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Abstract. In this paper, we introduce the concept of α-fuzzy ideals of a distributive lattice. We prove that the set of all α-fuzzy ideals of a lattice forms a complete distributive lattice. A set of equivalent conditions are derived for a fuzzy ideal of a lattice to become an α-fuzzy ideal. We also topologize the set of all prime α-fuzzy ideals of a distributive lattice. Properties of the space also studied. Moreover, a set of equivalent conditions are given the space of all prime α-fuzzy ideals of L to become Hausdorff and regular.

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1. Introduction

The concept of annulets and α-ideals in a distributive lattice with least element 0 is introduced by Cornish [3]. In 1986, Jayaram [5] studied prime α-ideals (in the sense of Cornish) in a 0-distributive lattice and he introduced a topology on the set of all prime α-ideals of a 0-distributive lattice.

On the other hand, many papers on fuzzy algebras have been published since Rosenfeld [12] introduced the concept of fuzzy group in 1971. In particular, W. J. Liu [11] initiated the study of fuzzy subrings, and fuzzy ideals of a ring; B. Yuan and Wu [14] introduced the notion of fuzzy ideals and fuzzy congruences of distributive lattices; Swamy and Raju [13] studied properties of fuzzy ideals and congruences of lattices. In [8], Rajesh Kumar topologized the set of all fuzzy prime ideals of a commutative ring with unity and studied some properties of the space. In 1994, Kumbhojkar [9], studied about the space of prime fuzzy ideals of a ring in different way. In [4], Hadji-Abadi and Zahedi extended the result of R. Kumar.

The aim of this paper is to introduce the concept of α-fuzzy ideals of a distributive lattice. We prove that the set of all α-fuzzy ideals of a distributive lattice forms a complete distributive lattice. Moreover, a set of equivalent conditions are derived for a fuzzy ideal of a distributive lattice to become an α-fuzzy ideal. We also study the space of all prime α-fuzzy ideals of a distributive lattice L. The set of prime α-fuzzy ideals of a lattice L is denoted by X_α. For an α-fuzzy ideal θ of L, open subset of X_α is of the form $X(\theta) = \{\mu \in X_\alpha : \theta \nsubseteq \mu\}$ and $V(\theta) = \{\mu \in X_\alpha : \theta \subseteq \mu\}$ is a closed set. Also we have shown that the set of all open sets of the form $X(x_\beta) = \{\mu \in X_\alpha : x_\beta \nsubseteq \mu,\ x \in L,\ \beta \in (0, 1]\}$ forms a base for the open sets of X_α. The set of all α-fuzzy ideals of L is isomorphic with the set of all open sets in X_α. Finally, we give necessary and sufficient condition for the space X_α to be Hausdorff and regular.

2. Preliminaries

We refer to G. Birkhoff [2] for the elementary properties of lattices.

In [3], W. H. Cornish observed that, for any $a \in L$, the ideal $(a]^* = \{x \in L : x \wedge a = 0\}$ is called an annulate. The set of all annulates denoted by $A_0(L)$. Each annulate is an annihilator ideal and hence for two annulets $(x]^*$ and $(y]^*$ their supremum and infimum in $A_0(L)$ are

$$(x]^* \veebar (y]^* = (x \wedge y]^* \text{ and } (x]^* \cap (y]^* = (x \vee y]^*$$

respectively.

In a distributive lattice L with 0 the set of all annulates $A_0(L)$ of L is a lattice $(A_0(L), \cap, \veebar)$ and a sublattice of the Boolean algebra of annihilator ideals of L.

For an ideal I in L

$$\alpha(I) = \{(x]^* : x \in I\}$$

is a filter in $A_0(L)$ and the set

$$\overleftarrow{\alpha}(F) = \{x \in L : (x]^* \in F\}$$

is an ideal of L when F is any filter in $A_0(L)$. An ideal I of L is called an α-ideal if $\overleftarrow{\alpha}\alpha(I) = I$.

Remember that, for any set A a function $\mu : A \to ([0, 1], \wedge, \vee)$ is called a fuzzy subset of A, where $[0, 1]$ is a unit interval, $\alpha \wedge \beta = min\{\alpha, \beta\}$ and $\alpha \vee \beta = max\{\alpha, \beta\}$ for all $\alpha, \beta \in [0, 1]$.

Definition 2.1 ([10])**.** Let $x \in L,\ 0 < \beta \leq 1$. A fuzzy point x_β of L is a fuzzy subset of L defined as:

$$x_\beta(a) = \begin{cases} \beta \text{ , if } \ a = x \\ 0, \quad \text{otherwise .} \end{cases}$$

Definition 2.2 ([12])**.** Let μ and θ be fuzzy subsets of a set A. Define the fuzzy subsets $\mu \cup \theta$ and $\mu \cap \theta$ of A as follows: for each $x \in A$,

$$(\mu \cup \theta)(x) = \mu(x) \vee \theta(x) \text{ and } (\mu \cap \theta)(x) = \mu(x) \wedge \theta(x).$$

Then $\mu \cup \theta$ and $\mu \cap \theta$ are called the union and intersection of μ and θ, respectively.

For any collection, $\{\mu_i : i \in I\}$ of fuzzy subsets of X, where I is a nonempty index set, the least upper bound $\bigcup_{i \in I} \mu_i$ and the greatest lower bound $\bigcap_{i \in I} \mu_i$ of the μ_i's are given by for each $x \in X$,

$$(\bigcup_{i\in I}\mu_i)(x)=\bigvee_{i\in I}\mu_i(x) \text{ and } (\bigcap_{i\in I}\mu_i)(x)=\bigwedge_{i\in I}\mu_i(x),$$

respectively.

We define the binary operations "$+$" and "$\cdot$" on the set of all fuzzy subsets of L as:

$$(\mu+\theta)(x)=Sup\{\mu(y)\wedge\theta(z) : y,z\in L,\ y\vee z=x\} \text{ and}$$
$$(\mu\cdot\theta)(x)=Sup\{\mu(y)\wedge\theta(z) : y,z\in L,\ y\wedge z=x\}.$$

If μ and θ are fuzzy ideals of L, then $\mu\cdot\theta=\mu\wedge\theta=\mu\cap\theta$ and $\mu+\theta=\mu\vee\theta$ is a fuzzy ideal generated by $\mu\cup\theta$.

If μ and θ are fuzzy filters of L, then $\mu+\theta=\mu\wedge\theta$ (the pointwise infimum of μ and θ) and $\mu\cdot\theta=\mu\vee\theta$ (the supremum of μ and θ).

For each $t\in[0,1]$ the set

$$\mu_t=\{x\in A:\mu(x)\geq t\}$$

is called the level subset of μ at t [15].

Regarding fuzzy ideals and fuzzy filters of lattices, we refer [13].

Definition 2.3 ([13])**.** A proper fuzzy ideal μ of L is called prime fuzzy ideal of L, if for any two fuzzy ideals θ,η of L, $\theta\cap\eta\subseteq\mu\Rightarrow\theta\subseteq\mu$ or $\eta\subseteq\mu$.

Remark 2.4 ([13])**.** μ is a prime fuzzy ideal of L if and only if $Im\ \mu=\{1,\beta\},\ \beta\in[0,1)$ and $\mu_*=\{x\in L:\mu(x)=1\}$ is a prime ideal of L.

Let μ be a fuzzy subset of a lattice L. The smallest fuzzy ideal of L containing μ is called a fuzzy ideal of L induced by μ and denoted by $\langle\mu\rangle$ and

$$\langle\mu\rangle=\bigcap\{\theta\in FI(L):\mu\subseteq\theta\}$$

This definition can be stated as follows.

Theorem 2.5 ([6])**.** *Let μ be a fuzzy subset of L. The fuzzy subset $\langle\mu\rangle$ of L define by $\langle\mu\rangle(x)=Sup\{t\in[0,1]:x\in\langle\mu_t\rangle\}$ for all $x\in L$ is the fuzzy ideal induced by μ.*

Note that a fuzzy subset μ of L is nonempty if there exists $x\in L$ such that $\mu(x)\neq 0$.

Definition 2.6 ([1])**.** Let μ be a nonempty fuzzy subset of L and θ be a fuzzy ideal of L. The fuzzy annihilator $\langle\mu,\theta\rangle$ of μ relative to θ is defined:

$$\langle\mu,\theta\rangle=Sup\{\eta:\eta\in[0,1]^L,\ \eta\cdot\mu\subseteq\theta\}.$$

Remark 2.7. In [1], Alaba and Norahun observed that for any nonempty fuzzy subset μ of L and fuzzy ideal θ of L,

$$\langle\mu,\theta\rangle=Sup\{\eta:\eta\in FI(L),\ \eta\cdot\mu\subseteq\theta\}$$

is a fuzzy ideal of L.

If $\theta=\chi_{\{0\}}$, then we denote $\langle\mu,\chi_{\{0\}}\rangle$ by μ^* and μ^* is a fuzzy annihilator of μ.

