

Put $M=K\cup L=\{m_1,m_2,\cdots,m_r\}$ with

$$(h_E)_{m_j}\quad=\quad\begin{cases}(f_E)_{m_j}\sqcap\tilde{E} & :\ m_j\in K-(K\cap L)\\ (g_E)_{m_j}\sqcap\tilde{E} & :\ m_j\in L-(K\cap L)\\ (f_E)_{m_j}\sqcap(g_E)_{m_j} & :\ m_j\in(K\cap L).\end{cases}$$

If $m_j\in K-(K\cap L)$, then

$$\begin{aligned}(\phi_\psi)_{m_j}^{-1}(\mathbf{I}_{m_j}(\psi_{m_j}(e),(f_E)_{m_j},r)) &= (\phi_\psi)_{m_j}^{-1}(\mathbf{I}_{m_j}(\psi_{m_j}(e),(f_E)_{m_j},r))\sqcap\tilde{E}\\ &= (\phi_\psi)_{m_j}^{-1}(\mathbf{I}_{m_j}(\psi_{m_j}(e),(f_E)_{m_j},r))\ \sqcap\\ &\quad\ (\phi_\psi)_{m_j}^{-1}(\mathbf{I}_{m_j}(\psi_{m_j}(e),\tilde{E},s))\\ &\sqsubseteq (\phi_\psi)_{m_j}^{-1}(\mathbf{I}_{m_j}(\psi_{m_j}(e),(f_E)_{m_j}\sqcap\tilde{E},r\wedge s))\\ &\sqsubseteq (\phi_\psi)_{m_j}^{-1}(\mathbf{I}_{m_j}(\psi_{m_j}(e),(h_E)_{m_j},r\wedge s)).\end{aligned}$$

Similarly, if $m_j\in L-(K\cap L)$, then

$$(\phi_\psi)_{m_j}^{-1}(\mathbf{I}_{m_j}(\psi_{m_j}(e),(g_E)_{m_j},s))\sqsubseteq(\phi_\psi)_{m_j}^{-1}(\mathbf{I}_{m_j}(\psi_{m_j}(e),(h_E)_{m_j},r\wedge s)).$$

Thus

$$\begin{aligned}&\bigsqcup_{\sqcap_{t=1}^{p}(\phi_\psi)_{k_t}^{-1}((f_E)_{k_t})\sqsubseteq f_E}\{\sqcap_{t=1}^{p}(\phi_\psi)_{k_t}^{-1}(\mathbf{I}_{k_t}(\psi_{k_t}(e),(f_E)_{k_t},r))\}\ \sqcap\\ &\bigsqcup_{\sqcap_{t=1}^{q}(\phi_\psi)_{l_t}^{-1}((g_E)_{l_t})\sqsubseteq g_E}\{\sqcap_{t=1}^{p}(\phi_\psi)_{l_t}^{-1}(\mathbf{I}_{l_t}(\psi_{l_t}(e),(g_E)_{l_t},s))\}\\ \sqsubseteq\ &\bigsqcup_{\sqcap_{t=1}^{r}(\phi_\psi)_{m_t}^{-1}((h_E)_{m_t})\sqsubseteq f_E\sqcap g_E}\{\sqcap_{t=1}^{r}(\phi_\psi)_{m_t}^{-1}(\mathbf{I}_{m_t}(\psi_{m_t}(e),(h_E)_{m_t},r\wedge s))\}\\ \sqsubseteq\ &\mathsf{I}(e,f_E\sqcap g_E,r\wedge s).\end{aligned}$$

So

$$\mathsf{I}(e,f_E,r)\sqcap\mathsf{I}(e,g_E,s)\ \sqsubseteq\ \mathsf{I}(e,f_E\sqcap g_E,r\wedge s).$$

For each $(f_E)_j\in\widetilde{(X_j,E_j)}$ and a family $\{(\phi_\psi)_j^{-1}((f_E)_j)\}_{j\in J}$, we have

$$\mathsf{I}(e,(\phi_\psi)_j^{-1}((f_E)_j),r)\sqsupseteq\ (\phi_\psi)_j^{-1}(\mathbf{I}_j(\psi_j(e),(f_E)_j,r)).$$

Hence for each $j\in J$, $(\phi_\psi)_j:(X,\mathsf{I})\to(X_j,\mathbf{I}_j)$ is a fuzzy soft I-map.

