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On soft compact and soft Lindelöf spaces via soft pre-open sets

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Abstract. In this study, the authors employ soft pre-open sets to define and discuss eight sorts of generalized soft compact spaces, namely soft pre-compact, soft pre-Lindelöf, almost (approximately, mildly) soft pre-compact and almost (approximately, mildly) soft pre-Lindelöf spaces. Then they characterize each one of these spaces and show the relationships among them with the help of examples. Also, they investigate the image of these spaces under soft pre-irresolute mappings. Furthermore, they present a soft pre-partition notion and point out this notion is sufficient for the equivalent among the four types of soft pre-compact spaces and for the equivalent among the four types of soft pre-Lindelöf spaces. They demonstrate the relationships between enriched soft topological spaces and the initiated spaces in different cases and obtain interesting results. Finally, they derive some findings which connect between some generalized soft compact spaces introduced in this work and some soft topological notions such as soft pre-connected spaces, soft pre-T_2-spaces and soft subspaces.

2010 AMS Classification: 54D20, 54D30, 54A05.

Keywords: Soft pre-compactness, Almost (Approximately, Mildly) soft pre-compactness, Soft pre-irresolute map and soft subspace.

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1. Introduction

Soft sets theory was proposed by Molodtsov [24] in the year 1999, as a new mathematical tool for handling problems which contain uncertainties. After he published his work, the theory of soft sets was developed rapidly by many researchers. A first attempt for formulating soft operators was in 2003 by Maji et al. [22]. He defined soft union and intersection between two soft sets, null and absolute soft set and a complement of a soft set. A significant contribution via the soft set theory was done by Ali et al. [6] in 2009. They redefined some soft operators like soft intersection between two soft sets and a complement of a soft set; and established newly soft operators between two soft sets like the restricted and extended intersection, the restricted union and the restricted difference. In this connection, Abbas et al. [1] relaxed conditions on a parameters set to introduce generalized operators on the soft set theory.

The reaction between the soft set theory and topologists was begun in the year 2011 by Shabir and Naz [30]. They utilized soft sets to construct the concept of soft topological spaces and focused on examining the basic properties of soft separation axioms. Min [23] continued Shabir and Naz's work by investigating the properties of soft regular spaces and correcting a relationship between soft T_2 and soft T_3-spaces. In 2012, Zorlutuna et al. [33] pointed out the connection between fuzzy sets and soft sets. Also, they came up with a brilliant idea, namely soft point in order to study some properties of soft interior points and soft neighborhood systems. In the same year, Aygünoğlu and Aygün [13] introduced a concept of soft compact spaces and concluded main features. They also presented a notion of enriched soft topological spaces and illuminated its role to verify some results which associate it with soft constant mappings and soft compact spaces. Hida [19] gave another definition for soft compact spaces, namely SCPT1 which is stronger than soft compact spaces given in [13]. The authors of [15] and [25] modified together the first shape of soft points to be more effective for studying soft limit points and soft metric spaces. [28, 29] did a comparative study on the definitions of soft points and discussed their applications via soft matrix. For seeking to generalize and apply the generalization of open sets via soft topology, Chen [14] defined and investigated a notion of soft semi-open sets. Then Arockiarani and Lancy [12] presented a concept of soft pre-open sets and studied its main properties. Depending on a soft pre-open sets concept, Akdag and Ozkan [5] carried out a detailed study on soft pre-separation axioms. Subhashinin and Sekar [32] did a paper on soft connected spaces and soft neighborhood systems via soft pre-open sets. Nazmul and Samanta [25] defined enriched soft topological spaces in terms of pseudo constant soft sets. Kandil et al. [20] presented different forms of soft continuous mappings and Ozkan et al. [27] explored new types of soft compact spaces utilizing soft b-open sets. Singh and Noorie [31] carried out a significant comparative study on soft axioms. They illustrated the relationships among the different soft axioms with the help of interesting examples. With regard to soft axioms, Al-shami [7, 8] and El-Shafei et al. [18] did some studies to correct some alleged results. Recently, Al-shami [9] explored soft somewhere dense sets as new class of generalized soft open sets; and Al-shami et al. [11] established a newly soft mathematical structure, namely soft topological ordered spaces. The authors of [2, 16] investigated new types of soft compactness via supra topological spaces. Some results concerning soft grill and soft bitopology were done in [3] and [4], respectively.

This study begins by presenting the fundamental definitions introduced and findings obtained in the soft set theory and soft topological spaces. The purpose of this study is to employ soft pre-open sets to define the concepts of soft pre-compact, soft pre-Lindelöf, almost (approximately, mildly) soft pre-compact and almost (approximately, mildly) soft pre-Lindelöf spaces. We provide various examples to elucidate the relationships among these spaces and to point out some properties of them. Also, we characterize each one of these concepts and deduce some findings which connect between enriched soft topological spaces and some of the given generalized soft compact spaces. Moreover, we offer some soft topological concepts such as soft pre-hyperconnected and soft pre-partition spaces and discuss some properties which associate these concepts with the introduced soft spaces. The sufficient conditions for the eight initiated soft spaces to be soft hereditary properties are investigated. Last but not least, we point out that the soft pre-irresolute mappings preserve all the types of generalized soft compact spaces introduced herein.

2. Preliminaries

Hereafter, we recall some definitions and results which help us to investigate and discuss our new sequels. In view of soft topological spaces are defined under a fixed parameters set, we draw attention of the readers to that the definitions and findings which given in the differently previous studies are mentioned herein with respect to a fixed parameters set.

Definition 2.1 ([24]). A pair (G, K) is called a soft set over X provided that G is a mapping of a parameters set K into 2^X. For short, it can be written as follows: $(G, K) = \{(k, G(k)) : k \in K \text{ and } G(k) \in 2^X\}$.

Definition 2.2 ([6]). (G, K) is a soft subset of (H, K), denoted by $(G, K)\widetilde{\subseteq}(H, K)$, provided that $G(k) \subseteq H(k)$, for each $k \in K$.

Definition 2.3 ([6]). The relative complement of a soft set (G, K), denoted by $(G, K)^c$, is given by $(G, K)^c = (G^c, K)$, where a mapping $G^c : K \to 2^X$ is defined by $G^c(k) = X - G(k)$, for each $k \in K$.

Definition 2.4 ([6]). Let (G, K) and (F, K) be two soft sets. Then:
(i) $(G, K)\widetilde{\bigcup}(F, K) = (H, K)$, where $H(k) = G(k) \bigcup F(k)$, for each $k \in K$,
(ii) $(G, K)\widetilde{\bigcap}(F, K) = (H, K)$, where $H(k) = G(k) \bigcap F(k)$, for each $k \in K$.

Definition 2.5 ([22]). A soft set (G, K) over X is called:
(i) an absolute soft set, if $G(k) = X$, for each $k \in K$. It is denoted by $\widetilde{X}$,
(ii) a null soft set, if $G(k) = \varnothing$, for each $k \in K$. It is denoted by $\widetilde{\varnothing}$.

Definition 2.6 ([17, 31]). Let (G, K) be a soft set cover X. Then:
(i) $x \in (G, K)$, if $x \in G(k)$, for each $k \in K$,
(ii) $x \Subset (G, K)$, if $x \in G(k)$, for some $k \in K$.

Definition 2.7 ([30]). A collection τ of soft sets over X with a fixed set of parameters K is called a soft topology on X, if it satisfies the following three axioms:
(i) the null soft set $\widetilde{\varnothing}$ and the absolute soft set $\widetilde{X}$ are members of τ,
(ii) the soft union of an arbitrary number of soft sets in τ is also a member of τ,
(iii) the soft intersection of a finite number of soft sets in τ is also a member of τ.
The triple (X, τ, K) is called a soft topological space. Each soft set in τ is called soft open and its relative complement is called soft closed.

Proposition 2.8 ([30]). *Let (X, τ, K) be a soft topological space. Then $\tau_k = \{G(k) : (G, K) \in \tau\}$ defines a topology on X, for each $k \in K$.*

Definition 2.9 ([30])**.** Let (F, K) be a soft subset of a soft topological space (X, τ, K). Then $(cl(F), K)$ is defined as $cl(F)(k) = cl(F(k))$, where $cl(F(k))$ is the closure of $F(k)$ in (X, τ_k) for each $k \in K$.

Proposition 2.10 ([30])**.** *Let (L, K) be a soft subset of a soft topological space (X, τ, K). Then:*
(i) $(cl(L), K)\widetilde{\subseteq} cl(L, K)$,
(ii) $(cl(L), K) = cl(L, K)$ if and only if $(cl(L), K)^c$ is soft closed.

Definition 2.11 ([25])**.** Let (X, τ, K) be a soft topological space and (Y, K) be a non-null soft subset of $\widetilde{X}$. Then $\tau_{(Y,K)} = \{(Y, K)\widetilde{\bigcap}(G, K) : (G, K) \in \tau\}$ is said to be a relative soft topology on (Y, K) and $((Y, K), \tau_{(Y,K)}, K)$ is called a soft subspace of (X, τ, K).

Definition 2.12. A soft set (P, K) over X is called:
(i) a soft point [15, 25], denoted by P_k^x, provided that there is $k \in K$ and $x \in X$ satisfies that $P(k) = \{x\}$ and $P(e) = \varnothing$, for each $e \in K \setminus \{k\}$,
(ii) a pseudo constant [26], provided that $P(k) = X$ or $\varnothing$, for each $k \in K$. A family of all pseudo constant soft sets is briefly denoted by $CS(X, K)$.

Definition 2.13 ([13])**.** A soft topology τ on X is said to be enriched, if (i) of Definition (2.7) is replaced by the following condition: $(G, K) \in \tau$, for all $(G, K) \in CS(X, K)$. In this case, the triple (X, τ, K) is called an enriched soft topological space over X.

Definition 2.14 ([15])**.** A soft set (H, K) over X is called countable (resp. finite), if $H(k)$ is countable (resp. finite), for each $k \in K$. A soft set is called uncountable (resp. infinite), if it is not countable (resp. finite).

