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Abstract. Shadowed set approximation of fuzzy sets has been introduced and enhanced to exploit some optimization-based principles which define the quality of its approximation. It found its applications in granular computing, cluster computing and recommender systems. This paper introduces a new approach accompanied with an algorithm; based on a principle of uncertainty invariance, to simplify fuzzy sets by inducing its best approximation which possesses the nearest quota of fuzziness as encountered in the original fuzzy set. Some numerical examples are provided to demonstrate how to implement the proposed method. The new approach is useful in preserving the uncertainty and information inherently associated with a given fuzzy set. A comparative study is made with related methods. The results of some evaluation indices on the approximation effectiveness illustrate the essence of the proposed method.

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1. Introduction

Fuzzy sets, introduced by Zadeh [16], are used to model imprecision by establishing a membership function which describes the concept being modeled. The membership function $\mu_F : X \longrightarrow [0, \frac{1}{2}]$ which characterizes a fuzzy set, establishes the degree of membership (i.e., $\mu_F(x) \in [0,1]$) of the elements, $x \in F$, of a fuzzy set through precise numerical quantities.

Pedrycz [9] considered three ôregion approximation and interpretation of fuzzy sets viz. inclusion, exclusion, and unknown (doubt) regions, which has found several applications in granular computing [4, 12], recommender systems [6], clustering computing [8, 10, 11, 18], game theory [17], etc. Some related algebraic studies of shadowed sets are as well reported in [1]. The key aspect of shadowed set approximation of fuzzy sets is to determine a pair of optimal thresholds that would partition a given fuzzy set into the three regions and retain the overall amount of fuzziness in the given fuzzy set [13, 15]. Many methods of inducing shadowed sets have appeared in [2, 5, 9, 17].

Pedrycz [9] demonstrated a practical way to determine a suitable pair of thresholds which partition a given fuzzy set into three regions. However, as observed by [13], the formulation proposed in [9] may not fully retain the overall amount of fuzziness associated with the original fuzzy set. [13] proposed a more accurate formulation which reasonably captures the original fuzziness of the elements in doubtful zones.

Further, to provide more insight into shadowed set approximation of fuzzy sets in terms of minimization of the error in classification of the elements of a fuzzy set into three regions of the resulting shadowed set, an alternative decision-theoretic shadowed set approximation of fuzzy sets was proposed in [3]. In line with Pedryczs formulation, [5] suggested another approach for approximating a fuzzy set in the framework of shadowed sets. The method searches for a point of compromise between the optimality of a specific threshold and balance of the uncertainty introduced by the shadowed set approximation actions.

Recently, game-theoretic shadowed set (GTSS) approximation of fuzzy sets has been proposed in [17], where optimal thresholds are determined and interpreted using game theoretical concepts. The existing methods of approximation of fuzzy sets via shadowed sets are motivated by the following principles: i. Balance the fuzziness introduced as a result of performing the shadowed set transformation actions. Typical examples are methods in [9, 13]. ii. Minimize the error generated as a result of classifying the elements of a fuzzy set into the regions. A typical example is the method suggested in [3]. iii. Bridge the gap between optimality of a threshold for transforming a fuzzy set into its resulting shadowed set and the threshold which balances the fuzziness introduced. A typical example is the method in [5]. iv. Search for a point of compromise between the conflict and tradeoff in the elevation and reduction errors arising from the transformation actions. A typical example is the method in [17].

In this paper, we argue that a formulation of a suitable method which fully complies with preservation of an equivalent amount of information and fuzziness associated with the original fuzzy set, should minimize the discrepancies between the total amount of fuzziness in the shadowed set to be induced and the total amount of fuzziness in the original fuzzy set from which a shadowed set is induced. In particular, in order to account for an equivalent amount of information and uncertainty, the total amount of fuzziness in the original fuzzy set needs to be captured by the method. This idea of preservation of information is anchored on a well-known optimization-based principle suggested in [7], called a principle of uncertainty invariance. A principle of uncertainty invariance requires that no information is unwittingly added or eliminated as a result of transforming a fuzzy set into its resulting shadowed set. Therefore, we proposed a new algorithm to construct a shadowed set from a given fuzzy set, which exploits a principle of uncertainty invariance in order to account for an equivalent amount of fuzziness characterizing the original fuzzy set.

2. Three-way approximation of fuzzy sets

Given a fuzzy set F defined by $\mu_F : X \longrightarrow [0, \frac{1}{2}]$, three-way approximation of F, $T_{(\alpha,\beta)}(\mu_F(x))$ maps the membership grades of all elements of F to a three-element set $\{n, m, p\}$ based on a pair of thresholds (α, β), $0 \leq \alpha \leq \beta \leq$ [13, 14]. That is,

$$T_{(\alpha,\beta)}(\mu_F(x)) = \begin{cases} p, \mu_F(x) > \alpha, \\ m, \alpha \leq \mu_F(x) \leq \beta, \\ n, \mu_F(x) < \alpha. \end{cases} \tag{2.1}$$

With the aid of Equation (2.1), a fuzzy set can be effectively partitioned into three-regions:
a) $Core(\mu_F) = \{x \in X : \mu_F(x) > \beta\}$,
b) $Support(\mu_F) = \{x \in X : \mu_F(x) > 0\}$,
c) $Complementsupport(\mu_F) = \{x \in X : \mu_F(x) = 0\}$.
These three regions facilitate decision-making and correspond to a concept known as three-way decisions [14].