Let $(\phi_\psi)_j:(X,\mathsf{I}^*)\to(X_j,\mathbf{I}_j)$ be a fuzzy soft I-map for each $j\in J$. Since for each $j\in J$ and $(f_E)_j\in\widetilde{(X_j,E_j)}$,

$$\mathsf{I}^*(e,(\phi_\psi)_j^{-1}((f_E)_j),r)\sqsupseteq\ (\phi_\psi)_j^{-1}(\mathbf{I}_j(\psi_j(e),(f_E)_j,r))$$

for all finite subsets $K = \{k_1, k_2, \cdots, k_n\}$ of J, we have

$$\begin{aligned}
\mathsf{I}(e, f_E, r) &= \bigsqcup_{\sqcap_{t=1}^{n}(\phi_\psi)_{k_t}^{-1}((f_E)_{k_t}) \sqsubseteq f_E} \{\sqcap_{t=1}^{n}(\phi_\psi)_{k_t}^{-1}(\mathbf{I}_{k_t}(\psi_{k_t}(e), (f_E)_{k_t}, r))\} \\
&\sqsubseteq \bigsqcup_{\sqcap_{t=1}^{n}(\phi_\psi)_{k_t}^{-1}((f_E)_{k_t}) \sqsubseteq f_E} \mathsf{I}^*(\psi_{k_t}(e), \sqcap_{t=1}^{n}(\phi_\psi)_{k_t}^{-1}((f_E)_{k_t}), \bigwedge_{t=1}^{n} r) \\
&= \bigsqcup_{\sqcap_{t=1}^{n}(\phi_\psi)_{k_t}^{-1}((f_E)_{k_t}) \sqsubseteq f_E} \mathsf{I}^*(\psi_{k_t}(e), \sqcap_{t=1}^{n}(\phi_\psi)_{k_t}^{-1}((f_E)_{k_t}), r) \\
&\sqsubseteq \mathsf{I}^*(e, f_E, r) \qquad (\text{ where } r = \bigwedge_{t=1}^{n} r).
\end{aligned}$$

□

The category of fuzzy soft interior spaces and fuzzy soft I-maps is denoted by **FSI**.

Theorem 2.2. *The forgetful functor* $\mathbf{W} : \mathbf{FSI} \to \mathbf{SET}$ *defined by* $\mathbf{W}(X, \mathsf{I}) = X$ *and* $\mathbf{W}(\phi_\psi) = \phi$ *is topological.*

Proof. From Theorem 2.1, every $\mathbf{W}$-structured source $((\phi_\psi)_j : X \to \mathbf{W}(X_j, (\mathbf{I})_j)_{j\in J})$ has a unique $\mathbf{W}$-initial left $((\phi_\psi)_j : (X, \mathsf{I}) \to \mathbf{W}(X_j, (\mathbf{I})_j)_{j\in J})$, where I is defined as in Theorem 2.1. □

Using Theorem 2.1 and Theorem 2.2, we can obtain the following definition.

Definition 2.3. Let $\{(X_j, (\mathbf{I}_j)_{E_j})\}_{j\in J}$ be a family of fuzzy soft interior spaces, for each $j \in J$, $X = \Pi_{j\in J} X_j$ and $E = \Pi_{j\in J} E_j$. Let $p_j : X \to X_j$ and $q_j : E \to E_j$ be projection maps for all $j \in J$. The initial fuzzy soft interior operator I, as given in Theorem 2.1, with respect to the parameter set E is the coarsest fuzzy soft interior operator on X for which all $(p_q)_j$, $j \in J$ are fuzzy soft I-maps.

Corollary 2.4. *Let* $\{(X_j, (\boldsymbol{I}_j)_{E_j})\}_{j\in J}$ *be a family of fuzzy soft interior spaces for each* $j \in J$, $\phi_j : X \to X_j$ *and* $\psi_j : E \to E_j$ *be maps for all* $j \in J$ *and* I *is the fuzzy soft interior operator as in Definition 1.1. Then*

(1) *if there exists* $j \in J$ *such that* $\boldsymbol{I}_j$ *is weakly stratified, then* I *is also weakly stratified on* X,

(2) *if* $\{(X_j, (\boldsymbol{I}_j)_{E_j})\}_{j\in J}$ *is a family of topological fuzzy soft interior spaces, then* (X, I_E) *is a topological fuzzy soft interior space,*

(3) *a map* $\phi_\psi : (Y, \mathsf{I}^*_F) \to (X, \mathsf{I}_E)$ *is fuzzy soft* I*-map iff for each* $j \in J$, $(\phi_\psi)_j \circ \phi_\psi : (Y, \mathsf{I}^*_F) \to (X_j, (\boldsymbol{I}_j)_{E_j})$ *is fuzzy soft* I*-map.*

Proof. (1) Let $(\mathbf{I}_j)_{E_j}$ be weakly stratified on X, for some $j \in J$. Since $\mathbf{I}(\psi_j(e), \tilde{E}^\alpha, r) \sqsupseteq \tilde{E}^\alpha$, for each $\tilde{\alpha} \in \widetilde{(X_j, E_j)}$, $e \in E, r \in I_0$ and $(\phi_\psi)_j^{-1}(\tilde{E}^\alpha) = \tilde{E}^\alpha$, we have for each

$\tilde{E}^\alpha \in \widetilde{(X,E)},\ e \in E, r \in I_0,$

$$\begin{aligned}\mathsf{I}(e,\tilde{E}^\alpha,r) &= \bigsqcup_{\sqcap_{k=1}^n(\phi_\psi)^{-1}_{j_k}((f_E)_{j_k})\sqsubseteq\tilde{E}^\alpha} \{\sqcap_{k=1}^n(\phi_\psi)^{-1}_{j_k}(\mathbf{I}_{j_k}(\psi_{j_k}(e),(f_E)_{j_k},r))\} \\ &\sqsupseteq (\phi_\psi)^{-1}_j(\mathbf{I}_j(\psi_j(e),\tilde{E}^\alpha,r)) \\ &\sqsupseteq \tilde{E}^\alpha.\end{aligned}$$

Thus I is also weakly stratified on X.