Lemma 2.8 ([1])**.** *Let μ and θ be nonempty fuzzy subsets of L. Then*

(1) $\chi_{\{0\}}\subseteq\mu^*$,
(2) $\mu\cdot\mu^*\subseteq\chi_{\{0\}}$,
(3) $\mu\cdot\mu^*=\chi_{\{0\}}$, *whenever* $\mu(0)=1$,
(4) $\mu^*\cap\mu^{**}=\chi_{\{0\}}$,

(5) $\mu \subseteq \theta \Rightarrow \theta^* \subseteq \mu^*$,
(6) $\theta \cdot \mu \subseteq \chi_{\{0\}} \Leftrightarrow \theta \subseteq \mu^*$,
(7) $\theta \cdot \mu = \chi_{\{0\}} \Leftrightarrow \theta \subseteq \mu^*$, *whenever* $\mu(0) = 1 = \theta(0)$,
(8) $\mu \subseteq \mu^{**}$,
(9) $\mu^* = \mu^{***}$.

Definition 2.9 ([1])**.** A fuzzy ideal μ of L is called a fuzzy annihilator ideal, if $\mu = \theta^*$, for some nonempty fuzzy subset θ of L, or equivalently, if $\mu = \mu^{**}$.

The set of all fuzzy ideals and fuzzy filters of L are denoted by $FI(L)$ and $FF(L)$ respectively.

3. α-fuzzy ideals

In this section, we introduce the concept of α-fuzzy ideals of a lattice. We study some basic properties of the class of α-fuzzy ideals. Throughout the rest of this paper a lattice L is distributive with least element 0 unless otherwise specified.

Theorem 3.1. *Let μ be a fuzzy ideal of L. Then the fuzzy subset $\alpha(\mu)$ of $A_0(L)$ defined by:*

$$\alpha(\mu)((x]^*) = Sup\{\mu(y) : (y]^* = (x]^*,\ y \in L\}$$

is a fuzzy filter of a lattice $A_0(L)$.

Proof. Let μ be a fuzzy ideal of L. Then clearly, $\alpha(\mu)((0]^*) = 1$. Let $(x]^*, (y]^* \in A_0(L)$. Then

$$\begin{aligned}
&\alpha(\mu)((x]^*) \wedge \alpha(\mu)((y]^*) \\
&\quad = Sup\{\mu(a) : (a]^* = (x]^*,\ a \in L\} \wedge Sup\{\mu(b) : (b]^* = (y]^*,\ b \in L\} \\
&\quad = Sup\{\mu(a) \wedge \mu(b) : (a]^* = (x]^*,\ (b]^* = (y]^*\} \\
&\quad \leq Sup\{\mu(a) \wedge \mu(b) : (a]^* \wedge (b]^* = (x]^* \wedge (y]^*\} \\
&\quad = Sup\{\mu(a \vee b) : (a \vee b]^* = (x \vee y]^*\} \\
&\quad \leq Sup\{\mu(c) : (c]^* = (x \vee y]^*\} \\
&\quad = \alpha(\mu)((x]^* \wedge (y]^*).
\end{aligned}$$

Thus $\alpha(\mu)((x]^* \wedge (y]^*) \geq \alpha(\mu)((x]^*) \wedge \alpha(\mu)((y]^*)$. On the other hand,

$$\begin{aligned}
\alpha(\mu)((x]^*) &= Sup\{\mu(a) : (a]^* = (x]^*\} \\
&\leq Sup\{\mu(a \wedge y) : (a]^* \underline{\vee} (y]^* = (x]^* \underline{\vee} (y]^*\} \\
&\leq Sup\{\mu(c) : (c]^* = (x \wedge y]^*\} \\
&= \alpha(\mu)((x]^* \underline{\vee} (y]^*).
\end{aligned}$$

Similarly, $\alpha(\mu)((y]^*) \leq \alpha(\mu)((x]^* \underline{\vee} (y]^*)$. So

$$\alpha(\mu)((x]^* \underline{\vee} (y]^*) \geq \alpha(\mu)((x]^*) \vee \alpha(\mu)((y]^*).$$

Hence $\alpha(\mu)$ is a fuzzy filter of $A_0(L)$. $\square$

Lemma 3.2. *Let θ be a fuzzy filter of $A_0(L)$. Then the fuzzy subset $\overleftarrow{\alpha}(\theta)$ of L defined by $\overleftarrow{\alpha}(\theta)(x) = \theta((x]^*)$ is a fuzzy ideal of a lattice L.*

Proof. Let θ be a fuzzy filter of $A_0(L)$. Since $(0]^*$ is the largest element of $A_0(L)$, we get $\overleftarrow{\alpha}(\theta)(0) = 1$. Again,

$$\begin{aligned}\overleftarrow{\alpha}(\theta)(x \vee y) &= \theta((x]^* \wedge (y]^*) \\ &= \theta((x]^*) \wedge \theta((y]^*) \\ &= \overleftarrow{\alpha}(\theta)(x) \wedge \overleftarrow{\alpha}(\theta)(y)\end{aligned}$$

Thus $\overleftarrow{\alpha}(\theta)$ is a fuzzy ideal of L. □

It can be easily observed that the set $FF(A_0(L))$ of all fuzzy filters of $A_0(L)$ also forms a complete distributive lattice with the inclusion ordering of fuzzy sets, in which the infimum of the set of fuzzy filters μ_i is $\bigwedge_{i\in I} \mu_i = \bigcap_{i\in I} \mu_i$, and the supremum is $\bigvee_{i\in I} \mu_i$.

The following lemma can be verified easily.

Lemma 3.3. *If μ and θ are fuzzy ideals of L, then $\mu \subseteq \theta$ implies $\alpha(\mu) \subseteq \alpha(\theta)$.*

Proposition 3.4. *If μ, θ are fuzzy filters of $A_0(L)$, then $\mu \subseteq \theta$ implies $\overleftarrow{\alpha}(\mu) \subseteq \overleftarrow{\alpha}(\theta)$.*

Theorem 3.5. *The mapping α is a homomorphism of $FI(L)$ into $FF(A_0(L))$.*

Proof. Let μ, θ be two fuzzy ideals of L. It is enough to prove that $\alpha(\mu\cap\theta) = \alpha(\mu)\cap\alpha(\theta)$ and $\alpha(\mu\vee\theta) = \alpha(\mu)\underline{\vee}\alpha(\theta)$. By Lemma 3.3, we have that $\alpha(\mu\cap\theta) \subseteq \alpha(\mu)\cap\alpha(\theta)$. For any $(x]^* \in A_0(L)$,

$$\begin{aligned}&\alpha(\mu)((x]^*) \wedge \alpha(\theta)((x]^*) \\ &= Sup\{\mu(a) : (a]^* = (x]^*\} \wedge Sup\{\theta(b) : (b]^* = (x]^*\} \\ &\leq Sup\{\mu(a\wedge b) : (a\wedge b]^* = (x]^*\} \wedge Sup\{\theta(a\wedge b) : (a\wedge b]^* = (x]^*\} \\ &= Sup\{\mu(a\wedge b) \wedge \theta(a\wedge b) : (a\wedge b]^* = (x]^*\} \\ &= Sup\{(\mu\cap\theta)(a\wedge b) : (a\wedge b]^* = (x]^*\} \\ &\leq Sup\{(\mu\cap\theta)(c) : (c]^* = (x]^*\} \\ &= \alpha(\mu\cap\theta)((x]^*).\end{aligned}$$

Thus $\alpha(\mu)\cap\alpha(\theta) \subseteq \alpha(\mu\cap\theta)$. So $\alpha(\mu)\cap\alpha(\theta) = \alpha(\mu\cap\theta)$.

Again clearly, $\alpha(\mu)\underline{\vee}\alpha(\theta) \subseteq \alpha(\mu\vee\theta)$. On the other hand,

$$\begin{aligned}&\alpha(\mu\vee\theta)((x]^*) \\ &= Sup\{(\mu\vee\theta)(a) : (a]^* = (x]^*\} \\ &= Sup\{Sup\{\mu(y)\wedge\theta(z) : a = y\vee z\} : (y\vee z]^* = (x]^*\} \\ &\leq Sup\{Sup\{\mu(b_1)\wedge\theta(b_2) : (b_1]^* = (y]^*,\ (b_2]^* = (z]^*\} : (y\vee z]^* = (x]^*\} \\ &= Sup\{Sup\{\mu(b_1) : (b_1]^* = (y]^*\} \wedge\ Sup\{\theta(b_2) : (b_2]^* = (z]^*\},\ (y\vee z]^* = (x]^*\} \\ &= Sup\{\alpha(\mu)(y]^* \wedge \alpha(\theta)(z]^* : (y\vee z]^* = (x]^*\} \\ &= Sup\{\alpha(\mu)(y]^* \wedge \alpha(\theta)(z]^* : (y]^* \wedge (z]^* = (x]^*\} \\ &= (\alpha(\mu)\underline{\vee}\alpha(\theta))((x]^*).\end{aligned}$$

Then $\alpha(\mu\vee\theta) \subseteq \alpha(\mu)\underline{\vee}\alpha(\theta)$. Thus $\alpha(\mu\vee\theta) = \alpha(\mu)\underline{\vee}\alpha(\theta)$. So α is a homomorphism. □

Corollary 3.6. *For any two fuzzy ideals μ and θ, we have*

$$\overleftarrow{\alpha}\alpha(\mu\cap\theta) = \overleftarrow{\alpha}\alpha(\mu) \cap \overleftarrow{\alpha}\alpha(\theta).$$

Proof. For any $x \in L$, $\overleftarrow{\alpha}\alpha(\mu\cap\theta)(x) = \alpha(\mu\cap\theta)((x]^*)$. Since $\alpha(\mu\cap\theta) = \alpha(\mu)\cap\alpha(\theta)$, we have $\overleftarrow{\alpha}\alpha(\mu\cap\theta)(x) = \overleftarrow{\alpha}\alpha(\mu)(x)\wedge\overleftarrow{\alpha}\alpha(\theta)(x)$. Thus $\overleftarrow{\alpha}\alpha(\mu\cap\theta) = \overleftarrow{\alpha}\alpha(\mu)\cap\overleftarrow{\alpha}\alpha(\theta)$. □

The proof of the following lemma is quite routine and will be omitted.

