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### Fuzzy soft upper and lower semi-continuous multifunctions

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ABSTRACT. In this paper, we define the upper and lower inverse of a fuzzy soft multifunction and prove some basic identities. Then by using these ideas we introduce the concept of fuzzy soft continuity and obtain many interesting properties of fuzzy soft upper and lower semicontinuous multifunctions. Also, we define the notions of fuzzy soft lower  $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{I})$ -continuity and fuzzy soft upper  $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{I})$ -continuity. Moreover, we present different cases of upper and lower fuzzy soft continuous multifunctions.

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#### 1. INTRODUCTION

Most of the existing mathematical tools for formal modeling, reasoning and computing are characteristically crisp, deterministic and precise. However, in real life, the problems in economics, engineering, environment, social science, medical science, etc., do not always involve crisp data. The reason for these difficulties is possibly, the inadequacy of the classical parameterization tool in general. Consequently, Molodtsov [17] initiated the concept of soft set theory as a new mathematical tool for dealing with vagueness and uncertainties which is free from the above difficulties. Molodtsov [18] successfully applied the soft set theory into several directions, such as smoothness of functions, game theory, Riemann integration, Perron integration and theory of measurement. Maji et al.[14] gave a practical application of soft sets in decisionmaking problems. They have also introduced the concept of fuzzy soft set (Maji et al. [13]), as more generalized concept, which is a combination of fuzzy set (Zadeh [28]) and soft set (Molodtsov [17]) and also studied some of its properties. Many of applications can be found in the works of Ahmad and Kharal [7], Kharal and Ahmad [12], Tanay and Kandemir [27], Aygunoglu et al. [8], Cetkin et al. [9], Metin et al. [15], Metin and Alkan [16], Abbas et al. [1, 2, 3], Gunduz and Bayramov [11], Abbas and Ibedou [6], Dizmana et al. [10], Qiu and Zhang [19], Qiu et al. [20, 21] and Šenel [22, 23, 24], Šenel and Cagman [25] and Serkan and Idris [26]. Metin et al. ([15]) defined the upper and lower inverse of a fuzzy soft multifunction from ordinary topological space to fuzzy soft topological space in Chang sense and proved some basic properties. Although there are some similarities of the definitions and theorems between fuzzy soft topological spaces and fuzzy topological spaces, there are a lot of differences. They arise especially when the cardinality |E|of the set E of parameters is greater than 1. If |E| = 1, then a fuzzy soft topological spaces can be treated as a fuzzy topological spaces and then the behavior of fuzzy soft topological space  $\tau_E$  is analogous to a fuzzy topological space  $\tau$ .

In this paper our purpose is two fold. First, we introduce the notions of fuzzy soft multifunction, upper and lower inverse of a fuzzy soft multifunction and study their various properties. Next, we introduce fuzzy soft upper and lower semi-continuous multifunctions. Also, we define the notions of fuzzy soft lower  $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{I})$ -continuous and fuzzy soft upper  $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{I})$ -continuous, then we present different cases of the fuzzy soft continuity multifunction.

#### 2. Preliminaries

Throughout this paper, X refers to an initial universe and E is the set of all parameters for X. A fuzzy soft ( simply, FS ) set  $f_E$  on X is called  $\lambda$ -absolute FS set and denoted by  $\tilde{\mathbf{E}}^{\lambda}$ , if  $f_e = \underline{\lambda}$ , for  $\lambda \in I$ ,  $\underline{\lambda}(x) = \lambda$ , for each  $x \in X$ and  $e \in E$  (where  $I = [0, 1], I_0 = (0, 1]$ ). For  $f_A, g_B \in (\widetilde{X, E}), f_A \overline{\wedge} g_B = \Phi$  if  $f_A \sqsubseteq g_B$  and  $f_A \overline{\wedge} g_B = f_A \sqcap g_B^c$  otherwise. The FS set  $f_A = e_x^t = \{(e, x_t)\}$  is called FS point where  $t \in I_0, x_t$  is fuzzy point and we say that  $e_x^t$  belongs to the FS set  $f_A$  if for  $e \in A$ ,  $t \leq f_e(x)$ . Also, the map  $\alpha : E \times (X, E) \times I_0 \to (X, E)$  is called a FS operator on X.

All definitions and properties of FS sets and FS topology are found in [1, 2, 8, 9, 10, 12, 14, 15, 21]. Recall that a FS ideal  $\mathcal{I}$  on X is a mapping  $\mathcal{I}: E \to I^{(\widetilde{X,E})}$ that satisfies the following conditions for each  $e \in E$ ,

- (i)  $\mathcal{I}_e(\Phi) = 1$ ,  $\mathcal{I}_e(\tilde{E}) = 0$ ,
- $\begin{array}{ll} (\mathrm{ii}) \ \ \mathcal{I}_e(f_{\scriptscriptstyle A} \sqcup g_{\scriptscriptstyle B}) \geq \mathcal{I}_e(f_{\scriptscriptstyle A}) \wedge \mathcal{I}_e(g_{\scriptscriptstyle B}, \ \mathrm{for \ each} \ \ f_{\scriptscriptstyle A}, g_{\scriptscriptstyle B} \in \widetilde{(X,E)}, \\ (\mathrm{iii}) \ \mathrm{if} \ \ f_{\scriptscriptstyle A} \sqsubseteq g_{\scriptscriptstyle B}, \ \mathrm{then} \ \ \mathcal{I}_e(f_{\scriptscriptstyle A}) \geq \mathcal{I}_e(g_{\scriptscriptstyle B}). \end{array}$

For  $f_A \in (X, E)$ ,  $\mathcal{I}_e^0(f_A) = 1$ , if  $f_A = \Phi$  and  $\mathcal{I}_e^0(f_A) = 0$ , otherwise.  $(X, \tau_E)$  is a FS topological space (See [9]), if a mapping  $\tau : E \to I^{(\widetilde{X,E})}$ 

satisfies the following conditions for each  $e \in E$ ,

(T1)  $\tau_e(\Phi) = \tau_e(\tilde{E}) = 1$ , (T2)  $\tau_e(f_A \sqcap g_B) \ge \tau_e(f_A) \land \tau_e(g_B), \ \forall f_A, g_B \in (X, E),$ (T3)  $\tau_e(\bigsqcup_{i\in\Gamma}(f_A)_i) \ge \bigwedge_{i\in\Gamma} \tau_e((f_A)_i), \quad \forall \ (f_A)_i \in \widetilde{(X,E)}, \ i\in\Gamma.$  Let  $(X, \tau_E)$  be a FS topological space. Then the mapping  $\tau^e : I^X \to I$  defined as: for each  $e \in E$ ,

$$\tau^{e}(\lambda) = \begin{cases} \quad \forall \{r \in I_0 : \tau_{e}(f_{\scriptscriptstyle A}) \geq r\}, \ \lambda = f_{e} \\ 0 \quad \text{otherwise} \end{cases}$$

is a fuzzy topology on X, so for each parameter  $e \in E$ , we have a fuzzy topology  $\tau^e$  on X. Thus, a FS topology on X gives a parameterized famely of fuzzy topologies on X.

#### 3. Fuzzy soft multifunctions.

In this section we introduce the same definition from FS topological space to FS topological space in  $\check{S}$  ostak sense which is a generalization of the same concepts introduced by Metin et al. 2015 ([17]).

**Definition 3.1.** Let  $\psi: E \to K$ . Then,  $F: X \to Y$  is called a FS multifunction, if  $F(x): K \to I^Y$  is a FS set on (Y, K), for each  $x \in X$ . Also, F is said to be onto, if for each FS set  $g_B \in (Y, K)$ , there is  $x \in X$  such that  $F(x) = g_B$ .

The degree of membership of y in F(x) with respect to a parameter  $k \in K$  is denoted by:

$$(F(x))_k(y) = (G_F)_k(x, y), \ k \in K,$$

where  $G_F: K \to I^{X \times Y}$ .