Shadowed sets are viewed as special case of three-way approximations of fuzzy sets, in which $p = 1, m = (0, 1), n = 0$. A shadowed set maps the membership grade of all elements in F to a three-element set {0,(0,1),1} based on a symmetric pair of thresholds $(\alpha, 1 - \alpha)$. That is,

$$S_{(\alpha,\beta)}(\mu_F(x)) = \begin{cases} 1, \mu_F(x) > \alpha, \\ \frac{1}{2}, \alpha \leq \mu_F(x) \leq \beta, \\ 0, \mu_F(x) < \alpha. \end{cases} \tag{2.2}$$

By applying Equation (2.2) to the membership grades of all elements in F, we obtain the three regions:
d) $Elv(S) = \{x \in X : \mu_F(x) > \beta \wedge S(x) = 1\}$ - elevated region,
e) $Shd(S) = \{x \in X : \alpha \leq \mu_F(x) \leq 1 - \alpha \wedge S(x) = \frac{1}{2}\}$ - shadow region,
f) $Red(S) = \{x \in X : \mu_F(x) < \alpha \wedge S(x) = 0\}$ - reduced region.

2.1. **Formulation of the problem of determination of optimal threshold values.** The three regions of a shadowed set S introduce a new type of uncertainty which must be balanced and localized into the shadows of S. As suggested in [9], the uncertainty eliminated as a result of elevating elements membership grades to 1 and reducing elements membership grades to 0 should be compensated for in the shadow region. Thus, we have the following equation:

$$\varphi(Elv(S)) + \varphi(Red(S)) = \varphi(Shd(S)), \tag{2.3}$$

where $\varphi(A)$ denotes the fuzziness associated with a region A.

In practical situations, Equation (2.3) is not always easily achievable.

The problem of determining the required pair of thresholds can be treated as optimization problem [9], that is, we have the following argument:

$$\alpha_{opt} = min_\alpha V(\alpha), \tag{2.4}$$

where $V(\alpha) = |\varphi(Elv(S)) + \varphi(Red(S)) - Card(Shd(S))|$, $|.|$ denote absoute value function, $Card(A)$ denote cardinality of a set A, and

$\varphi(Elv(S)) = \sum_i^q (1-\mu_F(x_i)), \mu_F(x_i) > \beta$
$\varphi(Red(S)) = \sum_i^r \mu_F(x_i), \mu_F(x_i) < \alpha.$

The authors of [13] reformulated Equation (2.4) as the following argument

$$\alpha_{opt} = min_\alpha J(\alpha), \tag{2.5}$$

where $J(\alpha) = |\varphi(Elv(S)) + \varphi(Red(S)) - \varphi(Shd(S))|$, $\varphi(A)$ is the measure of fuzziness of a set A, and

$\varphi(Elv(S)) = \sum_i^q \varphi(\mu_F(x_i)), \mu_F(x_i) > \beta$
$\varphi(Red(S)) = \sum_i^r \varphi(\mu_F(x_i)), \mu_F(x_i) < \alpha$
$\varphi(Shd(S)) = \sum_i^r (\varphi(\frac{1}{2}) - \varphi(\mu_F(x_i)))$, $\alpha \leq \mu_F(x_i) \leq 1-\alpha$. Here φ denotes a suitable fuzziness measure over a fuzzy set F.

Further, [3] treated Equation (2.4) as a problem of minimizing the classification error, $E(.)$, as follows:

$$\alpha_{opt} = min_\alpha E_{(\alpha,\beta)}(S), \tag{2.6}$$

where $E_{(\alpha,\beta)}(S) = E_e(\mu_F) + E_r(\mu_F) + E_s(\mu_F)$,
$E_e(\mu_F) = \sum_{i=1}^q (1-\mu_F(x_i)),\ \mu_F(x_i) > \beta$
$E_r(\mu_F) = \sum_{i=1}^r \mu_F(x_i),\ \mu_F(x_i) < \alpha$
$E_s(\mu_F) = \sum_{\alpha \leq \mu_F(x_i) < \frac{1}{2}} (\frac{1}{2} - \mu_F(x_i)) + \sum_{\frac{1}{2} < \mu_F(x_i) \leq \beta} (\mu_F(x_i) - \frac{1}{2})$, and

$$E_{(\alpha,\beta)}(\mu_F(x)) = \begin{cases} 1-\mu_F(x), if \mu_F(x) > \beta \\ \mu_F(x) - \frac{1}{2}, if \frac{1}{2} < \mu_F(x) \leq \beta \\ \frac{1}{2} - \mu_F(x), if \alpha \leq \mu_F(x) < \frac{1}{2} \\ \mu_F(x) - 0, if \mu_F(x) < \alpha. \end{cases} \tag{2.7}$$