Lemma 3.7. *For any fuzzy ideals μ, θ of L, we have the following:*

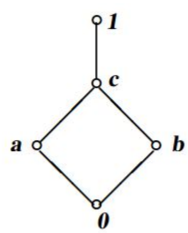
(1) $\mu \subseteq \overleftarrow{\alpha}\alpha(\mu)$,
(2) $\overleftarrow{\alpha}\alpha(\overleftarrow{\alpha}\alpha(\mu)) = \overleftarrow{\alpha}\alpha(\mu)$,
(3) $\mu \subseteq \theta \Rightarrow \overleftarrow{\alpha}\alpha(\mu) \subseteq \overleftarrow{\alpha}\alpha(\theta)$,
(4) $\overleftarrow{\alpha}\alpha(\mu) \vee \overleftarrow{\alpha}\alpha(\theta) \subseteq \overleftarrow{\alpha}\alpha(\mu \vee \theta)$.

Lemma 3.8. *For any fuzzy filter θ of $A_0(L)$, $\alpha\overleftarrow{\alpha}(\theta) = \theta$.*

Proof. Since θ is a fuzzy filter of $A_0(L)$, by Lemma 3.2, $\overleftarrow{\alpha}(\theta)$ is a fuzzy ideal of L and $\alpha\overleftarrow{\alpha}(\theta)$ is a fuzzy filter of $A_0(L)$. Now, $\alpha\overleftarrow{\alpha}(\theta)((x]^*) = Sup\{\overleftarrow{\alpha}(\theta)(a) : (a]^* = (x]^*\} = Sup\{\theta((a]^*) : (a]^* = (x]^*\} = \theta((x]^*)$. □

Definition 3.9. A fuzzy ideal μ of L is called an α-fuzzy ideal, if $\overleftarrow{\alpha}\alpha(\mu) = \mu$.

Example 3.10. Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given below.



Define a fuzzy subset μ of L as follows: $\mu(0) = 1$, $\mu(a) = 0.5$ and $\mu(b) = \mu(c) = \mu(1) = 0.4$. Then it can be easily verified that μ is an α-fuzzy ideal of L.

Example 3.11. If we define a fuzzy subset θ of L in the above example as: $\theta(0) = 1$, $\theta(a) = 0.5$, $\theta(b) = \theta(c) = 0.4$ and $\theta(1) = 0.3$. Then it can be easily verified that θ is a fuzzy ideal, but not an α-fuzzy ideal of L.

Now we define a multiplicatively closed fuzzy subset of L as follows.

Definition 3.12. A fuzzy subset μ of L is said to be multiplicatively closed, if

$$\mu(x \wedge y) \geq \mu(x) \wedge \mu(y), \text{ for all } x,\ y \in L.$$

Example 3.13. Let θ be a multiplicatively closed fuzzy subset of L with $Sup\{\theta(x) : x \in L\} = 1$. Then a fuzzy subset μ of L defined as:

$$\mu(x) = Sup\{\theta(a) : a \wedge x = 0,\ a \in L\}$$

is an α-fuzzy ideal of L.

Proof. First we have to show that μ is a fuzzy ideal of L. Clearly, $\mu(0) = 1$. For any $x, y \in L$,

$$\begin{aligned}
\mu(x) \wedge \mu(y) &= Sup\{\theta(a) : a \wedge x = 0\} \wedge Sup\{\theta(b) : b \wedge y = 0\} \\
&= Sup\{\theta(a) \wedge \theta(b) : a \wedge x = 0,\ b \wedge y = 0\} \\
&\leq Sup\{\theta(a \wedge b) : (a \wedge b) \wedge (x \vee y) = 0\} \\
&= \mu(x \vee y).
\end{aligned}$$

Then $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$. And

$$\mu(x) = Sup\{\theta(a) : a \wedge x = 0\} \leq Sup\{\theta(a) : a \wedge (x \wedge y) = 0\} = \mu(x \wedge y).$$

Similarly, $\mu(y) \leq \mu(x \wedge y)$. Thus $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$. So μ is a fuzzy ideal of L.
Now for each $a, x \in L$, if $(a]^* = (x]^*$, then

$$\mu(x) = Sup\{\theta(b) : b \wedge x = 0\} = Sup\{\theta(b) : b \wedge a = 0\} = \mu(a).$$

We proceed to show μ is an α-fuzzy ideal.

$$\overleftarrow{\alpha}\alpha(\mu)(x) = Sup\{\mu(a) : (a]^* = (x]^*\} = \mu(x).$$

Thus μ is an α-fuzzy ideal of L. □

Lemma 3.14. *For any fuzzy ideal μ of L, $\overleftarrow{\alpha}\alpha(\mu^*) = \mu^*$, where μ^* is a fuzzy annihilator of μ.*

Proof. Clearly, $\mu^* \subseteq \overleftarrow{\alpha}\alpha(\mu^*)$ and $\chi_{\{0\}}$ is an α-fuzzy ideal. To prove our claim, it is enough to show that $\overleftarrow{\alpha}\alpha(\mu^*) \cap \mu \subseteq \chi_{\{0\}}$. Since $\mu^* \cap \mu \subseteq \chi_{\{0\}}$, we have $\overleftarrow{\alpha}\alpha(\mu^*) \cap \overleftarrow{\alpha}\alpha(\mu) \subseteq \chi_{\{0\}}$. Then $\overleftarrow{\alpha}\alpha(\mu^*) \cap \mu \subseteq \chi_{\{0\}}$. Thus $\overleftarrow{\alpha}\alpha(\mu^*) \subseteq \mu^*$. So μ^* is an α-fuzzy ideal. □

Corollary 3.15. *Every fuzzy annihilator ideal is an α-fuzzy ideal.*

Proof. For any fuzzy annihilator ideal μ of L, we have $\mu = \mu^{**}$. By the above lemma, we get $\overleftarrow{\alpha}\alpha(\mu) = \mu$. Then a fuzzy annihilator ideal is an α-fuzzy ideal. □

Theorem 3.16. *For a fuzzy ideal μ of L. μ is an α-fuzzy ideal if and only if for each $x, y \in L$, $(x]^* = (y]^*$ imply $\mu(x) = \mu(y)$.*

Proof. Suppose μ is an α-fuzzy ideal of L and let $x, y \in L$ such that $(x]^* = (y]^*$. Then

$$\mu(x) = Sup\{\mu(a) : (a]^* = (x]^*,\ a \in L\} = Sup\{\mu(a) : (a]^* = (y]^*,\ a \in L\} = \mu(y).$$

Conversely, suppose that for each $x, y \in L$, $(x]^* = (y]^*$ imply $\mu(x) = \mu(y)$. For any $x \in L$, we have $\overleftarrow{\alpha}\alpha(\mu)(x) = Sup\{\mu(a) : (a]^* = (x]^*,\ a \in L\} = \mu(x)$. Thus μ is an α-fuzzy ideal of L. □

Recall that a lattice L is disjunctive if for any $a, b \in L$, $a < b$ implies there is an element $0 \neq c \in L$ such that $a \wedge c = 0$ and $0 < c < b$. However it is easy to see that a lattice L is disjunctive if and only if $(a]^* = (b]^*$ implies $a = b$, for any $a, b \in L$ [3]. We thus have the following Lemma.

Lemma 3.17. *If L is disjunctive, then every fuzzy ideal of L is an α-fuzzy ideal.*

Proof. Since L is disjunctive, $(a]^* = (b]^*$ implies $a = b$ for any $a, b \in L$. Then we always have $\overleftarrow{\alpha}\alpha(\mu)(x) = \mu(x)$. Thus every fuzzy ideal is an α-fuzzy ideal. □

Theorem 3.18. *For a nonempty fuzzy subset μ of L, μ is an α-fuzzy ideal if and only if each level subset of μ is an α-ideal of L.*

Proof. Suppose μ is an α-fuzzy ideal of L. Then $\mu_t = (\overleftarrow{\alpha}\alpha(\mu))_t$. To prove each level subset of μ is an α-ideal of L, it is enough to show $\overleftarrow{\alpha}\alpha(\mu_t) = (\overleftarrow{\alpha}\alpha(\mu))_t$, for all $t \in [0, 1]$. Clearly, $(\overleftarrow{\alpha}\alpha(\mu))_t \subseteq \overleftarrow{\alpha}\alpha(\mu_t)$. Let $x \in \overleftarrow{\alpha}\alpha(\mu_t)$. Then $(x]^* \in \alpha(\mu_t)$ and there is $y \in \mu_t$ such that $(x]^* = (y]^*$. Thus $Sup\{\mu(a) : (a]^* = (x]^*\} \geq t$. This shows that $x \in (\overleftarrow{\alpha}\alpha(\mu))_t$. So $\mu_t = \overleftarrow{\alpha}\alpha(\mu_t)$. Hence each level subset of μ is an α-ideal of L.