The inverse of F denoted by  $F^-: Y \multimap X$  is a FS multifunction defined by:

$$(F^{-}(y))_{e}(x) = (F(x))_{\psi(e)}(y) = (G_{F})_{\psi(e)}(x,y), \ e \in E.$$

The domain of F denoted by  $dom(F): E \to I^X$ , is defined as:

$$(dom(F))_e(x) = \bigvee_{y \in Y} (G_F)_{\psi(e)}(x,y), \text{ for } x \in X, y \in Y \text{ and } e \in E.$$

The range of F denoted by  $rng(F): K \to I^Y$ , is defined as:

$$(rng(F))_k(y) = \bigvee_{x \in X} (G_F)_k(x, y), \text{ for } x \in X, y \in Y \text{ and } k \in K.$$

**Definition 3.2.** FS multifunction  $F: X \multimap Y$  is called:

- (i) non-void, if  $F(x) \neq \Phi$ ,  $\forall x \in X$ ,
- (ii) surjective, if  $(rng(F))_k(y) = 1$ ,  $\forall y \in Y, k \in K, \psi : E \to K$ .
- (iii) normalized, if  $\forall x \in X, e \in E$  there is  $y_0 \in Y$  such that  $(G_F)_e(x, y_0) = 1$ .

**Definition 3.3.** Let  $F: X \to Y$  be a FS multifunction and  $\psi: E \to K$ . Then (i) the image  $F(f_A)$  of  $f_A \in (X, E)$  is FS set in (Y, K) defined as:

$$(F(f_A))_k(y) = \bigvee_{x \in X} [(G_F)_k(x, y) \wedge f_{\psi^{-1}(k)}(x)],$$

(ii) the lower inverse  $F^{l}(g_{B})$  of  $g_{B} \in (\widetilde{Y,K})$  is FS set in  $(\widetilde{X,E})$  defined as:  $(F^{l}(g_{B}))_{e}(x) = \bigvee_{y \in Y} [(G_{F})_{\psi(e)}(x,y) \land g_{\psi(e)}(y)],$ 

(iii) the upper inverse  $F^u(g_{\scriptscriptstyle B})$  of  $g_{\scriptscriptstyle B} \in \widetilde{(Y,K)}$  is FS set in  $\widetilde{(X,E)}$  defined as:  $\Lambda$ 

$$F^{ac}(g_{B})(e)(x) = \bigwedge_{y \in Y} [(G_{F}^{c})_{\psi(e)}(x,y) \lor g_{\psi(e)}(y)].$$

**Example 3.4.** Let  $X = \{x_1, x_2\}, Y = \{y_1, y_2\}$  and  $\psi : E = \{e_1, e_2\} \rightarrow \{k_1, k_2\} =$ K, where  $\psi(e_1) = k_1$ ,  $\psi(e_2) = k_2$ . And let  $F: X \multimap Y$  be a FS multifunction defined as:

$$\begin{aligned} & (G_F)_{k_1}(x_1,y_1) = 0.2, \ (G_F)_{k_1}(x_1,y_2) = 0.3, \ (G_F)_{k_1}(x_2,y_1) = 0.4, \\ & (G_F)_{k_1}(x_2,y_2) = 0.2, \ (G_F)_{k_2}(x_1,y_1) = 0.5, \ (G_F)_{k_2}(x_1,y_2) = 1.0, \\ & (G_F)_{k_2}(x_2,y_1) = 0.0, \ (G_F)_{k_2}(x_2,y_2) = 0.7. \end{aligned}$$

Then

$$dom(F) = \{(e_1, \{0.3, 0.4\}), (e_2, \{1.0, 0.7\})\}$$

and

$$rng(F) = \{(k_1, \{0.4, 0.3\}), (k_2, \{0.5, 1.0\})\}.$$

Thus for FS sets,

 $f_{F} = \{(e_1, \{0.2, 0.8\}), (e_2, \{0.3, 0.5\})\}$ 

and

$$g_K = \{(k_1, \{0.7, 0.1\}), (k_2, \{0.6, 0.5\})\},\$$

we have

$$\begin{split} F(f_{\scriptscriptstyle E}) &= \{(k_1, \{0.4, 0.2\}), (k_2, \{0.3, 0.5\})\}, \\ F^l(g_{\scriptscriptstyle K}) &= \{(k_1, \{0.2, 0.4\}), (k_2, \{0.5, 0.5\})\} \end{split}$$

and

$$F^{u}(g_{\kappa}) = \{(k_1, \underline{0.7}), (k_2, \underline{0.5})\}.$$

So  $F^u(g_B) \neq F^l(g_B)$ .

**Proposition 3.5.** Let  $F: X \multimap Y$  be a FS multifunction and  $\psi: E \to K$ . Then

- $\begin{array}{ll} (1) \quad F((f_A)_1) \sqsubseteq F((f_A)_2) \quad if \quad (f_A)_1 \sqsubseteq (f_A)_2, \\ (2) \quad F^l((g_B)_1) \sqsubseteq F^l((g_B)_2) \quad and \quad F^u((g_B)_1) \sqsubseteq F^u((g_B)_2), \quad if \quad (g_B)_1 \sqsubseteq (g_B)_2, \\ (3) \quad F^u(g_B) \sqsubseteq F^l(g_B) \quad and \quad (F^l(g_B))^c \sqsubseteq F^l(g_B^c), \quad if \ F \ is \ normalized, \\ (4) \quad (F(f_A))^c \sqsubseteq F(f_A^c), \quad if \ F \ is \ surjective, \end{array}$

- (5)  $(F^l(g_B^c))^c = F^u(g_B)$  and  $(F^u(g_B^c))^c = F^l(g_B)$ ,
- (6)  $F(\tilde{E}^{\lambda}) = \tilde{K}^{\lambda}$ , if F is surjective, (7)  $F^{l}(g_{B} \sqcap \tilde{K}^{\lambda}) = F^{l}(g_{B}) \sqcap \tilde{E}^{\lambda}$ .

*Proof.* (1) and (2) obvious from the definition.

(3) Since F is normalized, there exists  $y_0 \in Y$  such that  $(G_F)_{\psi(e)}(x, y_0) = 1$ . Then

$$(F^{l}(g_{B}))_{e}(x) = \bigvee_{\substack{y \in Y \\ y \in Y \\ e}} [(G_{F})_{\psi(e)}(x, y) \wedge g_{\psi(e)}(y)]$$
  
 
$$\geq [(G_{F})_{\psi(e)}(x, y_{0}) \wedge g_{\psi(e)}(y_{0})] = g_{\psi(e)}(y_{0}),$$

$$\begin{split} (F^u(g_{\scriptscriptstyle B}))_{\scriptscriptstyle e}(x) &= & \bigwedge_{y \in Y} [(G^c_F)_{\psi(e)}(x,y) \lor g_{\psi(e)}(y)] \\ &\leq & [(G^c_F)_{\psi(e)}(x,y_0) \lor g_{\psi(e)}(y_0)] = g_{\psi(e)}(y_0). \end{split}$$

Thus  $F^u(g_{\scriptscriptstyle B}) \sqsubseteq F^l(g_{\scriptscriptstyle B})$  and

$$\begin{aligned} ((F^{l}(g_{B}))_{e}(x))^{c} &= (\bigvee_{y \in Y} [(G_{F})_{\psi(e)}(x,y) \wedge g_{\psi(e)}(y)])^{c} \\ &= \bigwedge_{y \in Y} [(G_{F}^{c})_{\psi(e)}(x,y) \vee g_{\psi(e)}^{c}(y)] \\ &\leq [(G_{F}^{c})_{\psi(e)}(x,y_{0}) \vee g_{\psi(e)}^{c}(y_{0})] = g_{\psi(e)}^{c}(y_{0}), \end{aligned}$$

$$(F^{l}(g_{B}^{c}))_{e}(x) = \bigvee_{y \in Y} [(G_{F})_{\psi(e)}(x,y) \wedge g_{\psi(e)}^{c}(y)]$$
  
 
$$\geq [(G_{F})_{\psi(e)}(x,y_{0}) \wedge g_{\psi(e)}^{c}(y_{0})] = g_{\psi(e)}^{c}(y_{0}).$$