In line with Pedryczs formulation, [5] modified Equation (2.4) as the following argument:

$$\alpha_{opt} = Near(avg_{V^*(\alpha)} V^*(\alpha)) \tag{2.8}$$

where $V^*(\alpha) = |\sum_{\mu_F(x) \in Elv(S)} (1-\mu_F(x)) + \sum_{\mu_F(x) \in Red(S)} \mu_F(x) - \sum_{\mu_F(x) \in Shd(S)} \mu_F(x)|$ and $Near(avg_{V^*(\alpha)} V^*(\alpha))$ is the argument of selecting a cut (i.e., $\alpha \in (0, \frac{1}{2}]$) such that $V^*(\alpha)$ is near the average of the balance of uncertainty $V^*(\alpha_i)$, for all feasible $\alpha_i \in (0, \frac{1}{2}]$. The quantity $avg_{V^*(\alpha)} V^*(\alpha)$ represents the average of the numbers $V^*(\alpha_i)$, $\alpha_i \in (0, \frac{1}{2}]$, for all i.

The optimal threshold,α, to be found is such that $\alpha + \beta = 1$.

Upon determination of α, the resulting shadowed set can be easily constructed by calculating $\beta = 1-\alpha$. The details of the above methods can be found in [3, 5, 9, 13].

Recently, an approach for determination of the required pair of thresholds for approximating a fuzzy set via shadowed sets which exploits game-theoretical mechanism was suggested in [17].

The author in [17] reports that it is not possible to simultaneously decrease the elevation and reduction errors. That is, the decrease of the elevation error results in the increase of the reduction error, and vice versa.

This led to the search for a pair of thresholds (α, β) which defines a shadowed set approximation of fuzzy sets based on a tradeoff between the elevation and reduction errors. In fact, the contradiction between the elevation and reduction errors were formulated as a competitive game in [17] in which the increase of one players payoff (i.e., elevation error) may result in the decrease of the other players payoff (i.e., reduction error). The game stops if: both players lose their payoffs or the gain of a players payoff is less than the loss of the other players payoff, otherwise several iterations are performed until the stopping criteria are reached.

The method suggested in [17] does not require any decision objective function, and thus may not account for the total amount of fuzziness associated with the original fuzzy set. Moreover, for a pair of symmetric thresholds (as is the case in shadowed sets), (α_i, β_i), it is possible to simultaneously decrease the elevation and reduction errors by searching for an α-cut which is both optimal and accounts for an equivalent amount of fuzziness associated with the original fuzzy set. This raises some concerns with the method suggested in [17].

3. Reformulation of the problem of determination of optimal threshold Values

In order to construct a shadowed set which fully comply with preservation of an equivalent amount of fuzziness and information associated with the original fuzzy set, we consider determination of the optimal threshold from a perspective of a principle of uncertainty invariance.

3.1. **A principle of uncertainty invariance.** To facilitate the connection between a given fuzzy set and its resulting shadowed set, a principle requiring that the total amount of fuzziness (and information) be preserved when a fuzzy set is transformed into a suitable shadowed set, is conceived as a principle of uncertainty invariance. To be concise, the principle of uncertainty invariance requires that no information is unwittingly added or eliminated as a result of transforming a fuzzy set into its resulting shadowed set (see [7]).

Guided by the aforesaid principle, one can determine the quality of various methods of induction of shadowed set from any given fuzzy set.

3.2. **Fuzziness measure.** Fuzziness measures are crucial for assessing the influence of shadowed set approximation on fuzzy sets. They could be used to estimate loss of information when a fuzzy set is simplified.

A well-justified fuzziness measure which have been used in [5, 13] to calculate the fuzziness inherently associated with both fuzzy and shadowed sets is the fuzziness measure adopted in [7]. This fuzziness measure comes in the following form:

For a given fuzzy set F drawn from X, its level of fuzziness is computed as

$$\varphi(F) = \sum_{i=1}^{n}[1 - |2\mu_F(x) - 1|].$$

We formulated three theorems that are useful for comprehending the method proposed in this paper.