Definition 2.15. A soft subset (A, K) of a soft topological space (X, τ, K) is said to be:
(i) a soft pre-open [12], if $(A, K)\widetilde{\subseteq} int(cl(A, K))$ and its relative complement is called soft pre-closed,
(ii) a soft semi-open [14], if $(A, K)\widetilde{\subseteq} cl(int(A, K))$ and its relative complement is called soft semi-closed.

Definition 2.16 ([12])**.** Let (A, K) be a soft subset of a soft topological space (X, τ, K). Then $int_p(A, K)$ is the union of all soft pre-open subsets of (A, K) and $cl_p(A, K)$ is the intersection of all soft pre-closed supersets of (A, K).

Proposition 2.17 ([12])**.** *The union of an arbitrary family of soft pre-open sets is soft pre-open and the intersection an arbitrary family of soft pre-closed sets is soft pre-closed.*

Definition 2.18. A soft topological space (X, τ, K) is said to be:
(i) soft pre-connected [32], if the only soft pre-open and soft pre-closed subsets of (X, τ, K) are $\widetilde{\varnothing}$ and $\widetilde{X}$,
(ii) soft hyperconnected [21], if it does not contain disjoint soft open sets.

Definition 2.19 ([5])**.** A soft topological space (X, τ, K) is said to be soft pre T_2-space, if for every $x \neq y$ in X, there are two disjoint soft pre-open sets (G, K) and (F, K) such that $x \in (G, K)$ and $y \in (F, K)$.

Definition 2.20 ([17])**.** A soft subset (G,K) over X is called stable, if there exists a subset C of X such that $G(k)=C$, for each $k\in K$. And a soft topological space (X,τ,K) is called stable provided that all soft open subsets of $\widetilde{X}$ are stable.

Proposition 2.21 ([10])**.** *Consider* $((U,K),\tau_{(U,K)},K)$ *is a soft subspace of* (X,τ,K) *and let* cl_U *and* int_U *stand for the soft closure and soft interior operators, respectively, in* $((U,K),\tau_{(U,K)},K)$*. Then:*

(1) $cl_U(A,K)=cl(A,K)\widetilde{\bigcap}(U,K)$*, for each* $(A,K)\widetilde{\subseteq}(U,K)$*,*

(2) $int(A,K)=int_U(A,K)\widetilde{\bigcap}int(U,K)$*, for each* $(A,K)\widetilde{\subseteq}(U,K)$*.*

Through this work, we use a symbol S to refer a countable set.

3. Soft pre-compact spaces

Definition 3.1. (i) A collection $\{(G_i,K): i\in I\}$ of soft pre-open sets is called a soft pre-open cover of (X,τ,K), if $\widetilde{X}=\widetilde{\bigcup}_{i\in I}(G_i,K)$,

(ii) A soft topological space (X,τ,K) is called soft pre-compact (resp. soft pre-Lindelöf), if every soft pre-open cover of $\widetilde{X}$ has a finite (resp. countable) soft sub-cover of $\widetilde{X}$.

For the sake of brevity, the proofs of the following three propositions will be omitted.

Proposition 3.2. *Every soft pre-compact (resp. soft pre-Lindelöf) space is soft compact (resp. soft Lindelöf).*

Proposition 3.3. *Every soft pre-compact space is soft pre-Lindelöf.*

Proposition 3.4. *A finite (resp. countable) union of soft pre-compact (resp. soft pre-Lindelöf) subsets of* (X,τ,K) *is soft pre-compact (resp. soft pre-Lindelöf).*

The converse of Proposition (3.2) and Proposition (3.3) are not true in general as it is evident in the two examples below.

Example 3.5. Consider K is any set of parameters and let $\tau=\{\widetilde{\varnothing},\widetilde{\mathcal{R}},(G,K)$ such that $G(k)=\{1\}$, for each $k\in K\}$ be a soft topology on the set of real numbers $\mathcal{R}$. Obviously, $(\mathcal{R},\tau,K)$ is soft compact. On the other hand, a collection $\{(G,E): G(k)=\{1,x\}$, for each $k\in K$ and $x\in\mathcal{R}\}$ forms a soft pre-open cover of $\widetilde{\mathcal{R}}$. Since this collection has not a countable sub-cover of $\widetilde{\mathcal{R}}$, then $(\mathcal{R},\tau,K)$ is not soft pre-Lindelöf.

Example 3.6. Consider $K=\{k_1,k_2,...,k_n\}$ is a set of parameters and let $\tau=\{\widetilde{\varnothing},\widetilde{\mathcal{N}},(G,K)$ such that $G(k)=\{1\}$, for each $k\in K\}$ be a soft topology on the set of natural numbers $\mathcal{N}$. Since K and $\mathcal{N}$ are countable, then $(\mathcal{N},\tau,K)$ is soft pre-Lindelöf. On the other hand, a collection $\{(G,E): G(k)=\{1,x\}$, for each $k\in K$ and $x\in\mathcal{N}\}$ forms a soft pre-open cover of $\widetilde{\mathcal{N}}$. Since this collection has not a finite sub-cover of $\widetilde{\mathcal{N}}$, then $(\mathcal{N},\tau,K)$ is not soft pre-compact.

Proposition 3.7. *Every soft pre-closed subset* (D,K) *of a soft pre-compact (resp. soft pre-Lindelöf) space* (X,τ,K) *is soft pre-compact (resp. soft pre-Lindelöf).*

Proof. We will start with the proof for soft pre-Lindelöf spaces, as the proof for soft pre-compact spaces is analogous. Let (D, K) be a soft pre-closed subset of $\widetilde{X}$ and let $\{(H_i, K) : i \in I\}$ be a soft pre-open cover of (D, K). Then (D^c, K) is soft pre-open and $(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i, K)$. Thus $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(H_i, K)\widetilde{\bigcup}(D^c, K)$. Since $\widetilde{X}$ is soft pre-Lindelöf, $\widetilde{X} = \widetilde{\bigcup}_{i\in S}(H_i, K)\widetilde{\bigcup}(D^c, K)$. So $(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S}(H_i, K)$. Hence (D, K) is soft pre-Lindelöf. □

Corollary 3.8. *If (G, K) is a soft pre-compact (resp. soft pre-Lindelöf) subset of $\widetilde{X}$ and (D, K) is a soft pre-closed subset of $\widetilde{X}$, then $(G, K)\widetilde{\bigcap}(D, K)$ is soft pre-compact (resp. soft pre-Lindelöf).*

Proof. For the proof, let (G, K) be a soft pre-compact set and consider $\Lambda = \{(H_i, K) : i \in I\}$ is a soft pre-open cover of $(G, K)\widetilde{\bigcap}(D, K)$. Then $(G, K)\widetilde{\subseteq} \widetilde{\bigcup}_{i\in I}(H_i, K)\widetilde{\bigcup}(D^c, K)$. Since (G, K) is soft pre-compact, $(G, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}(H_i, K)\widetilde{\bigcup}\ (D^c, K)$. Thus (G, K) $\widetilde{\bigcap}(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}(H_i, K)$. So $(G, K)\widetilde{\bigcap}(D, K)$ is soft pre-compact.

A similar proof is given in case of a soft pre-Lindelöf space. □

In the following example, we show that the converse of the above proposition is not necessarily true.

Example 3.9. Let $K = \{k_1, k_2\}$ be a set of parameters and consider $\tau = \{\widetilde{\varnothing}, \widetilde{X}, (G, K),$ $(H, K), (L, K)$ such that $G(k_1) = \{x\}$, $G(k_2) = \varnothing$, $H(k_1) = \varnothing$, $H(k_2) = \{y\}\}$, $L(k_1) = \{x\}$ and $G(k_2) = \{y\}\}$ be a soft topology on $X = \{x, y\}$. Obviously, (X, τ, K) is soft pre-compact. On the other hand, a soft set (F, K), where $F(k_1) = \{x\}$, and $F(k_2) = X$, is soft pre-compact, but it is not soft pre-closed.

Theorem 3.10. *A soft topological space (X, τ, K) is soft pre-compact (resp. soft pre-Lindelöf) if and only if every soft collection of soft pre-closed subsets of (X, τ, K), satisfying the finite (resp. countable) intersection property, has, itself, a non-null soft intersection.*

Proof. We only prove the theorem when (X, τ, K) is soft pre-Lindelöf, the other case can be made similarly.

Let $\Lambda = \{(F_i, K) : i \in I\}$ be a soft pre-closed subsets of $\widetilde{X}$. Suppose that $\widetilde{\bigcap}_{i\in I}(F_i, K) = \widetilde{\varnothing}$. Then $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(F_i^c, K)$. Since (X, τ, K) is soft pre-Lindelöf, $\widetilde{\bigcup}_{i\in S}(F_i^c, K) = \widetilde{X}$. Thus $\widetilde{\bigcap}_{i\in S}(F_i, K) = \widetilde{\varnothing}$.

Conversely, let $\Lambda = \{(H_i, K) : i \in I\}$ be a soft pre-open cover of $\widetilde{X}$. Suppose that Λ has no a countable soft sub-collection which cover $\widetilde{X}$. Then $\widetilde{X}\setminus\bigcup_{i\in S}(H_i, K) \neq \widetilde{\varnothing}$, for any countable set S. Now, $\widetilde{\bigcap}_{i\in S}(H_i^c, K) \neq \widetilde{\varnothing}$ implies that $\{(H_i^c, K) : i \in I\}$ is a soft collection of soft pre-closed subsets of $\widetilde{X}$ which has the countable intersection property. Thus $\widetilde{\bigcap}_{i\in I}(H_i^c, K) \neq \widetilde{\varnothing}$. This implies that $\widetilde{X} \neq \widetilde{\bigcup}_{i\in I}(H_i, K)$. But this contradicts that Λ is a soft pre-open cover of $\widetilde{X}$. So (X, τ, K) is soft pre-Lindelöf. □

Definition 3.11. A soft map $g : (X, \tau, K) \to (Y, \theta, K)$ is called soft pre-irresolute, if the inverse image of each soft pre-open subset of $\widetilde{Y}$ is a soft pre-open subset of $\widetilde{X}$.