So 
$$(F^{l}(g_{B}))^{c} \subseteq F^{l}(g_{B}^{c})$$
.  
(4) By similar way as (3).  
(5)  $((F^{l}(g_{B}^{c}))_{e}(x))^{c} = (\bigvee_{y \in Y} [(G_{F})_{\psi(e)}(x, y) \land g_{\psi(e)}^{c}(y)])^{c}$   
 $= \bigwedge_{y \in Y} [(G_{F}^{c})_{\psi(e)}(x, y) \lor g_{\psi(e)}(y)] = (F^{u}(g_{B}))_{e})(x)$ .  
Then  $(F^{l}(g_{B}^{c}))^{c} = F^{u}(g_{B})$ .  
By similar way, we can prove that  $(F^{u}(g_{B}^{c}))^{c} = F^{l}(g_{B})$ .  
(6) Since  $(F(\tilde{E}^{\lambda}))_{k}(y) = \bigvee_{x \in X} [(G_{F})_{k}(x, y) \land \tilde{E}_{\psi^{-1}(k)}^{\lambda}(x)]$   
 $= \bigwedge_{y \in Y} [(G_{F}^{c})_{\psi(e)}(x, y) \lor g_{\psi(e)}(y)]$   
 $= (F^{u}(g_{B}))_{e})(x),$   
 $F(\tilde{E}^{\lambda}) = \tilde{K}^{\lambda}$ .  
(7) Since  $(F^{l}(g_{B} \sqcap \tilde{K}^{\lambda}))_{e}(x) = \bigvee_{y \in Y} [(G_{F})_{\psi(e)}(x, y) \land (g_{B} \sqcap \tilde{K}^{\lambda})_{\psi(e)}(y)]$   
 $= \bigvee_{y \in Y} [(G_{F})_{\psi(e)}(x, y) \land (g_{\psi(e)}(y)] \land \lambda$   
 $= [(F^{l}(g_{B}))_{e}(x)] \land \lambda,$   
 $F^{l}(g_{B} \sqcap \tilde{K}^{\lambda}) = F^{l}(g_{B}) \sqcap \tilde{E}^{\lambda}.$ 

**Proposition 3.6.** Let  $F: X \multimap Y$  be a FS multifunction and  $\psi: E \to K$ . Then

- $\begin{array}{lll} (1) & F(\sqcap_{i\in\Gamma}(f_A)_i) \sqsubseteq \sqcap_{i\in\Gamma} F((f_A)_i) & and & F(\sqcup_{i\in\Gamma}(f_A)_i) = \sqcup_{i\in\Gamma} F((f_A)_i), \\ (2) & F^l(\sqcap_{i\in\Gamma}(g_B)_i) \sqsubseteq \sqcap_{i\in\Gamma} F^l((g_B)_i) & and & F^l(\sqcup_{i\in\Gamma}(g_B)_i) = \sqcup_{i\in\Gamma} F^l((g_B)_i), \\ (3) & F^u(\sqcup_{i\in\Gamma}(g_B)_i) \sqsupseteq \sqcup_{i\in\Gamma} F^u((g_B)_i) & and & F^u(\sqcap_{i\in\Gamma}(g_B)_i) = \sqcap_{i\in\Gamma} F^u((g_B)_i). \\ & 5 \end{array}$

Proof.

Then  $F(\sqcap_{i\in\Gamma}(f_A)_i) \sqsubseteq \sqcap_{i\in\Gamma}F((f_A)_i).$ By the same way, we can prove (2) and (3).

**Proposition 3.7.** Let  $F: X \multimap Y$  be a FS multifunction and  $\psi: E \to K$ . Then

- $\begin{array}{lll} (1) & F(F^{l}(g_{_{B}})) \sqsupseteq g_{_{B}} & and & F(F^{u}(g_{_{B}})) \sqsubseteq g_{_{B}}, \ if \ F \ is \ surjective, \\ (2) & F^{l}(F(f_{_{A}})) \sqsupseteq f_{_{A}} & and \ F^{u}(F(f_{_{A}})) \sqsupseteq f_{_{A}}, \ if \ F \ is \ normalized, \\ (3) & F(F^{u}(g_{_{B}})) \neq g_{_{B}}, \ F^{u}(F(f_{_{A}})) \neq f_{_{A}} \ and \ F(F^{l}(g_{_{B}})) \neq g_{_{B}}. \end{array}$

*Proof.* (1) Since F is surjective, we have  $rng(F)_k(y) = 1$ , for all  $y \in Y$ ,  $k \in K$ . Then

$$\begin{split} (F(F^{l}(g_{B})))_{k}(y) &= \bigvee_{x \in X} \left[ (G_{F})_{k}(x,y) \wedge (F^{l}(g_{B}))_{\psi^{-1}(k)}(x) \right] \\ &= \bigvee_{x \in X} \left[ (G_{F})_{k}(x,y) \wedge (\bigvee_{y \in Y} \left[ (G_{F})_{k}(x,y) \wedge g_{k}(y) \right) \right] \right] \\ &\geq \bigvee_{x \in X} \left[ (G_{F})_{k}(x,y) \wedge ((G_{F})_{k}(x,y) \wedge g_{k}(y)) \right] \\ &= \bigvee_{x \in X} \left[ (G_{F})_{k}(x,y) \wedge g_{k}(y) \right] = g_{k}(y). \end{split}$$

Thus  $F(F^l(g_B)) \supseteq g_B$ .

The other case is similarly.

(2) Since F is normalized, there exists  $y_0 \in Y$  such that  $(G_F)_k(x, y_0) = 1$ . Then

$$\begin{split} (F^{l}(F(f_{A})))_{e}(x) &= \bigvee_{y \in Y} [(G_{F})_{\psi(e)}(x,y) \wedge (F(f_{A}))_{\psi(e)}(y)] \\ &= \bigvee_{y \in Y} [(G_{F})_{\psi(e)}(x,y) \wedge (\bigvee_{x \in X} ((G_{F})_{k}(x,y) \wedge f_{e}(x)))] \\ &\geq \bigvee_{y \in Y} [(G_{F})_{\psi(e)}(x,y) \wedge ((G_{F})_{k}(x,y_{0}) \wedge f_{e}(x))] \\ &= \bigvee_{y \in Y} [(G_{F})_{\psi(e)}(x,y) \wedge f_{e}(x)] = f_{e}(x). \end{split}$$

Thus  $F^l(F(f_A)) \supseteq f_A$ .

The other case is similarly.