Conversely, suppose each level subset of μ is an α-ideal. Then μ is a fuzzy ideal and $\mu \subseteq \overleftarrow{\alpha}\alpha(\mu)$. Let $t = \overleftarrow{\alpha}\alpha(\mu)(x) = Sup\{\mu(y) : (y]^* = (x]^*\}$. Then for each $\epsilon > 0$, there is $a \in L$ such that $(a]^* = (x]^*$ and $\mu(a) > t - \epsilon$. Thus $a \in \mu_{t-\epsilon}$, $(a]^* = (x]^*$ and $x \in \overleftarrow{\alpha}\alpha(\mu_{t-\epsilon}) = \mu_{t-\epsilon}$. So $x \in \bigcap_{\epsilon>0} \mu_{t-\epsilon} = \mu_t$. Hence $\overleftarrow{\alpha}\alpha(\mu) \subseteq \mu$. Therefore μ is an α-fuzzy ideal of L. □

Corollary 3.19. *For a nonempty subset I of L, I is an α-ideal if and only if χ_I is an α-fuzzy ideal of L.*

Proof. Suppose I is an α-idea of L. Then $\overleftarrow{\alpha}\alpha(I) = \{x \in L : (x]^* \in \alpha(I)\} = I$. Let $x \in L$. If $x \in I$, then $\overleftarrow{\alpha}\alpha(\chi_I)(x) = 1 = \chi_I(x)$. Let $x \notin I$ and assume that $\overleftarrow{\alpha}\alpha(\chi_I)(x) = 1$. Then there is $y \in I$ such that $(y]^* = (x]^* \in \alpha(I)$. Since I is an α-ideal, $x \in I$. This is a contradiction. Thus $\overleftarrow{\alpha}\alpha(\chi_I)(x) = 0$. So χ_I is an α-fuzzy ideal.

Conversely, suppose χ_I is an α-fuzzy ideal of L. Then clearly, $I \subseteq \overleftarrow{\alpha}\alpha(I)$. Let $x \in \overleftarrow{\alpha}\alpha(I)$. Since χ_I is an α-fuzzy ideal, $\overleftarrow{\alpha}\alpha(\chi_I)(x) = 1 = \chi_I(x)$. Thus $x \in I$. So I is an α-ideal of L. □

Theorem 3.20. *Let μ be a fuzzy ideal of L. The fuzzy subset*

$$\mu^{'}(x) = Sup\{\mu(a) : x \in (a]^{**}, x \in L\}$$

is a fuzzy ideal of L.

Proof. Let μ be a fuzzy ideal of L. Then clearly, $\mu^{'}(0) = 1$. For any $x, y \in L$,

$$\begin{aligned}
\mu^{'}(x) \wedge \mu^{'}(y) &= Sup\{\mu(a) : x \in (a]^{**}\} \wedge Sup\{\mu(b) : y \in (b]^{**}\} \\
&= Sup\{\mu(a) \wedge \mu(b) : x \in (a]^{**},\ y \in (b]^{**}\} \\
&\leq Sup\{\mu(a \vee b) : x \vee y \in (a \vee b]^{**}\} \\
&\leq Sup\{\mu(c) : x \vee y \in (c]^{**}\} \\
&= \mu^{'}(x \vee y)
\end{aligned}$$

and

$\mu^{'}(x) = Sup\{\mu(a) : x \in (a]^{**}\} \leq Sup\{\mu(a) : x \wedge y \in (a]^{**}\} = \mu^{'}(x \wedge y)$.

Similarly, $\mu^{'}(y) \leq \mu^{'}(x \wedge y)$. Thus $\mu^{'}(x \wedge y) \geq \mu^{'}(x) \vee \mu^{'}(y)$. So $\mu^{'}$ is a fuzzy ideal of L. □

Corollary 3.21. $\mu^{'}$ *is the smallest α-fuzzy ideal containing μ.*

Proof. Clearly, $\mu \subseteq \mu^{'}$. We proceed to show $\mu^{'}$ is an α-fuzzy ideal of L. For each $x, y \in L$, $(x]^{*} = (y]^{*}$, we need to show $\mu^{'}(x) = \mu^{'}(y)$. Now

$$\begin{aligned} \mu^{'}(x) &= Sup\{\mu(a) : x \in (a]^{**}, a \in L\} \\ &= Sup\{\mu(a) : (x]^{**} \subseteq (a]^{**}, a \in L\} \\ &= Sup\{\mu(a) : y \in (a]^{**}, a \in L\} \\ &= \mu^{'}(y). \end{aligned}$$

This shows that $\mu^{'}(x) = \mu^{'}(y)$, for each $x, y \in L$ whenever $(x]^{*} = (y]^{*}$. Thus by Theorem 3.16, $\mu^{'}$ is an α-fuzzy ideal of L.

To show $\mu^{'}$ is the smallest fuzzy ideal containing μ, let θ be any α-fuzzy ideal containing μ and $\mu^{'}(x) = t$. Then for each $\epsilon > 0$, there is $b \in L$ such that $x \in (b]^{**}$ and $\mu(b) > t - \epsilon$. Thus $b \in \theta_{t-\epsilon}$. Since $\theta_{t-\epsilon}$ is an α-ideal, $(b]^{**} \subseteq \theta_{t-\epsilon}$ and $x \in \theta_{t-\epsilon}$. So for each $\epsilon > 0$, $x \in \theta_{t-\epsilon}$. This shows that $x \in \bigcap_{\epsilon>0} \theta_{t-\epsilon} = \theta_t$ and $\theta(x) \geq t$. Thus $\mu^{'}(x) \leq \theta(x)$, for all $x \in L$. Hence $\mu^{'}$ is the smallest α-fuzzy ideal containing μ. □

Let us denote the set of all α-fuzzy ideals of L by $FI_{\alpha}(L)$.

Theorem 3.22. *The set $FI_{\alpha}(L)$ forms a compete distributive lattice with respect to inclusion ordering of fuzzy sets.*

Proof. Clearly, $(FI_{\alpha}(L), \subseteq)$ is a partially ordered set. For $\mu, \theta \in FI_{\alpha}(L)$, define

$$\mu \wedge \theta = \mu \cap \theta \text{ and } \mu \underline{\vee} \theta = \overleftarrow{\alpha}\alpha(\mu \vee \theta).$$

Then clearly, $\mu \cap \theta$, $\mu \underline{\vee} \theta \in FI_{\alpha}(L)$. We need to show $\mu \underline{\vee} \theta$ is the least upper bound of $\{\mu, \theta\}$. Since $\mu \vee \theta \subseteq \mu \underline{\vee} \theta$, it yields $\mu \underline{\vee} \theta$ is an upper bound of $\{\mu, \theta\}$. Let η be any upper bound for μ, θ in $FI_{\alpha}(L)$. Then $\mu \vee \theta \subseteq \eta$. Thus $\overleftarrow{\alpha}\alpha(\mu \vee \theta) \subseteq \overleftarrow{\alpha}\alpha(\eta) = \eta$. So $\mu \underline{\vee} \theta$ is the supremum of both μ and θ in $FI_{\alpha}(L)$. Hence $(FI_{\alpha}(L), \cap, \underline{\vee})$ is a lattice.

We now prove the distributivity. Let μ, θ, $\eta \in FI_{\alpha}(L)$. Then

$$\begin{aligned} \mu \underline{\vee} (\theta \cap \eta) &= \overleftarrow{\alpha}\alpha((\mu \vee \theta) \cap (\mu \vee \eta)) \\ &= \overleftarrow{\alpha}\alpha(\mu \vee \theta) \cap \overleftarrow{\alpha}\alpha(\mu \vee \eta) \\ &= (\mu \underline{\vee} \theta) \cap (\mu \underline{\vee} \eta) \end{aligned}$$

Thus $FI_{\alpha}(L)$ is a distributive lattice.

Next we prove the completeness. Since $\{0\}$ and L are α-ideals, $\chi_{\{0\}}$ and χ_{L} are least and greatest elements of $FI_{\alpha}(L)$, respectively. Let $\{\mu_i : i \in I\} \subseteq FI_{\alpha}(L)$. Then $\bigcap_{i \in I} \mu_i$ is a fuzzy ideal of L and $\bigcap_{i \in I} \mu_i \subseteq \overleftarrow{\alpha}\alpha(\bigcap_{i \in I} \mu_i)$.

$$\begin{aligned} \bigcap_{i \in I} \mu_i \subseteq \mu_i, \ \forall i \in I \quad &\Rightarrow \quad \overleftarrow{\alpha}\alpha(\bigcap_{i \in I} \mu_i) \subseteq \mu_i, \ \forall i \in I \\ &\Rightarrow \quad \overleftarrow{\alpha}\alpha(\bigcap_{i \in I} \mu_i) \subseteq \bigcap_{i \in I} \mu_i \end{aligned}$$

Thus $\overleftarrow{\alpha}\alpha(\bigcap_{i \in I} \mu_i) = \bigcap_{i \in I} \mu_i$. So $(FI_{\alpha}(L), \cap, \underline{\vee})$ is a complete distributive lattice. □

Theorem 3.23. *The set $FI_{\alpha}(L)$ is isomorphic to the lattice of fuzzy filters of $A_0(L)$.*

Proof. Define $f : FI_\alpha(L) \longrightarrow FF(A_0(L)),\ f(\mu) = \alpha(\mu),\ \forall\mu \in FI_\alpha(L)$. Let $\mu, \theta \in FI_\alpha(L)$ and $f(\mu) = f(\theta)$. Then $\alpha(\mu) = \alpha(\theta)$. Thus $\overleftarrow{\alpha}\alpha(\mu) = \overleftarrow{\alpha}\alpha(\theta)$. So $\mu = \theta$. Hence f is one to one.