We investigate the following theorem which will be useful to prove Theorem (4.11) and Theorem (5.15).

Theorem 3.12. *Consider $g : (X, \tau, K) \to (Y, \theta, K)$ is a soft map. Then the following statements are equivalent:*

(1) *g is soft pre-irresolute,*

(2) *The inverse image of each soft pre-closed subset of $\widetilde{Y}$ is a soft pre-closed subset of $\widetilde{X}$,*

(3) *$cl_p(g^{-1}(A, K)) \subseteq g^{-1}(cl_p(A, K))$, for each soft subset (A, K) of $\widetilde{Y}$,*

(4) *$g(cl_p(E, K)) \subseteq cl_p(g(E, K))$, for each soft subset (E, K) of $\widetilde{X}$,*

(5) *$g^{-1}(int_p(A, K)) \subseteq int_p(g^{-1}(A, K))$, for each soft subset (A, K) of $\widetilde{Y}$.*

Proof. (1)⇒ (2): Suppose that (F, K) is a soft pre-closed subset of $\widetilde{Y}$. Then (F^c, K) is soft pre-open. Thus $g^{-1}(F^c, K)$ is a soft pre-open subset of $\widetilde{X}$. It is well known that $g^{-1}(F^c, K) = \widetilde{X} - g^{-1}(F, K)$. So $g^{-1}(F, K)$ is a soft pre-closed subset of $\widetilde{X}$.

(2)⇒ (3): For any soft subset (A, K) of $\widetilde{Y}$, we get that $cl_p(A, K)$ is a soft pre-closed subset of $\widetilde{Y}$. Since $g^{-1}(cl_p(A, K))$ is a soft pre-closed subset of $\widetilde{X}$, $cl_p(g^{-1}(A, K))\widetilde{\subseteq}cl_p(g^{-1}(cl_p(A, K))) = g^{-1}(cl_p(A, K))$.

(3)⇒ (4): For any soft subset (E, K) of $\widetilde{X}$, we have $cl_p(E, K)\widetilde{\subseteq}cl_p(g^{-1}(g(E, K))$. By **(iii)**, we find that $cl_p(g^{-1}(g(E, K)))\widetilde{\subseteq}g^{-1}(cl_p(g(E, K)))$. Then $g(cl_p(E, K))\widetilde{\subseteq}$ $g(g^{-1}(cl_p(g(E, K))))\widetilde{\subseteq}cl_p(g(E, K))$.

(4)⇒ (5): Let (A, K) be any soft subset of $\widetilde{Y}$. Then $g(cl_p(X - g^{-1}(A, K)))$ $\widetilde{\subseteq}cl_p(g(X - g^{-1}(A, K)))$. Thus $g(\widetilde{X} - int_p(g^{-1}(A, K))) = g(cl_p(\widetilde{X} - g^{-1}(A, K)))$ $\widetilde{\subseteq}cl_p(\widetilde{Y} - (A, K)) = \widetilde{Y} - int_p(A, K)$. So $\widetilde{X} - int_p(g^{-1}(A, K))\widetilde{\subseteq}g^{-1}(\widetilde{Y} - int_p(A, K))$ $= g^{-1}(\widetilde{Y}) - g^{-1}(int_p(A, K))$. Hence $g^{-1}(int_p(A, K))\widetilde{\subseteq}int_p(g^{-1}(A, K))$.

(5)⇒(1): Suppose that (A, K) is any soft pre-open subset of $\widetilde{Y}$. Since $g^{-1}(int_p$ $(A, K))\widetilde{\subseteq}int_p(g^{-1}(A, K))$, $g^{-1}(A, K)\widetilde{\subseteq}int_p(g^{-1}(A, K))$. Since $int_p(g^{-1}(A, K)) \widetilde{\subseteq}$ $g^{-1}(A, K)$, $g^{-1}(A, K) = int_p(g^{-1}(A, K))$. Then $g^{-1}(A, K)$ is a soft pre-open set. Thus g is a soft pre-irresolute map. □

Proposition 3.13. *The soft pre-irresolute image of a soft pre-compact (resp. soft pre-Lindelöf) set is soft pre-compact (resp. soft pre-Lindelöf).*

Proof. For the proof, let $g : (X, \tau, K) \to (Y, \theta, K)$ be a soft pre-irresolute map and let (D, K) be a soft pre-Lindelöf subset of $\widetilde{X}$. Suppose that $\{(H_i, K) : i \in I\}$ is a soft pre-open cover of $g(D, K)$. Then $g(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i, K)$. Thus for each $i \in I$, $(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}g^{-1}(H_i, K)$ and $g^{-1}(H_i, K)$ is soft pre-open. By hypotheses, (D, K) is soft pre-Lindelöf. So $(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S}g^{-1}(H_i, K)$. Hence $g(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S}g(g^{-1}(H_i, K))$ $\widetilde{\subseteq}\ \widetilde{\bigcup}_{i\in S}(H_i, K)$. Therefore $g(D, K)$ is soft pre-Lindelöf.

A similar proof is given in case of a soft pre-compact space. □

Lemma 3.14. *If H is a pre-open subset of (X, τ_k), then there exists a soft pre-open subset (F, K) of (X, τ, K) such that $F(k) = H$.*

Proof. Without lose of generality, consider $K = \{k_1, k_2\}$ and let $H(k_1)$ be a pre-open subset of (X, τ_1). Then there exists an open subset $G(k_1)$ of (X, τ_1) such that $H(k_1) \subseteq G(k_1) \subseteq cl[H(k_1)]$. Now, there exists a soft open set (G, K). By putting $H(k_2) = G(k_2)$, we obtain that $H(k_2) \subseteq G(k_2) \subseteq cl[H(k_2)]$. So $(H, K)\widetilde{\subseteq}(G, K)\widetilde{\subseteq}(cl(H), K)$. By Proposition (2.10), we obtain that $(cl(H), K)\widetilde{\subseteq}cl(H, K)$. This completes the proof. □

Theorem 3.15. *If (X, τ, K) is an enriched soft pre-compact (resp. enriched soft pre-Lindelöf) space, then (X, τ_k) is pre-compact (resp. pre-Lindelöf), for each $k \in K$.*

Proof. We prove the theorem in case of an enriched soft pre-Lindelöf space and the other case is proven similarly.

Let $\{H_j(k) : j \in J\}$ be a pre-open cover of (X, τ_k). We construct a soft pre-open cover of (X, τ, K) consisting of the following soft pre-open sets:

(i) from the above lemma, we can choose all soft pre-open sets (F_j, K) in which $F_j(k) = H_j(k)$, for each $j \in J$,

(ii) since (X, τ, K) is enriched, we take a soft open set (G, K) which satisfies that $G(k) = \varnothing$ and $G(k_i) = X$, for all $k_i \neq k$.

Obviously, $\{(F_j, K)\widetilde{\bigcup}(G, K) : j \in J\}$ is a soft pre-open cover of (X, τ, K). As (X, τ, K) is soft pre-Lindelöf, $\widetilde{X} = \widetilde{\bigcup\limits_{j \in S}}(F_j, K)\widetilde{\bigcup}(G, K)$. Then $X = \bigcup\limits_{j \in S} F_j(k) = \bigcup\limits_{j \in S} H_j(k)$. Thus (X, τ_k) is a pre-Lindelöf space. □

Proposition 3.16. *If (X, τ, K) is an enriched soft pre-compact (resp. enriched soft pre-Lindelöf) space, then K is finite (resp. countable).*

Proof. Straightforward. □

Proposition 3.17. *If (U, K) is soft semi-open and (H, K) is soft pre-open subsets of (X, τ, K), then $(U, K)\widetilde{\bigcap}(H, K)$ is a soft pre-open subset of $((U, K), \tau_{(U,K)}, K)$.*

Proof. Since (U, K) is soft semi-open and (H, K) is soft pre-open subsets of (X, τ, K),

$$\begin{aligned}(U, K)\widetilde{\bigcap}(H, K) &\widetilde{\subseteq}(U, K)\widetilde{\bigcap}int(cl(H, K)) \\ &= int_U[(U, K)\widetilde{\bigcap}int(cl(H, K))] \\ &\widetilde{\subseteq}int_U[\ cl(int(U, K))\widetilde{\bigcap}int(cl(H, K)))] \\ &\widetilde{\subseteq}int_U[cl[int(U, K)\widetilde{\bigcap}int(cl(H, K))]] \\ &\widetilde{\subseteq}int_U[cl[int(U, K)\ \widetilde{\bigcap}cl(H, K)]] \\ &\widetilde{\subseteq}int_U[cl[(U, K)\widetilde{\bigcap}(H, K)]].\end{aligned}$$

Then

$$\begin{aligned}(U, K)\widetilde{\bigcap}(H, K)&\widetilde{\subseteq}int_U[cl[(U, K)\ \widetilde{\bigcap}(H, K)]]\ \widetilde{\bigcap}(U, K) \\ &= int_U[cl[(U, K)\widetilde{\bigcap}(H, K)]\widetilde{\bigcap}(U, K)] \\ &= int_U[cl_U[(U, K)\widetilde{\bigcap}(H, K)]].\end{aligned}$$

This completes the proof. □

Corollary 3.18. *If (U, K) is soft open and (H, K) is soft pre-open subsets of (X, τ, K), then $(U, K)\widetilde{\bigcap}(H, K)$ is a soft pre-open subset of $((U, K), \tau_{(U,K)}, K)$.*

Proposition 3.19. *For each soft open set (A, K) and soft set (B, K) in (X, τ, K), $(A, K)\widetilde{\bigcap}cl_p(B, K)\ \widetilde{\subseteq}cl_p((A, K)\widetilde{\bigcap}(B, K))$.*