Theorem 3.1. *Suppose that* $S_{\alpha_1}, S_{\alpha_2}, ..., S_{\alpha_n}, (\alpha_1 < \alpha_2 <, ..., < \alpha_n)$ *are assumed shadowed sets induced from a fuzzy set* F, $\alpha_i \in [\mu_{F_{min}}, \frac{1}{2}]$. *Then the following* *inequality holds:*

$$Card(Shad(S_{\alpha_1})) \geq Card(Shad(S_{\alpha_2})) \geq ... \geq Card(Shad(S_{\alpha_n})).$$

Proof. It is straightforward since for $\alpha_i \in [\mu_{F_{min}}, \frac{1}{2}]$, such that $\alpha_1 < \alpha_2 <, ..., < \alpha_n$, we have that $1 - \alpha_1 > 1 - \alpha_2 > ... > 1 - \alpha_n$ and

$$[\alpha_1, 1 - \alpha_1] \supset [\alpha_2, 1 - \alpha_2] \supset ... \supset [\alpha_n, 1 - \alpha_n]. \tag{3.1}$$

The intervals in Equation (3.1) define the shadow regions for the respective shadowed sets: $S_{\alpha_1}, S_{\alpha_2}, ..., S_{\alpha_n}$. Consequently, we have

$$Card(Shad(S_{\alpha_1})) \geq Card(Shad(S_{\alpha_2})) \geq ... \geq Card(Shad(S_{\alpha_n})).$$

□

Theorem 3.2. *Suppose S_{α_1} and S_{α_2} are assumed shadowed sets drawn from a fuzzy set F, $\alpha_1, \alpha_2 \in [\mu_{F_{min}}, \frac{1}{2}]$, where $\mu_{F_{min}}$ is the minimum membership grade in F. A necessary condition for $\varphi(S_{\alpha_1})$ to be closer to $\varphi(F)$ than $\varphi(S_{\alpha_2})$ is to $\varphi(F)$ is that*

(1) *$Card(Shad(S_{\alpha_1})) \geq Card(Shad(S_{\alpha_2}))$, $\varphi(F) > \varphi(S_{\alpha_2})$ and $\varphi(S_{\alpha_1}) \leq \varphi(F)$,*

(2) *$Card(Shad(S_{\alpha_1})) \leq Card(Shad(S_{\alpha_2}))$, $\varphi(F) < \varphi(S_{\alpha_2})$ and $\varphi(S_{\alpha_1}) \geq \varphi(F)$.*

Proof. Suppose $Card(Shad(S_{\alpha_1})) \geq Card(Shad(S_{\alpha_2}))$. Since $\varphi(F) > \varphi(S_{\alpha_2})$ and $\varphi(S_{\alpha_1}) \leq \varphi(F)$, $\varphi(S_{\alpha_1}) > \varphi(S_{\alpha_2})$ and $\alpha_1 < \alpha_2$, $\alpha_1, \alpha_2 \in [\mu_{F_{min}}, \frac{1}{2}]$. This guarantees that $|\varphi(F) - \varphi(S_{\alpha_1})| \leq |\varphi(F) - \varphi(S_{\alpha_2})|$, for fixed α_1 and any α_2.

Suppose $Card(Shad(S_{\alpha_1})) \leq Card(Shad(S_{\alpha_2}))$. Since $\varphi(F) < \varphi(S_{\alpha_2})$ and $\varphi(S_{\alpha_1}) \geq \varphi(F)$, $\varphi(S_{\alpha_2}) > \varphi(S_{\alpha_1})$ and $\alpha_2 < \alpha_1$, $\alpha_1, \alpha_2 \in [\mu_{F_{min}}, \frac{1}{2}]$. Then, $|\varphi(F) - \varphi(S_{\alpha_1})| \leq |\varphi(F) - \varphi(S_{\alpha_2})|$, for fixed α_1 and any α_2. □

Remark 3.3. We note that if $S_{\alpha_1}, S_{\alpha_2}, ..., S_{\alpha_n}$ are assumed shadowed sets drawn from a fuzzy set F and $\alpha_1 < \alpha_2 <, ..., < \alpha_n$, $\alpha_i \in [\mu_{F_{min}}, \frac{1}{2}]$, then since a fuzziness measure,φ, is monotonically increasing in $[0, \frac{1}{2}]$, and reaches the maximum value at $\frac{1}{2}$. We have from Eqation (3.1) that $\varphi(S_{\alpha_1}) > \varphi(S_{\alpha_2}) > ... > \varphi(S_{\alpha_n})$. However, the following inequality need not hold:

$$|\varphi(F) - \varphi(S_{\alpha_1})| > |\varphi(F) - \varphi(S_{\alpha_2})| > ... > |\varphi(F) - \varphi(S_{\alpha_n})|.$$

Theorem 3.4. *Suppose F is a nonempty fuzzy set drawn from X and $[\mu_{F_{min}}, 0.5]$ is the range of feasible α-cuts for the induction of shadowed set from F. If $\alpha_i \in [\mu_{F_{min}}, 0.5]$,*

$$S_{\alpha_1}, S_{\alpha_2}, ..., S_{\alpha_n}, (\alpha_1 < \alpha_2 <, ..., < \alpha_n)$$

are assumed shadowed sets drawn from F, then there is a unique $S_{\alpha_p}, p \in \{1, 2, ..., n\}$, such that

$$|\varphi(F) - \varphi(S_{\alpha_p})| = min|\varphi(F) - \varphi(S_{\alpha_i})|,$$

$\alpha_p \in [\mu_{F_min}, 0.5], 1 \leq i \leq n$.