(2) For all finite subsets $K=\{k_1,k_2,\cdots,k_n\}$ of J, we have

$$\begin{aligned}&\mathsf{I}(e,f_E,r)\\ &= \bigsqcup_{\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}((f_E)_{k_t})\sqsubseteq f_E} \{\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}(\mathbf{I}_{k_t}(\psi_{k_t}(e),(f_E)_{k_t},r))\}\\ &= \bigsqcup_{\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}((f_E)_{k_t})\sqsubseteq f_E} \{\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}(\mathbf{I}_{k_t}(\psi_{k_t}(e),\mathbf{I}_{k_t}(\psi_{k_t}(e),(f_E)_{k_t},r),r))\}\\ &\sqsubseteq \bigsqcup_{\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}(\mathbf{I}_{k_t}(\psi_{k_t}(e),(f_E)_{k_t},r))\sqsubseteq\mathsf{I}(e,f_E,r)} \{\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}(\mathbf{I}_{k_t}(\psi_{k_t}(e),\mathbf{I}_{k_t}(\psi_{k_t}(e),(f_E)_{k_t},r),r))\}\\ &\sqsubseteq \mathsf{I}(e,\mathsf{I}(e,f_E,r),r),\end{aligned}$$

because $\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}((f_E)_{k_t})\sqsubseteq f_E$ implies that

$$\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}(\mathbf{I}_{k_t}(\psi_{k_t}(e),(f_E)_{k_t},r))\sqsubseteq\mathsf{I}(e,f_E,r).$$

(3) For all finite subsets $K=\{k_1,k_2,\cdots,k_n\}$ of J, we have for each $e\in F$,

$$\begin{aligned}&(\phi_\psi)^{-1}(\mathsf{I}(\psi(e),f_E,r))\\ &= (\phi_\psi)^{-1}[\bigsqcup_{\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}((f_E)_{k_t})\sqsubseteq f_E} \{\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}(\mathbf{I}_{k_t}(\psi_{k_t}(\psi(e)),(f_E)_{k_t},r))\}]\\ &= \bigsqcup_{\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}((f_E)_{k_t})\sqsubseteq f_E} \{(\phi_\psi)^{-1}[\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}(\mathbf{I}_{k_t}(\psi_{k_t}(\psi(e)),(f_E)_{k_t},r))]\}\\ &= \bigsqcup_{\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}((f_E)_{k_t})\sqsubseteq f_E} \{\sqcap_{t=1}^n(\phi_\psi)^{-1}(\phi_\psi)^{-1}_{k_t}(\mathbf{I}_{k_t}(\psi_{k_t}(\psi(e)),(f_E)_{k_t},r))\}\\ &\sqsubseteq \bigsqcup_{\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}((f_E)_{k_t})\sqsubseteq f_E} \{\sqcap_{t=1}^n\mathsf{I}^*(\psi_{k_t}(\psi(e)),(\phi_\psi)^{-1}(\phi_\psi)^{-1}_{k_t}(f_E)_{k_t},r)\}\\ &\sqsubseteq \bigsqcup_{\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}((f_E)_{k_t})\sqsubseteq f_E} \{\mathsf{I}^*(\psi_{k_t}(\psi(e)),(\phi_\psi)^{-1}(\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}(f_E)_{k_t}),\bigwedge_{t=1}^n r)\}\\ &= \bigsqcup_{\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}((f_E)_{k_t})\sqsubseteq f_E} \{\mathsf{I}^*(\psi_{k_t}(\psi(e)),(\phi_\psi)^{-1}(\sqcap_{t=1}^n(\phi_\psi)^{-1}_{k_t}(f_E)_{k_t}),r)\}\\ &\sqsubseteq \mathsf{I}^*(\psi_{k_t}(\psi(e)),(\phi_\psi)^{-1}(f_E),r).\end{aligned}$$

□

From Theorem 2.1, we can prove the following corollaries.

Corollary 2.5. *Let I be a fuzzy soft interior operator on Y, $\phi: X\to Y$ and $\psi: E\to F$ be maps. We define a map $I: E\times\widetilde{(X,E)}\times I_0\to\widetilde{(X,E)}$ as*

$$I(e,f_E,r)=\bigsqcup_{(\phi_\psi)^{-1}(g_E)\sqsubseteq f_E}(\phi_\psi)^{-1}(\mathsf{I}^*(\psi(e),g_E,r)).$$

Then I is the coarsest fuzzy soft interior operator on X for which ϕ_ψ is a fuzzy soft I-map.

