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Abstract. Molodtsov initiated the concept of soft set theory as a general mathematical tool for dealing with uncertainties and imprecision about vague concepts. Decision making based on hybrid soft sets has found great importance due to their resourcefulness in real life problems. In this paper, we apply the adjustable approach of fuzzy soft set based decision making. Using concrete and illustrative examples, we present the adjustable approach to hesitant fuzzy soft multisets based decision making by using level soft sets of hesitant fuzzy soft multisets. We also discuss weighted hesitant fuzzy soft multiset and systematically applied it to decision making problems.

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1. Introduction

Set theory formulated by George Cantor is fundamental in virtually the whole of mathematics. However, in real life, most of the complicated problems faced in engineering, economics, environment, medical, management and social sciences have various levels of uncertainties and imprecision embedded in them. In an attempt to find solution to the problems of uncertainties and vagueness, researchers postulated a number of theories, such as theory of probability [22], interval mathematics [3, 14], fuzzy set theory [27], intuitionistic fuzzy set [4], rough set [21] and vague sets [13] but could not successfully proffer solutions to the problems of uncertainties due to the inadequacies of the parameterization tools. Molodtsov [19] in 1999 initiated soft set theory as a new mathematical tool for dealing with uncertainties and imprecision without underlying problems of parameterization. This makes soft set theory very convenient and easy to apply in theory and practice.

After the adventurous discovery of soft set by Molodtsov, Maji et al. [16, 17]] and Ali et al. [1] defined some basic terms of the theory such as equality of two soft sets, subset and super set of a soft set, complement of a soft set, null soft set, and absolute soft set with examples. Binary operations like AND, OR, union and intersection were also defined. De Morgan's laws and a number of results are verified in soft set theory context.

It is important to note that in classical soft set based decision making, the important notion is called choice value that precisely equals the number of good qualities or characteristics possessed by an object [16]. Therefore, it is simply reasonable to select the object with the maximum choice value as the optimal option. The situation becomes more complicated when we consider decision making that involves hesitant fuzzy soft multisets. The "induced fuzzy choice value" of an object is the sum total of all membership values with respect to different attributes or characteristics. Consequently, it does not represent the number of fair qualities possessed by that object.

To address this issue, instead of using choice values the Roy-Maji method [23] builds upon a series of new concepts, their method involves the construction of comparison table from the resultant fuzzy soft set and the optimal decision is taken on the maximum score computed from the comparison table. Neutrosophic soft set and its applications to decision making can be obtain [5, 7, 8]. Broumi et al. defined some new operations on intuitionistic fuzzy soft set [10] and over interval valued intuitionistic hesitant fuzzy set [11]. They also worked on fuzzy soft matrix based on reference function [9] and intuitionistic neutrosophic soft sets over rings [6].

In this paper, we study the adjustable approach to fuzzy soft set based decision making by Feng et al. [12] and extend it to hesitant fuzzy soft multiset, since the concept of choice values designed for standard soft set is not fit to solve decision making problems involving hesitant fuzzy soft multiset. The definitions of level soft sets of hesitant fuzzy soft multiset are derived from that of [12]. This research work is an extension work of Onyeozili et al. [20].

2. Preliminary concepts

2.1 Soft set

We first recall some basic notions in soft set theory. Let U be an initial universe set, E be a set of parameters or attributes with respect to U, $P(U)$ be the power set of U and $A \subseteq E$.

Definition 2.1.1 [19]. A pair $(F,\ A)$ is called a soft set over U, where F is a mapping given by $F : A \to P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U. For $x \in A$, $F(x)$ may be considered as the set of x-elements or as the set of x-approximate elements of the soft set $(F,\ A)$. The soft set $(F,\ A)$ can be represented as a set of ordered pairs as follows:

$$(F,\ A) = \{(x,\ F(x)) : x \in A,\ F(x) \in P(U)\}.$$

Example 2.1.1 Let $U = \{\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3, \mathrm{S}_4, \mathrm{S}_5, \mathrm{S}_6\}$ consisting of six students and $A = \{a_1,\ a_2,\ a_3\}$ be the set of parameters under consideration, where each parameter

e_i, i= 1, 2, 3 stands for, brilliant, average and healthy, respectively. In this case, we define a soft set means to point out brilliant students, average students and healthy students such that $F(a_1) = \{S_1, S_2, S_5\}$, $F(a_2) = \{S_3, S_4, S_6\}$, $F(a_3) = \{S_1, S_4, S_5, S_6\}$. The soft set (F, A) over U is thus given by:

$$(F, A) = \{(a_1, \{\ \mathrm{S}_1,\ \mathrm{S}_2, \mathrm{S}_5\}), (a_2, \{\ \mathrm{S}_3,\ \mathrm{S}_4, \mathrm{S}_6\}), (a_3, \{\ \mathrm{S}_1,\ \mathrm{S}_4, \mathrm{S}_5,\ \mathrm{S}_6\})\}.$$

2.2. Fuzzy soft set

Let U be an initial universe set and E be a set of parameters (which are fuzzy words or sentences involving fuzzy words). Let $P(U)$ denote the set of all fuzzy subsets of U, and $A \subseteq E$.

Definition 2.2.1 [18]. A pair $(\widetilde{\Gamma},\ A)$ is called a fuzzy soft set over U, where $\widetilde{\Gamma}$ is a mapping given by $\widetilde{\Gamma} : A \ \longrightarrow P(U)$. The mapping $\widetilde{\Gamma}$ is called fuzzy approximation function of the fuzzy set $(\widetilde{\Gamma},\ A)$ and the values $\widetilde{\Gamma}(x)$ are fuzzy subsets of U, $\forall x \in A$. Therefore, a fuzzy soft set $(\widetilde{\Gamma},\ A)$ over U can be represented by the set of ordered pairs

$$\left(\widetilde{\Gamma},\ A\right) = \left\{\left(x,\ \widetilde{\Gamma}\,(x)\right) : x \in A,\ \widetilde{\Gamma}(x) \in P(U)\right\}.$$

Example 2.2.1. Suppose that $U = \{h_1, h_2, h_3, h_4, h_5\}$ is a universe and $E = \{x_1, x_2, x_3, x_4\}$ is a set of parameters where $A = \{x_1, x_2, x_3\} \subseteq E$, $\widetilde{\Gamma}\,(x_1) = \left\{\frac{h_2}{0.8}, \frac{h_4}{0.6}\right\}$, $\widetilde{\Gamma}\,(x_2) = U$ and $\widetilde{\Gamma}\,(x_3) = \left\{\frac{h_1}{0.3}, \frac{h_4}{0.4}, \frac{h_5}{0.9}\right\}$. Then the fuzzy soft set $\left(\widetilde{\Gamma},\ A\right)$ is written as

$$\left(\widetilde{\Gamma},\ A\right) = \ \left\{\left(x_1, \left\{\frac{h_2}{0.8}, \frac{h_4}{0.6}\right\}\right), (x_2,\ U)\,, \left(x_3, \left\{\frac{h_1}{0.3}, \frac{h_4}{0.4}, \frac{h_5}{0.9}\right\}\right)\right\}.$$

2.3. Hesitant fuzzy set

Definition 2.3.1 [24]. A hesitant fuzzy set on U is in terms of a function that when applied to U returns a subset of [0, 1] which can be represented with the following mathematical symbol.

$$\tilde{A} = \ \{\langle u,\ h_{\tilde{A}}(u)\rangle : u \in U\}\,,$$

where $h_{\tilde{A}}(u)$ is a set of values in [0, 1], denoting the possible membership degrees of the element $u \in U$ to the set $\tilde{A}$. For convenience, we call $h_{\tilde{A}}(u)$ a hesitant fuzzy element and H the set of all hesitant fuzzy elements.