Proof. Let $P_k^x \in (A,K)\widetilde{\bigcap} cl_p(B,K)$. Then $P_k^x \in (A,K)$ and $P_k^x \in cl_p(B,K)$. Thus for each soft pre-open set (U,K) containing P_k^x, we have $(U,K)\widetilde{\bigcap}(B,K) \neq \widetilde{\varnothing}$. Since $(U,K)\widetilde{\bigcap}(A,K)$ is a non-null soft pre-open set and $P_k^x \in (U,K)\widetilde{\bigcap}(A,K)$, $((U,K)\widetilde{\bigcap}(A,K))\widetilde{\bigcap}(B,K) \neq \widetilde{\varnothing}$. Thus $P_k^x \in cl_p((A,K)\widetilde{\bigcap}(B,K))$. So

$$(A,K)\widetilde{\bigcap} cl_p(B,K)\widetilde{\subseteq} cl_p((A,K)\widetilde{\bigcap}(B,K)).$$

□

Lemma 3.20. *If (U,K) is a soft open subset of (X,τ,K) and (H,K) is soft pre-open subset of $((U,K),\tau_{(U,K)},K)$, then (H,K) is a soft pre-open subset of (X,τ,K).*

Proof. Since (H,K) is soft pre-open subset of $((U,K),\tau_{(U,K)},K)$,

$$\begin{aligned}(H,K)&\widetilde{\subseteq} int_U(\ cl_U(H,K))\\ &\widetilde{\subseteq}\ int_U[cl((H,K)\widetilde{\bigcap}(U,K))]\widetilde{\bigcap}(U,K)\\ &= int[cl((H,K)\widetilde{\bigcap}(U,K))]\widetilde{\subseteq} int[cl(H,\ K)].\end{aligned}$$

Then (H,K) is a soft pre-open subset of (X,τ,K). □

Now, we are in a position to verify the following result.

Theorem 3.21. *A soft open subset (A,K) of (X,τ,K) is soft pre-compact (resp. soft pre-Lindelöf) if and only if a soft open subspace $((A,K),\tau_{(A,K)},K)$ is soft pre-compact (resp. soft pre-Lindelöf).*

Proof. We prove the theorem in case of soft pre-compactness and the proof of the case between parentheses is made similarly.

Necessity: Let $\{(H_i,K) : i \in I\}$ be a soft pre-open cover of $((A,K),\tau_{(A,K)},K)$. Since (A,K) is soft open containing (H_i,K), it follows, by the above lemma, that (H_i,K) is soft pre-open subsets of (X,τ,K). By hypothesis, $(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}(H_i,K)$. Then a soft subspace $((A,K),\tau_{(A,K)},K)$ is soft pre-compact.

Sufficiency: Let $\{(G_i,K) : i \in I\}$ be a soft pre-open cover of (A,K) in (X,τ,K). Now, $(A,K)\widetilde{\bigcap}(G_i,K)$ is soft pre-open subset of (X,τ,K). Then by Corollary (3.18), we find that $(A,K)\widetilde{\bigcap}(G_i,K)$ is soft pre-open subset of $((A,K),\tau_{(A,K)},K)$. As a soft subspace $((A,K),\tau_{(A,K)},K)$ is soft pre-compact, $(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}((A,K)\widetilde{\bigcap}(G_i,K))$. Thus $(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}(G_i,K)$. So (A,K) is a soft pre-compact subset of (X,τ,K). □

Definition 3.22. A soft topological space (X,τ,K) is said to be soft pre T_2'-space, if for every two distinct soft points P_k^x and P_e^y ($x \neq y$ or $k \neq e$), there are two disjoint soft pre-open sets (G,K) and (F,K) such that $P_k^x \in (G,K)$ and $P_e^y \in (F,K)$.

Lemma 3.23. *The soft intersection of finite soft pre-open subsets of a soft hyperconnected space is soft pre-open.*

Proof. Let (H,K) and (F,K) be two soft pre-open subsets of $\widetilde{X}$. If (H,K) or (F,K) are null soft pre-open sets, then the proof is trivial. Thus we suppose that (H,K) and (F,K) are two non-null soft pre-open sets. So (H^c,K) and (F^c,K) are two proper soft pre-closed subsets of $\widetilde{X}$. This automatically means that $cl(int(H^c,K))\widetilde{\subseteq}(H^c,K)$ and $cl(int(F^c,K))\widetilde{\subseteq}(F^c,K)$. If $int(H^c,K) \neq \widetilde{\varnothing}$. It follows, by hypotheses (X,τ,K)

is soft hyperconnected, that $cl(int(H^c, K)) = \widetilde{X}$. This contradicts that (H^c, K) is soft pre-closed. So $int(H^c, K) = \widetilde{\varnothing}$. This means that $cl(H, K) = \widetilde{X}$.

Similarly, $cl(F, K) = \widetilde{X}$. Now, we have

$$\begin{aligned}(H, K)\widetilde{\bigcap}(F, K) &\widetilde{\subseteq} int(cl(H, K))\widetilde{\bigcap} int(cl(F, K)) \\ &= int[cl(H, K)\widetilde{\bigcap} cl(F, K)] \\ &= \widetilde{X}.\end{aligned}$$

Hence $(H, K)\widetilde{\bigcap}(F, K)$ is a soft pre-open set. □

Proposition 3.24. *If (A, K) is a soft pre-compact subset of a soft hyperconnected soft pre T_2'-space (X, τ, K), then (A, K) is soft pre-closed.*

Proof. Let the given conditions be satisfied and let $P_k^x \in (A, K)^c$. Then for each $P_e^y \in (A, K)$, there are two disjoint soft pre-open sets (G_i, K) and (W_i, K) such that $P_k^x \in (G_i, K)$ and $P_e^y \in (W_i, K)$. It follows that $\{(W_i, K) : i \in I\}$ forms a soft pre-open cover of (A, K). Consequently, $(A, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}(W_i, K)$. Since (X, τ, K) is soft hyperconnected, $\widetilde{\bigcap}_{i=1}^{i=n}(G_i, K) = (H, K)$ is a soft pre-open set. $(H, K)\widetilde{\subseteq}(A, K)^c$, since $(H, K)\widetilde{\bigcap}[\widetilde{\bigcup}_{i=1}^{i=n}(W_i, K)] = \widetilde{\varnothing}$. Thus $(A, K)^c$ is a soft pre-open set. So (A, K) is soft pre-closed. □

Corollary 3.25. *If (A, K) is a soft pre-compact stable subset of a soft hyperconnected soft pre T_2-space (X, τ, K), then (A, K) is soft pre-closed.*

Proof. Since (A, K) is stable, then $P_k^x \in (A, K)$ if and only if $x \in (A, K)$. Then by using similar technique of the above proof, the corollary holds. □

4. Almost soft pre-compact spaces

Definition 4.1. A soft topological space (X, τ, K) is called almost soft pre-compact (resp. almost soft pre-Lindelöf) if every soft pre-open cover of $\widetilde{X}$ has a finite (resp. countable) soft sub-collection the soft pre-closures of whose members cover $\widetilde{X}$.

Definition 4.2. A soft subset (D, K) of (X, τ, K) is said to be:
(i) a soft pre-clopen provided that it is soft pre-open and soft pre-closed,
(ii) a soft pre-dense provided that $cl_p(F, K) = \widetilde{X}$.

For the sake of brevity, the proofs of the following three propositions will be omitted.

Proposition 4.3. *Every almost soft pre-compact space is almost soft pre-Lindelöf.*

Proposition 4.4. *A finite (resp. countable) union of almost soft pre-compact (resp. almost soft pre-Lindelöf) subsets of (X, τ, K) is almost soft pre-compact (resp. almost soft pre-Lindelöf).*

Proposition 4.5. *Every soft pre-compact (resp. soft pre-Lindelöf) space is almost soft pre-compact (resp. almost soft pre-Lindelöf).*

The converse of Proposition (4.5) is not true as it is evident in the example below.

Example 4.6. Consider a set of parameters K is singleton and let $\tau = \{\widetilde{\varnothing}, (G_i, K)\widetilde{\subseteq} \widetilde{\mathcal{R}} : 1 \in (G_i, K)\}$ be a soft topology on $\mathcal{R}$. Since any soft pre-open set is soft pre-dense, $(\mathcal{R}, \tau, K)$ is almost soft pre-compact. On the other hand, a collection $\{(G, E) : G(k) = \{1, x\}\}$ forms a soft pre-open cover of $\widetilde{X}$. Since this collection has not a countable sub-cover of $\widetilde{X}$, $(\mathcal{R}, \tau, K)$ is not soft pre-Lindelöf.

Proposition 4.7. *Every soft pre-clopen subset (D, K) of an almost soft pre-compact (resp. almost soft pre-Lindelöf) space (X, τ, K) is almost soft pre-compact (resp. almost soft pre-Lindelöf).*

Proof. Let us prove the proposition in case of (X, τ, K) is almost soft pre-compact, the case between parentheses can be achieved similarly. Let (D, K) be a soft pre-clopen subset of $\widetilde{X}$ and $\{(H_i, K) : i \in I\}$ be a soft pre-open cover of (D, K). Then (D^c, K) is soft pre-clopen. Thus $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(H_i, K)\widetilde{\bigcup}(D^c, K)$. Since $\widetilde{X}$ is almost soft pre-compact, $\widetilde{X} = \widetilde{\bigcup}_{i=1}^{i=n} cl_p(H_i, K)\ \widetilde{\bigcup}(D^c, K)$. So $(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_p(H_i, K)$. Hence (D, K) is almost soft pre-compact. □

Corollary 4.8. *If (G, K) is an almost soft pre-compact (resp. almost soft pre-Lindelöf) subset of $\widetilde{X}$ and (D, K) is a soft pre-clopen subset of $\widetilde{X}$, then $(G, K)\widetilde{\bigcap}(D, K)$ is almost soft pre-compact (resp. almost soft pre-Lindelöf).*

Proof. To prove the proposition in case of almost soft pre-compactness, let $\Lambda = \{(H_i, K) : i \in I\}$ be a soft pre-open cover of $(G, K)\widetilde{\bigcap}(D, K)$. Then

$$(G, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i, K)\widetilde{\bigcup}(D^c, K).$$

Because (G, K) is almost soft pre-compact,

$$(G, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_p(H_i, K)\widetilde{\bigcup}(D^c, K).$$

Thus $(G, K)\widetilde{\bigcap}(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_p(H_i, K)$. sO $(G, K)\widetilde{\bigcap}(D, K)$ is almost soft pre-compact.