Let $\eta \in FF(A_0(L))$. Then by Lemma 3.2 $\overleftarrow{\alpha}(\eta)$ is a fuzzy ideal of L. We show that $\overleftarrow{\alpha}(\eta)$ is an α-fuzzy ideal of L. Let $x \in L$. Then $\overleftarrow{\alpha}\alpha(\overleftarrow{\alpha}(\eta))(x) = \alpha\overleftarrow{\alpha}(\eta)((x]^*)$. Thus by Lemma 3.8, we get that $\alpha\overleftarrow{\alpha}(\eta)((x]^*) = \eta((x]^*) = \overleftarrow{\alpha}(\eta)(x)$. So $\overleftarrow{\alpha}(\eta) = \overleftarrow{\alpha}\alpha(\overleftarrow{\alpha}(\eta))$. Hence for each $\eta \in FF(A_0(L))$, $f(\overleftarrow{\alpha}(\eta)) = \eta$. Therefore f is onto.

Now for any $\mu, \theta \in FI_\alpha(L)$, $f(\mu\underline{\vee}\theta) = f(\overleftarrow{\alpha}\alpha(\mu\vee\theta)) = \alpha(\overleftarrow{\alpha}\alpha(\mu\vee\theta)) = \alpha(\mu\vee\theta) = \alpha(\mu)\underline{\vee}\alpha(\theta) = f(\mu)\underline{\vee}f(\theta)$. Similarly, $f(\mu\cap\theta) = f(\mu)\cap f(\theta)$. Then f is an isomorphism of $FI_\alpha(L)$ onto the lattice of fuzzy filters of $A_0(L)$. □

Theorem 3.24. *In L the following are equivalent:*

1. *Each fuzzy ideal is an α-fuzzy ideal,*
2. *Each prime fuzzy ideal is an α-fuzzy ideal,*
3. *L is disjunctive.*

Proof. The proof of (1) ⇒ (2) and (3) ⇒ (1) is straightforward. To show (2) ⇒ (3), suppose that every prime fuzzy ideal of L is an α-fuzzy ideal. Let $x, y \in L$ such that $(x]^* = (y]^*$. Assume that $x \neq y$. Without loss of generality, we can assume that $(x] \cap [y) = \phi$. We know that $\chi_{(x]}$ and $\chi_{[y)}$ are fuzzy ideal and fuzzy filter of L respectively such that $\chi_{(x]} \cap \chi_{[y)} = \chi_\phi$(the constant fuzzy subset attaining, value 0), by Corollary 1.5 in [13] there exists a prime fuzzy ideal θ of L such that

$$\chi_{(x]} \subseteq \theta \text{ and } \theta \cap \chi_{[y)} = \chi_\phi.$$

Since $\chi_{(x]} \subseteq \theta$, we get $\theta(x) = 1$. Again, $\theta(y) \wedge \chi_{[y)}(y) = 0$. Since every element in $[0, 1]$ is a meet irreducible element, we get $\theta(y) = 0$. Which is a contradiction. Thus L is disjunctive. □

Theorem 3.25. *Let μ be an α-fuzzy ideal of L and λ be a fuzzy filter of L such that $\mu \cap \lambda \leq \beta$, $\beta \in [0, 1)$. Then there exists a prime α-fuzzy ideal θ of L such that $\mu \subseteq \theta$ and $\theta \cap \lambda \leq \beta$.*

Proof. Put $\mathcal{P} = \{\eta \in FI_\alpha(L) : \mu \subseteq \eta \texttt{ and } \eta \cap \lambda \leq \beta\}$. Since $\mu \in \mathcal{P}$, $\mathcal{P}$ is nonempty and it forms a poset together with the inclusion ordering of fuzzy sets. Let $\mathcal{A} = \{\mu_i\}_{i\in I}$ be any chain in $\mathcal{P}$. We need to prove $\bigcup_{i\in I}\mu_i \in \mathcal{A}$. Clearly, $(\bigcup_{i\in I}\mu_i)(0) = 1$. For any $x, y \in L$,

$$\begin{aligned}(\bigcup_{i\in I}\mu_i)(x) \wedge (\bigcup_{i\in I}\mu_i)(y) &= Sup\{\mu_i(x) : i \in I\} \wedge Sup\{\mu_j(y) : j \in I\} \\ &= Sup\{\mu_i(x) \wedge \mu_j(y) : i, j \in I\} \\ &\leq Sup\{(\mu_i \cup \mu_j)(x) \wedge (\mu_i \cup \mu_j)(y) : i, j \in I\}.\end{aligned}$$

Since $\mathcal{A}$ is a chain, either $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$. Without loss of generality, we can assume that $\mu_i \subseteq \mu_j$. Then $\mu_i \cup \mu_j = \mu_j$. Thus

$$\begin{aligned} (\bigcup_{i\in I}\mu_i)(x) \wedge (\bigcup_{i\in I}\mu_i)(y) &\leq Sup\{\mu_j(x) \wedge \mu_j(y) : j \in I\} \\ &= Sup\{\mu_j(x \vee y) : j \in I\} \\ &= (\bigcup_{i\in I}\mu_i)(x \vee y). \end{aligned}$$

Again, $(\bigcup_{i\in I}\mu_i)(x) = Sup\{\mu_i(x) : i \in I\} \leq Sup\{\mu_i(x \wedge y) : i \in I\} = (\bigcup_{i\in I}\mu_i)(x \wedge y)$. Similarly, $(\bigcup_{i\in I}\mu_i)(y) \leq (\bigcup_{i\in I}\mu_i)(x \wedge y)$. Then $\bigcup_{i\in I}\mu_i$ is a fuzzy ideal of L. It remains to show $\bigcup_{i\in I}\mu_i$ is an α-fuzzy ideal.

$$\begin{aligned} \overleftarrow{\alpha}\alpha(\bigcup_{i\in I}\mu_i)(x) &= Sup\{(\bigcup_{i\in I}\mu_i)(a) : (a]^* = (x]^*\} \\ &= Sup\{Sup\{\mu_i(a) : i \in I\} : (a]^* = (x]^*\} \\ &= Sup\{Sup\{\mu_i(a) : (a]^* = (x]^*\} : i \in I\} \\ &= Sup\{\mu_i(x) : i \in I\} \\ &= (\bigcup_{i\in I}\mu_i)(x). \end{aligned}$$

Thus $\bigcup_{i\in I}\mu_i$ is an α-fuzzy ideal. Since $\mu_i \cap \eta \leq \beta$, for each $i \in I$,

$$\begin{aligned} ((\bigcup_{i\in I}\mu_i) \cap \eta)(x) &= (\bigcup_{i\in I}\mu_i)(x) \wedge \eta(x) \\ &= Sup\{\mu_i(x) \wedge \eta(x) : i \in I\} \\ &= Sup\{(\mu_i \cap \eta)(x) : i \in I\} \leq \beta. \end{aligned}$$

So $(\bigcup_{i\in I}\mu_i) \cap \eta \leq \beta$. Hence $\bigcup_{i\in I}\mu_i \in \mathcal{A}$. By applying Zorn's lemma, we get a maximal element, let say $\theta \in \mathcal{P}$, that is, θ is an α-fuzzy ideal of L such that $\mu \subseteq \theta$ and $\theta \cap \eta \leq \beta$.