Definition 2.3.2 [26]. For a hesitant fuzzy element h, $S\,(h) = \ \left(\frac{1}{l(h)}\right) \sum_{\gamma \in h} \gamma$ is called the score function of h, where $l(h)$ is the number of values in h.

2.4. Hesitant fuzzy soft set

Definition 2.4.1 [25]. Let $\tilde{H}(U)$ be the set of all hesitant fuzzy sets in U; a pair $(\tilde{F},\ A)$ is called a hesitant fuzzy soft set over U, where $\tilde{F}$ is a mapping given by

$$\tilde{F} : A \ \longrightarrow \ \tilde{H}(U).$$

A hesitant fuzzy soft set is a mapping from a set of parameters to $\tilde{H}(U)$. It is a parameterized family of hesitant fuzzy subsets of U. For $e \in A$, $\tilde{F}(e)$ may be considered as the set of $e-$approximate elements of the hesitant fuzzy soft set $(\tilde{F},\ A)$.

Example 2.4.1. Suppose that $U = \{h_1,\ h_2, h_3, h_4, h_5, h_6\}$ is a set of houses and $A = \{e_1,\ e_2, e_3, e_4, e_5\}$ is a set of parameters, which stands for the parameters "cheap," "beautiful," "size," "location" and "surrounding environment," respectively. Then a hesitant fuzzy soft set $(\tilde{F},\ A)$ can describe the characteristics of the houses under hesitant fuzzy information. Consider

$$\begin{aligned}
\tilde{F}(e_1) &= \left\{\frac{h_1}{\{0.2,0.3\}}, \frac{h_2}{\{0.5,0.6\}}, \frac{h_3}{\{0.3\}}, \frac{h_4}{\{0.3,0.5\}}, \frac{h_5}{\{0.4,0.5\}}, \frac{h_6}{\{0.6,0.7\}}\right\},\\
\tilde{F}(e_2) &= \left\{\frac{h_1}{\{0.4,0.6,0.7\}}, \frac{h_2}{\{0.5,0.7,0.8\}}, \frac{h_3}{\{0.6,\ 0.8\}}, \frac{h_4}{\{0.7,0.9\}}, \frac{h_5}{\{0.3,0.4,0.5\}}, \frac{h_6}{\{0.3\}}\right\},\\
\tilde{F}(e_3) &= \left\{\frac{h_1}{\{0.2,0.4\}}, \frac{h_2}{\{0.6,0.7\}}, \frac{h_3}{\{0.8,0.9\}}, \frac{h_4}{\{0.3,0.5\}}, \frac{h_5}{\{0.4,0.6\}}, \frac{h_6}{\{0.7\}}\right\},\\
\tilde{F}(e_4) &= \left\{\frac{h_1}{\{0.3,0.5,0.6\}}, \frac{h_2}{\{0.2\}}, \frac{h_3}{\{0.5\}}, \frac{h_4}{\{0.6,0.7\}}, \frac{h_5}{\{0.5,0.6\}}, \frac{h_6}{\{0.8\}}\right\},\\
\tilde{F}(e_5) &= \left\{\frac{h_1}{\{0.6\}}, \frac{h_2}{\{0.2,0.3,0.5\}}, \frac{h_3}{\{0.5,0.7\}}, \frac{h_4}{\{0.2,0.4\}}, \frac{h_5}{\{0.5,0.7\}}, \frac{h_6}{\{0.3,0.5\}}\right\}.
\end{aligned}$$

We can view the hesitant fuzzy soft set $(\tilde{F},\ A)$ as consisting of the following collection of approximations.

$$\left(\tilde{F},\ A\right) = \left\{\begin{array}{c}
\left(e_1, \left(\left\{\frac{h_1}{\{0.2,0.3\}}, \frac{h_2}{\{0.5,0.6\}}, \frac{h_3}{\{0.3\}}, \frac{h_4}{\{0.3,0.5\}}, \frac{h_5}{\{0.4,0.5\}}, \frac{h_6}{\{0.6,0.7\}}\right\}\right)\right),\\
\left(e_2, \left(\left\{\frac{h_1}{\{0.4,0.6,0.7\}}, \frac{h_2}{\{0.5,0.7,0.8\}}, \frac{h_3}{\{0.6,\ 0.8\}}, \frac{h_4}{\{0.7,0.9\}}, \frac{h_5}{\{0.3,0.4,0.5\}}, \frac{h_6}{\{0.3\}}\right\}\right)\right),\\
\left(e_3, \left(\left\{\frac{h_1}{\{0.2,0.4\}}, \frac{h_2}{\{0.6,0.7\}}, \frac{h_3}{\{0.8,0.9\}}, \frac{h_4}{\{0.3,0.5\}}, \frac{h_5}{\{0.4,0.6\}}, \frac{h_6}{\{0.7\}}\right\}\right)\right),\\
\left(e_4, \left(\left\{\frac{h_1}{\{0.3,0.5,0.6\}}, \frac{h_2}{\{0.2\}}, \frac{h_3}{\{0.5\}}, \frac{h_4}{\{0.6,0.7\}}, \frac{h_5}{\{0.5,0.6\}}, \frac{h_6}{\{0.8\}}\right\}\right)\right),\\
\left(e_5, \left(\left\{\frac{h_1}{\{0.6\}}, \frac{h_2}{\{0.2,0.3,0.5\}}, \frac{h_3}{\{0.5,0.7\}}, \frac{h_4}{\{0.2,0.4\}}, \frac{h_5}{\{0.5,0.7\}}, \frac{h_6}{\{0.3,0.5\}}\right\}\right)\right)
\end{array}\right\}.$$

2.5. Soft multiset

Let $\{U_i : i \in I\}$ be a collection of universes such that $\bigcap_{i\in I} U_i = \varnothing$ and let $\{E_{U_i} : i \in I\}$ be a collection of sets of parameters. Let $U = \biguplus_{\boldsymbol{i\in I}} P\left(U_{\boldsymbol{i}}\right)$, where $P\left(U_i\right)$ denotes the power sets of U_i's, $E = \biguplus_{\boldsymbol{i\in I}} E_{U_i}$ and $A \subseteq E$.

Definition 2.5.1 [2]. A pair (F, A) is called a soft multiset over U, where F is a mapping given by $F : A \longrightarrow U$. In other words, a soft multiset over U is a parameterized family of subsets of U. For $\varepsilon \in A, F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft multiset (F, A). Based on the definition, any change in the order of the universes will produce a different soft multiset.