**Example 3.8.** Let  $X = \{x_1, x_2\}$ ,  $Y = \{y_1, y_2\}$  and  $\psi : E = \{e_1, e_2\} \rightarrow \{k_1, k_2\} = K$ , where  $\psi(e_1) = k_1$ ,  $\psi(e_2) = k_2$ . And let  $F : X \multimap Y$  be a FS multifunction defined by:

 $\begin{array}{l} (G_F)_{k_1}(x_1,y_1)=0.1, \ (G_F)_{k_1}(x_1,y_2)=1.0, \ (G_F)_{k_1}(x_2,y_1)=0.0, \\ (G_F)_{k_1}(x_2,y_2)=0.5, \ (G_F)_{k_2}(x_1,y_1)=0.3, \ (G_F)_{k_2}(x_1,y_2)=0.5, \\ (G_F)_{k_2}(x_2,y_1)=0.2, \ (G_F)_{k_2}(x_2,y_2)=0.0. \end{array}$  If  $f_E=\{(e_1,\{0.3,0.4\}),(e_2,\underline{0.7})\}, \ \text{then} \end{array}$ 

$$F^{l}(F(f_{E})) = \{(e_{1}, \underline{0.4}), (e_{2}, \{0.5, 0.2\})\} \neq f_{E}$$

and

$$F^u(F(f_E)) = \{(e_1, \{0.4, 0.5\}), (e_2, \{0.5, 0.8\})\} \neq f_E.$$

If  $g_{\kappa} = \{(k_1, \{0.6, 0.4\}), (k_2, \{1.0, 0.3\})\}$ , then

$$F(F^{l}(g_{\kappa})) = \{(k_{1}, \{0.1, 0.4\}), (k_{2}, \underline{0.3})\} \neq g_{\kappa}$$

and

$$F(F^{u}(g_{\kappa})) = \{(k_{1}, \{0.1, 0.5\}), (k_{2}, \{0.3, 0.5\})\} \neq g_{\kappa}$$

•

**Definition 3.9.** Let  $\psi : E \to K$ ,  $\varpi : K \to M$  and FS multifunctions  $F : X \multimap Y$ ,  $H : Y \multimap Z$ . Then the composition  $H \circ F$  defined as:

$$((H \circ F)(x))_{m}(z) = \bigvee_{y \in Y} [(G_{H})_{m}(y, z) \wedge (G_{F})_{\varpi^{-1}(m)}(x, y)].$$

**Theorem 3.10.** Let  $\psi: E \to K$ ,  $\varpi: K \to M$  and FS multifunctions  $F: X \multimap Y$ ,  $H: Y \multimap Z$ . Then

(1) 
$$((H \circ F) = H(F),$$
  
(2)  $(H \circ F)^{l} = F^{l}(H^{l}),$   
(2)  $(H \circ F)^{u} = F^{u}(H^{u}),$ 

 $(3) (H \circ F)^u = F^u(H^u).$ 

*Proof.* (1) Let  $f_A \in (\widetilde{X, E})$ . Then

$$\begin{split} ((H \circ F)(f_A))_m(z) &= \bigvee_{x \in X} \left[ (G_{H \circ F})_m(x, z) \wedge f_{\psi^{-1}(\varpi^{-1}(m))}(x) \right] \\ &= \bigvee_{x \in X} \left[ \bigvee_{y \in Y} ((G_H)_m(y, z) \wedge (G_F)_{\varpi^{-1}(m)}(x, y)) \wedge f_{\psi^{-1}(\varpi^{-1}(m))}(x) \right] \\ &= \bigvee_{y \in Y} \left[ G_H \right]_m(y, z) \wedge \left( \bigvee_{x \in X} ((G_F)_{\varpi^{-1}(m)}(x, y) \wedge f_{\psi^{-1}(\varpi^{-1}(m))}(x))) \right] \\ &= \bigvee_{y \in Y} \left[ (G_H)_m(y, z) \wedge (F(f_A))_{\varpi^{-1}(m)}(y) \right] \\ &= (H(F(f_A)))_m(z). \end{split}$$

Thus  $((H \circ F) = H(F))$ .

(2) Let 
$$w_D \in (\widetilde{Z}, \widetilde{M})$$
. Then  

$$((H \circ F)^l(w_D))_e(x) = \bigvee_{z \in Z} [(G_{H \circ F})_{\varpi(\psi(e))}(x, z) \wedge w_{\varpi(\psi(e))}(z)]$$

$$= \bigvee_{z \in Z} [\bigvee_{y \in Y} ((G_F)_{\psi(e)}(x, y) \wedge (G_H(y, z))_{(\varpi(\psi(e))}) \wedge w_{\varpi(\psi(e))}(z)]$$

$$= \bigvee_{y \in Y} [G_F)_{\psi(e)}(x, y) \wedge (\bigvee_{z \in Z} [(G_H)_{\varpi(\psi(e))}(y, z) \wedge w_{\varpi(\psi(e))}(z))]$$

$$= \bigvee_{y \in Y} [G_F)_{\psi(e)}(x, y) \wedge (H^l(w_D))_{\psi(e)}(y)]$$

$$= (F^l(H^l(w_D)))_e(x).$$
The output of  $U^l(W_D)$ 

Thus  $(H \circ F)^l = F^l(H^l)$ .

(3) The proof is similar to (2).

#### 4. Continuity of fuzzy soft multifunctions.

In this section, we introduce FS upper and lower semi-continuous multifunctions which a generalization of FS continuous multifunctions introduced by Metin et al. 2015 ([17]).

**Definition 4.1.** Let  $\psi: E \to K$  and  $F: X \multimap Y$  be a FS multifunction between two FS topological spaces  $(X, \tau_E)$ ,  $(Y, \eta_K)$ ,  $e \in E$  and  $r \in I_0$ . Then F is called:

- (i) FS upper semi-continuous at a FS point  $e_x^t \in dom(F)$ , if  $e_x^t \in F^u(g_B)$  for each  $g_B \in \widetilde{(Y,K)}$  and  $\eta_{\psi(e)}(g_B) \ge r$ , there exists  $f_A \in \widetilde{(X,E)}$ ,  $\tau_e(f_A) \ge r$ and  $e_x^t \in f_A$  such that  $f_A \sqcap dom(F) \sqsubseteq F^u(g_B)$ ,
- (ii) FS upper semi-continuous, if it is FS upper semi-continuous at every  $e_x^t \in dom(F)$ ,
- (iii) FS lower semi-continuous at a FS point  $e_x^t \in dom(F)$ , if  $e_x^t \in F^l(g_B)$  for each  $g_B \in \widetilde{(Y,K)}$  and  $\eta_{\psi(e)}(g_B) \ge r$ , there exists  $f_A \in \widetilde{(X,E)}$ ,  $\tau_e(f_A) \ge r$ and  $e_x^t \in f_A$  such that  $f_A \sqsubseteq F^l(g_B)$ , (iv) FS lower semi-continuous, if it is FS lower semi-continuous at every  $e_x^t \in$
- (iv) FS lower semi-continuous, if it is FS lower semi-continuous at every  $e_x^t \in dom(F)$ .

**Remark 4.2.** If F is normalized, then F is FS upper semi-continuous at a FS point  $e_x^t \in dom(F)$  iff  $e_x^t \in F^u(g_B)$  for each  $g_B \in \widetilde{(Y,K)}$  and  $\eta_{\psi(e)}(g_B) \ge r$ , there exists  $f_A \in \widetilde{(X,E)}, \ \tau_e(f_A) \ge r$  and  $e_x^t \in f_A$  such that  $f_A \sqsubseteq F^u(g_B)$ .

**Theorem 4.3.** Let  $F: X \multimap Y$  be a FS multifunction between two FS topological spaces  $(X, \tau_E)$ ,  $(Y, \eta_K)$ ,  $\psi: E \to K$ ,  $g_B \in (\widecheck{Y, K})$  and  $e \in E$ .

(1) If F is normalized, then F is FS upper semi-continuous iff

$$\tau_e(F^u(g_{\scriptscriptstyle B})) \ge \eta_{\psi(e)}(g_{\scriptscriptstyle B}).$$

(2) F is FS lower semi-continuous iff

$$\tau_e(F^l(g_B)) \ge \eta_{\psi(e)}(g_B).$$

- (3) If  $F: (X, \tau_E) \multimap (Y, \eta_K)$  is FS lower semi-continuous multifunction, then  $F: (X, \tau^e) \multimap (Y, \eta^{\psi(e)})$  is fuzzy lower semi-continuous multifunction, for each  $e \in E$  [4].
- (4) If the normalized  $F : (X, \tau_E) \multimap (Y, \eta_K)$  is FS upper semi-continuous multifunction, then  $F : (X, \tau^e) \multimap (Y, \eta^{\psi(e)})$  is fuzzy upper semi-continuous multifunction, for each  $e \in E$ .