A similar proof is given in case of almost soft pre-Lindelöfness. □

In Example (4.6), let (H, K) be a soft subset of $(\mathcal{R}, \tau, K)$, where $H(k) = \{1, 4\}$. Then a soft set (H, K) is almost soft pre-compact, but it is not soft pre-clopen. So the converse of the above proposition is not necessarily true.

Definition 4.9. A collection $\Lambda = \{(F_i, K) : i \in I\}$ of soft sets is said to have the first type of finite (resp. countable) pre-intersection property, if $\widetilde{\bigcap}_{i=1}^{i=n} int_p(F_i, K) \neq \widetilde{\varnothing}$, for any $n \in \mathcal{N}$ (resp. $\widetilde{\bigcap}_{i\in S} int_p(F_i, K) \neq \widetilde{\varnothing}$, for any countable set S).

It is clear that any collection satisfies the first type of finite (resp. countable) pre-intersection property, it also satisfies the finite (resp. countable) intersection property.

Theorem 4.10. *A soft topological space (X, τ, K) is almost soft pre-compact (resp. almost soft pre-Lindelöf) if and only if every soft collection of soft pre-closed subsets of (X, τ, K), satisfying the first type of finite (resp. countable) pre-intersection property, has, itself, a non-null soft intersection.*

Proof. We will start with the proof for almost soft pre-compactness, because the proof for almost soft pre-Lindelöfness is analogous. Let $\Lambda = \{(F_i, K) : i \in I\}$ be a soft pre-closed subsets of $\widetilde{X}$. Suppose that $\widetilde{\bigcap}_{i\in I}(F_i, K) = \widetilde{\varnothing}$. Then $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(F_i^c, K)$. As (X, τ, K) is almost soft pre-compact, $\widetilde{X} = \widetilde{\bigcup}_{i=1}^{i=n} cl_p(F_i^c, K)$. ThUS $\widetilde{\varnothing} = (\widetilde{\bigcup}_{i=1}^{i=n} cl_p(F_i^c, K))^c = \widetilde{\bigcap}_{i=1}^{i=n} int_p(F_i, K)$. sO the necessary condition holds.

Conversely, let Λ be a soft pre-closed subsets of $\widetilde{X}$ which satisfies the first type of finite pre-intersection property. Then it also satisfies the finite intersection property. Since Λ has a non-null soft intersection, (X, τ, K) is a soft pre-compact space. It follows, by Proposition (4.5), that (X, τ, K) is almost soft pre-compact. □

Theorem 4.11. *The soft pre-irresolute image of an almost soft pre-compact (resp. almost soft pre-Lindelöf) set is almost soft pre-compact (resp. almost soft pre-Lindelöf).*

Proof. For the proof, let $g : X \to Y$ be a soft pre-irresolute map and (D, K) be an almost soft pre-Lindelöf subset of $\widetilde{X}$. Suppose that $\{(H_i, K) : i \in I\}$ is a soft pre-open cover of $g(D, K)$. Then $g(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i, K)$. Now, $(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I} g^{-1}(H_i, K)$ and $g^{-1}(H_i, K)$ is soft pre-open, for each $i \in I$. By hypotheses, (D, K) is almost soft pre-Lindelöf. Thus $(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S} cl_p(g^{-1}(H_i, K))$. So $g(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S} g(cl_p(g^{-1}(H_i, K)))$. From (iv) of Theorem (3.12), we obtain that

$$g(cl_p(g^{-1}(H_i, K)))\widetilde{\subseteq} cl_p(g(g^{-1}(H_i, K)))\widetilde{\subseteq} cl_p(H_i, K).$$

So $g(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S} cl_p(H_i, K)$. Hence $g(D, K)$ is almost soft pre-Lindelöf.

A similar proof is given in case of an almost soft pre-compact space. □

Definition 4.12. A soft topological space is said to be soft pre-hyperconnected, if it does not contain disjoint soft pre-open sets.

Proposition 4.13. *Every soft pre-hyperconnected space is almost soft pre-compact.*

Proof. Since any soft pre-open set in a soft pre-hyperconnected space is soft pre-dens, the space is almost soft pre-compact. □

The converse of this proposition is not necessarily true as it is evident in the following example.

Example 4.14. Assume that (X, τ, K) is the same as in Example (3.9). Obviously, (X, τ, K) is almost soft pre-compact. On the other hand, (G, K) and (H, K) are two disjoint soft pre-open sets. Then (X, τ, K) is not soft pre-hyperconnected.

Definition 4.15. Let (F, K) be a soft subset of a soft topological space (X, τ, K). Then $(cl_p(F), K)$ is defined as $cl_p(F)(k) = cl_p(F(k))$, where $cl_p(F(k))$ is the pre-closure of $F(k)$ in (X, τ_k), for each $k \in K$.

Proposition 4.16. *Let (L, K) be a soft subset of a soft topological space (X, τ, K). Then:*

(1) $(cl_p(L), K)\widetilde{\subseteq} cl_p(L, K)$,

(2) $(cl_p(L), K) = cl_p(L, K)$ *if and only if* $(cl_p(L), K)$ *is soft pre-closed.*

Proof. (1) For any $k \in K$, $cl_p(L(k))$ is the smallest pre-closed subset of (X, τ_k) containing $L(k)$. Putting $cl_p(L, K) = (F, K)$. Then $F(k)$ is a pre-closed subset of (X, τ_k) containing $L(k)$ as well. This means that $(cl_p(L))(k) = cl_p(L(k)) \subseteq F(k)$. Thus $(cl_p(L), K)\widetilde{\subseteq} cl_p(L, K)$.

(2) If $(cl_p(L), K) = cl_p(L, K)$, then $(cl_p(L), K)$ is a soft pre-closed set.

Conversely, let $(cl_p(L), K)$ be a soft pre-closed set. Obviously, $(cl_p(L), K)$ containing (L, K). Then from the definition of soft pre-closure of (L, K), we infer that $cl_p(L, K)\widetilde{\subseteq}(cl_p(L), K)$. Thus from (1), we obtain that $(cl_p(L), K)\widetilde{\subseteq} cl_p(L, K)$. So $(cl_p(L), K) = cl_p(L, K)$. □

Lemma 4.17. *Let (H, K) be a soft subset of an enriched soft topological space (X, τ, K). If $H(k)$ is a non-empty subset of (X, τ_k) and $H(k_j) = \varnothing$, for each $k_j \neq k$, then $(cl_p(H), K)$ is a soft pre-closed set.*

Proof. Assume that $k = k_1$ and $(cl_p(H), K) = \{(k_1, cl_p(H(k_1)), (k_2, \varnothing), ..., (k_n, \varnothing), ...\}$. Let $P_k^x \in cl_p(H, K)$. As (X, τ, K) is enriched, $k = k_1$. Now, for each soft pre-open set (W, K) containing $P_{k_1}^x$, we have $(W, K)\widetilde{\bigcap}(H, K) \neq \widetilde{\varnothing}$. Then $W(k_1) \bigcap H(k_1) \neq \varnothing$. Thus it follows, by Lemma (3.14), that for each pre-open set $L(k_1)$ in (X, τ_{k_1}) containing x, we have that $L(k_1) \bigcap H(k_1) \neq \varnothing$. This implies that $x \in cl_p(H(k_1))$. So $P_k^x \in (cl_p(H), K)$. Hence $cl_p(H, K)\widetilde{\subseteq}(cl_p(H), K)$. Therefore it follows, from Proposition (4.16), that $cl_p(H, K) = (cl_p(H), K)$. This completes the proof. □

Theorem 4.18. *If (X, τ, K) is an enriched almost soft pre-compact (resp. enriched almost soft pre-Lindelöf) space, then (X, τ_k) is almost pre-compact (resp. almost pre-Lindelöf), for each $k \in K$.*

Proof. We prove the theorem in case of an almost soft pre-compact space and the other proof follows similar lines. Let $\{H_j(k) : j \in J\}$ be a pre-open cover for (X, τ_k). We construct a soft pre-open cover for $\widetilde{X}$ like the introduced soft pre-open cover in the proof of Theorem (3.15). Now, (X, τ, K) is almost soft pre-compact implies that $\widetilde{X} = \bigcup_{j=1}^{j=n} cl_p(F_j, K)\widetilde{\bigcup}(G, K) = \bigcup_{j=1}^{j=n} (cl_p(F_j), K)\ \widetilde{\bigcup}(G, K)$. Then

$$X = \bigcup_{j=1}^{j=n} cl_p(F_j(k)) = \bigcup_{j=1}^{j=n} cl_p(H_j(k)).$$

Thus (X, τ_k) is an almost pre-compact space. □

Proposition 4.19. *If (X, τ, K) is an enriched almost soft pre-compact (resp. enriched almost soft pre-Lindelöf) space, then K is finite (resp. countable).*

Proof. Straightforward. □

Proposition 4.20. *Consider $((U, K), \tau_{(U,K)}, K)$ is a soft open subspace of (X, τ, K). Let cl_p and int_p stand for the soft pre-closure and soft pre-interior operators, respectively, in (X, τ, K) and let cl_{pU} and int_{pU} stand for the soft pre-closure and soft pre-interior operators, respectively, in $((U, K), \tau_{(U,K)}, K)$. Then:*

(1) $cl_{pU}(A, K) = cl_p(A, K)\widetilde{\bigcap}(U, K)$, *for each* $(A, K)\widetilde{\subseteq}(U, K)$,

(2) $int_p(A, K) = int_{pU}(A, K)$, *for each* $(A, K)\widetilde{\subseteq}(U, K)$.