Example 2.5.1 Suppose that there are three universes U_1, U_2 and U_3. Let us consider a soft multiset (F, A) which describes the "attractiveness of houses", "cars" and "hotels" that Mr. X is considering for accommodation purchase, transportation purchase, and venue to hold a wedding celebration, respectively.
Let $U_1 = \{h_1, h_2, h_3, h_4, h_5, h_6\}, U_2 = \{c_1, c_2, c_3, c_4, c_5\}$ and $U_3 = \{v_1, v_2, v_3, v_4\}$

Let $E_U = \{E_{U_1}, E_{U_2}, E_{U_3}\}$ be a collection of sets of decision parameters related to the above universes, where

$$E_{U_1} = \left\{ \begin{array}{c} e_{U_1}, 1 = expensive,\ e_{U_1}, 2 = cheap,\ e_{U_1}, 3 = beautiful, \\ e_{U_1}, 4 = wooden,\ e_{U_1}, 5 = in\ green\ surroundings \end{array} \right\},$$

$$E_{U_2} = \left\{ \begin{array}{c} e_{U_2}, 1 = expensive,\ e_{U_2}, 2 = cheap,\ e_{U_2}, 3 = model\ 2000\ and\ above, \\ e_{U_2}, 4 = black,\ e_{U_2}, 5 = made\ in\ Japan,\ e_{U_2}, 6 = Made\ in\ Malaysia \end{array} \right\},$$

$$E_{U_3} = \left\{ \begin{array}{c} e_{U_3}, 1 = expensive,\ e_{U_3}, 2 = cheap,\ e_{U_3}, 3 = majestic, \\ e_{U_3}, 4 = in\ Kuala\ Lumpur,\ e_{U_3}, 5 = in\ Kajang \end{array} \right\}.$$

Let $U = \biguplus_{i=1}^{3} P(U_{\boldsymbol{i}})$, $\quad E = \biguplus_{i=1}^{3} E_{U_i}$ and $A \subseteq E$, such that $A =$

$\{a_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1)$, $a_2 = (e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1)$, $a_3 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1)$, a_4 $= (e_{U_1}, 5, e_{U_2}, 4, e_{U_3}, 2)$, $a_5 = (e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 3)$, $a_6 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 2)$, $a_7 = (e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 1)$, $a_8 = (e_{U_1}, 1, e_{U_2}, 3, e_{U_3}, 2)\}$.

Suppose that

$$\begin{aligned}
&F(a_1) = (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\}), \\
&F(a_2) = (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2\}), \\
&F(a_3) = (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \varnothing), \\
&F(a_4) = (\{h_1, h_4, h_6\}, \varnothing, \{v_1, v_4\}), \\
&F(a_5) = (\{h_1, h_4\}, \{c_1, c_3\}, \{v_1\}), \\
&F(a_6) = (\{h_1, h_4, h_5\}, \{c_1, c_3\}, U_3), \\
&F(a_7) = (\{h_1, h_4\}, \varnothing, \{v_3\}), \\
&F(a_8) = (\{h_2, h_3, h_6\}, \{c_1, c_3\}, \{v_1, v_4\}).
\end{aligned}$$

We can view the soft multiset (F, A) as consisting of the following collection of approximations:

$(F, A) = \{(a_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})), (a_2, (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2\})), (a_3, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \varnothing)), (a_4, (\{h_1, h_4, h_6\}, \varnothing, \{v_1, v_4\})), (a_5, (\{h_1, h_4\}, \{c_1, c_3\}, \{v_1\})), (a_6, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, U_3)), (a_7, (\{h_1, h_4\}, \varnothing, \{v_3\})), (a_8, (\{h_2, h_3, h_6\}, \{c_1, c_3\}, \{v_1, v_4\}))\}$.

3. Hesitant fuzzy soft multiset

Let $\{U_i : i \in I\}$ be a collection of universes such that $\bigcap_{i \in I} U_i = \varnothing$ and let $\{E_{U_i} : i \in I\}$ be a collection of sets of parameters or attributes related to the universes. Let $U = \biguplus_{\boldsymbol{i \in I}} HFS(U_i)$, where $HFS\ (U_i)$ denotes the set of all hesitant fuzzy submultisets of the U_i's, $E = \biguplus_{\boldsymbol{i \in I}} E_{U_i}$ and $A \subseteq E$.

Definition 3.1 [20]. A pair $(\widetilde{\Gamma},\ A)$ is called a hesitant fuzzy soft multiset over U, where $\widetilde{\Gamma}$ is a mapping given by $\widetilde{\Gamma} : A \longrightarrow U$. In other words, a hesitant fuzzy soft multiset over U is a parameterized family of hesitant fuzzy submultisets of U. For $e \in A$, $\widetilde{\Gamma}(e)$ may be considered as the set of $e-$approximate elements of the hesitant fuzzy soft Multiset $(\widetilde{\Gamma},\ A)$. Based on the above definition, any change in the order of universes will produce a different hesitant fuzzy soft Multiset.

Example 3.1 Suppose that there are three universes U_1, U_2 and U_3. Suppose that Mr. X has a budget to purchase a house, a car and rent a venue to hold a wedding celebration. Let us consider a hesitant fuzzy soft Multiset $(\widetilde{\Gamma},\ A)$ which describes the "houses," "Cars"and "hotels" that Mr. X with enough budget is considering for accommodation, transportation and venue to hold a wedding celebration with hesitant fuzzy elements, respectively.

Let $U_1 = \{h_1, h_2,\ h_3, h_4\}$, $U_2 = \{c_1, c_2,\ c_3\}$ and $U_3 = \{v_1, v_2\}$. Let $\{E_{U_1},\ E_{U_2}, E_{U_3}\}$ be a collection of sets of decision parameters or attributes related to the above universes, where

$$E_{U_1} = \left\{ \begin{array}{c} e_{U_1},\ 1 = expensive,\ \ e_{U_1},\ 2 = cheap,\ \ e_{U_1},\ 3 = beautiful, \\ e_{U_1},\ 4 = in\ green\ surrounding \end{array} \right\},$$

$$E_{U_2} = \{e_{U_2},\ 1 = expensive,\ e_{U_2},\ 2 = cheap,\ e_{U_2},\ 3 = sporty\},$$

$$E_{U_3} = \{e_{U_3},\ 1 = expensive, e_{U_3},\ 2 = cheap, e_{U_3},\ 3 = in\ Kuala\ Lumpur, e_{U_3}, 4 = Majestic\}.$$

Let $U = \biguplus_{i=1}^{3} HFS(U_i)$, $E = \biguplus_{i=1}^{3} E_{U_i}$ and $A \subseteq E$, such that $A =$

$$\{a_1 = (e_{U_1}, 1, e_{U_2}, 1,\ e_{U_3}, 1),\ a_2 = (e_{U_1}, 1, e_{U_2}, 2,\ e_{U_3}, 1),\ a_3 = (e_{U_1}, 2, e_{U_2}, 2,\ e_{U_3}, 1),$$
$$a_4 = (e_{U_1}, 4, e_{U_2}, 3,\ e_{U_3}, 2),\ a_5 = (e_{U_1}, 4, e_{U_2}, 2,\ e_{U_3}, 2),\ \ a_6 = (e_{U_1}, 2, e_{U_2}, 2,\ e_{U_3}, 2)\}$$

Suppose that $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ where $a_1 = expensive$, $a_2 = cheap$, $a_3 = beautiful$, $a_4 = majestic$, $a_5 = in\ Kuala\ Lumpur$, $a_6 = in\ green\ surrounding$, then

$$\widetilde{\Gamma}(a_1) = \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.5,0.8,0.3\}}, \frac{h_2}{\{0.4,0.5\}}, \frac{h_3}{\{0.9,0.7,0.5\}}, \frac{h_4}{\{0.6,0.4\}} \right\}, \\ \left\{ \frac{c_1}{\{0.5,0.3\}}, \frac{c_2}{\{0.7,0.6,0.8\}}, \frac{c_3}{\{0.9,0.2,0.3\}} \right\}, \left\{ \frac{v_1}{\{0.7,0.5,0.8\}}, \frac{v_2}{\{0.9,0.7,0.8\}} \right\} \end{array} \right),$$

$$\widetilde{\Gamma}(a_2) = \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.7,0.7,0.9\}}, \frac{h_2}{\{0.3,0.6,0.1\}}, \frac{h_3}{\{0.8,0.6,0.2\}}, \frac{h_4}{\{0.3,0.2,\ 0.2\}} \right\}, \\ \left\{ \frac{c_1}{\{0.7,0.6,0.2\}}, \frac{c_2}{\{0.2,0.4,0.8\}}, \frac{c_3}{\{0.7,0.9,0.8\}} \right\}, \left\{ \frac{v_1}{\{0.8,0.9,0.6\}}, \frac{v_2}{\{0.5,0.6,0.9\}} \right\} \end{array} \right),$$