Corollary 2.6. *Let $\{(\boldsymbol{I}_j)_{E_j})\}_{j\in J}$ be a family of fuzzy soft interior operators on X. Define a map $I: E\times\widetilde{(X,E)}\times I_0\to\widetilde{(X,E)}$ by:*

$$I(e,f_E,r)=\bigsqcup_{(f_E)_{k_1}\sqcap(f_E)_{k_2}\sqcap\cdots\sqcap(f_E)_{k_n}\ \sqsubseteq\ f_E}\boldsymbol{I}_{k_1}(\psi_{k_1}(e),(f_E)_{k_1},r)\sqcap\cdots\sqcap\boldsymbol{I}_{k_n}(\psi_{k_n}(e),(f_E)_{k_n},r),$$

for all finite subsets $K=\{k_1,k_2,\cdots,k_n\}$ of J.
Then I is the coarsest fuzzy soft interior operator on X finer than $\boldsymbol{I}_j$, for each $j\in J$.

3. Initial fuzzy soft closure spaces

Theorem 3.1. *Let $\{(X_j,(\boldsymbol{C}_j)_{E_j})\}_{j\in J}$ be a family of fuzzy soft closure spaces, X be a set, $\phi_j: X\to X_j$ and $\psi_j: E\to E_j$ be maps for each $j\in J$. Define a map $\mathsf{C}: E\times\widetilde{(X,E)}\times I_1\to\widetilde{(X,E)}$ on X by:*

$$\mathsf{C}(e,f_E,r)=\sqcap\{\bigsqcup_{j=1}^{n}(\sqcap_{j\in J}(\phi_\psi)_j^{-1}(\boldsymbol{C}_j(\psi_j(e),(\phi_\psi)_j((f_E)_j),r)))\},$$

for all $f_E\in\widetilde{(X,E)}$, $e\in E, r\in I_1$, where the first $\sqcap$ is taken over all finite families $\{(f_E)_j: f_E=\bigsqcup_{j=1}^{n}(f_E)_j\}$. Then

(1) *C is the coarsest fuzzy soft closure operator on X, for which all $(\phi_\psi)_j$ are fuzzy soft C-maps,*

(2) *if $\{(X_j,(\boldsymbol{C}_j)_{E_j})\}_{j\in J}$ is a family of topological fuzzy soft closure spaces, then (X,C_E) is a topological fuzzy soft closure space,*

(3) *a map $\phi_\psi:(Y,\mathsf{C}^*_F)\to(X,\mathsf{C}_E)$ is a fuzzy soft C-map iff for each $j\in J$, $(\phi_\psi)_j\circ\phi_\psi:(Y,\mathsf{C}^*_F)\to(X_j,(\boldsymbol{C}_j)_{E_j})$ is a fuzzy soft C-map.*

Proof. (1) Firstly, we will show that C is a fuzzy soft closure operator on X.

(C1), (C2) and (C5): it is easily proved from the definition of C.

(C3): Let $f_E\sqsubseteq g_E$ be given. For each family $\{(g_E)_j: g_E=\bigsqcup_{j=1}^{n}(g_E)_j\}$, there exists a finite family $\{f_E\sqcap(g_E)_j: f_E=\bigsqcup_{j=1}^{n}(f_E\sqcap(g_E)_j)\}$ such that

$$\begin{aligned}\mathsf{C}(e,f_E,r)&\sqsubseteq\bigsqcup_{j=1}^{n}[\sqcap_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e),(\phi_\psi)_j(f_E\sqcap(g_E)_j),r))]\\&\sqsubseteq\bigsqcup_{j=1}^{n}[\sqcap_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e),(g_E)_j,r))].\end{aligned}$$

Then $\mathsf{C}(e,f_E,r)\sqsubseteq\mathsf{C}(e,g_E,r)$.

(C4): From (C3), we have

$$\mathsf{C}(e,f_E,r)\sqcup\mathsf{C}(e,g_E,r)\sqsubseteq\mathsf{C}(e,f_E\sqcup g_E,r).$$

Now we prove that $\mathsf{C}(e,f_E,r)\sqcup\mathsf{C}(e,g_E,r) \sqsupseteq \mathsf{C}(e,f_E\sqcup g_E,r)$. For all finite families $\{(f_E)_j : f_E = \bigsqcup_{j=1}^{m}(f_E)_j\}$ and $\{(g_E)_j : g_E = \bigsqcup_{j=1}^{n}(g_E)_j\}$, there exists a finite family $\{(f_E)_j,(g_E)_j : f_E \sqcup g_E = (\bigsqcup_{j=1}^{m}(f_E)_j) \sqcup (\bigsqcup_{j=1}^{n}(g_E)_j)\}$ such that

$$\begin{aligned}\mathsf{C}(e,f_E\sqcup g_E,r) \quad &\sqsubseteq \quad [\bigsqcup_{j=1}^{m}(\sqcap_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e),(\phi_\psi)_j((f_E)_j),r)))] \ \sqcup \\ &\quad\ \ [\bigsqcup_{j=1}^{n}(\sqcap_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e),(\phi_\psi)_j((g_E)_j),r)))].\end{aligned}$$