Proof. (1) Let $P_k^x \in cl_{pU}(A,K)$. Then for any soft pre-open set (D,K) in $\tau_{(U,K)}$ such that $P_k^x \in (D,K)$, we have $(D,K)\widetilde{\bigcap}(A,K) \neq \widetilde{\varnothing}$. If (G,K) is a soft pre-open set in τ such that $P_k^x \in (G,K)$, then by Corollary (3.18), $(H,K) = (G,K)\widetilde{\bigcap}(U,K)$ is a soft pre-open set in $\tau_{(U,K)}$ and $(H,K)\widetilde{\bigcap}(A,K) \neq \widetilde{\varnothing}$. Obviously, we find that $(G,K)\widetilde{\bigcap}(A,K) \neq \widetilde{\varnothing}$. SThus $P_k^x \in cl_p(A,K)$. So $cl_{pU}(A,K)\widetilde{\subseteq}\ cl_p(A,K)\ \widetilde{\bigcap}(U,K)$.

On the other hand, let $P_k^x \in [cl_p(A,K)\widetilde{\bigcap}(U,K)]$ and consider (G,K) is soft pre-open set in $\tau_{(U,K)}$ such that $P_k^x \in (G,K)$. Then by Lemma (3.20), (G,K) is soft pre-open set in τ as well. Since $P_k^x \in cl_p(A,K)$, $(G,K)\widetilde{\bigcap}(A,K) \neq \widetilde{\varnothing}$. Thus $P_k^x \in cl_{pU}(A,K)$. So $cl_p(A,K)\widetilde{\bigcap}(U,K)\widetilde{\subseteq}cl_{pU}(A,K)$. Hence $cl_{pU}(A,K) = cl_p(A,K)\widetilde{\bigcap}(U,K)$.

(2) Let $P_k^x \in int_p(A,K)$. Then there exists a soft pre-open set (G,K) in τ such that $P_k^x \in (G,K)\widetilde{\subseteq}(A,K)$. Thus by Corollary (3.18), we have $(G,K)\widetilde{\bigcap}(U,K)\widetilde{\subseteq}(G,K)$ is a soft pre-open set in $\tau_{(U,K)}$. This implies that $P_k^x \in int_{pU}(A,K)$. Thus $int_p(A,K)$ $\widetilde{\subseteq}int_{pU}(A,K)$.

On the other hand, let $P_k^x \in int_{pU}(A,K)\widetilde{\subseteq}(U,K)$. Then there exists a soft pre-open set (G,K) in $\tau_{(U,K)}$ such that $P_k^x \in (G,K)\widetilde{\subseteq}(A,K)$. Thus from Lemma (3.20), (G,K) is soft pre-open set in τ as well. So $P_k^x \in int_p(A,K)$. Hence $int_{pU}(A,K)\widetilde{\subseteq}int_p(A,K)$. Therefore $int_p(A,K) = int_{pU}(A,K)$. □

Theorem 4.21. *A soft open subset (A,K) of (X,τ,K) is almost soft pre-compact (resp. almost soft pre-Lindelöf) if and only if a soft open subspace $((A,K),\tau_{(A,K)},K)$ is almost soft pre-compact (resp. almost soft pre-Lindelöf).*

Proof. Necessity: Let $\{(H_i,K) : i \in I\}$ be a soft pre-open cover of $((A,K),\tau_{(A,K)},K)$. Since (A,K) is a soft open set containing (H_i,K), it follows, from Lemma (3.20), that (H_i,K) is soft pre-open subsets of (X,τ,K). By hypotheses,

$$(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_p(H_i,K) = \widetilde{\bigcup}_{i=1}^{i=n}[cl_p(H_i,K)\widetilde{\bigcap}(A,K)] = \widetilde{\bigcup}_{i=1}^{i=n} cl_{pU}(H_i,K).$$

Then a soft open subspace $((A,K),\tau_{(A,K)},K)$ is almost soft pre-compact.

Sufficiency: Let $\{(G_i,K) : i \in I\}$ be a soft pre-open cover of (A,K) in (X,τ,K). Now, $(A,K)\widetilde{\bigcap}(G_i,K)$ is a soft pre-open subset of (X,τ,K). Then by Corollary (3.18), we find that $(A,K)\widetilde{\bigcap}(G_i,K)$ is a soft pre-open subset of $((A,K),\tau_{(A,K)},K)$. As a soft open subspace $((A,K),\tau_{(A,K)},K)$ is soft pre-compact,

$$(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_{pU}[(A,K)\widetilde{\bigcap}(G_i,K)]\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_{pU}(G_i,K).$$

Thus $(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_p(G_i,K)$. So (A,K) is an almost soft pre-compact subset of (X,τ,K).

A case between parentheses can be proven similarly. □

Proposition 4.22. *If (A,K) is an almost soft pre-compact subset of a soft hyperconnected soft pre T_2'-space (X,τ,K), then (A,K) is soft pre-closed.*

Proof. Let the given conditions be satisfied and let $P_k^x \in (A,K)^c$. Then for each $P_e^y \in (A,K)$, there are two disjoint soft pre-open sets (G_i,K) and (W_i,K) such that $P_k^x \in (G_i, K)$ and $P_e^y \in (W_i, K)$. It follows that $\{(W_i, K) : i \in I\}$ forms a soft pre-open cover of (A, K). Consequently, $(A, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_p(W_i, K)$. Since (X, τ, K) is soft hyperconnected, $\widetilde{\bigcap}_{i=1}^{i=n}(G_i, K) = (H, K)$ is a soft pre-open set. Since $(H, K)\widetilde{\bigcap}[\widetilde{\bigcup}_{i=1}^{i=n}(W_i, K)] = \widetilde{\varnothing}$, by Proposition (3.19), we find that

$$(H, K)\widetilde{\bigcap}[\widetilde{\bigcup}_{i=1}^{i=n} cl_p(W_i, K)] = \widetilde{\varnothing}.$$

Thus $(H, K)\widetilde{\subseteq}\widetilde{(A, K)^c}$. Thus $(A, K)^c$ is a soft pre-open set. So (A, K) is soft pre-closed. □

Corollary 4.23. *If (A, K) is an almost soft pre-compact stable subset of a soft hyperconnected soft pre T_2-space (X, τ, K), then (A, K) is soft pre-closed.*

5. Approximately soft pre-compact spaces

Definition 5.1. A soft topological space (X, τ, K) is called approximately soft pre-compact (resp. approximately soft pre-Lindelöf) space, if every soft pre-open cover of $\widetilde{X}$ has a finite (resp. countable) soft sub-collection in which its soft pre-closure cover $\widetilde{X}$.

Proposition 5.2. *Every approximately soft pre-compact space is approximately soft pre-Lindelöf*

Proof. Straightforward. □

The following example shows that the converse of the above proposition is not true.

Example 5.3. Consider $(\mathcal{R}, \tau, K)$ is a soft topological space such that $K = \{k_1, k_2\}$ is a set of parameters and $\tau = \{\widetilde{\varnothing}, (G_i, K)\widetilde{\subseteq}\widetilde{\mathcal{R}}$ such that for each $k \in K$, $G_i(k) = \{n : n \in \mathcal{N}\}$ or their soft union$\}$. Then any soft set (G, K) is soft pre-open if and only if there exists $n \in \mathcal{N}$ such that $n \Subset (G_i, K)$. Taking a soft pre-open cover Λ of $\widetilde{\mathcal{R}}$ as follows, $\Lambda = \{(G, K)$ is countable such that there is only one parameter $k \in K$ satisfies that $G(k)$ contains only one natural number $n \in \mathcal{N}\}$. This soft pre-open cover has not a finite sub-cover which its soft pre-closure cover $\widetilde{\mathcal{R}}$. Then $(\mathcal{R}, \tau, K)$ is not approximately soft pre-compact.

On the other hand, for any soft pre-open cover of $\widetilde{\mathcal{R}}$, we can find countable soft pre-open subsets of Λ contains a soft open set $\{(G(k_1), \mathcal{N}), (G(k_2), \mathcal{N})\}$. This a soft pre-open set is soft pre-dense. Yhus $(\mathcal{R}, \tau, K)$ is approximately soft pre-Lindelöf.

Proposition 5.4. *A finite (resp. countable) union of approximately soft pre-compact (resp. approximately soft pre-Lindelöf) subsets of (X, τ, K) is approximately soft pre-compact (resp. approximately soft pre-Lindelöf).*

Proof. Let $\{(A_s, K) : s \in S\}$ be approximately soft pre-Lindelöf subsets of (X, τ, K) and let $\{(G_i, K) : i \in I\}$ be a soft pre-open cover of $\widetilde{\bigcup}_{s \in S}(A_s, K)$. Then there exist countable sets M_s such that

$$(A_1, K)\widetilde{\subseteq} cl_p(\widetilde{\bigcup}_{i \in M_1}(G_i, K)), \ldots, (A_n, K)\widetilde{\subseteq} cl_p(\widetilde{\bigcup}_{i \in M_n}(G_i, K)), \ldots.$$

Thus

$$\widetilde{\bigcup}_{s\in S}(A_s,K)\widetilde{\subseteq}cl_p(\widetilde{\bigcup}_{i\in M_1}(G_i,K))\widetilde{\bigcup}\dots\widetilde{\bigcup}cl_p(\widetilde{\bigcup}_{i\in M_n}(G_i,K))\widetilde{\bigcup}\dots$$
$$\widetilde{\subseteq}cl_p\ (\widetilde{\bigcup}_{i\in \bigcup\limits_{s\in S} M_s}(G_i,K))$$

and $\bigcup\limits_{s\in S} M_s$ is countable, as required.