$$\widetilde{\Gamma}(a_3) = \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.2,0.3,0.7\}}, \frac{h_2}{\{0.7,0.9,0.2\}}, \frac{h_3}{\{0.2,0.3,0.4\}}, \frac{h_4}{\{0.6,0.5,0.2\}} \right\}, \\ \left\{ \frac{c_1}{\{0.7,0.8,0.9\}}, \frac{c_2}{\{0.2,0.4,0.5\}}, \frac{c_3}{\{0.9,0.7,0.2\}} \right\}, \left\{ \frac{v_1}{\{0.9,0.8\}}, \frac{v_2}{\{0.4,0.2,0.5\}} \right\} \end{array} \right),$$

$$\widetilde{\Gamma}(a_4) = \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.5,0.4,0.6\}}, \frac{h_2}{\{0.4,0.8,0.9\}}, \frac{h_3}{\{0.9,0.8,0.1\}}, \frac{h_4}{\{0.5,0.2,0.3\}} \right\}, \\ \left\{ \frac{c_1}{\{0.2,0.8,0.1\}}, \frac{c_2}{\{0.6,0.2,0.8\}}, \frac{c_3}{\{0.1,0.2,0.7\}} \right\}, \left\{ \frac{v_1}{\{0.8,0.9}, \frac{v_2}{0.2,0.4\}} \right\} \end{array} \right),$$

$$\widetilde{\Gamma}(a_5) = \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.2,0.3,0.7\}}, \frac{h_2}{\{0.3,0.2,0.5\}}, \frac{h_3}{\{0.7,0.8\}}, \frac{h_4}{\{0.7,0.8,0.9\}} \right\}, \\ \left\{ \frac{c_1}{\{0.9,0.2\}}, \frac{c_2}{\{0.7,0.8,0.9\}}, \frac{c_3}{\{0.2,0.7\}} \right\}, \left\{ \frac{v_1}{\{0.5,0.7\}}, \frac{v_2}{\{0.7,0.8,0.9\}} \right\} \end{array} \right),$$

$$\widetilde{\Gamma}\left(a_6\right)=\left(\begin{array}{c}\left\{\frac{h_1}{\{0.2,0.4,0.5\}},\frac{h_2}{\{0.9,0.2\}},\frac{h_3}{\{0.4,0.6,0.7\}},\frac{h_4}{\{0.9,0.8,0.3\}}\right\},\\ \left\{\frac{c_1}{\{0.4,0.5\}},\frac{c_2}{\{0.9,0.2,0.3\}},\frac{c_3}{\{0.4,0.6\}}\right\},\left\{\frac{v_1}{\{0.9,0.2\}},\frac{v_2}{\{0.9,0.7,0.2\}}\right\}\end{array}\right).$$

Therefore, we can view the hesitant fuzzy soft Multiset $(\widetilde{\Gamma},\ A)$ as consisting of the following collection of approximations: $\left(\widetilde{\Gamma},\ A\right)=$

$$\left\{\begin{array}{c}\left(a_1,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.5,0.8,0.3\}},\frac{h_2}{\{0.4,0.5\}},\frac{h_3}{\{0.9,0.7,0.5\}},\frac{h_4}{\{0.6,0.4\}}\right\},\\ \left\{\frac{c_1}{\{0.5,0.3\}},\frac{c_2}{\{0.7,0.6,0.8\}},\frac{c_3}{\{0.9,0.2,0.3\}}\right\},\left\{\frac{v_1}{\{0.7,0.5,0.8\}},\frac{v_2}{\{0.9,0.7,0.8\}}\right\}\end{array}\right)\right),\\
\left(a_2,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.7,0.7,0.9\}},\frac{h_2}{\{0.3,0.6,0.1\}},\frac{h_3}{\{0.8,0.6,0.2\}},\frac{h_4}{\{0.3,0.2,\ 0.2\}}\right\},\\ \left\{\frac{c_1}{0.7,0.6,0.2},\frac{c_2}{0.2,0.4,0.8},\frac{c_3}{0.7,0.9,0.8}\right\},\left\{\frac{v_1}{\{0.8,0.9,0.6\}},\frac{v_2}{\{0.5,0.6,0.9\}}\right\}\end{array}\right)\right),\\
\left(a_3,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.2,0.3,0.7\}},\frac{h_2}{\{0.7,0.9,0.2\}},\frac{h_3}{\{0.2,0.3,0.4\}},\frac{h_4}{\{0.6,0.5,0.2\}}\right\},\\ \left\{\frac{c_1}{0.7,0.8,0.9},\frac{c_2}{0.2,0.4,0.5},\frac{c_3}{0.9,0.7,0.2}\right\},\left\{\frac{v_1}{0.9,0.8},\frac{v_2}{0.4,0.2,0.5}\right\}\end{array}\right)\right),\\
\left(a_4,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.5,0.4,0.6\}},\frac{h_2}{\{0.4,0.8,0.9\}},\frac{h_3}{\{0.9,0.8,0.1\}},\frac{h_4}{\{0.5,0.2,0.3\}}\right\},\\ \left\{\frac{c_1}{\{0.2,0.8,0.1\}},\frac{c_2}{\{0.6,0.2,0.8\}},\frac{c_3}{\{0.1,0.2,0.7\}}\right\},\left\{\frac{v_1}{\{0.8,0.9},\frac{v_2}{0.2,0.4\}}\right\}\end{array}\right)\right),\\
\left(a_5,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.2,0.3,0.7\}},\frac{h_2}{\{0.3,0.2,0.5\}},\frac{h_3}{\{0.7,0.8\}},\frac{h_4}{\{0.7,0.8,0.9\}}\right\},\\ \left\{\frac{c_1}{\{0.9,0.2\}},\frac{c_2}{\{0.7,0.8,0.9\}},\frac{c_3}{\{0.2,0.7\}}\right\},\ \left\{\frac{v_1}{\{0.5,0.7\}},\frac{v_2}{\{0.7,0.8,0.9\}}\right\}\end{array}\right)\right),\\
\left(a_6,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.2,0.4,0.5\}},\frac{h_2}{\{0.9,0.2\}},\frac{h_3}{\{0.4,0.6,0.7\}},\frac{h_4}{\{0.9,0.8,0.3\}}\right\},\\ \left\{\frac{c_1}{\{0.4,0.5\}},\frac{c_2}{\{0.9,0.2,0.3\}},\frac{c_3}{\{0.4,0.6\}}\right\},\left\{\frac{v_1}{\{0.9,0.2\}},\frac{v_2}{\{0.9,0.7,0.2\}}\right\}\end{array}\right)\right)\end{array}\right\}.$$

Each approximation has two parts, namely a predicate name and an approximate value set. We can logically explain the above example as follows. We know that $a_1=(e_{U_1},1,e_{U_2},1,\ e_{U_3},1)$, where $e_{U_1},\ 1=\textit{expensive houses}$, $e_{U_2},\ 1=\textit{expensive car}$, and $e_{U_3},\ 1=\textit{expensive venue}$.

$$\widetilde{\Gamma}\left(a_1\right)=\left(\begin{array}{c}\left\{\frac{h_1}{\{0.5,0.8,0.3\}},\frac{h_2}{\{0.4,0.5\}},\frac{h_3}{\{0.9,0.7,0.5\}},\frac{h_4}{\{0.6,0.4\}}\right\},\\ \left\{\frac{c_1}{\{0.5,0.3\}},\frac{c_2}{\{0.7,0.6,0.8\}},\frac{c_3}{\{0.9,0.2,0.3\}}\right\},\left\{\frac{v_1}{\{0.7,0.5,0.8\}},\frac{v_2}{\{0.9,0.7,0.8\}}\right\}\end{array}\right).$$