Put $h_E = \bigsqcup_{j=1}^{n}(\sqcap_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e),(\phi_\psi)_j((g_E)_j),r)))$ Then

$$\begin{aligned}\mathsf{C}(e,f_E\sqcup g_E,r) \quad &\sqsubseteq \quad \sqcap[\bigsqcup_{j=1}^{m}(\sqcap_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e),(\phi_\psi)_j((f_E)_j),r))) \ \sqcup \ h_E] \\ &= \quad [\sqcap(\bigsqcup_{j=1}^{m}(\sqcap_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e),(\phi_\psi)_j((f_E)_j),r)))) \ \sqcup \ h_E] \\ &= \quad \mathsf{C}(e,f_E,r) \ \sqcup \ h_E,\end{aligned}$$

where $\sqcap$ is taken over all finite families $\{(f_E)_j : f_E = \bigsqcup_{j=1}^{m}(f_E)_j\}$. Again,

$$\begin{aligned}\mathsf{C}(e,f_E\sqcup g_E,r) \quad &\sqsubseteq \quad \sqcap(\mathsf{C}(e,f_E,r) \ \sqcup \ h_E) \\ &= \quad \mathsf{C}(e,f_E,r)\sqcup[\sqcap(\bigsqcup_{j=1}^{n}(\sqcap_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e),(\phi_\psi)_j((g_E)_j),r))))] \\ &= \quad \mathsf{C}(e,f_E,r) \ \sqcup \ \mathsf{C}(e,g_E,r),\end{aligned}$$

where $\sqcap$ is taken over all finite families $\{(g_E)_j : g_E = \bigsqcup_{j=1}^{n}(g_E)_j\}$.

Secondly, from the definition of C, we have the following, for a family $\{f_E : f_E = \bigsqcup_{j=1}^{n}(f_E)_j\}$,

$$\begin{aligned}\mathsf{C}(e,f_E,r) \quad &\sqsubseteq \quad \sqcap_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e),(\phi_\psi)_j(f_E)_j,r)) \\ &\sqsubseteq \quad (\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e),(\phi_\psi)_j(f_E)_j,r)).\end{aligned}$$

It implies that

$$\begin{aligned}\phi_\psi(\mathsf{C}(e,f_E,r)) \quad &\sqsubseteq \quad (\phi_\psi)_j((\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e),(\phi_\psi)_j(f_E)_j,r))) \\ &\sqsubseteq \quad \mathbf{C}_j(\psi_j(e),(\phi_\psi)_j(f_E)_j,r).\end{aligned}$$

Thus for each $j\in J$, $(\phi_\psi)_j : (X,\mathsf{C}_E) \to (X_j,(\mathbf{C}_j)_{E_j})$ is a fuzzy soft C-map.

If $(\phi_\psi)_j : (X,\mathsf{C}^*_F) \to (X_j,(\mathbf{C}_j)_{E_j})$ is a fuzzy soft C-map for every $j\in J,\ e\in F$, then we have

$$(\phi_\psi)_j(\mathsf{C}^*(e,f_E,r)) \sqsubseteq \mathbf{C}_j(\psi_j(e),(\phi_\psi)_j(f_E)_j,r).$$

It implies that

$$\begin{aligned} \mathsf{C}^*(e, f_E, r) &\sqsubseteq (\phi_\psi)_j^{-1}((\phi_\psi)_j(\mathsf{C}^*(e, f_E, r))) \\ &\sqsubseteq (\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e), (\phi_\psi)_j(f_E)_j, r)). \end{aligned}$$

So we have

$$\mathsf{C}^*(e, f_E, r) \sqsubseteq \sqcap(\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e), (\phi_\psi)_j(f_E)_j, r)).$$

We have the following, for all finite families $\{(f_E)_j : f_E = \bigsqcup_{j=1}^{m}(f_E)_j\}$,

$$\begin{aligned} \mathsf{C}(e, f_E, r) &= \sqcap[\bigsqcup_{j=1}^{m}(\sqcap_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e), (\phi_\psi)_j((f_E)_j), r)))] \\ &\sqsupseteq \sqcap[\bigsqcup_{j=1}^{m}\mathsf{C}^*(e, (f_E)_j, r)] \\ &= \sqcap[\mathsf{C}^*(e, \bigsqcup_{j=1}^{m}(f_E)_j, r)] \\ &= \mathsf{C}^*(e, f_E, r). \end{aligned}$$

Hence C is the coarsest fuzzy soft closure operator on X.

(2) We will show that $\mathsf{C}(e, \mathsf{C}(e, f_E, r), r) = \mathsf{C}(e, f_E, r)$, for all $f_E \in \widetilde{(X, E)}$, $e \in E, r \in I_1$.

For all finite families $\{(f_E)_j : f_E = \bigsqcup_{j=1}^{m}(f_E)_j\}$, we have

$$\begin{aligned} \mathsf{C}(e, f_E, r) &= \sqcap[\bigsqcup_{j=1}^{m}(\sqcap_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e), (\phi_\psi)_j((f_E)_j), r)))] \\ &= \sqcap[\bigsqcup_{j=1}^{m}(\sqcap_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e), \mathbf{C}_j(\psi_j(e), (\phi_\psi)_j((f_E)_j), r), r)))] \\ &\sqsupseteq \sqcap[\bigsqcup_{j=1}^{m}(\sqcap_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e), (\phi_\psi)_j(\mathsf{C}(e, (f_E)_j, r)), r)))] \\ &\sqsupseteq \mathsf{C}(e, \mathsf{C}(e, f_E, r), r). \end{aligned}$$

From (C2) in (1), we have $\mathsf{C}(e, \mathsf{C}(e, f_E, r), r) = \mathsf{C}(e, f_E, r)$.