Now we proceed to show θ is a prime fuzzy ideal. Assume that θ is not prime fuzzy ideal. Let $\gamma_1 \cap \gamma_2 \subseteq \theta$ such that $\gamma_1 \nsubseteq \theta$ and $\gamma_2 \nsubseteq \theta$, γ_1, $\gamma_2 \in FI(L)$. If we put $\theta_1 = \overleftarrow{\alpha}\alpha(\gamma_1 \vee \theta)$ and $\theta_2 = \overleftarrow{\alpha}\alpha(\gamma_2 \vee \theta)$, then both θ_1 and θ_2 are α-fuzzy ideals of L properly containing θ. Since θ is maximal in $\mathcal{P}$, we get $\theta_1 \notin \mathcal{P}$ and $\theta_2 \notin \mathcal{P}$. Thus $\theta_1 \cap \eta \nleq \beta$ and $\theta_2 \cap \eta \nleq \beta$. This implies there exist $x, y \in L$ such that $(\theta_1 \cap \eta)(x) > \beta$ and $(\theta_2 \cap \eta)(y) > \beta$. So $((\theta_1 \cap \theta_2) \cap \eta)(x \wedge y) > \beta \Rightarrow (\overleftarrow{\alpha}\alpha(\theta \vee (\gamma_1 \wedge \gamma_2))(x \wedge y) \wedge \eta(x \wedge y) > \beta$. This shows that $(\theta \cap \eta)(x \wedge y) > \beta$. This is a contradiction. Hence θ is prime α-fuzzy ideal of L. □

Corollary 3.26. *Let μ be an α-fuzzy ideal of L, $a \in L$ and $\beta \in [0, 1)$. If $\mu(a) \leq \beta$, then there exists a prime α-fuzzy ideal θ of L such that $\mu \subseteq \theta$ and $\theta(a) \leq \beta$.*

Corollary 3.27. *Every α-fuzzy ideal of L is the intersection of all prime α-fuzzy ideals containing it.*

Proof. Let μ be α-fuzzy ideal of L. Consider the following.

$$\mu_0 = \bigcap\{\eta : \eta \textit{ is a prime } \alpha\textit{-fuzzy ideal and } \mu \subseteq \eta\}.$$

Then clearly, $\mu \subseteq \mu_0$. Now we proceed to show that $\mu_0 \subseteq \mu$. Suppose not. Then there is $a \in L$ such that $\mu_0(a) > \mu(a)$. Let $\mu(a) = \beta$. Consider the set $\mathcal{P} = \{\eta \in FI_\alpha(L) : \mu \subseteq \eta \text{ and } \eta(a) \leq \beta\}$. By the above Corollary, we can find a prime α-fuzzy ideal θ of L such that $\mu \subseteq \theta$ and $\theta(a) \leq \beta$. Then $\mu_0 \subseteq \theta$. Thus $\mu_0(a) \leq \beta$. This is a contradiction. So $\mu_0 \subseteq \mu$. Hence $\mu_0 = \mu$. □

4. The space of prime α-fuzzy ideals

In this section, we study the space of prime α-fuzzy ideals of a distributive lattice. Some properties of the space also studied. The set of equivalent conditions are given for the space of all prime α-fuzzy ideals of L to become regular.

Now we recall some topological concepts in [7].

A topology on a set X is a family of $\mathcal{T}$ of sets which satisfies the following conditions: the intersection of any finite members of $\mathcal{T}$ is a member of $\mathcal{T}$, the union of the members of each subfamily of $\mathcal{T}$ is a member of $\mathcal{T}$ and $\phi, X \in \mathcal{T}$. If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an open set of X if $U \in \mathcal{T}$. A subfamily $\mathcal{B}$ of a topology $\mathcal{T}$ is a base for $\mathcal{T}$ if and only if each member of $\mathcal{T}$ is the union of members of $\mathcal{B}$.

A topological space is a T_0-space if, for any distinct points x and y of X there exists an open set containing one but not the other. A topological space is a T_1-space if and only if each set which consists of a single point is closed. A topological space X is a regular space if, given any nonempty closed set F and any point x that does not belong to F, there exists a neighborhood U of x and a neighborhood V of F that are disjoint.

Let X_α be the set of all prime α-fuzzy ideals of a distributive lattice. Let $V(\theta) = \{\mu \in X_\alpha : \theta \subseteq \mu\}$ where θ is a fuzzy ideal of L and $X(\theta) = \{\mu \in X_\alpha : \theta \nsubseteq \mu\} = X_\alpha - V(\theta)$. We let $\mu_* = \mu_1$, i.e. $\mu_* = \{x \in L : \mu(x) = 1\}$.

Lemma 4.1. *For any fuzzy ideals μ and θ of L, we have*

(1) $\mu \subseteq \theta \Rightarrow X(\mu) \subseteq X(\theta)$,
(2) $X(\mu \vee \theta) = X(\mu) \cup X(\theta)$,
(3) $X(\mu \cap \theta) = X(\mu) \cap X(\theta)$.

Proof. (1) Let $\mu \subseteq \theta$ and $\eta \in X(\mu)$. Then $\mu \nsubseteq \eta$ and $\theta \nsubseteq \eta$. Thus $\eta \in X(\theta)$.

(2) By (1), we have $X(\mu) \cup X(\theta) \subseteq X(\mu \vee \theta)$. Again if $\eta \in X(\mu \vee \theta)$, then $\mu \vee \theta \nsubseteq \eta$. Thus either $\mu \nsubseteq \eta$ or $\theta \nsubseteq \eta$. So $\eta \in X(\mu) \cup X(\theta)$. Hence $X(\mu \vee \theta) = X(\mu) \cup X(\theta)$.

(3) Clearly $X(\mu \cap \theta) \subseteq X(\mu) \cap X(\theta)$. Again if $\eta \in X(\mu) \cap X(\theta)$, then $\mu \nsubseteq \eta$ and $\theta \nsubseteq \eta$. Since η is a prime fuzzy ideal, we have $\mu \cap \theta \nsubseteq \eta$. Thus $\eta \in X(\mu \cap \theta)$. So $X(\mu \cap \theta) = X(\mu) \cap X(\theta)$. □

Lemma 4.2. *Let θ be a fuzzy subset of L. Then $X(\theta) = X(\langle\theta\rangle)$.*

Proof. Clearly, $X(\theta) \subseteq X(\langle\theta\rangle)$. Let $\mu \in X(\langle\theta\rangle)$. Then $\langle\theta\rangle \nsubseteq \mu$. We now to show $\theta \nsubseteq \mu$. Suppose $\theta \subseteq \mu$. Then $\langle\theta\rangle \subseteq \langle\mu\rangle = \mu$. Which is impossible. Thus $\theta \nsubseteq \mu$. So $X(\theta) = X(\langle\theta\rangle)$. □

Theorem 4.3. *Let $x, y \in L$ and $\beta \in (0, 1]$. Then*

(1) $X((x \wedge y)_\beta) = X(x_\beta) \cap X(y_\beta)$,
(2) $X((x \vee y)_\beta) = X(x_\beta) \cup X(y_\beta)$,

(3) $\bigcup_{x\in L,\ \beta\in(0,1]} X(x_\beta) = X_\alpha$.

Proof. (1) If $\mu \in X(x_\beta)\cap X(y_\beta)$, then $x_\beta \nsubseteq \mu$ and $y_\beta \nsubseteq \mu$. This implies $\beta > \mu(x)$ and $\beta > \mu(y)$. Thus $x, y \notin \mu_*$. Since μ is prime fuzzy ideal, card Im $\mu = 2$ and $\mu_* = \{x \in L : \mu(x) = 1\}$ is prime. So $x\wedge y \notin \mu_*$. Which implies $\beta > \mu(x\wedge y)$. Hence $(x\wedge y)_\beta \nsubseteq \mu$. Therefore $X(x_\beta)\cap X(y_\beta) \subseteq X((x\wedge y)_\beta)$.

Again, let $\mu \in X((x\wedge y)_\beta)$. Then $(x\wedge y)_\beta \nsubseteq \mu$. Which implies $\beta > \mu(x\wedge y) \geq \mu(x)\vee\mu(y)$. Thus $\beta > \mu(x)$ and $\beta > \mu(y)$. This shows that $x_\beta \nsubseteq \mu$ and $y_\beta \nsubseteq \mu$. So $\mu \in X(x_\beta)\cap X(y_\beta)$. Hence $X((x\wedge y)_\beta) \subseteq X(x_\beta)\cap X(y_\beta)$. Therefore $X((x\wedge y)_\beta) = X(x_\beta)\cap X(y_\beta)$.

(2) If $\mu \in X(x_\beta)\cup X(y_\beta)$, then either $x_\beta \nsubseteq \mu$ or $y_\beta \nsubseteq \mu$. Which implies either $\beta > \mu(x)$ or $\beta > \mu(y)$. This shows that $\beta > \mu(x)\wedge\mu(y) = \mu(x\vee y)$. Thus $(x\vee y)_\beta \nsubseteq \mu$. So $\mu \in X((x\vee y)_\beta)$.

Again, let $\mu \in X((x\vee y)_\beta)$. Then $\beta > \mu(x\vee y) = \mu(x)\wedge\mu(y)$. Thus either $x_\beta \nsubseteq \mu$ or $y_\beta \nsubseteq \mu$. So $\mu \in X(x_\beta)\cup X(y_\beta)$.

(3) Clearly, $\bigcup_{x\in L,\ \beta\in(0,1]} X(x_\beta) \subseteq X_\alpha$. Let $\mu \in X_\alpha$. Then $Im\mu = \{1,\gamma\}$, $\gamma \in [0,1)$. This implies there is $x \in L$ such that $\mu(x) = \gamma$. Let us take some $\beta \in (0,1]$ such that $\beta > \gamma$. Then $x_\beta \nsubseteq \mu$. Thus $\mu \in \bigcup_{x\in L,\ \beta\in(0,1]} X(x_\beta)$. So $X_\alpha \subseteq \bigcup_{x\in L,\ \beta\in(0,1]} X(x_\beta)$. Hence $X_\alpha = \bigcup_{x\in L,\ \beta\in(0,1]} X(x_\beta)$. □

Lemma 4.4. *Let $\beta_1,\ \beta_2 \in (0,1]$; $\beta = min\{\beta_1,\beta_2\}$ and $x,\ y \in L$. Then*

$$X(x_{\beta_1})\cap X(y_{\beta_2}) = X((x\wedge y)_\beta).$$

Proof. If $\mu \in X(x_{\beta_1})\cap X(y_{\beta_2})$, then $x_{\beta_1} \nsubseteq \mu$ and $x_{\beta_2} \nsubseteq \mu$. This implies that $\beta_1 > \mu(x)$ and $\beta_2 > \mu(y)$. Since μ_* is prime ideal and $x, y \notin \mu_*$, we have $x\wedge y \notin \mu_*$ and $\mu(x) = \mu(y) = \mu(x\wedge y)$. This shows that $\beta > \mu(x\wedge y)$. Thus $(x\wedge y)_\beta \nsubseteq \mu$. So $\mu \in X((x\wedge y)_\beta)$.