Proof. (1) Assume that, there is  $g_B \in (Y, \widetilde{K})$  and  $r \in I_0$  such that  $\tau_e(F^u(g_B)) \leq r < \eta_{\psi(e)}(g_B).$ 

Since F is normalized and FS upper semi-continuous, for  $e_x^t \in F^u(g_B)$  with  $\eta_{\psi(e)}(g_B) \ge r$ , there is  $(f_A)_{e_x^t} \in \widetilde{(X,E)}$  with  $\tau_e((f_A)_{e_x^t}) \ge r$  and  $e_x^t \in (f_A)_{e_x^t}$  such that  $(f_A)_{e_x^t} \sqsubseteq F^u(g_B)$ . Then  $e_x^t \in F^u(g_B) = \bigsqcup_{e_x^t \in F^u(g_B)} (f_A)_{e_x^t}$ . Thus

$$\tau_e(F^u(g_B)) = \tau_e(\bigsqcup_{e_x^t \in F^u(g_B)} (f_A)_{e_x^t}) \ge \bigwedge_{e_x^t \in F^u(g_B)} \tau_e((f_A)_{e_x^t}) \ge r.$$

It is a contradiction. So  $\tau_e(F^u(g_B)) \ge \eta_{\psi(e)}(g_B)$ .

Conversely, Suppose that  $\tau_e(F^u(g_B)) \ge \eta_{\psi(e)}(g_B)$ , for any  $g_B \in (Y, K)$ . Consider  $e_x^t \in dom(F)$  and  $\eta_{\psi(e)}(g_B) \ge r$  such that  $e_x^t \in F^u(g_B)$ . Then  $\tau_e(F^u(g_B)) \ge \eta_{\psi(e)}(g_B) \ge r$ . Thus F is FS upper semi-continuous at  $e_x^t$ . So F is FS upper semi-continuous.

(2) The proof can be proved as (1).

(3) and (4) follow directly from (1), (2) and  $(X, \tau_e)$  and  $(Y, \eta_{\psi(e)})$  are two fuzzy topologies.

The following example shows generally that F is FS upper semi-continuous but not normalized and  $\tau_e(F^u(g_\kappa)) \geq \eta_{\psi(e)}(g_B)$ .

**Example 4.4.** Let  $X = \{x_1, x_2\}$ ,  $Y = \{y_1, y_2\}$  and  $\psi : E = \{e_1, e_2\} \rightarrow \{k_1, k_2\} = K$ , where  $\psi(e_1) = k_1$ ,  $\psi(e_2) = k_2$ . And let  $F : X \multimap Y$  be a FS multifunction defined by:

- $(G_F)_{k_1}(x_1, y_1) = 0.1, \ (G_F)_{k_1}(x_1, y_2) = 0.4, \ (G_F)_{k_1}(x_2, y_1) = 0.3, \ (G_F)_{k_1}(x_1, y_2) = 0.4, \ (G_F)_{k_1}(x_2, y_1) = 0.6, \ (G_F)_{k_1}(x_1, y_2) = 0.4, \ (G_F)_{k_1}(x_2, y_1) = 0.6, \ (G_F)_{k_1}(x_1, y_2) = 0.4, \ (G_F)_{k_1}(x_2, y_1) = 0.6, \ (G_F)_{k_1}(x_1, y_2) = 0.4, \ (G_F)_{k_1}(x_2, y_1) = 0.6, \ (G_F)_{k_1}(x_1, y_2) = 0.4, \ (G_F)_{k_1}(x_2, y_1) = 0.6, \ (G_F)_{k_1}(x_1, y_2) = 0.4, \ (G_F)_{k_1}(x_2, y_1) = 0.4, \ (G_F)_{k_1}(x_2, y_1) = 0.6, \ (G_F)_{k_1}(x_2, y_1) = 0.6,$
- $(G_F)_{k_1}(x_2, y_2) = 0.2, \ (G_F)_{k_2}(x_1, y_1) = 0.4, \ (G_F)_{k_2}(x_1, y_2) = 0.6,$

 $(G_F)_{k_2}(x_2, y_1) = 0.5, \ (G_F)_{k_2}(x_2, y_2) = 0.9.$ 

And let  $f_E = \{(e_1, \{0.5, 0.8\}), (e_2, \underline{0.6})\}, g_K = \{(k_1, \{0.4, 0.6\}), (k_2, \{0.4, 0.5\})\}$  and the FS topologies  $(X, \tau_E), (Y, \eta_K)$  are defined as:

$$\begin{split} \tau_{e}(u_{\scriptscriptstyle E}) \; = \; \left\{ \begin{array}{ll} 1 & \mathrm{if}\, u_{\scriptscriptstyle E} = \Phi \ \mathrm{or} \ \ \check{\mathrm{E}} \\ \frac{1}{2} & \mathrm{if} \ u_{\scriptscriptstyle E} = f_{\scriptscriptstyle E} \\ 0 & \mathrm{otherwise}, \end{array} \right. \\ \eta_{\scriptscriptstyle k}((w_{\scriptscriptstyle K}) \; = \; \left\{ \begin{array}{ll} 1 & \mathrm{if} \ (w_{\scriptscriptstyle K} = \Phi \ \mathrm{or} \ \ \check{\mathrm{K}} \\ \frac{1}{2} & \mathrm{if} \ ; w_{\scriptscriptstyle K} = g_{\scriptscriptstyle K} \\ 0 & \mathrm{otherwise}. \end{array} \right. \end{split} \end{split}$$

Then F is FS upper semi-continuous, where  $F^u(g_K) = \{(k_1, \{0.6, 0.7\}), (k_2, \underline{0.5})\}$ and  $dom(F) = \{(e_1, \{0.4, 0.3\}), (e_2, \{0.6, 0.9\})\}$ . But  $\tau_e(F^u(g_K)) = 0 \not\geq \eta_{\psi(e)}(g_B) = 0.5$ and F is not normalized. The following example shows generally that F is fuzzy upper semi-continuous but not FS upper semi-continuous.

**Example 4.5.** Let  $X = \{x_1, x_2\}$ ,  $Y = \{y_1, y_2\}$  and  $\psi : E = \{e_1, e_2\} \rightarrow \{k_1, k_2\} = K$ , where  $\psi(e_1) = k_1$ ,  $\psi(e_2) = k_2$ . And let  $F : X \multimap Y$  be a FS multifunction defined by:

$$(G_F)_{k_1}(x_1, y_1) = 0.6, \ (G_F)_{k_1}(x_1, y_2) = 1.0, \ (G_F)_{k_1}(x_2, y_1) = 1.0, (G_F)_{k_1}(x_2, y_2) = 0.1, \ (G_F)_{k_2}(x_1, y_1) = 1.0, \ (G_F)_{k_2}(x_1, y_2) = 0.5, (G_F)_{k_2}(x_2, y_1) = 0.4, \ (G_F)_{k_2}(x_2, y_2) = 1.0$$

Let  $f_E = \{(e_1, \{0.4, 0.3\}), (e_2, \{0.5, 0.4\})\}, g_E = \{(e_1, \{0.3, 0.2\}), (e_2, \{0.4, 0.3\})\}, h_K = \{(k_1, \{0.3, 0.4\}), (k_2, \{0.4, 0.3\})\}$  and FS topologies  $(X, \tau_E), (Y, \eta_K)$  are defined as:

$$\tau_{e}(u_{E}) = \begin{cases} 1 & \text{if } u_{E} = \Phi \text{ or } E \\ \frac{1}{2} & \text{if } u_{E} = f_{E} \\ \frac{1}{2} & \text{if } u_{E} = g_{E} \\ 0 & \text{otherwise,} \end{cases} \\ \eta_{k}((w_{K}) = \begin{cases} 1 & \text{if } (w_{K} = \Phi \text{ or } \tilde{K} \\ \frac{1}{2} & \text{if } ; w_{K} = h_{K} \\ 0 & \text{otherwise.} \end{cases}$$