A similar proof is given in case of an approximately soft pre-compact space. □

Proposition 5.5. *Every almost soft pre-compact (resp. almost soft pre-Lindelöf) space is approximately soft pre-compact (resp. approximately soft pre-Lindelöf).*

Proof. Since $\widetilde{\bigcup}_{i\in I}cl_p(G_i,K)\widetilde{\subseteq}cl_p(\widetilde{\bigcup}_{i\in I}(G_i,K))$, the result is satisfied. □

Corollary 5.6. *Every soft pre-hyperconnected space is approximately soft pre-Lindelöf.*

The following example shows that the converse of the above proposition is not true in general.

Example 5.7. Consider $(\mathcal{R},\tau,K)$ is a soft topological space such that $K=\{k_1,k_2\}$ is a set of parameters and $\tau=\{\widetilde{\varnothing},\widetilde{\mathcal{R}},(G_1,K),(G_2,K),(G_3,K)\}$, where the soft open sets are defined as follows, for each $k\in K, G_1(k)=\{1\},G_2(k)=\{2\},G_3(k)=\{1,2\}$. Then any soft set (G,K) is soft pre-open if and only if $1\Subset(G,K)$ or $2\Subset(G,K)\}$. Taking a soft pre-open cover Λ of $\widetilde{\mathcal{R}}$ as follows, $\Lambda=\{(G,K)$ is finite such that there exists only one parameter $k\in K$ satisfies that $1\in G(k)$ or $2\in G(k)\}$. This soft pre-open cover has not a countable sub-cover in which its soft pre-closure of whose members cover $\widetilde{\mathcal{R}}$, hence $(\mathcal{R},\tau,K)$ is not almost soft pre-Lindelöf.

On the other hand, for any soft pre-open cover of $\widetilde{\mathcal{R}}$, we can find four soft pre-open subsets of Λ contains a soft open set (G_3,K). A soft pre-open set (G_3,K) is soft pre-dense. Thus $(\mathcal{R},\tau,K)$ is approximately soft pre-compact.

Definition 5.8. A collection $\Lambda=\{(F_i,K):i\in I\}$ of soft sets is said to have the second type of finite (resp. countable) pre-intersection property, if $int_p[\widetilde{\bigcap}_{i=1}^{i=n}(F_i,K)]\neq\widetilde{\varnothing}$, for any $n\in\mathcal{N}$ (resp. $int_p[\widetilde{\bigcap}_{i\in S}(F_i,K)]\neq\widetilde{\varnothing}$, for any countable set S).

It is clear that any collection satisfies the second type of finite (resp. countable) pre-intersection property, it also satisfies the first type of finite (resp. countable) pre-intersection property.

Theorem 5.9. *A soft topological space (X,τ,K) is approximately soft pre-compact (resp. approximately soft pre-Lindelöf) if and only if every soft collection of soft pre-closed subsets of (X,τ,K), satisfying the second type of finite (resp. countable) pre-intersection property, has, itself, a non-null soft intersection.*

Proof. We only prove the theorem when (X,τ,K) is approximately soft pre-compact, the other case can be made similarly. Let $\Lambda=\{(F_i,K):i\in I\}$ be a soft pre-closed subsets of $\widetilde{X}$. Suppose that $\widetilde{\bigcap}_{i\in I}(F_i,K)=\widetilde{\varnothing}$. Then $\widetilde{X}=\widetilde{\bigcup}_{i\in I}(F_i^c,K)$. As (X,τ,K) is approximately soft pre-compact, $\widetilde{X}=cl_p(\widetilde{\bigcup}_{i=1}^{i=n}(F_i^c,K))$. Thus $\widetilde{\varnothing}=(cl_p(\widetilde{\bigcup}_{i=1}^{i=n}(F_i^c,K)))^c=int_p(\widetilde{\bigcap}_{i=1}^{i=n}(F_i,K))$. So the necessary condition holds.

Conversely, let Λ be a soft pre-closed subsets of $\widetilde{X}$ which satisfies the second type of finite pre-intersection property. Then it also satisfies the first type of finite pre-intersection property. Since Λ has a non-null soft intersection, (X, τ, K) is an almost soft pre-compact space. It follows, by Proposition (5.5), that (X, τ, K) is approximately soft pre-compact. □

Definition 5.10. A topological space (X, τ) is called approximately pre-compact (resp. approximately pre-Lindelöf) space, if every pre-open cover of X has a finite (resp. countable) sub-cover in which its pre-closure cover X.

Theorem 5.11. *A soft open subset (A, K) of (X, τ, K) is approximately soft pre-compact (resp. approximately soft pre-Lindelöf) if and only if a soft subspace $((A, K), \tau_{(A,K)}, K)$ is approximately soft pre-compact (resp. approximately soft pre-Lindelöf).*

Proof. The proof is similar of that Theorem (4.21). □

Definition 5.12. A soft topological space (X, τ, K) is called soft pre-separable provided that it contains a countable, pre-dense soft set.

Proposition 5.13. *If there exists a finite (resp. countable) soft pre-dense subset of a soft topological space (X, τ, K) such that K is finite (resp. countable), then (X, τ, K) is approximately soft pre-compact (resp. approximately soft pre-Lindelöf).*

Proof. Let $\{(G_i, K) : i \in I\}$ be a soft pre-open cover of a soft topological space (X, τ, K) and let (B, K) be a finite (countable) soft pre-dense subset of (X, τ, K). Then for each $P_{k_s}^{x_s} \in (B, K)$, there exists (G_{x_s}, K) containing $P_{k_s}^{x_s}$. This implies that $\widetilde{X} = cl_p[\widetilde{\bigcup}(G_{x_s}, K)]$. Since (B, K) and K are finite (countable), the collection $\{(G_s, K)\}$ is finite (countable). Thus the proof is completed. □

Corollary 5.14. *Every soft pre-separable with a countable set of parameters K is approximately soft pre-Lindelöf.*

Theorem 5.15. *The soft pre-irresolute image of an approximately soft pre-compact (resp. approximately soft pre-Lindelöf) set is approximately soft pre-compact (resp. approximately soft pre-Lindelöf).*

Proof. We prove the theorem by using a similar technique of the proof of Theorem (4.11) and employing (3) of Theorem (3.12). □

Proposition 5.16. *If (A, K) is an approximately soft pre-compact subset of a soft hyperconnected soft pre T_2'-space (X, τ, K), then (A, K) is soft pre-closed.*

Proof. The proof is similar of that Proposition (4.22). □

Corollary 5.17. *If (A, K) is an approximately soft pre-compact stable subset of a soft hyperconnected soft pre T_2-space (X, τ, K), then (A, K) is soft pre-closed.*

6. Mildly soft pre-compact spaces

Definition 6.1. A soft topological space (X, τ, K) is called mildly soft pre-compact (resp. mildly soft pre-Lindelöf), if every soft pre-clopen cover of $\widetilde{X}$ has a finite (resp. countable) soft subcover.

The proofs of the next two propositions are easy and so will be omitted.

Proposition 6.2. *A finite (resp. countable) union of mildly soft pre-compact (resp. mildly soft pre-Lindelöf) subsets of a soft topological space (X, τ, K) is mildly soft pre-compact (resp. mildly soft pre-Lindelöf).*

Proposition 6.3. *Every mildly soft pre-compact space is mildly soft pre-Lindelöf.*

It can be seen from Example (5.3) that the converse of the above proposition fails.

Proposition 6.4. *Every almost soft pre-compact (resp. almost soft pre-Lindelöf) space (X, τ, K) is mildly soft pre-compact (resp. mildly soft pre-Lindelöf).*

Proof. Suppose (X, τ, K) is almost soft pre-Lindelöf and let $\Lambda = \{(H_i, K) : i \in I\}$ be a soft pre-clopen cover of (X, τ, K). Then $\widetilde{X} = \widetilde{\bigcup}_{s \in S} cl_p(H_i, K)$. Now, $cl_p(H_i, K) = (H_i, K)$. Thus (X, τ, K) is mildly soft pre-Lindelöf.

A similar proof is given when (X, τ, K) is mildly soft pre-compact. □

Corollary 6.5. *Every soft pre-compact (resp. soft pre-Lindelöf) space (X, τ, K) is mildly soft pre-compact (resp. mildly soft pre-Lindelöf).*

Proposition 6.6. *If (X, τ, K) is soft pre-hyperconnected, then the following six concepts are equivalent:*

(1) *almost soft pre-compact,*
(2) *almost soft pre-Lindelöf,*
(3) *approximately soft pre-compact,*
(4) *approximately soft pre-Lindelöf,*
(5) *mildly soft pre-compact,*
(6) *mildly soft pre-Lindelöf.*

Proof. The proof comes immediately from the fact states that a soft pre-hyperconnected spaces (X, τ, K) implies that the soft pre-closure of any non-null soft open set is $\widetilde{X}$ and the only soft pre-clopen sets are $\widetilde{X}$ and $\widetilde{\varnothing}$. □

Proposition 6.7. *Every soft pre-connected space (X, τ, K) is mildly soft pre-compact.*

Proof. Because (X, τ, K) is soft pre-connected, the only soft pre-clopen subsets of (X, τ, K) are $\widetilde{X}$ and $\widetilde{\varnothing}$. Then (X, τ, K) is mildly soft pre-compact. □

One can be easily seen from Example (3.9) that the converse of the above proposition fails.

In the next example, we illuminate that an approximately soft pre-compact space need not be mildly soft pre-Lindelöf in general.

Example 6.8. Assume that $(\mathcal{R}, \tau, K)$ is the same as in Example (5.7). It was illustrated that $(\mathcal{R}, \tau, K)$ is an approximately soft pre-Lindelöf space.

On the other hand, the given soft collection Λ forms a pre-clopen cover of $\widetilde{\mathcal{R}}$. Since that a soft collection has not a countable sub-cover, then $(\mathcal{R}, \tau, K)$ is not a mildly soft pre-Lindelöf space.