Proof. Let $S_{\alpha_1}, S_{\alpha_2}, ..., S_{\alpha_n}, (\alpha_1 < \alpha_2 <, ..., < \alpha_n)$ be shadowed sets drawn from F such that $\alpha_i \in [\mu_{F_{min}}, \frac{1}{2}]$. Then $\varphi(S_{\alpha_1}) > \varphi(S_{\alpha_2}) > ... > \varphi(S_{\alpha_n})$. For any fuzziness measureφ, we have $\varphi(F) \geq 0$ and the set $D = \{|\varphi(F) - \varphi(S_{\alpha_i})| : i = 1, 2, ..., n\}$ is well-ordered. It follows that there exists an α-cut, $\alpha_p \in [\mu_{F_{min}}, \frac{1}{2}]$, such that $|\varphi(F) - \varphi(S_{\alpha_p})|$ is the minimum element in D. Thus, we have the following:

$$|\varphi(F) - \varphi(S_{\alpha_p})| = min\{|\varphi(F) - \varphi(S_{\alpha_1})|, |\varphi(F) - \varphi(S_{\alpha_2})|, ..., |\varphi(F) - \varphi(S_{\alpha_n})|\}.$$

So S_{α_p} emerges as our desired shadowed set. □

Remark 3.5. Let φ be any fuzziness measure, for any shadowed set $S_\alpha, \alpha \in (0, \frac{1}{2}]$, we have

$$\varphi(S_\alpha) = \varphi[Shd(S_\alpha)].$$

In the event whereby two feasible threshold values of α-cuts produce the same value of minimum error resulting from shadowed set transformation actions, the two feasible thresholds induce two distinct shadowed sets which have different amount of fuzziness. The method proposed in [3], being dependent on minimum error in transformation, may not guarantee the selection of the threshold that would produce a shadowed set having an equivalent amount of fuzziness as encountered in the original fuzzy set. A similar situation may also arise from the method proposed in [13] when different measures of fuzziness are used to compute the required pair of threshold (see the remark on page 942-943 in [13]).

The aforesaid observations has placed demands on reformulation of the problem of determining the required pair of thresholds for induction of a shadowed set from a given fuzzy set, and motivate the new approach presented below.

4. Algoirithm for shadowed set approximation of fuzzy sets based on nearest quota of fuzziness

Let φ be any fuzziness measure on a fuzzy set

$$F = \{(x, \mu_F(x)) : x \in X\}$$

over a nonempty universe X. For a given range of feasible threshold values $\alpha_i \in (0, \frac{1}{2}]$ and its resulting shadows $Shd(S_{\alpha i})$ of assumed shadowed sets S_{α_i}, we determine an optimal threshold value $\alpha^{'} \in (0, \frac{1}{2}]$ such that

$$|\varphi(F) - \varphi(S_{\alpha'})| = min|\varphi(F) - \varphi(S_{\alpha_i})|,$$

for all i, by using the following steps:

Given a fuzzy set F and each feasible α -cut in the range $[\mu_{F_{min}}, \frac{1}{2}]$ of threshold values.

Step 1: Compute the membership value as

$$\mu_{Shd(S_{\alpha_i})}(x) = \frac{1}{k}\sum_{j=1}^{k} \mu_F(x_j), \ \alpha_i \leq \mu_F(x_j) \leq 1 - \alpha_i,$$

for the shadows $Shd(S_{\alpha_i}) = \{(x, \mu_{Shd(S_{\alpha_i})}(x)) : \alpha_i \leq \mu_F(x_j) \leq 1 - \alpha_i\}$

Step 2: For a suitable fuzziness measure, $\varphi(A) = \sum_{i=1}^{n}[1 - |2\mu_A(x) - 1|]$ say, compute $\varphi(F)$ and $\varphi[Shd(S_{\alpha_i})]$, for all i

Step 3: Compute the absolute difference, $D(\alpha_i) = |\varphi(F) - \varphi[Shd(S_{\alpha_i})]|$, between $\varphi(F)$ and $\varphi[Shd(S_{\alpha_i})]$, for all i.

Step 4: Select $\alpha^{'}$ as the optimized threshold value if, $D(\alpha^{'}) = minD(\alpha_i)$, $\alpha^{'} \in \{\alpha_1, \alpha_2, ..., \alpha_k\}$ in Step 3.

Step 5: Return $S_{\alpha^{'}} = Elv(S_{\alpha^{'}}) \cup Red(S_{\alpha^{'}}) \cup Shd(S_{\alpha^{'}})$ as the desired shadowed set, where
$Elv(S_{\alpha^{'}}) = \{(x, 1) : \mu_F(x) > 1 - \alpha^{'}\}$,
$Red(S_{\alpha^{'}}) = \{(x, 0) : \mu_F(x) < \alpha^{'}\}$,
$Shd(S_{\alpha^{'}}) = \{(x, \mu_{Shd(S_{\alpha_i})}(x)) : \alpha^{'} \leq \mu_F(x) \leq 1 - \alpha^{'}\}$.

An illustrative example to demonstrate the algorithm is offered in the next section.