Then fuzzy topologies  $(X,\tau_{\scriptscriptstyle e_1}),~(X,\tau_{\scriptscriptstyle e_2}),~(Y,\eta_{\scriptscriptstyle k_1}),~(Y,\eta_{\scriptscriptstyle k_2})$  are defined as:

$$\begin{split} \tau_{e_1}(\lambda) \ &= \ \begin{cases} 1 & \text{if } \lambda = \underline{0}, \underline{1} \\ \frac{1}{2} & \text{if } \lambda = f_E(e_1) = \{0.4, 0.3\} \\ \frac{1}{2} & \text{if } \lambda = g_E(e_1) = \{0.3, 0.2\} \\ 0 & \text{otherwise}, \end{cases} \\ \tau_{e_2}(\lambda) \ &= \ \begin{cases} 1 & \text{if } \lambda = \underline{0}, \underline{1} \\ \frac{1}{2} & \text{if } \lambda = f_E(e_2) = \{0.5, 0.4\} \\ \frac{1}{2} & \text{if } \lambda = g_E(e_2) = \{0.4, 0.3\} \\ 0 & \text{otherwise}, \end{cases} \\ \eta_{k_1}(\mu) \ &= \ \begin{cases} 1 & \text{if } \mu = \underline{0}, \underline{1} \\ \frac{1}{2} & \text{if } \mu = h_K(k_1) = \{0.3, 0.4\} \\ 0 & \text{otherwise}, \end{cases} \\ \eta_{k_2}(\mu) \ &= \ \begin{cases} 1 & \text{if } \mu = \underline{0}, \underline{1} \\ \frac{1}{2} & \text{if } \mu = h_K(k_2) = \{0.4, 0.3\} \\ 0 & \text{otherwise}, \end{cases} \\ \eta_{k_2}(\mu) \ &= \ \begin{cases} 1 & \text{if } \mu = \underline{0}, \underline{1} \\ \frac{1}{2} & \text{if } \mu = h_K(k_2) = \{0.4, 0.3\} \\ 0 & \text{otherwise}, \end{cases} \end{split}$$

Thus  $dom(F) = \tilde{E}$  and F is fuzzy upper semi-continuous, because

$$(F^u(h_{\scriptscriptstyle K}))(e_1) = \{0.4, 0.3\} = f_{\scriptscriptstyle E}(e_1)$$

and

$$\tau_{e_1}((F^u(h_{\kappa}))(e_1)) = \frac{1}{2} \ge \eta_{\psi(e_1)}(h_{\kappa}(\psi(e_1))) = \frac{1}{2}.$$

Also

$$(F^u(h_{\scriptscriptstyle K}))(e_2) = \{0.4, 0.3\} = g_{\scriptscriptstyle E}(e_2)$$

and

$$\tau_{e_2}((F^u(h_{\kappa}))(e_2)) = \frac{1}{2} \ge \eta_{\psi(e_2)}(h_{\kappa}(\psi(e_2))) = \frac{1}{2}.$$

But F is not FS upper semi-continuous, because

$$F^u(h_{\scriptscriptstyle K}) = \{(e_1, \{0.4, 0.3\}), (e_2, \{0.4, 0.3\})\}$$

and

$$\tau_{\scriptscriptstyle e}(F^u(h_{\scriptscriptstyle K}))=0 \not\geq \eta_{_{\psi(e)}}(h_{\scriptscriptstyle K})=\frac{1}{2}, \ \forall \ e\in E.$$

**Theorem 4.6.** Let  $\varphi : X \to Y$ ,  $\psi : E \to K$ ,  $\vartheta : Y \to Z$  and  $\varpi : K \to M$ . Let  $F : X \multimap Y$ ,  $H : Y \multimap Z$  be two FS multifunctions and let  $(X, \tau_E)$ ,  $(Y, \eta_K)$  and  $(Z, \sigma_M)$  be three FS topological spaces. Then we have the following:

- (1) If F and H are normalized FS upper semi-continuous, then  $H \circ F$  is FS upper semi-continuous,
- (2) If F and H are FS lower semi-continuous, then  $H \circ F$  is FS lower semicontinuous.

*Proof.* (1) Let F, H be normalized FS upper semi-continuous and  $w_D \in (Z, M)$ ,  $e \in E$ . Then from Theorem 3.10, we have

$$\tau_e((H \circ F)^u(w_{\scriptscriptstyle D})) = \tau_e((F^u(H^u(w_{\scriptscriptstyle D})))) \ge \eta_{\psi(e)}(H^u(g_{\scriptscriptstyle B})) \ge \sigma_{_{\varpi(\psi(e))}}(w_{\scriptscriptstyle D}).$$

Thus  $H \circ F$  is FS upper semi-continuous.

(2) The proof is similar to (1).

**Definition 4.7.** Let  $\psi : E \to K$  and  $F : (X, \tau_E, \mathcal{I}) \to (Y, \eta_K)$  be a FS multifunction, where  $\mathcal{I}$  is a FS ideal on X. For  $\mathcal{A}, \mathcal{B}$  are FS operators on  $(X, \tau_E)$  and  $\mathcal{C}, \mathcal{D}$  are FS operators on  $(Y, \eta_K)$ , respectively. Then,  $\forall g_B \in (Y, K), r \in I_0$  and  $e \in E$ ,

(i) F is called FS lower  $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{I})$ -continuous, if

 $\mathcal{I}_e[\mathcal{A}(e, F^l(\mathcal{D}(\psi(e), g_{\scriptscriptstyle B}, r)), r) \bar{\wedge} \mathcal{B}(e, F^l(\mathcal{C}(\psi(e), g_{\scriptscriptstyle B}, r)), r)] \geq \eta_{\psi(e)}(g_{\scriptscriptstyle B}).$ 

(ii) suppose F is a normalized, then F is called FS upper  $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{I})$ -continuous, if

$$\mathcal{I}_e[\mathcal{A}(e,F^u(\mathcal{D}(\psi(e),g_{_B},r)),r) \bar{\wedge} \, \mathcal{B}(e,F^u(\mathcal{C}(\psi(e),g_{_B},r)),r)] \geq \eta_{\psi(e)}(g_{_B}).$$

We can see that, the above definition is generalized of the concept of FS lower (resp. upper) semi-continuous multifunction (Theorem 4.3), when we choose,  $\mathcal{A} =$  identity operator,  $\mathcal{B} =$  interior operator,  $\mathcal{D} =$  identity operator,  $\mathcal{C} =$  identity operator and  $\mathcal{I} = \mathcal{A}^0$ .

From the above definition, we can present different cases of the FS continuity multifunction as follow:

(i) F is FS lower almost continuous (resp. normalized FS upper almost continuous) multifunction: for every  $g_B \in \widetilde{(Y,K)}$ ,  $r \in I_0$  and  $e \in E$  with  $\eta_{\psi(e)}(g_B) \geq r$ , then

 $F^{l}(g_{B}) \sqsubseteq \operatorname{int}_{\tau}(e, F^{l}(\operatorname{int}_{n}(\psi(e), \operatorname{cl}_{n}(\psi(e), g_{B}, r), r)), r)$ 

(resp.  $F^u(g_B) \sqsubseteq \operatorname{int}_{\tau}(e, F^u(\operatorname{int}_n(\psi(e), \operatorname{cl}_n(\psi(e), g_B, r), r)), r))$ 

Here,  $\mathcal{A}$  =identity operator,  $\mathcal{B}$  = interior operator,  $\mathcal{C}$  =interior closure operator,  $\mathcal{D}$  =identity operator and  $\mathcal{I} = \mathcal{I}^0$ .