(3) Necessity of the composition condition is clear since the composition of fuzzy soft C-maps is a fuzzy soft C-map.

Conversely, suppose $(\phi_\psi)_j \circ \phi_\psi$ is a fuzzy soft C-map, for each $g_E \in \widetilde{(X, E)}$. Then we have

$$\begin{aligned} (\phi_\psi)_j(\phi_\psi(\mathsf{C}^*(e, (\phi_\psi)^{-1}(g_E), r))) &\sqsubseteq \mathbf{C}_j(\psi_j(e), (\phi_\psi)_j(\phi_\psi(\phi_\psi^{-1}(g_E)))_j, r) \\ &\sqsubseteq \mathbf{C}_j(\psi_j(e), (\phi_\psi)_j(g_E)_j, r). \end{aligned}$$

It follows, for all $j \in J$, that

$$\phi_\psi(\mathsf{C}^*(e, (\phi_\psi)^{-1}(g_E), r)) \sqsubseteq (\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e), (\phi_\psi)_j(g_E)_j, r)).$$

Thus we have

$$\phi_\psi(\mathsf{C}^*(e,(\phi_\psi)^{-1}(g_E),r)) \sqsubseteq \sqcap_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e),(\phi_\psi)_j(g_E)_j,r)).$$

For all finite families $\{(g_E)_j : \phi_\psi(f_E)=\bigsqcup_{j=1}^{m}(g_E)_j\}$, we have

$$\begin{aligned}
\phi_\psi(\mathsf{C}^*(e,f_E,r)) &\sqsubseteq \phi_\psi(\mathsf{C}^*(e,(\phi_\psi)^{-1}(\phi_\psi(f_E)),r))\\
&= \phi_\psi(\mathsf{C}^*(e,(\phi_\psi)^{-1}(\bigsqcup_{j=1}^{m}(g_E)_j),r))\\
&= \phi_\psi(\mathsf{C}^*(e,\bigsqcup_{j=1}^{m}(\phi_\psi)^{-1}((g_E)_j),r))\\
&= \phi_\psi(\bigsqcup_{j=1}^{m}(\mathsf{C}^*(e,(\phi_\psi)^{-1}((g_E)_j),r)))\\
&= \bigsqcup_{j=1}^{m}\ \phi_\psi(\mathsf{C}^*(e,(\phi_\psi)^{-1}((g_E)_j),r))\\
&\sqsubseteq \bigsqcup_{j=1}^{m}[\sqcap_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_j(\psi_j(e),(\phi_\psi)_j((g_E)_j),r))].
\end{aligned}$$

So $\phi_\psi(\mathsf{C}^*(e,f_E,r)) \sqsubseteq \mathsf{C}(\psi(e),\phi_\psi(f_E),r)$, for all $f_E\in\widetilde{(Y,F)}$, $e\in E, r\in I_1$. Hence $\phi_\psi:(Y,\mathsf{C}^*_F)\to(X,\mathsf{C}_E)$ is a fuzzy soft C-map. □

The category of fuzzy soft closure spaces and fuzzy soft C-maps is denoted by **FSC**.

Theorem 3.2. *The forgetful functor* $\mathbf{W}:\mathbf{FSC}\to\mathbf{SET}$ *defined by* $\mathbf{W}(X,\mathcal{C})=X$ *and* $\mathbf{W}(\phi_\psi)=\phi$ *is topological.*

Proof. The proof is straightforward from Theorem 3.1, and every $\mathbf{W}$-structured source $((\phi_\psi)_j : X\to\mathbf{W}(X_j,(\mathbf{C})_j)_{j\in J})$ has a unique $\mathbf{W}$-initial left $((\phi_\psi)_j:(X,\mathsf{C})\to\mathbf{W}(X_j,(\mathbf{C})_j)_{j\in J})$, where C is defined as in Theorem 3.1. □

Using Theorem 3.1 and Theorem 3.2, we obtain the following definition.

Definition 3.3. Let $\{(X_j,(\mathbf{C}_j)_{E_j})\}_{j\in J}$ be a family of fuzzy soft closure spaces, for each $j\in J$, $X=\Pi_{j\in J}X_j$ and $E=\Pi_{j\in J}E_j$. Let $p_j:X\to X_j$ and $q_j:E\to E_j$ be projection maps for all $j\in J$. The initial fuzzy soft closure operator C as given in Theorem 3.1, with respect to the parameter set E is the coarsest fuzzy soft closure operator on X for which all $(p_q)_j,\ j\in J$ are fuzzy soft C-maps.

Using Theorem 3.1, we have the following corollary.