We can see that the membership score functional value of h_1, h_2 and h_4 is 0.5, so these houses are fairly expensive for Mr X; also we can see that the membership score functional value of h_3 is 0.7, this means that the house h_3 is expensive. Since the first set is concerning expensive houses, then we can explain the second set as follows: the membership score functional value for c_1 is 0.4, this means this car is not expensive for him. The membership score functional value of c_2 is 0.7, so this car is expensive (however, this car may not be expensive, if the first set is concerning cheap houses), also we can see that the membership score functional value of c_3 is 0.4, this means that this car is not expensive for him. Now, since the first set is concerning expensive houses and the second set is concerning expensive cars, then we can also explain the third set as follows: since the membership score functional value for v_1 is 0.6, then this venue is quite expensive, also we can see that, the membership score functional value of v_2 is 0.8, so this venue is expensive (this venue may not be expensive, if the first set is concerning cheap houses or / and the second set is concerning cheap cars). Depending on the previous explanation we can say the following:

If $\left\{\frac{h_1}{\{0.5,0.8,0.3\}}, \frac{h_2}{\{0.4,0.5\}}, \frac{h_3}{\{0.9,0.7,0.5\}}, \frac{h_4}{\{0.6,0.4\}}\right\}$ is a hesitant fuzzy set of expensive houses, then the hesitant fuzzy set of relatively expensive cars is $\left\{\frac{c_1}{\{0.5,0.3\}}, \frac{c_2}{\{0.7,0.6,0.8\}}, \frac{c_3}{\{0.9,0.2,0.3\}}\right\}$, and if $\left\{\frac{h_1}{\{0.5,0.8,0.3\}}, \frac{h_2}{\{0.4,0.5\}}, \frac{h_3}{\{0.9,0.7,0.5\}}, \frac{h_4}{\{0.6,0.4\}}\right\}$ is the hesitant fuzzy set of expensive houses and $\left\{\frac{c_1}{\{0.5,0.3\}}, \frac{c_2}{\{0.7,0.6,0.8\}}, \frac{c_3}{\{0.9,0.2,0.3\}}\right\}$ is the hesitant fuzzy set of relatively expensive cars, then the hesitant fuzzy set of relatively expensive venue is $\left\{\frac{v_1}{\{0.7,0.5,0.8\}}, \frac{v_2}{\{0.9,0.7,0.8\}}\right\}$. It is obvious that the relation in a hesitant fuzzy soft multiset is a conditional relation.

4. Application of hesitant fuzzy soft multiset in decision making problems

Like most of the decision making problems, hesitant fuzzy soft multiset based decision making involves the evaluation of all the objects which are decision options. Most of these problems are essentially humanistic and therefore subjective in nature (that is based on human understanding and ability to see). In general, there actually does not exist a unique or uniform criterion for the evaluation of decision options.

4.1 Level soft sets of hesitant fuzzy soft multiset

In this subsection, we present an approach to hesitant fuzzy soft multiset based decision making problems. This is based on the following concept called level soft set.

Definition 4.1.1. Let $\varpi = \left(\widetilde{\Gamma},\ A\right)$ be a hesitant fuzzy soft multiset over U, where $A \subseteq E$ and E is the parameter set. For $t \in [0,\ 1]$, the $t-level$ soft set of the hesitant fuzzy soft multiset ϖ is a crisp soft set $L(\varpi; t) = (\Gamma_t,\ A)$ defined by:

$$\Gamma_t(a) = \ L\left(\widetilde{\Gamma}(a); t\right) = \{x \in U : \widetilde{\Gamma}(a)(x) \geq t\ \}, \text{ for all } a \in A.$$

In the definition above, $t \in [0,\ 1]$ can be viewed as a given threshold on membership values. For real life applications of hesitant fuzzy soft multiset based decision making, usually these thresholds are chosen in advance by the decision makers and represent their requirements on membership levels. In the definition of $t-level$ soft set, the level (or threshold) assigned to each parameter is always a constant value $t \in [0,\ 1]$. But in some decision making problems, it is possible for decision makers to impose different thresholds on different decision parameters. To cope with such problems, we use a function rather than a constant number as the threshold on membership values.

Definition 4.1.2. Let $\varpi = \left(\widetilde{\Gamma},\ A\right)$ be a hesitant fuzzy soft multiset over U, where $A \subseteq E$ and E is the parameter set. Let $\lambda : A \to [0, 1]$ be a fuzzy set in A which is called a threshold fuzzy set. The level soft set of the hesitant fuzzy soft multiset ϖ with respect to the fuzzy set λ is a crisp soft set $L(\varpi; \lambda) = (\Gamma_\lambda,\ A)$ defined by:

$$\Gamma_\lambda(a) = \quad L\left(\widetilde{\Gamma}(a); \lambda(a)\right) = \{x \in U : \widetilde{\Gamma}(a)(x) \geq \lambda(a)\ \},\ \text{for}\ \ \text{all}\ a \in A.$$

It is obvious that a level soft set with respect to a fuzzy set generalize $t-$level soft sets by substituting a function on the parameter set A, namely a fuzzy set $\lambda : A \to [0, 1]$, for a constant $t \in [0,\ 1]$.

Definition 4.1.3 (The mid-level soft set of a hesitant fuzzy soft multiset). Let $\varpi = \left(\widetilde{\Gamma},\ A\right)$ be a hesitant fuzzy soft multiset over U, where $A \subseteq E$ and E is the parameter set. Based on the hesitant fuzzy soft multiset $\varpi = \left(\widetilde{\Gamma},\ A\right)$, we can define a fuzzy set $\widetilde{mid}_\varpi : A\ \to [0,\ 1]$ by:

$$\widetilde{mid}_\varpi(a) = \ \frac{1}{|U|}\ \sum_{x \in U} \widetilde{\Gamma}(a)(x),$$

for all $a \in A$. The fuzzy set $\widetilde{mid}_\varpi$ is called the mid-threshold of the fuzzy soft multiset ϖ. In addition, the level soft set of ϖ with respect to the mid-threshold fuzzy set $\widetilde{mid}_\varpi$, namely $L(\varpi; \widetilde{mid}_\varpi)$ is called the mid-level soft set of ϖ and simply denoted by $L(\varpi; mid)$. In what follows the mid-level decision rule will mean using the mid-threshold and considering the mid-level soft set in hesitant fuzzy soft multiset based decision making.

Definition 4.1.4 (The Top-level soft set of a fuzzy soft multiset). Let $\varpi = \left(\widetilde{\Gamma},\ A\right)$ be a hesitant fuzzy soft multiset over U, where $A \subseteq E$ and E is the parameter set. Based on the hesitant fuzzy soft multiset $\varpi = \left(\widetilde{\Gamma},\ A\right)$, we can define a fuzzy set $\widetilde{max}_\varpi : A\ \to [0,\ 1]$ by:

$$\widetilde{max}_\varpi(a) = \ \max_{x \in U} \widetilde{\Gamma}(a)(x),$$

for all $a \in A$. The fuzzy set $\widetilde{max}_\varpi$ is called the max-threshold of the hesitant fuzzy soft multiset ϖ. In addition, the level soft set of ϖ with respect to the max-threshold $\widetilde{max}_\varpi$, namely $L(\ \varpi,\ \widetilde{max}_\varpi)$ is called the *top-level decision rule* will mean using *the max-threshold* and considering the top-level soft set in hesitant fuzzy soft multiset based decision making.