(ii) F is FS lower weakly continuous (resp. normalized FS upper weakly continuous) multifunction: for every  $g_B \in \widetilde{(Y,K)}$ ,  $r \in I_0$  and  $e \in E$  with  $\eta_{\psi(e)}(g_B) \geq r$ , then

$$F^{l}(g_{B}) \sqsubseteq \operatorname{int}_{\tau}(e, F^{l}(\operatorname{cl}_{\eta}(\psi(e), g_{B}, r)), r)$$

 $(\text{ resp. } F^u(g_{\scriptscriptstyle B}) \sqsubseteq \operatorname{int}_{_\tau}(e, F^u(\operatorname{cl}_{_\eta}(\psi(e), g_{\scriptscriptstyle B}, r)), r))$ 

Here,  $\mathcal{A}$ =identity operator,  $\mathcal{B}$ =interior operator,  $\mathcal{C}$ = closure operator,  $\mathcal{D}$ =identity operator and  $\mathcal{I} = \mathcal{I}^0$ .

(iii) F is FS lower almost weakly continuous (resp. normalized FS upper almost weakly continuous) multifunction: for every  $g_B \in \widetilde{(Y,K)}$ ,  $r \in I_0$  and  $e \in E$  with  $\eta_{\psi(e)}(g_B) \geq r$ , then

$$F^{l}(g_{B}) \sqsubseteq \operatorname{int}_{\tau}(e, \operatorname{cl}_{\tau}(e, F^{l}(\operatorname{cl}_{\eta}(\psi(e), g_{B}, r)), r), r))$$

(resp.  $F^u(g_B) \sqsubseteq \operatorname{int}_{\tau}(e, \operatorname{cl}_{\tau}(e, F^u(\operatorname{cl}_n(\psi(e), g_B, r)), r), r))$ 

Here,  $\mathcal{A}$  =identity operator,  $\mathcal{B}$  = interior closure operator,  $\mathcal{C}$  = closure operator,  $\mathcal{D}$  =identity operator and  $\mathcal{I} = \mathcal{I}^0$ .

(iv) F is FS lower precontinuous (resp. normalized FS upper precontinuous) multifunction: for every  $g_B \in (\widetilde{Y}, \widetilde{K}), r \in I_0$  and  $e \in E$  with  $\eta_{\psi(e)}(g_B) \ge r$ , then  $F^{l}(e_B) = \overline{F} + (e_B + e_B + e_B)$ 

$$F^{\iota}(g_{\scriptscriptstyle B}) \sqsubseteq \operatorname{int}_{\tau}(e, \operatorname{cl}_{\tau}(e, F^{\iota}(g_{\scriptscriptstyle B}), r), r)$$

 $( \ \text{resp.} \quad F^u(g_{\scriptscriptstyle B}) \sqsubseteq \operatorname{int}_{_\tau}(e,\operatorname{cl}_{_\tau}(e,F^u(g_{\scriptscriptstyle B}),r),r) \ )$ 

Here,  $\mathcal{A}$  =identity operator,  $\mathcal{B}$  = interior closure operator,  $\mathcal{C}$  = identity operator,  $\mathcal{D}$  = identity operator and  $\mathcal{I} = \mathcal{I}^0$ .

(v) F is FS lower strongly precontinuous (resp. normalized FS upper strongly precontinuous) multifunction: for every  $g_B \in \widetilde{(Y,K)}$ ,  $r \in I_0$  and  $e \in E$  with  $\eta_{\psi(e)}(g_B) \geq r$ , then

$$F^{l}(g_{B}) \sqsubseteq \operatorname{int}_{\tau}(e, \operatorname{Pcl}_{\tau}(e, F^{l}(g_{B}), r), r)$$

(resp.  $F^u(g_B) \sqsubseteq \operatorname{int}_{\tau}(e, \operatorname{Pcl}_{\tau}(e, F^u(g_B), r), r))$ 

Here,  $\mathcal{A}$  =identity operator,  $\mathcal{B}$  = interior preclosure operator,  $\mathcal{C}$  = identity operator,  $\mathcal{D}$  = identity operator and  $\mathcal{I} = \mathcal{I}^0$ .

(vi) F is FS lower  $\alpha$ -continuous (resp. normalized FS upper  $\alpha$ -continuous) multifunction: for every  $g_B \in \widetilde{(Y,K)}$ ,  $r \in I_0$  and  $e \in E$  with  $\eta_{\psi(e)}(g_B) \geq r$ , then;

 $F^{l}(g_{B}) \sqsubseteq \operatorname{int}_{\tau}(e, \operatorname{cl}_{\tau}(e, \operatorname{int}_{\tau}(e, F^{l}(g_{B}), r), r), r))$ 

(resp.  $F^u(g_B) \sqsubseteq \operatorname{int}_{\tau}(e, \operatorname{cl}_{\tau}(e, \operatorname{int}_{\tau}(e, F^l(g_B), r), r), r))$ 

Here,  $\mathcal{A}$  =identity operator,  $\mathcal{B}$  = interior closure interior operator,  $\mathcal{C}$  = identity operator,  $\mathcal{D}$  = identity operator and  $\mathcal{I} = \mathcal{I}^0$ .

(vii) F is FS lower almost  $\alpha$ -continuous (resp. normalized FS upper almost  $\alpha$ -continuous) multifunction:  $\forall g_B \in (Y, K), r \in I_0 \text{ and } e \in E \text{ with } \eta_{\psi(e)}(g_B) \geq r$ , then

$$\begin{split} F^{l}(g_{\scriptscriptstyle B}) &\sqsubseteq \alpha \operatorname{int}_{\tau}(e, F^{l}(\operatorname{Scl}_{\eta}(\psi(e), g_{\scriptscriptstyle B}, r)), r) \\ (\text{ resp. } F^{u}(g_{\scriptscriptstyle B}) &\sqsubseteq \alpha \operatorname{int}_{\tau}(e, F^{u}(\operatorname{Scl}_{\eta}(\psi(e), g_{\scriptscriptstyle B}, r)), r)) \\ 12 \end{split}$$

Here,  $\mathcal{A}$  =identity operator,  $\mathcal{B} = \alpha$  interior operator,  $\mathcal{C}$  =semi-closure operator,  $\mathcal{D}$  = identity operator and  $\mathcal{I} = \mathcal{I}^0$ .

(viii) F is FS lower weakly  $\alpha$ -continuous (resp. normalized FS upper weakly  $\alpha$ -continuous) multifunction:  $\forall g_B \in (Y, K), r \in I_0 \text{ and } e \in E \text{ with } \eta_{\psi(e)}(g_B) \geq r$ , then

 $F^{l}(g_{\scriptscriptstyle B}) \sqsubseteq \alpha \operatorname{int}_{\tau}(e, F^{l}(\operatorname{cl}_{\eta}(\psi(e), g_{\scriptscriptstyle B}, r)), r)$ 

(resp.  $F^u(g_B) \sqsubseteq \alpha \operatorname{int}_\tau(e, F^u(\operatorname{cl}_n(\psi(e), g_B, r)), r))$ 

Here,  $\mathcal{A}$  =identity operator,  $\mathcal{B} = \alpha$  interior operator,  $\mathcal{C}$  =closure operator,  $\mathcal{D}$  = identity operator and  $\mathcal{I} = \mathcal{I}^0$ .

(ix) F is FS lower  $\beta$ -continuous (resp. normalized FS upper  $\beta$ -continuous) multifunction: for every  $g_B \in (Y, K)$ ,  $r \in I_0$  and  $e \in E$  with  $\eta_{\psi(e)}(g_B) \geq r$ , then;

$$F^{\iota}(g_{\scriptscriptstyle B}) \sqsubseteq \operatorname{cl}_{\tau}(e, \operatorname{int}_{\tau}(e, \operatorname{cl}_{\tau}(e, F^{\iota}(g_{\scriptscriptstyle B}), r), r), r))$$

 $( \text{ resp. } F^u(g_{\scriptscriptstyle B}) \sqsubseteq \operatorname{cl}_{\scriptscriptstyle \tau}(e, \operatorname{int}_{\scriptscriptstyle \tau}(e, \operatorname{cl}_{\scriptscriptstyle \tau}(e, F^u(g_{\scriptscriptstyle B}), r), r), r))$ 

Here,  $\mathcal{A}$  =identity operator,  $\mathcal{B}$  = closure interior closure operator,  $\mathcal{C}$  = identity operator,  $\mathcal{D}$  = identity operator and  $\mathcal{I} = \mathcal{I}^0$ .