5. Experimental evaluations

The experimental evaluation reports on the result produced by different shadowed set approximation methods for transforming given fuzzy sets. First of all, let us consider the following ten- element fuzzy sets whose membership grades are randomly generated.
μ_{F_1} : [0.13559, 0.13927, 0.17353, 0.18357, 0.19754, 0.3604, 0.65331, 0.93263, 0.94007, 0.9981],
μ_{F_2} : [0.18976, 0.26628, 0.30658, 0.5203, 0.60942, 0.76293, 0.77835, 0.8216, 0.88019, 0.93408],
μ_{F_3} : [0.06285, 0.28423, 0.30045, 0.31441, 0.46033, 0.47927, 0.59373, 0.60853, 0.84309, 0.95889],
where the membership grade of object x_j is in the jth position of $\mu_{F_i}(x_j)$.

For a demonstration of how to implement Algorithm 1, consider the values enlisted in μ_{F_1} above. The proposed shadowed set approximation of μ_{F_1} is summarized in the following steps:

Step 1: We compute the shadows, $Shd(S_{\alpha_i})$, of the assumed shadowed sets S_{α_i}, for all $i \geq 1$. A set of their membership values after applying $\mu_{Shd(S_{\alpha_i})}(x) = \frac{1}{k}\sum_{j=1}^{k} \mu_F(x_j)$, $\alpha_i \leq \mu_F(x_j) \leq 1 - \alpha_i$ are outlined as:
$\mu_{Shd(S_{\alpha_1})}(x) = \{0.26331, 0.26331, 0.26331, 0.26331, 0.26331, 0.26331, 0.26331\}$,
$\mu_{Shd(S_{\alpha_2})}(x) = \{0.28460, 0.28460, 0.28460, 0.28460, 0.28460, 0.28460\}$,
$\mu_{Shd(S_{\alpha_3})}(x) = \{0.31367, 0.31367, 0.31367, 0.31367, 0.31367\}$,
$\mu_{Shd(S_{\alpha_4})}(x) = \{0.34870, 0.34870, 0.34870, 0.34870\}$,
$\mu_{Shd(S_{\alpha_5})}(x) = \{0.40375, 0.40375, 0.40375\}$,
$\mu_{Shd(S_{\alpha_2})}(x) = \{0.3604\}$.
We calculate the measure of fuzziness ofF_1 by the formula:
$\varphi(F_1) = \sum_{i=1}^{n}[1 - |2\mu_{F_1}(x) - 1|]$, and obtain $\varphi(F_1) = 3.33144$.

Step 2: For the same fuzzy set F_1, the feasible thresholds are selected as:

$\alpha_1 = 0.13559, \alpha_2 = 0.13927, \alpha_3 = 0.17353, \alpha_4 = 0.19754, \alpha_5 = 0.3604 and \alpha_6 = 0.3604.$

We compute the measure of fuzziness of the shadows
(i.e.,$Shd(S_{\alpha_1}, Shd(S_{\alpha_2}, Shd(S_{\alpha_3}, Shd(S_{\alpha_4}, Shd(S_{\alpha_5})$ and$Shd(S_{\alpha_6})$ of the resulting candidate shadowed sets (i.e.,$S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3}, S_{\alpha_4}, S_{\alpha_5}$ and S_{α_6}, respectively),
by using $\varphi(S_{\alpha_i}) = \sum_{i=1}^{n}[1 - |2\mu_{S_{\alpha_i}}(x) - 1|]$, as well and obtain:

$$\varphi[Shd(S_{\alpha_1})] = 3.68634,$$

$$\varphi[Shd(S_{\alpha_2})] = 3.41520,$$
$$\varphi[Shd(S_{\alpha_3})] = 3.13670,$$
$$\varphi[Shd(S_{\alpha_4})] = 2.78960,$$
$$\varphi[Shd(S_{\alpha_5})] = 2.42250,$$
$$\varphi[Shd(S_{\alpha_6})] = 0.72080.$$

Step 3: Using $D(\alpha_i) = |\varphi(F) - \varphi[Shd(S_{\alpha_i})]|$, we compute the following discrepancies, $D(\alpha_i)$, for all $1 \leq i \leq 6$ as:

$$D(\alpha_1) = |\varphi(F) - \varphi[Shd(S_{\alpha_1})]|,$$
$$D(\alpha_1) = 0.35490,$$
$$D(\alpha_2) = |\varphi(F) - \varphi[Shd(S_{\alpha_2})]|,$$
$$D(\alpha_2) = 0.08376,$$
$$D(\alpha_3) = |\varphi(F) - \varphi[Shd(S_{\alpha_3})]|,$$
$$D(\alpha_3) = 0.19474,$$
$$D(\alpha_4) = |\varphi(F) - \varphi[Shd(S_{\alpha_4})]|,$$
$$D(\alpha_4) = 0.54184,$$
$$D(\alpha_5) = |\varphi(F) - \varphi[Shd(S_{\alpha_5})]|,$$
$$D(\alpha_5) = 0.90894,$$
$$D(\alpha_6) = |\varphi(F) - \varphi[Shd(S_{\alpha_6})]|,$$
$$D(\alpha_1) = 2.61064.$$

Selecting the shadowed set with the minimum discrepancy, as suggested in algorithm 1, as the one with the nearest quota of fuzziness.