Again, let $\mu \in X((x\wedge y)_\beta)$. Then $\beta > \mu(x\wedge y) \geq \mu(x)\vee\mu(y)$. This implies $\beta_1 > \mu(x)$ and $\beta_2 > \mu(y)$. Thus $x_{\beta_1} \nsubseteq \mu$ and $y_{\beta_2} \nsubseteq \mu$. So $\mu \in X(x_{\beta_1})\cap X(y_{\beta_2})$. Hence $X(x_{\beta_1})\cap X(y_{\beta_2}) = X((x\wedge y)_\beta)$. □

Theorem 4.5. *The collection $\mathcal{T} = \{X(\theta) : \theta$ is a fuzzy ideal of $L\}$ is a topology on X_α.*

Proof. Consider the fuzzy subsets η_1, η_2 of L defined as: $\eta_1(x) = 0$ and $\eta_2(x) = 1$, for all $x \in L$. Clearly, $< \eta_1 >$ and η_2 are fuzzy ideals of L. Again, $< \eta_1 > \subseteq \mu$, for all $\mu \in X_\alpha$. Then $V(< \eta_1 >) = X_\alpha$ and thus $X(\eta_1) = \phi$. Since each $\mu \in X_\alpha$ is non-constant, $\eta_2 \nsubseteq \mu$, for all $\mu \in X_\alpha$. So $X(\eta_2) = X_\alpha$. Hence $\phi,\ X_\alpha \in \mathcal{T}$.

Next, let $X(\theta_1), X(\theta_2) \in \mathcal{T}$. Since θ_1 and θ_2 are fuzzy ideals of L, by Lemma 4.1, we get that $X(\theta_1)\cap X(\theta_2) = X(\theta_1\cap\theta_2)$. Then $\mathcal{T}$ is closed under finite intersection.

Finally, let $\{\theta_i : i \in I\}$ be any family of fuzzy ideals of L. It can be easily verified that

$$\bigcap_{i\in I} V(\theta_i) = V(\langle\bigcup_{i\in I}\theta_i\rangle).$$

Then $\bigcup_{i\in I} X(\theta_i) = X(\langle\bigcup_{i\in I}\theta_i\rangle)$. Thus by Lemma 4.2, we have $X(\bigcup_{i\in I}\theta_i) = X(\langle\bigcup_{i\in I}\theta_i\rangle)$. So $\mathcal{T}$ is closed under arbitrary unions. Consequently, $\mathcal{T}$ is a topology on X_α. The space $(X_\alpha, \mathcal{T})$ will be called the space of prime α-fuzzy ideals in L. □

Theorem 4.6. *The subfamily $\mathcal{B} = \{X(x_\beta) : x \in L,\ \beta \in (0,1]\}$ of $\mathcal{T}$ is a base for $\mathcal{T}$.*

Proof. Let $X(\theta)$ be any open set in X_α and $\mu \in X(\theta)$. Then $\theta \nsubseteq \mu$ and there is $x \in L$ such that $\theta(x) > \mu(x)$. Put $\theta(x) = \beta$. Then $x_\beta \subseteq \theta$ and $\mu \in X(x_\beta)$. To show $X(x_\beta) \subseteq X(\theta)$, let $\eta \in X(x_\beta)$. Then $x_\beta \nsubseteq \eta$ and $\theta(x) > \eta(x)$. Which implies $\eta \in X(\theta)$. Thus $\mu \in X(x_\beta) \subseteq X(\theta)$. So for any open set $X(\theta)$ in X_α, we can find $X(x_\beta)$ in $\mathcal{B}$ such that $X(x_\beta) \subseteq X(\theta)$. Hence $\mathcal{B}$ is a base for $\mathcal{T}$. $\square$

Theorem 4.7. *The space X_α is a T_0-space.*

Proof. Let $\mu,\ \theta \in X_\alpha$ such that $\mu \neq \theta$. Then either $\mu \nsubseteq \theta$ or $\theta \nsubseteq \mu$. Without loss of generality, we can assume that $\mu \nsubseteq \theta$. Then $\theta \in X(\mu)$ and $\mu \notin X(\mu)$. Thus X_α is a T_0-space. $\square$

Theorem 4.8. *For any fuzzy ideal μ of L, $X(\mu) = X(\overleftarrow{\alpha}\alpha(\mu))$.*

Proof. Let μ be any fuzzy ideal of L. Then by Lemma 3.7, we have that $\mu \subseteq \overleftarrow{\alpha}\alpha(\mu)$. Thus $X(\mu) \subseteq X(\overleftarrow{\alpha}\alpha(\mu))$.

Conversely, let $\theta \in X(\overleftarrow{\alpha}\alpha(\mu))$. Then $\overleftarrow{\alpha}\alpha(\mu)) \nsubseteq \theta$. Suppose $\theta \notin X(\mu)$. Then $\mu \subseteq \theta$. Thus $\overleftarrow{\alpha}\alpha(\mu) \subseteq \theta$. This is impossible. So $\theta \in X(\mu)$. Hence $X(\mu) = X(\overleftarrow{\alpha}\alpha(\mu))$. $\square$

Theorem 4.9. *For any fuzzy ideal μ of L, $X(\mu) = \bigcup_{x_\beta \subseteq \mu} X(x_\beta)$.*

Proof. Let $\theta \in X(\mu)$. Then $\mu \nsubseteq \theta$. Thus there is $x \in L$ such that $\mu(x) > \theta(x)$. Put $\mu(x) = \beta$. Then $x_\beta \subseteq \mu$ and $x_\beta \nsubseteq \theta$. Thus $\theta \in \bigcup_{x_\beta \subseteq \mu} X(x_\beta)$ and $X(\mu) \subseteq \bigcup_{x_\beta \subseteq \mu} X(x_\beta)$.

Again, let $\theta \in \bigcup_{x_\beta \subseteq \mu} X(x_\beta)$. Then $\theta \in X(x_\beta)$, for some $x_\beta \subseteq \mu$. Thus $\beta > \theta(x)$ and $\mu(x) > \theta(x)$. So $\theta \in X(\mu)$ and $\bigcup_{x_\beta \subseteq \mu} X(x_\beta) \subseteq X(\mu)$. Hence $\bigcup_{x_\beta \subseteq \mu} X(x_\beta) = X(\mu)$. $\square$

Theorem 4.10. *The lattice $FI_\alpha(L)$ is isomorphic with the lattice of all open sets in X_α.*

Proof. The lattice of all open sets in X_α is $(\mathcal{T}, \cap, \cup)$. Define the mapping

$$f : FI_\alpha(L) \longrightarrow \mathcal{T} \text{ by } f(\mu) = X(\mu), \text{ for all } \mu \in X_\alpha.$$

Since $X(\mu) = X(\overleftarrow{\alpha}\alpha(\mu))$ and $\overleftarrow{\alpha}\alpha(\mu)$ is an α-fuzzy ideal, every open subset of X_α is of the form $X(\theta)$, for some $\theta \in FI_\alpha(L)$. Then the mapping is onto.

Let $f(\mu) = f(\theta)$. If $\mu \neq \theta$, then there is $x \in L$ such that either $\mu(x) < \theta(x)$ or $\theta(x) < \mu(x)$. Without loss of generality, we can assume that $\mu(x) < \theta(x)$. Put $\mu(x) = \beta$. Then by Corollary 3.26, we can find a prime α-fuzzy ideal η such that $\mu \subseteq \eta$ and $\eta(x) \leq \beta$. Thus $\eta \notin X(\mu)$ and $\theta \nsubseteq \eta$. So $\eta \notin X(\mu)$ and $\eta \in X(\theta)$. This is a contradiction. Hence $\mu = \theta$.