(x) F is FS lower almost  $\beta$ -continuous (resp. normalized FS upper almost  $\beta$ -continuous) multifunction:  $\forall g_B \in (Y, K), r \in I_0 \text{ and } e \in E \text{ with } \eta_{\psi(e)}(g_B) \geq r$ , then

$$F^{l}(g_{B}) \sqsubseteq \beta \operatorname{int}_{\tau}(e, F^{l}(\operatorname{Scl}_{\eta}(\psi(e), g_{B}, r)), r)$$

(resp.  $F^{u}(g_{B}) \sqsubseteq \beta \operatorname{int}_{\tau}(e, F^{u}(\operatorname{Scl}_{\eta}(\psi(e), g_{B}, r)), r))$ 

Here,  $\mathcal{A} = \text{identity operator}$ ,  $\mathcal{B} = \beta$  interior operator,  $\mathcal{C} = \text{semi-closure oper$  $ator}$ ,  $\mathcal{D} = \text{identity operator}$  and  $\mathcal{I} = \mathcal{I}^0$ .

(xi) F is FS lower weakly  $\beta$ -continuous (resp. normalized FS upper weakly  $\beta$ -continuous) multifunction:  $\forall g_B \in (Y, K), r \in I_0 \text{ and } e \in E \text{ with } \eta_{\psi(e)}(g_B) \geq r$ , then

$$F^{l}(g_{B}) \sqsubseteq \beta \operatorname{int}_{\tau}(e, F^{l}(\operatorname{cl}_{n}(\psi(e), g_{B}, r)), r)$$

resp. 
$$F^{u}(g_{B}) \subseteq \beta \operatorname{int}_{z}(e, F^{u}(\operatorname{cl}_{n}(\psi(e), g_{B}, r)), r))$$

Here,  $\mathcal{A}$  =identity operator,  $\mathcal{B} = \beta$  interior operator,  $\mathcal{C}$  =closure operator,  $\mathcal{D}$  = identity operator and  $\mathcal{I} = \mathcal{I}^0$ .

Where,  $\operatorname{int}_{\tau}$ ,  $\operatorname{cl}_{\tau}$ ,  $\operatorname{Scl}_{n}$ ,  $\operatorname{Pcl}_{n}$ ,  $\alpha \operatorname{int}_{\tau}$ ,  $\beta \operatorname{int}_{\tau}$  and  $\beta \operatorname{cl}_{\tau}$  are found in ([1, 16]).

#### 5. Conclusions

In the present work, we have continued to study the properties of fuzzy soft topological spaces. We introduce the notions of fuzzy soft multifunction, upper and lower inverse of a fuzzy soft multifunction and study their various properties. Next, we use these ideas to introduce upper and lower fuzzy soft semi-continuous multifunctions which are generalization of the concepts introduced in Abbas et al. [4, 5]. We hope that the findings in this paper will help researcher enhance and promote the further study on fuzzy soft topology to carry out a general framework for their applications in practical life.

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#### References

- S. E. Abbas, E. El-sanowsy and A. Atef, On fuzzy soft irresolute functions, J. Fuzzy Math. 24 (2) (2016) 465–482.
- [2] S. E. Abbas, E. El-sanowsy and A. Atef, fuzzy soft  $(\alpha, \beta, \theta, \delta, \mathcal{I})$ -continuous functions, Journal of the Egyptian Mathematical Society 25 (2017) 59–64.
- [3] S. E. Abbas, E. El-sanowsy and A. Atef, Stratified modeling in soft fuzzy topological structures, Soft Computing 22 (2018) 1603–1613.
- [4] S. E. Abbas, M. A. Hebeshi and I. M. Taha, On fuzzy upper and lower semi-continuous multifunctions, J. Fuzzy Math. 28 (6) (2014) 951–962.
- [5] S. E. Abbas, M. A. Hebeshi and I. M. Taha, On upper and lower almost weakly continuous fuzzy multifunctions, Iranian Journal of Fuzzy Systems 12 (1) (2015) 101–115.
- [6] S. E. Abbas and I. Ibedou, fuzzy soft filter convergence, Filomat (To appear).
- [7] B. Ahmad and A. Kharal, On fuzzy soft sets, Advances in Fuzzy Systems, Article ID 586507 (2009) 6 pages.
- [8] A. Aygnoglu, V. Šostak and H. Aygun, An Introduction to fuzzy soft topological spaces, Hacettepe Journal of Mathematics and Statistics 43 (2) (2014) 193–204.
- [9] V. Cetkin, A. p. Sostak and H. Aygun, An Approach to the Concept of Soft Fuzzy Proximity, Hindawi Publishing Corparation, Article ID 782583 (2009) 6 pages.
- [10] T. S. Dizmana, V. Šostak and S. Yukse, Soft Ditopological Spaces, Filomat 30 (1) (2016) 209–222.
- [11] C. Gunduz and S. Bayramov, Some Results on Fuzzy Soft Topological Spaces, Mathematical Problems in Engineering, Article ID 835308 (2013) 10 pages.
- [12] A. Kharal and B. Ahmad, Mappings on fuzzy soft classes, Advances in Fuzzy Systems Article ID 407890 (2009) 6 pages.
- [13] P. K. Maji, R. Biswas and A. R. Roy, Fuzzy soft sets, J. Fuzzy Math. 9 (3) (2001) 589–602.
- [14] P. K. Maji, A. R. Roy and R. Biswas, An application of soft sets in a decision making problem, Computers and Mathematics with Applications 44 (8-9) (2002) 1077–1083.
- [15] A. Metin, k. Serkan and E. Fethullah, Upper and Lower Continuity of Fuzzy Soft Multifunctions, Journal of New Theory 1 (2015) 59–68.
- [16] A. Metin and O. Alkan, On Soft β-Open Sets and Soft β-Continuous Functions, Scientific World Journal doi: 10.1155/2014/843456 PMCID: PMC4087269 (2014).
- [17] D. Molodtsov, Soft set theory-first results, Journal of Comput. Math. Appl. 37 (1999) 19–31.
- [18] D. Molodtsov, Describing dependences using soft sets, Journal of Computer and Systems Sciences International 40 (6) (2001) 975–982.
- [19] D. Qiu and W. Zhang, Symmetric fuzzy numbers and additive equivalence of fuzzy numbers, Soft Computing 17 (2013) 1471–1477.
- [20] D. Qiu, C. Lu, W. Zhang and Y. Lan, Algebraic properties and topological properties of the quotient space of fuzzy numbers based on Mares equivalence relation, Fuzzy Sets and Systems 245 (2014) 63–82.
- [21] D. Qiu, W. Zhang and C. Lu, On fuzzy differential equations in the quotient space of fuzzy numbers, Fuzzy Sets and Systems 295 (2016) 72–98.
- [22] G. Senel, The Theory of Soft Ditopological Spaces, International Journal of Computer Applications 150 (4) (2016) 1–5.
- [23] G. Senel, A New Approach to Hausdorff Space Theory via the Soft Sets, Mathematical Problems in Engineering 9 (2016) 1–6.
- [24] G. Senel, A Comparative Research on the Definition of Soft Point, International Journal of Computer Applications 163 (2) (2017) 1–4.
- [25] G. Senel and N. Cagman, Soft Closed Sets on Soft Bitopological Space, Journal of New Results in Science 3 (5) (2014) 57–66.
- [26] A. Serkan and Z. Idris, On fuzzy soft topological spaces, Ann. Fuzzy Math. Inform. 5 (2) (2013) 377–386.

[27] B. Tanay and M. B. Kandemir, Topological structure of fuzzy soft sets, Computer and Mathematics with applications 61 (2011) 2952–2957.

[28] L. A. Zadeh, Fuzzy sets, Inform and Control 8 (1965) 338-353.

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