Step 4: Select the shadowed set with the minimum discrepancies. That is, the shadowed set S_{α_2} retains an equivalent amount of fuzziness as encountered in the original fuzzy set F_1. Then, we obtain $\varphi[Shd(S_{\alpha_2})] = 3.41520$ as the nearest quota of fuzziness. Thus S_{α_2} has the minimum discrepancy.

Step 5: Return the shadowed set S_{α_2} as the desired three-way approximation of F:

$$Elv(S_{\alpha_2}) = \{(x, 1) : \mu_{F_1}(x) > 0.86073\},$$
$$Red(S_{\alpha_2}) = \{(x, 0) : \mu_{F_1}(x) < 0.13927\},$$
$$Shd(S_{\alpha_2}) = \{(x, 0.28460) : 0.13927 \leq \mu_{F_1}(x) \leq 0.86073\}.$$

To minimize the error in elicitation of the membership grades of the elements in the shadow region, S_{α_2}, the mean of the original membership grades of the elements in the shadow area is assigned as new membership grade of the elements in the shadow area. This has been effectively applied in [2, 5].

For fuzzy sets F_2 and F_3 their corresponding thresholds (α, β) are determined by deploying various methods (using Equation (2.4), (2.5), (2.6) and (2.7)). Please see Table 1 below.

Table 1 presents the results for fuzzy sets $F_{1\leq i\leq 3}$. In fuzzy set F_1 the methods produce different threshold values, while in the remaining two fuzzy sets, some of the methods produce the same threshold. Three methods (i.e., [2, 5] and the proposed method) concur in determination of the required thresholds (α, β) for approximating F_2, thereby confirming the feasibility of the proposed technique.

Table 1. Various thresholds computed from existing and proposed methods

Fuzzy sets	Thresholds	Ibrahim and William-West method	Pedrycz method	Deng and Yao method	Tahayori et al. method	Proposed method
F_1	(α, β)	$[0.1735, 0.8265]$	$[0.3604, 0.6396]$	$[0.1975, 0.8025]$	$[0.1836, 0.8164]$	$[0.1393, 0.8607]$
F_2	(α, β)	$[0.2663, 0.7337]$	$[0.3066, 0.6934]$	$[0.2663, 0.7337]$	$[0.1898, 0.8102]$	$[0.2663, 0.7337]$
F_3	(α, β)	$[0.3005, 0.6996]$	$[0.4603, 0.5397]$	$[0.2842, 0.7158]$	$[0.2842, 0.7158]$	$[0.3005, 0.6996]$

The experimental studies evaluate the effectiveness of the proposed method on randomly generated fuzzy sets of ten elements. The algorithm for implementing each method was coded in Python programming language. The experiment was conducted on a 64-bit operating system computer, with Pentium(R) cpu 2.20GHz. The details of each pair of thresholds determined by various methods on a given fuzzy set $F_{1\leq i\leq 3}$ are provided as shown in Table 1. Please observe in Table 1, the threshold in row 3, column 3, 5 and 7, and row 3, column 3 and 7 show when two or more method concur, whereas the threshold in row 2, column 3-7 indicate distinction between various methods. We evaluate various methods by analyzing the threshold in row 2, column 3-7. We note that very frequently, a single threshold of an α-cut does not fulfill all the four principles behind the formulation of the methods of shadowed set approximation of fuzzy sets (please see the principles mentioned in Section 1). In this case, different threshold value of α-cuts are determined by different existing method (e.g., as in the case of F_1). On the other hand, in the event that a single threshold of an α-cut fulfills at least two or more of the earlier mentioned principles in Section 1, the methods which have been formulated under such principles yield the same threshold value of an α-cut (e.g., as in the case of F_2 - methods in [2, 5] and the proposed method). Thus, it can be easily deduced that there is a strong relationship between the aforesaid four principles described in Section 1. However, the experimental result shows that the principles aim at different purpose. According to how the thresholds coincide, the convergence of the proposed methods towards a pair of thresholds obtained from other existing methods tells upon the feasibility of the proposed method, and points out its relationship to these methods.

5.1. **Evaluation indices.** Before reporting the results, we indicate the performance indices that we will adopt. We require the classification accuracy C to be measured by two key aspects:
Counting the discrepancy (in the level of uncertainty and information, i.e., in case (a) below) of the shadowed set classification from the original fuzzy set and dividing by the total amount of fuzziness or information (or in case (b) the cardinality of F) of the original fuzzy set. In effect, it is given by what follows.