Now we prove f is homomorphism. Let $\mu,\ \theta \in FI_\alpha(L)$. Then $f(\mu \underline{\vee} \theta) = X(\overleftarrow{\alpha}\alpha(\mu \vee \theta)) = X(\mu \vee \theta) = X(\mu) \cup X(\theta) = f(\mu) \cup f(\theta)$. Similarly, $f(\mu \cap \theta) = f(\mu) \cap f(\theta)$. Thus f is a homomorphism. So f is an isomorphism. $\square$

Theorem 4.11. *For any family $\mathcal{F} \subseteq X_\alpha$, closure of $\mathcal{F}$ is given by $\overline{\mathcal{F}} = V(\bigcap_{\mu \in \mathcal{F}} \mu)$.*

Proof. We know that closure of $\mathcal{F}$ is the smallest closed set containing $\mathcal{F}$. To prove our claim, it is enough to show that $V(\bigcap_{\mu\in\mathcal{F}}\mu)$ is the smallest closed set containing $\mathcal{F}$. Since the set of all α-fuzzy ideal is a complete distributive lattice, $\bigcap_{\mu\in\mathcal{F}}\mu$ is an α-fuzzy ideal and $V(\bigcap_{\mu\in\mathcal{F}}\mu)$ is a closed set in X_α. If $\eta\in\mathcal{F}$, then $\bigcap_{\mu\in\mathcal{F}}\mu\subseteq\eta$. Thus $\eta\in V(\bigcap_{\mu\in\mathcal{F}}\mu)$. This implies that $\mathcal{F}\subseteq V(\bigcap_{\mu\in\mathcal{F}}\mu)$. Let $V(\theta)$ be any closed set in X_α containing $\mathcal{F}$. Then $\theta\subseteq\mu$, for each $\mu\in\mathcal{F}$. Thus $\theta\subseteq\bigcap_{\mu\in\mathcal{F}}\mu$ and $V(\bigcap_{\mu\in\mathcal{F}}\mu)\subseteq V(\theta)$. So $V(\bigcap_{\mu\in\mathcal{F}}\mu)$ is the smallest closed set containing $\mathcal{F}$. Hence $\overline{\mathcal{F}}=V(\bigcap_{\mu\in\mathcal{F}}\mu)$. □

Theorem 4.12. *X_α is a T_1- space if and only if every prime α-fuzzy ideal of L is maximal.*

Proof. Suppose that the space X_α is a T_1 space. Let $\mu\in X_\alpha$ and μ is not maximal. Then there exists a maximal fuzzy ideal θ of L such that $\mu\subset\theta$. Since X_α is T_1 space, $\{\mu\}$ and $\{\theta\}$ are closed. Thus $\theta\in X_\alpha-\{\mu\}$ and $\mu\in X_\alpha-\{\theta\}$ are open sets. So there exist two basic open sets $\theta\in X(a_\beta)$ and $\mu\in X(b_\gamma)$ such that $\mu\notin X(a_\beta)$ and $\theta\notin X(b_\gamma)$. Since $\mu\subset\theta$ and $a_\beta\subseteq\mu$, we get that $a_\beta\subseteq\theta$. Which is impossible. Hence μ is maximal. Therefore every prime α-fuzzy ideal is maximal. Conversely, suppose that every prime α-fuzzy ideal is maximal. To show each singleton subset of X_α is closed, let $\theta\in X_\alpha-\{\mu\}$. Then $\mu\neq\theta$ and by assumption, μ is not properly contained in θ and θ is not properly contained in μ. This implies that there is $a\in L$ such that $\mu(a)>\theta(a)$. Put $\mu(a)=\beta$. Then $a_\beta\subseteq\mu$ and $a_\beta\nsubseteq\theta$. Thus $X(a_\beta)$ is an open set containing θ but not μ. So $X_\alpha-\{\mu\}$ is open. Hence X_α is a T_1 space. □

Theorem 4.13. *If X_α is a Hausdorff space containing more than one element, then there exist $a,\ b\in L$ such that $X_\alpha=X(a_\beta)\cup X(b_\gamma)\cup V(\mu)$ where μ is a fuzzy ideal generated by $a_\beta\cup b_\gamma$.*

Proof. Suppose that X_α is a Hausdorff space containing more than one element. Let μ and θ be any two elements of X_α such that $\mu\neq\theta$. Then there exist two open sets $\mu\in X(a_\beta)$ and $\theta\in X(b_\gamma)$ such that $X(a_\beta)\cap X(b_\gamma)=\phi$. Let η be the ideal generated by $a_\beta\cup b_\gamma$. But we can not find any $\sigma\in X_\alpha$ such that $a_\beta\nsubseteq\sigma$ and $b_\gamma\nsubseteq\sigma$. Let us consider the following cases. If either $a_\beta\nsubseteq\sigma$ or $b_\gamma\nsubseteq\sigma$ but not both, then $\sigma\in X(a_\beta)\cup X(b_\gamma)\cup V(\eta)$.

And again, if $a_\beta\subseteq\sigma$ and $b_\gamma\subseteq\sigma$, then $a_\beta\cup b_\gamma\subseteq\sigma$ and $\sigma\in X(a_\beta)\cup X(b_\gamma)\cup V(\eta)$. Thus $X_\alpha\subseteq X(a_\beta)\cup X(b_\gamma)\cup V(\eta)$. So $X_\alpha=X(a_\beta)\cup X(b_\gamma)\cup V(\eta)$. □

Theorem 4.14. *The following are equivalent in L.*

(1) *X_α is a Hausdorff space*
(2) *For any two elements μ and θ of X_α there exist $a_\beta\nsubseteq\mu$, $b_\alpha\nsubseteq\theta$ and there does not exist any element η of X_α such that $a_\beta\nsubseteq\eta$ and $b_\gamma\nsubseteq\eta$.*

Proof. (1) $\Rightarrow$ (2): Suppose that the space X_α is Hausedorff. Then for any two distinct elements μ and θ of X_α, there exist two open sets $\mu\in X(a_\beta)$ and $\theta\in X(b_\gamma)$ such that $X(a_\beta)\cap X(b_\gamma)=\phi$. Thus $a_\beta\nsubseteq\mu$ and $b_\gamma\nsubseteq\theta$. To show our claim, assume that there is an element η in X_α such that $a_\beta\nsubseteq\eta$ and $b_\gamma\nsubseteq\eta$. Then $X(a_\beta)\cap X(b_\gamma)\neq\phi$. This is a contradiction. Thus there is no $\eta\in X_\alpha$ which satisfies $a_\beta\nsubseteq\eta$ and $b_\gamma\nsubseteq\eta$.

(2) $\Rightarrow$ (1): Suppose (2) holds. Let μ and θ be two distinct elements of X_α. Then by assumption, there exist a_β and b_γ such that $a_\beta \nsubseteq \mu$ and $b_\gamma \nsubseteq \theta$. Thus $\mu \in X(a_\beta)$ and $\theta \in X(b_\gamma)$.

Again by assumption, we can't find any η in X_α such that $a_\beta \nsubseteq \eta$ and $b_\alpha \nsubseteq \eta$. So $X(a_\beta) \cap X(b_\gamma) = \phi$. Hence X_α is a Hausdorff space. □

As a characterization of a regular space X_α, we have.

Theorem 4.15. *X_α is a regular space if and only if for any $\mu \in X_\alpha$ and $a_\beta \nsubseteq \mu$, there exist a fuzzy ideal θ of L and b_γ such that $\mu \in X(b_\gamma) \subseteq V(\theta) \subseteq X(a_\beta)$.*

Proof. Let X_α be a regular space. Let $\mu \in X_\alpha$ and $a_\beta \nsubseteq \mu$, for some $a \in L$. Then $\mu \in X(a_\beta)$. Since X_α is a regular space, there exists a nbd $X(\eta)$ of μ such that $\mu \in X(\eta) \subseteq \overline{X(\eta)} \subseteq X(a_\beta)$. Since $\overline{X(\eta)}$ is closed in X_α, there exists some fuzzy ideal θ of L such that $\overline{X(\eta)} = V(\theta)$. Thus $\mu \in X(\eta) \subseteq V(\theta) \subseteq X(a_\beta)$. Since $\mu \in X(\eta)$ and $X(\eta)$ is open in X_α, there exists a basic element $X(b_\gamma)$ such that $\mu \in X(b_\gamma) \subseteq X(\eta)$. So $\mu \in X(b_\gamma) \subseteq V(\theta) \subseteq X(a_\beta)$.

Conversely, suppose that for any $\mu \in X_\alpha$ and $a_\beta \nsubseteq \mu$, there exist a fuzzy ideal θ of L and b_γ such that $\mu \in X(b_\gamma) \subseteq V(\theta) \subseteq X(a_\beta)$. To show that the space X_α is regular. Let $\mu \in X_\alpha$ and $V(\eta)$ be any closed set of X_α such that $\mu \notin V(\eta)$. This gives $\eta \nsubseteq \mu$ and there is $a \in L$ such that $\eta(a) > \mu(a)$. Put $\eta(a) = \beta$. Then $a_\beta \subseteq \eta$ and $a_\beta \nsubseteq \mu$. Thus $\mu \in X(a_\beta)$. Since $a_\beta \nsubseteq \mu$, by assumption, there exists a fuzzy ideal θ of L and b_γ such that $\mu \in X(b_\gamma) \subseteq V(\theta) \subset X(a_\beta)$. So $X(b_\gamma) \cap X(\theta) = \phi$.

Now we prove $V(\eta) \subseteq X(\theta)$. Since $a_\beta \subseteq \eta$, we have $V(\eta) \subseteq V(a_\beta)$. Then $V(\eta) \subseteq X_\alpha - X(a_\beta) \subseteq X_\alpha - V(\theta)$. Thus there exist two disjoint open sets $X(a_\beta)$ and $X(\theta)$ such that $\mu \in X(a_\beta)$ and $V(\eta) \subseteq X(\theta)$ (or in other words, for any $\mu \in X_\alpha$ and a closed set $V(\eta)$ not containing μ, we can find a nbd $X(b_\gamma)$ of μ and open $X(\theta) \supseteq V(\eta)$ such that $X(b_\gamma) \cap X(\theta) = \phi$). Thus X_α is a regular space. □

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