Corollary 3.4. *Let* $\{(X_j,(\boldsymbol{C}_j)_{E_j})\}_{j\in J}$ *be a family of fuzzy soft closure spaces. Let* $X=\Pi_{j\in J}X_j$ *and* $E=\Pi_{j\in J}E_j$ *be the product sets and for each* $j\in J$, $p_j:X\to X_j$ *and* $q_j:E\to E_j$ *are projections. The structure* $\boldsymbol{C}=\otimes\boldsymbol{C}_j$ *on* X *is defined for each*

$e \in E$ by:

$$\boldsymbol{C}(e, f_E, r) = \sqcap[\bigsqcup_{j=1}^{m}(\sqcap_{j\in J}(p_q)_j^{-1}(\boldsymbol{C}_j(q_j(e), (p_q)_j((f_E)_j), r)))],$$

where $\sqcap$ is taken over all finite families $\{(f_E)_j : f_E = \bigsqcup_{j=1}^{m}(f_E)_j\}$. Then

(1) $\boldsymbol{C}$ is the coarsest fuzzy soft closure operator on X for which all $(p_q)_j,\ j \in J$ are fuzzy soft $\boldsymbol{C}$-maps,

(2) A map $\phi_\psi : (Y, \boldsymbol{C}^*) \to (X, \boldsymbol{C})$ is fuzzy soft $\boldsymbol{C}$-map iff $(p_q)_j \circ \phi_\psi : (Y, \boldsymbol{C}^*) \to (X_j, \boldsymbol{C}_j)$ is a fuzzy soft $\boldsymbol{C}$-map for each $j \in J$.

Definition 3.5. Let $X = \Pi_{j\in J}X_j,\ E = \Pi_{j\in J}E_j$ be the product of sets from the family $\{(X_j, (\mathbf{C}_j)_{E_j})\}_{j\in J}$ of fuzzy soft closure spaces. The initial fuzzy soft closure structure $\mathbf{C} = \otimes\mathbf{C}_j$ on X with respect to the families of all projection maps $p_j : X \to X_j$ and $q_j : E \to E_j,\ j \in J$ is called the product fuzzy soft closure structure of $(\mathbf{C}_j)_{E_j} : j \in J$, and $(X, \otimes\mathbf{C}_j)$ is called the product fuzzy soft closure space.

From the following theorem, an initial structure of fuzzy soft topological spaces could be obtained by the initial structure of fuzzy soft closure spaces.

Theorem 3.6. *Let $\{(X_j, (\tau_j)_{E_j})\}_{j\in J}$ be a family of fuzzy soft topological spaces. Let X be a set, and for each $j \in J$, $\phi_j : X \to X_j$ and $\psi_j : E \to E_j$ are functions. Define the map $\mathsf{C} : E \times \widetilde{(X, E)} \times I_1 \to \widetilde{(X, E)}$ on X by:*

$$\mathsf{C}(e, f_E, r) = \sqcap\{\bigsqcup_{t=1}^{m}(\sqcap_{j\in J}(\phi_\psi)_j^{-1}(\boldsymbol{C}_{\tau_j}(\psi_j(e), (\phi_\psi)_j((f_E)_t), r)))\},$$

for all $f_E \in \widetilde{(X, E)}$, $e \in E, r \in I_1$, where $\sqcap$ is taken over all finite families $\{(f_E)_t : f_E = \bigsqcup_{t=1}^{m}(f_E)_t\}$. Then we have $\tau_{\mathsf{C}} = \tau_\beta$ where the fuzzy soft topology τ_{C} is induced by C and the fuzzy soft topology τ_β is defined as in Theorem 1.9.

Proof. Suppose that there exists $f_E \in \widetilde{(X, E)}$ such that for each $e \in E$,

$$(\tau_\beta)_e(f_E) \ngeq (\tau_{\mathsf{C}})_e(f_E).$$

By the definition of τ_{C} from Theorem 1.5, there exists $r_0 \in I_1$ such that

$$\mathsf{C}(e, f_E^c, r_0) = f_E^c \ \text{ and } \ (\tau_\beta)_e(f_E) \ngeq r_0.$$

Since $\mathsf{C}(e, f_E^c, r_0) = f_E^c$, we have

$$\begin{aligned}
f_E &= (\mathsf{C}(e, f_E^c, r_0))^c \\
&= (\sqcap\{\bigsqcup_{t=1}^{m}(\sqcap_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_{\tau_j}(\psi_j(e), (\phi_\psi)_j((g_E)_t), r_0)))\})^c \\
&= \sqcup\{\sqcap_{t=1}^{m}(\sqcup_{j\in J}(\phi_\psi)_j^{-1}(\mathbf{C}_{\tau_j}(\psi_j(e), (\phi_\psi)_j((g_E)_t), r_0)))\}^c \\
&= \sqcup\{\sqcap_{t=1}^{m}(\sqcup_{j\in J}((\phi_\psi)_j^{-1}(\mathbf{C}_{\tau_j}(\psi_j(e), (\phi_\psi)_j((g_E)_t), r_0)))^c)\} \\
&= \sqcup\{\sqcap_{t=1}^{m}(\sqcup_{j\in J}(\phi_\psi)_j^{-1}((\mathbf{C}_{\tau_j}(\psi_j(e), (\phi_\psi)_j((g_E)_t), r_0))^c))\},
\end{aligned}$$