Theorem 6.9. *A soft topological space (X, τ, K) is mildly soft pre-compact (resp. mildly soft pre-Lindelöf) if and only if every soft collection of soft pre-clopen subsets of (X, τ, K), satisfying the finite (resp. countable) intersection property, has, itself, a non-null soft intersection.*

Proof. We only prove the theorem when (X,τ,K) is mildly soft pre-compact, the other case can be made similarly.

Let $\Lambda = \{(F_i,K) : i \in I\}$ be a soft pre-clopen subsets of $\widetilde{X}$. Suppose that $\widetilde{\bigcap}_{i\in I}(F_i,K) = \widetilde{\varnothing}$. Then $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(F_i^c,K)$. As (X,τ,K) is mildly soft pre-compact, $\widetilde{\bigcup}_{i=1}^{i=n}(F_i^c,K) = \widetilde{X}$. Thus $\widetilde{\bigcap}_{i=1}^{i=n}(F_i,K) = \widetilde{\varnothing}$. Hence the necessary condition holds.

Conversely, let $\Lambda = \{(H_i,K) : i \in I\}$ be a soft pre-clopen cover of $\widetilde{X}$. Suppose Λ has no finite soft sub-collection which cover $\widetilde{X}$. Then $\widetilde{X}\backslash \widetilde{\bigcup}_{i=1}^{i=n}(H_i,K) \neq \widetilde{\varnothing}$, for any $n \in \mathcal{N}$. Now, $\widetilde{\bigcap}_{i=1}^{i=n}(H_i^c,K) \neq \widetilde{\varnothing}$ implies that $\{(H_i^c,K) : i \in I\}$ is a soft collection of soft pre-clopen subsets of $\widetilde{X}$ which has the finite intersection property. Thus $\widetilde{\bigcap}_{i\in I}(H_i^c,K) \neq \widetilde{\varnothing}$. This implies that $\widetilde{X} \neq \widetilde{\bigcup}_{i\in I}(H_i,K)$. But this contradicts that Λ is a soft pre-clopen cover of $\widetilde{X}$. So (X,τ,K) is mildly soft pre-compact. □

Proposition 6.10. *The soft pre-irresolute image of a mildly soft pre-compact (resp. mildly soft pre-Lindelöf) set is mildly soft pre-compact (resp. mildly soft pre-Lindelöf).*

Proof. By using a similar technique of the proof of Proposition (3.13), the proposition holds. □

For the sake of economy, the proofs of the following two propositions will be omitted.

Proposition 6.11. *If (D,K) is a soft pre-clopen subset of a mildly soft pre-compact (resp. mildly soft pre-Lindelöf) space (X,τ,K), then (D,K) is mildly soft pre-compact (resp. mildly soft pre-Lindelöf).*

Corollary 6.12. *If (G,K) is a mildly soft pre-compact (resp. mildly soft pre-Lindelöf) subset of $\widetilde{X}$ and (D,K) is a soft pre-clopen subset of $\widetilde{X}$, then $(G,K)\widetilde{\bigcap}(D,K)$ is mildly soft pre-compact (resp. mildly soft pre-Lindelöf).*

Definition 6.13. A soft topological space (X,τ,K) is said to be soft pre-partition provided that a soft set is soft pre-open, if it is soft pre-closed.

Theorem 6.14. *Let (X,τ,K) be soft pre-partition space. Then the following four statements are equivalent:*

(1) *(X,τ,K) is soft pre-Lindelöf (resp. soft pre-compact),*
(2) *(X,τ,K) is almost soft pre-Lindelöf (resp. almost soft pre-compact),*
(3) *(X,τ,K) is approximately soft pre-Lindelöf (resp. approximately soft pre-compact),*
(4) *(X,τ,K) is mildly soft pre-Lindelöf (resp. mildly soft pre-compact).*

Proof. (1)⇒ (2): It follows from Proposition (4.5).

(2)⇒ (3): It follows from Proposition (5.5).

(3)⇒ (4): Let $\{(G_i,K) : i \in I\}$ be a pre-clopen cover of $\widetilde{X}$. Since (X,τ,K) is approximately soft pre-Lindelöf, $\widetilde{X}\widetilde{\subseteq} cl_p(\widetilde{\bigcup}_{s\in S}(G_i,K))$. Since (X,τ,K) is soft pre-partition, $cl_p(\widetilde{\bigcup}_{s\in S}(G_i,K)) = \widetilde{\bigcup}_{s\in S}(G_i,K)$. Then (X,τ,K) is mildly soft pre-Lindelöf.

(4)⇒ (1): Let $\{(G_i, K) : i \in I\}$ be a soft pre-open cover of $\widetilde{X}$. Since (X, τ, K) is soft pre-partition, $\{(G_i, K) : i \in I\}$ is a pre-clopen cover of $\widetilde{X}$. Since (X, τ, K) is mildly soft pre-Lindelöf, $\widetilde{X} = \widetilde{\bigcup}_{s\in S}(G_i, K)$.

A similar proof can be given for the case between parentheses. □

Definition 6.15. Let (F, K) be a soft subset of a soft topological space (X, τ, K). Then $(int_p(F), K)$ is defined as $int_p(F)(k) = int_p(F(k))$, where $int_p(F(k))$ is the pre-interior of $F(k)$ in (X, τ_k), for each $k \in K$.

Proposition 6.16. *Let (L, K) be a soft subset of a soft topological space (X, τ, K). Then:*

(1) $int_p(L, K)\widetilde{\subseteq}(int_p(L), K)$,

(2) $int_p(L, K) = (int_p(L), K)$ *if and only if* $(int_p(L), K)$ *is soft pre-open.*

Proof. (1) For any $k \in K$, $int_p(L(k))$ is the largest pre-open subset of (X, τ_k) contained in $L(k)$. Putting $int_p(L, K) = (F, K)$. Then $F(k)$ is a pre-open subset of (X, τ_k) contained in $L(k)$ as well. This means that $F(k) \subseteq int_p(L(k)) = int_p(L)(k)$. Thus $int_p(L, K)\widetilde{\subseteq}(int_p(L), K)$.

(2) If $(int_p(L), K) = int_p(L, K)$, then $(int_p(L), K)$ is a soft pre-open set.

Conversely, let $(int_p(L), K)$ be a soft pre-open set. Obviously, $(int_p(L), K)$ contained in (L, K). Then from the definition of soft pre-interior of (L, K), we infer that $(int_p(L), K) \widetilde{\subseteq} int_p(L, K)$. Thus from (1), we obtain that $int_p(L, K)\widetilde{\subseteq} (int_p(L), K)$. So $(int_p(L), K) = int_p(L, K)$. □

Proposition 6.17. *If (X, τ, K) is an enriched soft mildly pre-compact (resp. enriched soft mildly pre-Lindelöf) space, then K is finite (resp. countable).*

Proof. The proof is easy. □

Definition 6.18. A collection β of soft pre-open sets is called soft pre-base of a soft topological space (X, τ, K), if every soft pre-open subset of $\widetilde{X}$ can be written as a soft union of members of β

Theorem 6.19. *Consider (X, τ, K) has a soft pre-base consists of soft pre-clopen sets. Then (X, τ, K) is soft pre-compact (resp. soft pre-Lindelöf) if and only if it is mildly soft pre-compact (resp. mildly soft pre-Lindelöf).*

Proof. The necessary condition is obvious.

To verify the sufficient condition, assume that Λ is a soft pre-open cover of a mildly soft pre-compact space (X, τ, K). Since $\widetilde{X}$ is a soft union of members of the soft pre-base and $\widetilde{X}$ is mildly soft pre-compact, we can find a finite member (H_s, K) of the soft pre-base satisfies that $\widetilde{X} = \widetilde{\bigcup}_{s=1}^{s=n}(H_s, K)$. Then for each member (G_s, K) of Λ, there exists a member (H_s, K) of the soft pre-base such that $(H_s, K)\widetilde{\subseteq}(G_s, K)$. Thus $\widetilde{X} = \widetilde{\bigcup}_{s=1}^{s=n}(G_s, K)$. So (X, τ, K) is soft pre-compact.

The proof in case of a mildly soft pre-Lindelöf space is similar. □

Proposition 6.20. *If (A, K) is a mildly soft pre-compact subset of a soft hyperconnected soft pre T_2'-space (X, τ, K), then (A, K) is soft pre-closed.*

Proof. The proof is similar to that of Proposition (3.24). □

Corollary 6.21. *If (A, K) is a mildly soft pre-compact stable subset of a soft hyperconnected soft T_2-space (X, τ, K), then (A, K) is soft pre-closed.*

7. Conclusion

Depending on soft pre-open sets, we establish and study eight generalized sorts of soft compact spaces, namely soft pre-compactness, soft pre-Lindelöfness, almost (approximately, mildly) soft pre-compactness and almost (approximately, mildly) soft pre-Lindelöfness. We show the relationships among these concepts with the help of illustrative examples and we discuss the image of these spaces under soft pre-irresolute mappings. Also, we deduce some properties of soft semi-open and soft pre-open sets which enable us to prove certain of our results. We point out the relationships which associate some of the introduced spaces with soft pre T_2-spaces and soft pre T_2'-spaces. With regard to the finite intersection property, we present two new types of the finite pre-intersection property and utilize them to give the equivalent conditions for almost (approximately) soft pre-compact and almost (approximately) soft pre-Lindelöf spaces. Furthermore, we study under what conditions the four types of soft pre-compact (the four types of soft pre-Lindelöf) spaces are equivalent and discuss the relationships between enriched soft topological spaces and the initiated spaces in different cases. Finally, the eight introduced concepts are compared in relation with many soft topological notions such as soft pre-connectedness, soft pre-irresolute mappings and soft subspaces. The concepts presented in this study are fundamental for further researches and will open a way to improve more applications on soft topology.

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