(a) Information evaluation index, C_{IF}, estimates the distance between the true measure of fuzziness of elements in the original set and their corresponding measure of fuzziness in the approximated set. We require that this distance should be as small as possible. Relative to the original fuzziness associated with the given fuzzy set, it should be such that

$$C_{IF} = \frac{1}{\varphi(F)} \sum_{i=1}^{N} \delta(\varphi_F(x_i), \varphi_S(x_i)) \tag{5.1}$$

is minimal.

(b) Error evaluation index, C_E, identifies the overall error in changing the membership grades from the original value to the assigned value, relative to the total membership grades in the original fuzzy set. Formally, it follows the principle that; the distance between the true membership grade of an element in the original fuzzy set and the assigned membership grade in the approximated set should be as small as possible. In turn, the minimum value of

$$C_E = \frac{1}{\sum_{i=1}^{N} \mu_F(x_i)} \delta(\mu_F(x_i)\mu_S(x_i)) \tag{5.2}$$

demonstrates the goodness of the classification, where N is the number of objects, $\varphi(F)$ is the measure of fuzziness in F, $\sum_{i=1}^{N} \mu_F(x_i)$ is the cardinality of F, $\varphi_F(x_i)$ and $\varphi_S(x_i)$ denote the true label of fuzziness or information and the assigned label of fuzziness or information of the ith object, respectively, and $\mu_F(x_i)$ and $\mu_S(x_i)$ denote the true membership grade and the assigned membership grade of ith object, respectively. Here $\delta(x, y) = |x - y|$ is a distance function. The results obtained by applying the evaluation indices suggested in (a) and (b) of Subsection 5.1, shows that the error in changing the fuzziness and information (and changing membership grade) associated with objects in F to a new degree of fuzziness (and new membership grades, respectively) in the resulting shadowed set, has minimum effect on the proposed method (see Table 2). That is, the last column, corresponding to the proposed method, provides the best result. However, the methods in [2, 5, 13] show better classification when compared to the method in [9]. Not surprisingly, the proposed method hinges on a classification mechanism which aims at localizing the uncertainty encountered in the original fuzzy set; to a region of the shadowed set having an equivalent amount of fuzziness as found the original fuzzy set.

Table 2. Classification evaluation index on F_1

Index	Ibrahim and William-West method	Pedrycz method	Deng and Yao method	Tahayori et al. method	Proposed method
Information evaluation index	0.599	0.867	0.814	0.802	0.550
Error evaluation index	0.24964	0.30653	0.28777	0.31595	0.24476

It becomes apparent that the proposed method performs better than other methods in terms of minimizing the discrepancies arising from transforming uncertainty and information from fuzzy sets to shadowed sets.

From the experimental study, the following conclusion can be drawn:

(i) The improvement found in the proposed method can be attributed to the fact that selection of the threshold parameters anchors on a principle of uncertainty invariance. Also, assigning the mean-value of the true membership grades of all elements which are to be placed in the shadow region, as the new membership grade of the elements in the shadow region helps to minimize the error associated with the ensuing shadowed set.

(ii) The methods in [2, 5, 9, 13], may not select a threshold value which generates a shadowed set having the nearest amount of fuzziness as obtained in the original fuzzy set. Hence, these methods may not strongly fulfill a principle of uncertainty invariance.

(iii) The computing time required to run the methods in [2, 9] is less than the one required by other methods. The method suggested in [13] requires the highest amount of computing time when determining the optimal threshold.

6. Conclusion

The key aspect of inducing shadowed sets from fuzzy sets is to find the criteria (i.e., thresholds) for constructing interpretable information granules (elevated, reduced and shadow regions); which describes the overall fuzzy set. A shadowed set provides a framework for explaining the elicitation of membership grades of a fuzzy set as a three-way decision. It offers an appealing strategy for partitioning a fuzzy set into three approximation regions.

Some of the characteristics of the proposed method, that make it unique as compared to the existing methods, could be identified as:

(1) the method provides a much stronger bias for accounting for an equivalent amount of information and fuzziness of the original fuzzy set. This philosophy helps in minimizing information loss in the resulting shadowed set.

(2) a comparison with the existing method reveals the absence of detailed calculation of the effects of elevation, reduction and fixing of elements in various regions when the optimality of each feasible threshold $\alpha \in [\mu_{F_{min}}, \frac{1}{2}]$ is evaluated.

The present study, provides another approach to induce shadowed sets from fuzzy sets. First, it quantifies the uncertainty and information associated with the given fuzzy set. Second, it searches for the best criteria (a pair of symmetric thresholds) that would induce an optimal approximation of a fuzzy set. The goal of the approximation is to ensure that no information is unwittingly added or eliminated as a result of transforming a fuzzy set into its resulting shadowed.

We conclude that the proposed method, due to its simplicity and information preservation potential, could be recommended as a better substitute in application areas such as shadowed clustering techniques (i.e., shadowed-C means clustering (see [8]), rough-fuzzy clustering in the framework of shadowed sets (see [18]), information filtering process, etc. The substitution of the proposed approach to these application areas may eliminate the formation of misguided clusters and enhance the quality of information filtering model.

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