Algorithm 1

(1) Input the hesitant fuzzy soft multiset $\left(\widetilde{\Gamma},\ A\right)$.

(2) Compute the induced fuzzy soft multiset $\Delta_{\tilde{F}} = \left(\widetilde{\Gamma},\ A\right)$.

(3) Input a threshold fuzzy set $\lambda : A \to [0,\ 1]$, (or give a threshold value $t \in [0,\ 1]$; or choose the mid-level decision rule; or choose the top-level decision rule) for decision making.
(4) Compute the level soft set $L(\Delta_{\tilde{F}}; \lambda\ )$ of $\Delta_{\tilde{F}}$ with respect to the threshold fuzzy set λ (or the $t-$level soft set $L(\Delta_{\tilde{F}}; t\ )$; or the mid-level soft set $L(\Delta_{\tilde{F}}; mid\ )$ or the top-level soft set $L(\Delta_{\tilde{F}}; max\ )$).
(5) Present the level soft set $L(\Delta_{\tilde{F}}; \lambda\ )$(or $L\,(\Delta_{\tilde{F}}; t\ )$; or $L(\Delta_{\tilde{F}}; mid)$ or $L(\Delta_{\tilde{F}}; max)$) in tabular form and compute the choice value c_i for all i.
(6) The optimal decision is to select o_k if $c_k = max_i c_i$, from each U_i.
(7) If there are more than one k, then any one of o_k may be chosen, from each U_i.

Remark: In order to get a unique optimal choice according to the algorithm, the decision makers can go back to the third step and change the threshold (or decision criteria) in case there exist more than one optimal choice that can be obtained in the last step. Moreover, the final optimal decision can be adjusted according to the decision maker's preferences.

TABLE 4.1. Tabular representation of the hesitant fuzzy soft multiset $(\widetilde{\Gamma}, A)$ in Example 3.1

U/A	a_1	a_2	a_3	a_4	a_5	a_6
h_1	$\{0.5, 0.8, 0.3\}$	$\{0.7, 0.7, 0.9\}$	$\{0.2, 0.3, 0.7\}$	$\{0.5, 0.4, 0.6\}$	$\{0.2, 0.3, 0.7\}$	$\{0.2, 0.4, 0.5\}$
h_2	$\{0.4, 0.5\}$	$\{0.3, 0.6, 0.1\}$	$\{0.7, 0.9, 0.2\}$	$\{0.4, 0.8, 0.9\}$	$\{0.3, 0.2, 0.5\}$	$\{0.9, 0.2\}$
h_3	$\{0.9, 0.7, 0.5\}$	$\{0.8, 0.6, 0.2\}$	$\{0.2, 0.3, 0.4\}$	$\{0.9, 0.8, 0.1\}$	$\{0.7, 0.8\}$	$\{0.4, 0.6, 0.7\}$
h_4	$\{0.6, 0.4\}$	$\{0.3, 0.2, 0.2\}$	$\{0.6, 0.5, 0.2\}$	$\{0.5, 0.2, 0.3\}$	$\{0.7, 0.8, 0.9\}$	$\{0.9, 0.8, 0.3\}$
c_1	$\{0.5, 0.3\}$	$\{0.7, 0.6, 0.2\}$	$\{0.7, 0.8, 0.9\}$	$\{0.2, 0.8, 0.1\}$	$\{0.9, 0.2\}$	$\{0.4, 0.5\}$
c_2	$\{0.7, 0.6, 0.8\}$	$\{0.2, 0.4, 0.8\}$	$\{0.2, 0.4, 0.5\}$	$\{0.6, 0.2, 0.8\}$	$\{0.7, 0.8, 0.9\}$	$\{0.9, 0.2.0.3\}$
c_3	$\{0.9, 0.2, 0.3\}$	$\{0.7, 0.9, 0.8\}$	$\{0.9, 0.7, 0.2\}$	$\{0.1, 0.2, 0.7\}$	$\{0.2, 0.7\}$	$\{0.4, 0.6\}$
v_1	$\{0.7, 0.5, 0.8\}$	$\{0.8, 0.9, 0.6\}$	$\{0.9, 0.8\}$	$\{0.8, 0.9\}$	$\{0.5, 0.7\}$	$\{0.9, 0.2\}$
v_2	$\{0.9, 0.7, 0.8\}$	$\{0.5, 0.6, 0.9\}$	$\{0.4, 0.2, 0.5\}$	$\{0.2, 0.4\}$	$\{0.7, 0.8, 0.9\}$	$\{0.9, 0.7, 0.2\}$

$\widetilde{mid}_{\boldsymbol{\Delta}_{\tilde{F}}} = \{(a_1,\ 0.58),\ (a_2,\ 0.56),\ (a_3,\ 0.52),\ (a_4,\ 0.5),\ (a_5,\ 0.61),\ (a_6,\ 0.53)\}$. Furthermore, we can obtain the mid-level soft set $L(\Delta_{\tilde{F}}; mid)$ of $\Delta_{\tilde{F}}$ whose tabular form is in Table 4.3, CV= $Choice\ Values$ (c_i).

From Table 4.3, the maximum choice value $c_3 = 4$, from U_1, $c_6 = 3$, from U_2 and $c_8 = 5$, from U_3. Therefore, the optimum decision is for Mr. X to select house h_3, car c_2 and venue v_1 for his wedding celebration.

Consider Example 3.1 with induced tabular representation in Table 4.2

TABLE 4.2. Tabular representation of the induced fuzzy soft multiset $\Delta_{\widetilde{F}} = (\widetilde{\Gamma}, A)$

U/A	a_1	a_2	a_3	a_4	a_5	a_6
h_1	0.53	0.77	0.4	0.5	0.4	0.37
h_2	0.45	0.33	0.6	0.7	0.33	0.55
h_3	0.7	0.53	0.3	0.6	0.75	0.57
h_4	0.5	0.23	0.43	0.33	0.8	0.67
c_1	0.4	0.5	0.8	0.37	0.55	0.45
c_2	0.7	0.47	0.37	0.53	0.8	0.47
c_3	0.47	0.8	0.6	0.33	0.45	0.5
v_1	0.67	0.77	0.85	0.85	0.6	0.55
v_2	0.8	0.67	0.37	0.3	0.8	0.6

TABLE 4.3. Tabular representation of mid-level soft set $L(\Delta_{\widetilde{F}}; mid)$ with choice values (c_i)

U/A	a_1	a_2	a_3	a_4	a_5	a_6	CV
h_1	0	1	0	1	0	0	2
h_2	0	0	1	1	0	1	3
h_3	1	0	0	1	1	1	4
h_4	0	0	0	0	1	1	2
c_1	0	0	1	0	0	0	1
c_2	1	0	0	1	1	0	3
c_3	0	1	1	0	0	0	2
v_1	1	1	1	1	0	1	5
v_2	1	1	0	0	1	1	4

$\widetilde{max}_{\Delta_{\widetilde{F}}} = \{(a_1,\ 0.8),\ (a_2,\ 0.77),\ (a_3,\ 0.85),\ (a_4,\ 0.85),\ (a_5,\ 0.8),\ (a_6,\ 0.67)\}$. Furthering we can obtain the top-level soft set $L(\Delta_{\tilde{F}}; max)$ of $\Delta_{\tilde{F}}$ whose tabular form is in Table 4.4, CV= $Choice\ Values\ (c_i)$.