where the first $\sqcup$ is taken over all families $\{(g_E)_t : f_E^c = \bigsqcup_{t=1}^{m}(g_E)_t\}$. Since

$$\mathbf{C}_{\tau_j}(\psi_j(e),(\phi_\psi)_j((g_E)_t),r_0) = \mathbf{C}_{\tau_j}(\psi_j(e),\mathbf{C}_{\tau_j}(\psi_j(e),(\phi_\psi)_j((g_E)_t),r_0),r_0),$$

for each $j \in J$, $t = 1,2,\cdots,m$, and using the result $\tau_j = \tau_{\mathsf{C}_{\tau_j}}$ in Theorem 1.7, we have

$$(\tau_j)_{\psi_j(e)}((\mathbf{C}_{\tau_j}(\psi_j(e),(\phi_\psi)_j((g_E)_t),r_0))^c) \ \geq \ r_0.$$

Put

$$(h_E)_{j_t} = (\phi_\psi)_j^{-1}((\mathbf{C}_{\tau_j}(\psi_j(e),(\phi_\psi)_j((g_E)_t),r_0))^c).$$

Then from Theorem 1.10, we have

$$\beta_{\psi_j(e)}((h_E)_{j_t}) \ \geq \ (\tau_j)_{\psi_j(e)}((\mathbf{C}_{\tau_j}(\psi_j(e),(\phi_\psi)_j((g_E)_t),r_0))^c) \ \geq \ r_0 \ \forall j \in J.$$

Thus by the definition of τ_β from Theorem 1.9, we have

$$(\tau_\beta)_e(\bigsqcup_{j\in J}(h_E)_{j_t}) \ \geq \ \bigwedge_{j\in J}\beta_e((h_E)_{j_t}) \ \geq \ r_0.$$

So $(\tau_\beta)_e(f_E) \ \geq \ r_0$ from (O2) and (O3) of Definition 1.3. It is a contradiction. Hence $(\tau_\beta)_e(g_E) \ \geq (\tau_{\mathsf{C}})_e(g_E)$, for all $g_E \in \widetilde{(X,E)}$.

We show that $(\tau_\beta)_e(g_E) \ \leq \ (\tau_{\mathsf{C}})_e(g_E)$, for every $g_E \in \widetilde{(X,E)}$, equivalently, the identity function $(id_X)_{id_E} : (X,(\tau_{\mathsf{C}})_E) \to (X,(\tau_\beta)_E)$ is fuzzy soft continuous. From Theorem 1.10, we only show that $(\phi_\psi)_j \circ (id_X)_{id_E} : (X,(\tau_{\mathsf{C}})_E) \to (X_j,(\tau_j)_{E_j})$ is fuzzy soft continuous, that is,

$$(\tau_j)_{\psi(e)}((g_E)_j) \ \leq \ (\tau_{\mathsf{C}})_e((\phi_\psi)_j^{-1}((g_E)_j)),$$

for every $(g_E)_j \in \widetilde{(X_j,E_j)}$. If $(\tau_j)_{\psi(e)}((g_E)_j) \ \geq \ r$ for $r \in I_{01}$, then by Theorem 1.4, we have $\mathbf{C}_{\tau_j}(\psi_j(e),(g_E)_j^c,r) = (g_E)_j^c$. Thus from the definition of C, it follows that:

$$\begin{aligned}\mathsf{C}(e,((\phi_\psi)_j^{-1}((g_E)_j))^c,r) \ &\sqsubseteq \ (\phi_\psi)_j^{-1}(\mathbf{C}_{\tau_j}(\psi_j(e),(\phi_\psi)_j((\phi_\psi)_j^{-1}((g_E)_j))^c,r)) \\ &\sqsubseteq \ (\phi_\psi)_j^{-1}(\mathbf{C}_{\tau_j}(\psi_j(e),(\phi_\psi)_j((\phi_\psi)_j^{-1}((g_E)_j)^c),r)) \\ &\sqsubseteq \ (\phi_\psi)_j^{-1}(\mathbf{C}_{\tau_j}(\psi_j(e),((g_E)_j)^c,r)) \\ &= \ (\phi_\psi)_j^{-1}((g_E)_j)^c.\end{aligned}$$

So from (C2) of Definition 1.2, we have

$$\mathsf{C}(e,((\phi_\psi)_j^{-1}((g_E)_j))^c,r) = ((\phi_\psi)_j^{-1}((g_E)_j))^c.$$

Hence by Theorem 1.5, $(\tau_{\mathsf{C}})_e((\phi_\psi)_j^{-1}((g_E)_j)) \ \geq \ r.$ □

4. Conclusions

The dual concept of Final fuzzy soft closure spaces is a typical study and systematically ensured following the same results proved in this paper, and we think that no need to add special study for the final fuzzy soft closure structures.

Acknowledgements. The authors would like to thank the referees for their valuable comments and suggestions which have improved this paper. The authors are scientifically supported by Deanship of Scientific Research, Jazan University, Saudi Arabia.

Conflict of interest The authors declare that they have no conflict of interest.

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