TABLE 4.4. Tabular representation of top-level soft set $L(\Delta_{\widetilde{F}}; max)$ with choice value

U/A	a_1	a_2	a_3	a_4	a_5	a_6	CV
h_1	0	1	0	0	0	0	1
h_2	0	0	0	0	0	0	0
h_3	0	0	0	0	0	0	0
h_4	0	0	0	0	1	1	2
c_1	0	0	0	0	0	0	0
c_2	0	0	0	0	1	0	1
c_3	0	0	0	0	0	0	0
v_1	1	1	1	1	0	0	3
v_2	1	0	0	0	1	0	2

From Table 4.4, it follows that, the maximum choice value from U_1 is $c_4 = 2$, from U_2 is $c_6 = 1$ and from U_3 is $c_8 = 3$. Therefore, the optimal decision is for Mr. X is to select house h_4, car c_2 and venue v_1 for his wedding celebration.

Example 4.2. Suppose that there are two universes U_1 and U_2, let us consider a hesitant fuzzy soft multiset $\left(\widetilde{\Gamma},\ A\right)$ which describes the attractiveness of "cloths" and "shoes" that Mrs Y is considering to purchase in a boutique to wear for a job interview with hesitant fuzzy elements, respectively. Let $U_1 = \{c_1, c_2, c_3, c_4\}$ be a set of cloths and $U_2 = \{s_1, s_2, s_3\}$ be a set of shoes under consideration. Let $E_U = \{E_{U_1},\ E_{U_2}\}$ be a collection of sets of attribute or decision parameters related to the above universes, where

$$E_{U_1} = \ \{e_{U_1}, 1 = expensive, e_{U_1}, 2 = cheap, e_{U_1}, 3 = beautiful\ \},$$
$$E_{U_2} = \ \{e_{U_2}, 1 = expensive, e_{U_2}, 2 = made\ in\ Italy, e_{U_2}, 3 = black, e_{U_2}, 4 = high\ hill\ \}.$$

Let $U = \biguplus_{i=1}^{2} P(U_i)$, $E = \biguplus_{i=1}^{2} E_{U_i}$ and $A \subseteq E$ such that
$A = \{a_1 = (e_{U_1}, 1, e_{U_2}, 1)\,, a_2 = (e_{U_1}, 2, e_{U_2}, 2)\,, a_3 = (e_{U_1}, 2, e_{U_2}, 3)\,,\quad a_4 = (e_{U_1}, 3, e_{U_2}, 2)\}$

Suppose that, $a_1 = expensive$, $a_2 = black$, $a_3 = beautiful$, $\ a_4 = madeinItaly$

$$\widetilde{\Gamma}\left(a_1\right) = \left(\left\{\frac{c_2}{\{0.5, 0.4, 0.7\}}, \frac{c_3}{\{0.4, 0.8, 0.6\}}\right\}, \left\{\frac{s_1}{\{0.9, 0.3, 0.7\}}, \frac{s_3}{\{0.4, 0.8, 0.9\}}\right\}\right),$$
$$\widetilde{\Gamma}\left(a_2\right) = \left(\left\{\frac{c_1}{\{0.3, 0.6, 0.4\}}, \frac{c_4}{\{0.5, 0.6, 0.8\}}\right\}, \left\{\frac{s_2}{\{0.4, 0.9, 0.3\}}, \frac{s_3}{\{0.7, 0.2, 0.4\}}\right\}\right),$$
$$\widetilde{\Gamma}\left(a_3\right) = \left(\left\{\frac{c_1}{\{0.3, 0.5, 0.6\}}, \frac{c_3}{\{0.2, 0.3, 0.8\}}, \frac{c_4}{\{0.8, 0.1, 0.7\}}\right\}, \left\{\frac{s_2}{\{0.9, 0.5, 0.6\}}\right\}\right),$$
$$\widetilde{\Gamma}\left(a_4\right) = \left(\left\{\frac{c_3}{\{0.9, 0.6, 0.5\}}\right\}, \left\{\frac{s_1}{\{0.5, 0.6, 0.3\}}, \frac{s_2}{\{0.4, 0.6, 0.5\}}, \frac{s_3}{\{0.9, 0.3, 0.6\}}\right\}\right).$$
$$\left(\widetilde{\Gamma},\ A\right) = \left\{\begin{array}{l} \left(a_1, \left(\left\{\frac{c_2}{\{0.5,0.4,0.7\}}, \frac{c_3}{\{0.4,0.8,0.6\}}\right\}, \left\{\frac{s_1}{\{0.9,0.3,0.7\}}, \frac{s_3}{\{0.4,0.8,0.9\}}\right\}\right)\right), \\ \left(a_2, \left(\left\{\frac{c_1}{\{0.3,0.6,0.4\}}, \frac{c_4}{\{0.5,0.6,0.8\}}\right\}, \left\{\frac{s_2}{\{0.4,0.9,0.3\}}, \frac{s_3}{\{0.7,0.2,0.4\}}\right\}\right)\right), \\ \left(a_3, \left(\left\{\frac{c_1}{\{0.3,0.5,0.6\}}, \frac{c_3}{\{0.2,0.3,0.8\}}, \frac{c_4}{\{0.8,0.1,0.7\}}\right\}, \left\{\frac{s_2}{\{0.9,0.5,0.6\}}\right\}\right)\right), \\ \left(a_4, \left(\left\{\frac{c_3}{\{0.9,0.6,0.5\}}\right\}, \left\{\frac{s_1}{\{0.5,0.6,0.3\}}, \frac{s_2}{\{0.4,0.6,0.5\}}, \frac{s_3}{\{0.9,0.3,0.6\}}\right\}\right)\right) \end{array}\right\}.$$

Taking $t = 0.55$, we obtain the 0.55-level sets of the fuzzy set $\widetilde{\Gamma}(a_1)$, $\widetilde{\Gamma}(a_2)$, $\widetilde{\Gamma}(a_3)$ and $\widetilde{\Gamma}(a_4)$ as follows:

$$L\left(\widetilde{\Gamma}\left(a_1\right), 0.55\right) = \{\{c_3\}\,, \{s_1, s_3\}\},$$
$$L\left(\widetilde{\Gamma}\left(a_2\right), 0.55\right) = \{\{c_4\}\},$$
$$L\left(\widetilde{\Gamma}\left(a_3\right), 0.55\right) = \{\{s_2\}\},$$
$$L\left(\widetilde{\Gamma}\left(a_4\right), 0.55\right) = \{\{c_3\}\,, \{s_3\}\}.$$

Table 4.5. Tabular representation of the hesitant fuzzy soft multiset $(\widetilde{\Gamma}, A)$

U/A	a_1	a_2	a_3	a_4
c_1	$\varnothing$	$\{0.3, 0.6, 0.4\}$	$\{0.3, 0.5, 0.6\}$	$\varnothing$
c_2	$\{0.5, 0.4, 0.7\}$	$\varnothing$	$\varnothing$	$\varnothing$
c_3	$\{0.4, 0.8, 0.6\}$	$\varnothing$	$\{0.2, 0.3, 0.8\}$	$\{0.9, 0.6, 0.5\}$
c_4	$\varnothing$	$\{0.5, 0.6, 0.8\}$	$\{0.8, 0.1, 0.7\}$	$\varnothing$
s_1	$\{0.9, 0.3, 0.7\}$	$\varnothing$	$\varnothing$	$\{0.5, 0.6, 0.3\}$
s_2	$\varnothing$	$\{0.4, 0.9, 0.3\}$	$\{0.9, 0.5, 0.6\}$	$\{0.4, 0.6, 0.5\}$
s_3	$\{0.4, 0.8, 0.9\}$	$\{0.7, 0.2, 0.4\}$	$\varnothing$	$\{0.9, 0.3, 0.6\}$

Table 4.6. Tabular representation of the induced fuzzy soft multiset $\Delta_{\widetilde{\Gamma}} = (F, A)$

U/A	a_1	a_2	a_3	a_4
c_1	0	0.45	0.47	0
c_2	053	0	0	0
c_3	0.6	0	0.43	0.67
c_4	0	0.63	0.53	0
s_1	0.63	0	0	0.47
s_2	0	0.53	0.67	0.5
s_3	0.7	0.43	0	0.6

Table 4.7. Tabular representation of the t-level soft set

U/A	a_1	a_2	a_3	a_4	Choice Value (c_i)
c_1	0	0	1	0	1
c_2	0	0	0	0	0
c_3	1	0	0	1	2
c_4	0	1	0	0	1
s_1	1	0	0	0	1
s_2	0	0	1	0	1
s_3	1	0	0	1	2

According to Table 4.7, the maximum choice value from U_1 is $c_3 = 2$ and the optimal decision is to select c_3; similarly, the maximum choice value from U_2 is also $c_7 = 2$ and the optimal decision is to select s_3. Therefore, Mrs Y should select cloth c_3 and shoe s_3 to put on for the job interview.

5. Weighted hesitant fuzzy soft multiset based decision making

In 1996, Lin [15] defined a new theory of mathematical analysis, namely the weighted soft sets (W-soft sets). In accordance with Lin's style, Maji et al. [9] defined the weighted table of a soft set. A weighted table of a soft set is presented by having

$d_{ij} = w_j \times h_{ij}$ instead of 0 and 1 only, where h_{ij} are entries in the table of the soft set and w_j are the weights of the attributes e_j. The weighted choice value of an object o_i is $\bar{c}_i$, given by $\bar{c}_i = \sum_j d_{ij}$. By imposing weights on choice parameters, a revised algorithm for arriving at the final optimal decisions was established in [9]. In line with this idea, we introduce the notion of weighted hesitant fuzzy soft multisets and present its application to decision making problems.

Let $\{U_i : i \in I\}$ be a collection of universes such that $\bigcap_{i\in I} U_i = \varnothing$ and let $\{E_{U_i} : i \in I\}$ be a collection of sets of parameters or attributes related to the universes. Let $U = \biguplus_{\boldsymbol{i\in I}} HFS(U_i)$, where $HFS\ (U_i)$ denotes the set of all hesitant fuzzy sub-multisets of the U_i's, $E = \biguplus_{\boldsymbol{i\in I}} E_{U_i}$ and $A \subseteq E$.

Definition 5.1 [12]. A weighted hesitant fuzzy soft multiset is a triple $\eta = (\widetilde{\Gamma},\ A,\ w)$, where $(\widetilde{\Gamma},\ A)$ is a hesitant fuzzy soft multiset over U, and $w : A \longrightarrow [0,\ 1]$ is a weight function specifying the weight $w_j = w(e_j)$, for each attribute $e_j \in A$.

By definition, every hesitant fuzzy soft multiset can be considered as a weighted hesitant fuzzy soft multiset. Obviously, the notion of weighted hesitant fuzzy soft multiset provides a mathematical framework for modeling and analyzing the decision making problems in which all the choice parameters may not be of equal importance. The difference between the importance of parameter are characterize by the weight function in a weighted hesitant fuzzy soft multiset.

Algorithm 1 can be revised to deal with decision making problems based on weighted hesitant fuzzy soft multisets (see algorithm 2). In the revised algorithm, we take the weights of parameters in to consideration and compute the weighted choice values $\bar{c}_i$ instead of the choice values c_i. Note that for a weighted hesitant fuzzy soft multiset $\eta = (\widetilde{\Gamma},\ A,\ w)$ the weight function $w : A \longrightarrow [0,\ 1]$ can be used as a threshold fuzzy set, which implies that one can consider the level soft set $L\left(\left(\widetilde{\Gamma},\ A\right); w\right)$. This will be called decision making based on the weight function decision rule in what follows. Sometimes it is much reasonable to use this decision rule since the decision maker may need higher membership levels on the parameters he puts on more emphasis.

Algorithm 2

(1) Input the weighted hesitant fuzzy soft multiset $\eta = (\widetilde{\Gamma},\ A,\ w)$.
(2) Compute the induced fuzzy soft multiset $\Delta_{\tilde{F}} = \left(\widetilde{\Gamma},\ A\right)$.
(3) Input a threshold fuzzy set $\lambda : A \to [0,\ 1]$, (or give a threshold value $t \in [0,\ 1]$; or choose the mid-level decision rule; or choose the top-level decision rule or choose the weight function decision rule) for decision making.
(4) Compute the level soft set $L(\Delta_{\tilde{F}}; \lambda)$ of $\Delta_{\tilde{F}}$ with respect to the threshold fuzzy set λ (or the $t-$level soft set $L(\Delta_{\tilde{F}}; t)$; or the mid-level soft set $L(\Delta_{\tilde{F}}; mid)$ or the top-level soft set $L\left(\Delta_{\tilde{F}}; max\right)$ or $L\left(\left(\widetilde{\Gamma},\ A\right),\ w\right)$).

(5) Present the level soft set
$L(\Delta_{\tilde{F}};\lambda\ )$ or $L\left(\Delta_{\tilde{F}};t\ \right)$; or $L(\Delta_{\tilde{F}};mid)$ or $L\left(\Delta_{\tilde{F}};max\right)$ or $L\left(\left(\widetilde{\Gamma},\ A\right),\ w\right)$
in tabular form and compute the weighted choice value $\overline{c}_i$ of o_i for all i.
(6) The optimal decision is to select o_k if $\overline{c}_k =\ max_i\overline{c}_i$, from each U_i.
(7) If k has more than one value, then any one of o_k may be chosen, from each U_i.

Note that in the last step of algorithm 2, if too many optimal choices are obtained, one can go back to the third step and change the threshold (or decision rule) previously used so as to adjust the final optimal decision.

To illustrate the above idea, we reconsider Example 4.2. Now assume that Mrs Y have imposed the following weights for the parameters in A: for the parameter "expensive", $w_1 = 0.5$; for the parameter "black", $w_2 = 0.45$; for the parameter "beautiful", $w_3 = 0.6$; for the parameter "made in Italy", $w_4 = 0.5$.

Table 5.1. Tabular representation of the level soft set $L\left(\left(\widetilde{\Gamma}, A\right), w\right)$

$\boldsymbol{U}$	$\boldsymbol{a_1, w = 0.5}$	$\boldsymbol{a_2, w = 0.45}$	$\boldsymbol{a_3, w = 0.6}$	$\boldsymbol{a_4, w = 0.5}$	Choice Value (c_i)
c_1	0	1	0	0	0.45
c_2	1	0	0	0	0.5
c_3	1	0	0	1	1.0
c_4	0	1	0	0	0.45
s_1	1	0	0	0	0.5
s_2	0	1	1	1	1.55
s_3	1	0	0	1	1.0

From Table 5.1, it is clear that the maximum weighted choice value is c_3 from U_1 with a weighted choice value of 1.0 and s_2 from U_2 with a weighted choice value of 1.55. Therefore, Mrs Y will choose cloth c_3 and shoe s_2 to wear for the job interview.

6. Conclusion

In this paper, we studied and used the approach introduced by Feng et al.[12] and presented an adjustable approach to hesitant fuzzy soft multisets based decision making problems using level soft set of hesitant fuzzy soft multisets with relevant and illustrative examples. Also, weighted hesitant fuzzy soft multiset based decision making problem was presented and supported with concrete example.

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